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WARING'S PROBLEM FOR POLYNOMIAL BIQUADRATES OVER A FINITE FIELD OF ODD CHARACTERISTIC

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To Jean-Marc Deshouillers,
friendly, for his sixty years

Abstract: Let q be a power of an odd prime p and let k be a finite field with q elements. Our main result is: If $q \notin \{3, 9, 5, 13, 17, 25, 29\}$, every polynomial $P \in k[t]$ of degree $\geqslant 269$ is a strict sum of 11 biquadrates. We first decompose P as a strict mixed sum of biquadrates.
Keywords: Waring's problem, biquadrates, polynomials, finite fields, odd characteristic.

1. Introduction

Waring's problem for biquadrates of polynomials in $\mathbb{F}[t]$ over some field $\mathbb{F}$, is the analogue of the same problem over the positive integers $\mathbb{N}$. We can represent an integer $n > 0$ as

$$n = n_1^4 + \ldots + n_g^4,$$

for some positive integer g such that, necessarily, the integers n_i satisfy $n_i^4 \leqslant n$ for all $i = 1, \ldots, g$. In particular no cancellation of any terms occurs in the above sum.

Waring's problem for biquadrates over $\mathbb{N}$ consists of determining or at least bounding the minimal such g, say $g(4, \mathbb{N})$. Wieferich [16], proved that $g(4, \mathbb{N}) \leqslant 37$ while, later Balasubramanian et al., [1] obtained $g(4, \mathbb{N}) = 19$.

Two related problems arise.

a) The (so called) "easy" Waring problem in which we allow sums and differences, instead of merely sums of biquadrates to appear in the decomposition above; so that we can represent now all integers. Let $v(4, \mathbb{Z})$ represent the analogue of $g(4, \mathbb{N})$. The exact value of $v(4, \mathbb{Z})$ is unknown, however:

$$9 \leqslant v(4, \mathbb{Z}) \leqslant 10.$$

(See, e.g. [8, Theorem 5.6, p. 505] and [10]).

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b) The “asymptotic” Waring problem in which we restrict our attention to represent only “sufficiently large” integers, i.e. we try to represent all integers bigger than some bound b. Let $G(4,\mathbb{Z})$ denote the analogue of $g(4,\mathbb{Z})$ for these large integers. The inequality $G(4,\mathbb{Z}) \geqslant 16$ is due to Kempner [7], and the bound $G(4,\mathbb{Z}) \leqslant 16$ has been established by Davenport [3]. Moreover, in [4] Deshouillers et al. proved that the largest integer that requires seventeen biquadrates for its representation is $\leqslant 13792$.

Let $\mathbb{F}$ be a commutative field and let $P \in \mathbb{F}[t]$ be a polynomial such that

$$P = P_1^4 + \ldots + P_s^4$$

for some polynomials $P_1, \ldots, P_s \in \mathbb{F}[t]$. Cancellation may occur and it is possible to have $\big(\deg(P_i{}^4) - \deg P\big)$ large. We want to write polynomials as sums of biquadrates with the least possible number of cancellations. Thus, we ask that the polynomials $P_1, \ldots, P_s$ appearing in the above sum satisfy the most restrictive degree conditions, i.e, $4 \deg P_i < 4 + \deg P$. When this condition is satisfied, we say that P is a **strict** sum of s biquadrates.

Similarly, we say that P is a **mixed strict** sum of m biquadrates if

$$P = s_1 P_1^4 + \ldots + s_m P_m^4$$

for some polynomials $P_1, \ldots, P_m \in \mathbb{F}[t]$ such that $\deg(P_i^4) < \deg(P) + 4$ for all $i = 1, \ldots, m$ and $s_j \in \{1, -1\}$ for all j while, at least for some i, one has $s_i = -1$ and in that case not all the s_i's are equal to -1.

In other words, we may say that in the “mixed strict” problem as before, we want conditions on degrees, but we relax on the condition that all coefficients must be equal to 1. So, this problem has no an exact analogue for the integers indeed it is “in between” the “easy” and the“classical” Waring's problem for biquadrates.

For any field $\mathbb{F}$ we denote by

$$G(4, \mathbb{F}[t]),$$

the least positive integer G, if it exists such that every polynomial $P \in \mathbb{F}[t]$ of degree large enough, is a strict sum of G biquadrates. If it does not exist, then we put $G(4, \mathbb{F}[t]) = \infty$. Similarly, we denote by

$$V(4, \mathbb{F}[t]),$$

the least positive integer g, if it exists, such that every polynomial $P \in \mathbb{F}[t]$ of degree large enough, is a mixed strict sum of g biquadrates. If it does not exist, then we put $V(4, \mathbb{F}[t]) = \infty$.

For $q \in \{3, 9\}$ some congruence obstructions appear (see [6, section 1.1]), and $G(4, \mathbb{F}[t]) = \infty$. In this paper we restrict our attention to $q \notin \{3, 9\}$.

Roughly speaking, the object of the paper is to prove (see Theorem 5.1 and section Identities) that:

a) For odd $q \notin \{3, 9, 5, 13, 17, 25, 29\}$,

$$V(4, \mathbb{F}_q[t]) \leqslant 10, \quad G(4, \mathbb{F}_q[t]) \leqslant 11,$$

b) For odd $q \notin \{17, 29\}$ and congruent to 1 modulo 8,

$$G(4, \mathbb{F}_q[t]) \leqslant 10.$$

In fact, we prove effective results, that is to say:

a) For odd $q \notin \{3, 9, 5, 13, 17, 25, 29\}$, every polynomial $P \in \mathbb{F}_q[t]$ of degree $\deg P \geqslant 269$ is a mixed strict sum of 10 biquadrates.

b) For odd $q \notin \{3, 9, 5, 13, 17, 25, 29\}$ every polynomial $P \in \mathbb{F}_q[t]$ of degree $\deg P \geqslant 269$ is a strict sum of 11 biquadrates.

In order to express effective results, we introduce new notations. For any field $\mathbb{F}$ and any non negative integer $d \geqslant 0$ we denote by

$$g(4, d, \mathbb{F}[t])$$

the minimal positive integer g such that 0 and every P of degree $\geqslant d$ is a strict sum of g biquadrates. Similarly, we denote by $v(4, d, \mathbb{F}[t])$ the minimal positive integer v such that 0 and every P of degree $\geqslant d$ is a mixed strict sum of v biquadrates.

In the special cases $q \in \{5, 13, 17, 25, 29\}$ more biquadrates are necessary to represent a polynomial. Our method improves Gallardo's (see [9]) that obtained for most $q's$ $v(4, \mathbb{F}_q[t]) \leqslant 11$ so that $g(4, \mathbb{F}_q[t]) \leqslant 16$.

Results without degree conditions were obtained by Vaserstein as a special case in [14, 15], where it is proven that the minimal length $w(4, \mathbb{F}_q[t]) = w$ necessary to represent every sum of biquadrates in $\mathbb{F}_q[t]$, as a sum of w biquadrates in $\mathbb{F}_q[t]$, satisfy

$$3 \leqslant w(4, \mathbb{F}_q[t]) \leqslant 7, \quad \text{for} \quad q \neq 5 \quad \text{and} \quad 3 \leqslant w(4, \mathbb{F}_q[t]) \leqslant 9, \quad \text{for} \quad q = 5.$$

In this paper, by using our new identity, we improve (see Corollary 3.4) on this result as follows:

$$3 \leqslant w(4, \mathbb{F}_q[t]) \leqslant 4, \quad \text{for odd} \quad q \notin \{3, 5, 9, 13, 29\}.$$

A word on some classical notation used in the paper: Given some field $\mathbb{F}$, we say that a polynomial $P \in \mathbb{F}[t]$ is *monic* if its leading coefficient equals 1. Moreover, let denote by i a 2-root of -1 in a fixed algebraic closure of $\mathbb{F}_q$. We also put $-\infty$ for the degree of the 0 polynomial so that $\deg(0) < n$ for all positive integers n.

2. Method of proof

We choose, again, a wholly elementary method (linear algebra and identities) already used in [2], [9] to get our results. The reason is that it works! Indeed, mathematically more interesting and powerful method as the circle method (see [6]) seems to produce only weaker results on the Waring's problem for biquadrates over $\mathbb{F}_q[t]$.

Given a polynomial $P \in \mathbb{F}_q[t]$, say monic and of degree $4n > 8$,

$$P = t^{4n} + ... + a_0,$$

to be decomposed, say as a strict sum of biquadrates.

The method consists, roughly, of:

a) Find a biquadrate A^4 such that P and A^4 have a maximum of equal consecutive coefficients beginning by the leading coefficient.

b) Repeat a) with P replaced by $P - A^4$ till get a polynomial R which degree be less than $n+1$. Care is taken so that this can be done.

c) Apply some polynomial identities (more precisely either Norrie's identity or our new identity) to R in order to show R equal to a sum of biquadrates of polynomials S^4 in which the polynomials S, R have the same degree.

Parts a) and b) are covered in section "Descent" and part c) in section "Identities".

Roughly, the improvements on the upper bound are obtained by applying exactly the above procedure.

We leave for further study:

a) The representation of some polynomials of small degree (more precisely the polynomials with degree $\leqslant 28$) for the q's considered here;

b) The special values of q not considered here (i.e., $q \in \{3, 9, 5, 13, 29\}$)

c) The question of the lower bounds.

3. Identities

First of all, we recall the classical Norrie's identity ([12], [5, p. 279], [9, Lemma 1]):

Lemma 3.1. *Let F be a field of odd characteristic. Let $b, c \in F$ be such that $bc(b^8 - c^8) \neq 0$ and let $d = c^8 - b^8$.*

Then we have Norrie's identity:

$$t = \left(\frac{c^2(d+2t)}{2d}\right)^4 - \left(\frac{c^2(d-2t)}{2d}\right)^4 + \left(\frac{2c^4t - b^4d}{2bcd}\right)^4 - \left(\frac{2c^4t + b^4d}{2bcd}\right)^4. \quad (1)$$

Our new identity is based on:

Theorem 3.2. *Let F be a finite field with odd characteristic and order $q \notin \{3,5,9,13,29\}$ such that -1 is not a biquadrate in F. Then, there exist $a,b,c,d \in F$ such that*

1) $abcd \neq 0$
2) $b^4 = -1 - a^4$
3) $d^4 = 1 - c^4$
4) $g = (ad)^2(a^4 + c^4 + 1) + (bc)^2(a^4 + c^4 - 1) \neq 0.$

The proof of this theorem is delayed to a special subsection below. We begin by proving some important corollaries.

Corollary 3.3. *Let F be a finite field of order $q \notin \{3,5,9,13,29\}$ and odd characteristic. Then there exist four polynomials $L_1, L_2, L_3, L_4 \in F[t]$ of degree at most 1 such that*

$$t = L_1^4 + L_2^4 + L_3^4 + L_4^4. \tag{2}$$

Proof. If -1 is a biquadrate in F then apply Lemma 3.1. Suppose that -1 is not a biquadrate in F. According to the previous Theorem 3.2, there exist $a,b,c,d \in F$ satisfying identities 1), 2), 3) and 4). Set $A = at$, $B = bt + cd((bd)^2 - (ac)^2)$, $C = ct + bd((ab)^2 + (cd)^2)$, $D = dt + bc(a^4 - c^4 + 1)$, and $\alpha = 4a^4bcd(a^4 - c^4 + 1)g$. Then $\alpha \in F$ is nonzero and there exists $\lambda \in F$ depending polynomially on the parameters $a,b,c,d \in F$ above, such that:

$$\alpha t + \lambda = A^4 + B^4 + C^4 + D^4.$$

In this expression the parameters b, d occur with exponents at most equal to 3. It is essential to have $\alpha \neq 0$. This is true since from equality 3) one obtains $a^4 - c^4 + 1 = d^4 + a^4$. ■

The following corollary improves on some results of Vaserstein:

Corollary 3.4.

$$3 \leqslant w(4, \mathbb{F}_q[t]) \leqslant 4,$$

for odd q with $q \notin \{3,5,9,13,29\}$.

Proof. The upper bound follows from Corollary 3.3 while the lower bound follows from the fact that the polynomial t is not a sum of 2 biquadrates. ■

3.1. Proof of the Theorem 3.2. Firstly, we assume that -1 is not a square in $\mathbb{F}_q$.

The proof of the following proposition is easy and we leave it to the reader:

Proposition 3.5. *Suppose q congruent to 3 modulo 4. Then, the set S_2 of squares of $\mathbb{F}_q$ equals the set S_4 of biquadrates of $\mathbb{F}_q$. Moreover the map $\psi : x \mapsto x^2$ from S_2 to S_4 is bijective.*

Consider the system:

$$\alpha^2+\beta^2=-1, \tag{3}$$
$$\gamma^2+\delta^2=1, \tag{4}$$
$$\gamma\delta\neq 0, \tag{5}$$
$$\alpha\delta(\alpha^2+\gamma^2+1)+\beta\gamma(\alpha^2+\gamma^2-1)\neq 0. \tag{6}$$

in which $\alpha,\beta,\gamma,\delta\in\mathbb{F}_q$

Corollary 3.6. *If $q\equiv 3 \pmod 4$ then there is a one to one correspondance between the solutions $(a,b,c,d)\in\mathbb{F}_q^4$ of the system $I:\ a^4+b^4=-1,\quad c^4+d^4=1,$ $abcd\neq 0$ and $(ad)^2(a^4+c^4+1)+(bc)^2(a^4+c^4-1)\neq 0,$ and the solutions $(\alpha,\beta,\gamma,\delta)\in\mathbb{F}_q^4$ of the system $II:$ (3), (4), $\alpha\beta\gamma\delta\neq 0$ and (6).*

Proposition 3.7. *Suppose q congruent to 3 modulo 4. Then, among the $(\alpha,\beta,\gamma,\delta)$ $\in\mathbb{F}_q^4$ that satisfy (3), (4), (5), at least one satisfies also (6).*

Proof. Let $(\alpha,\beta,\gamma,\delta)\in\mathbb{F}_q^4$ satisfy (3), (4) and (5). Then $\alpha\beta\neq 0$ since -1 is not a square in $\mathbb{F}_q$. We note that $(\alpha,-\beta,\gamma,\delta)$ also satisfy (3), (4) and (5). Assume that both of them do not satisfy (6). Then,

$$\alpha\delta(\alpha^2+\gamma^2+1)=0=\beta\gamma(\alpha^2+\gamma^2-1)$$

and

$$\alpha^2+\gamma^2+1=0=\alpha^2+\gamma^2-1$$

so that we get a contradiction. ■

Consider the system:

$$a^4+b^4=-1, \tag{7}$$
$$c^4+d^4=1, \tag{8}$$
$$abcd\neq 0, \tag{9}$$
$$(ad)^2(a^4+c^4+1)+(bc)^2(a^4+c^4-1)\neq 0. \tag{10}$$

in which we are searching for $a,b,c,d\in\mathbb{F}_q$.

We are ready to give a solution in the case when $q\equiv 3 \pmod 4$.

Proposition 3.8. *If $q>3$ is congruent to 3 modulo 4, then there exist $(a,b,$ $c,d)\in\mathbb{F}_q^4$ such that (7), (8), (9) and (10) hold.*

Proof. Since -1 is not a square in $\mathbb{F}_q$, it is well known (see e.g. [11, Lemma 6.24, p. 282]) that for all nonzero $b\in\mathbb{F}_q$, the equation

$$x^2+y^2=b$$

has $q+1$ solutions $(x,y) \in \mathbb{F}_q^2$. On the other hand, the same equation for $b = 1$ has only 4 solutions with $xy = 0$ so, the system (3), (4) and (5) has $(q+1)(q+1-4)$ solutions. Thus, when $q > 3$ the system (3), (4) and (5) has a solution. This implies by Proposition 3.7 that the system (3), (4) (5) and (6) has a solution. For these solution, (3) implies that $\alpha\beta \neq 0$. Now, the system II of Corollary 3.6 has a solution so that the system I of the same corollary has also a solution. This finishes the proof. ∎

We observe that if $q = 3$, (8) implies $cd = 0$ so there is no solution in the field $\mathbb{F}_3$.

Now, we assume that -1 is a square in $\mathbb{F}_q$ but that it is not a biquadrate in $\mathbb{F}_q$.

We consider the same system (7), (8), (10) as before but now with the condition

$$cd \neq 0; \tag{11}$$

instead of (9). This comes from the observation that -1 is not a biquadrate in $\mathbb{F}_q$ and that (7) implies $ab \neq 0$.

Proposition 3.9. *Suppose that -1 is a square in $\mathbb{F}_q$. Then, among the $(a,b,c,d) \in \mathbb{F}_q^4$ satisfying (7), (8), (11), at least one satisfies also (10).*

Proof. Observe that if $(a,b,c,d) \in \mathbb{F}_q^4$ is a solution of (7), (8), (11), then (a, ib, d, c) is also a solution where $-1 = i^2$. The rest of the proof is analogous to the proof of Proposition 3.7. ∎

For $a \in \mathbb{F}_q$ we denote by $N(a)$ the number of solutions $(x,y) \in \mathbb{F}_q^2$ of the equation

$$a = x^4 + y^4. \tag{12}$$

Proposition 3.10. *Let $a \in \mathbb{F}_q$ be different from 0. Then,*

$$N(a) \geqslant q - 3 - 6\sqrt{q}. \tag{13}$$

Proof. The inequality is a special case of [11, Example 6.38, p. 295]. ∎

Our main result follows:

Proposition 3.11. *If $q \geqslant 37$ is congruent to 1 modulo 4 but it is not congruent to 1 modulo 8, then there exist $a,b,c,d \in \mathbb{F}_q$ such that (7), (8), (9) and (10) hold simultaneously.*

Proof. The equation $x^4 + y^4 = 1$ has 8 solutions $(x,y) \in \mathbb{F}_q^2$ with $xy = 0$. From Proposition 3.10 if $q > 55$ then $N(1) \geqslant 9$, $N(-1) \geqslant 9$. By a simple check we get also the result when $q \in \{37, 53\}$. So, the equation $x^4 + y^4 = 1$ has a solution $(x,y) \in \mathbb{F}_q^2$ with $xy \neq 0$ and the equation $x^4 + y^4 = -1$ has $N(-1) \geqslant 9$ solutions. So, the conditions (7), (8), (9) hold simultaneously. It follows from Proposition 3.9 that condition (10) also holds. ∎

This finishes the proof of Theorem 3.2.

Remark 3.12. In the case when $q \in \{5, 13, 29\}$, the system (7), (8), (9), and (10) has no solution.

This is obtained by direct computations.

3.2. Sums of biquadrates in $\mathbb{F}_q$. The following lemma (see also [9, Lemma 3]) follows from [11, Example 6.38, p. 295] for $q \geqslant 43$ and from a check for the other values of q.

Lemma 3.13. *Let $\mathbb{F}$ be a finite field of order q and odd characteristic. If $q \geqslant 31$ or if $q \in \{7, 11, 19, 23, 27\}$, then every non-zero element of $\mathbb{F}$ is a sum of 2 biquadrates.*

4. Descent

Lemma 4.1. *Let $\mathbb{F}$ be a finite field of odd characteristic in which each element is a sum of 2 biquadrates. Let $n \geqslant 0$ be an integer and let $P \in \mathbb{F}[t]$ be of degree $d \in \{4n, 4n-1, 4n-2, 4n-3\}$. Then,*

(I) *There exist $A, B, C, R \in \mathbb{F}[t]$ such that $AC = 0$ and:*

a) $P = -A^4 + B^4 + C^4 + R$,

b) $\deg(A^4) < d + 4$, $\deg(B^4) < d + 4$ *and* $\deg(C^4) < d + 4$.

c) *R is monic and $\deg(R)$ is the least multiple of 4 such that $\deg R \geqslant 3n$.*

d) *If 4 divides d then $A = 0$.*

e) *If 4 does not divides d then $C = 0$.*

f) *If P is monic and 4 divides d then $A = 0$ and $C = 0$.*

(II) *There exist $A, B, C, R \in \mathbb{F}[t]$ such that $AC = 0$ and such that* a), b), d), e) *and* f) *are satisfied and such that*

g) $\deg R < 3n$.

Proof. If 4 does not divide d, we set $A = t^n, Q = P + A^4$. If 4 divides d, we set $A = 0, Q = P$. In the two cases, Q has degree $4n$. The leading coefficient of Q is a sum, in $\mathbb{F}$, of two biquadrates $b^4 + c^4$, with, say, $b \neq 0$. Set $C = ct^n$. If 4 divides d, Q is monic and we take $b = 1, c = 0$. Set now r equal to the least multiple of 4 such that $r \geqslant 3n$ and let $B = bt^n + b_{n-1}t^{n-1} + \ldots + b_0$, with unknowns $b_{n-1}, \ldots, b_0$ in $\mathbf{F}$ to determine in such a manner that all coefficients of $R = Q - B^4$, from the coefficient of t^{4n-1}, to those of t^{r+1}, be equal to zero and such that the coefficient of t^r in R be equal to 1. This results on a triangular linear system over $\mathbb{F}$ in at most n unknowns $b_{n-1}, \ldots, b_0$ soluble since $b \neq 0$. This proves the first part of the lemma. If we do not ask R to be monic and have degree multiple of 4, we solve the system in exactly n unknowns $b_{n-1}, \ldots, b_0$ and we obtain that $\deg R = \deg(Q - B^4) < 3n$. ∎

5. Representation by biquadrates

Here we prove our main result:

Theorem 5.1. a) *For odd* $q \notin \{3, 9, 5, 13, 17, 25, 29\}$ *every polynomial* $P \in \mathbb{F}_q[t]$ *of degree* $d \geqslant 269$ *is a mixed strict sum of* 10 *biquadrates, i.e:*

$$v(4, 272, \mathbb{F}_q[t]) \leqslant 10.$$

b) *For odd* $q \notin \{3, 9, 5, 13, 17, 25, 29\}$ *every polynomial* $P \in \mathbb{F}_q[t]$ *of degree* $d \geqslant 272$ *is a strict sum of* 11 *biquadrates, i.e:*

$$g(4, 269, \mathbb{F}_q[t]) \leqslant 11.$$

Proof. Let $P \in \mathbb{F}[t]$ of degree $d \in \{4n, \ldots, 4n-3\}$.

First step: Applying Lemma 4.1, we get the existence of A, B, C and $R_1 \in \mathbb{F}[t]$ such that $AC = 0$ and

$$P = -A^4 + B^4 + C^4 + R_1 \tag{1}$$

with $\deg A^4, \deg B^4, \deg C^4 < d + 4$, with R_1 monic of degree the least multiple of 4 $\geqslant 3n$. Set

$$C_0 = C, n_0 = n, 4n_1 = \deg R_1.$$

Hence,

$$4n_1 \geqslant 3n_0 > 4n_1 - 4. \tag{2}$$

Second, third, fourth steps: Applying recursively 3 times Lemma 4.1, part (f), we get the existence of $C_1, C_2, C_3, R_2, R_3, R_4 \in \mathbb{F}[t]$ such that

$$R_i = (C_i)^4 + R_{i+1}, \tag{3}$$

where for $i = 1, \ldots, 3$,

$$\deg C_i = n_i, \deg R_i = 4n_i, \tag{4}$$

$$4n_{i+1} \geqslant 3n_i > 4n_{i+1} - 4. \tag{5}$$

Fifth step: We apply Lemma 4.1, part (g), and we get the existence of $C_5, R_5 \in \mathbb{F}[t]$ such that

$$R_4 = (C_4)^4 + R_5, \tag{6}$$

$$\deg R_5 < 3n_4. \tag{7}$$

By (2) and (5),

$$n_4 \leqslant (\frac{3}{4})^4 n_0 + 3\left(1 - (\frac{3}{4})^4\right).$$

Thus, if n_0 is such that

$$n_0 \geqslant 3\left((\frac{3}{4})^4 n_0 + 3\left(1 - (\frac{3}{4})^4\right)\right) - 1,$$

then $\deg R_5 \leqslant n_0$. A sufficient condition under which this occurs is

$$n_0 \geqslant \frac{1319}{13} \Leftrightarrow n_0 \geqslant 102.$$

By direct computation one finds that if $n_0 \geqslant 68$, then $3n_4 - 1 \leqslant n_0$.

Finally, we apply our Corollary 3.3, in the identities section, to R to obtain 4 more biquadrates for the representation of P.

Observe that we obtained that P is a mixed strict sum of 10 biquadrates with, at most, only one of them with the sign -1. This implies b) since, by Lemma 3.13, -1 is a sum of two biquadrates. This construction works for polynomials P such that $\deg P \in \{4n_0, \ldots, 4n_0 - 3\}$ with $n_0 \geqslant 68$.

Remark 5.2. For odd $q \notin \{3, 9, 5, 13, 17, 25, 29\}$ and congruent to 1 modulo 8,

$$g(4, 269, \mathbb{F}_q[t]) \leqslant 10.$$

Proof. If q is congruent to 1 modulo 8, then -1 is a biquadrate in the field $\mathbb{F}_q$. So, a mixed strict sum of biquadrates is a strict sum of biquadrates. ∎

The descent process runs for smaller degrees. More steps are necessary to get the expected degree.

Proposition 5.3. *Assume that* q *is odd and that* $q \notin \{3, 9, 5, 13, 17, 25, 29\}$. *Then:*

a) Polynomials with degree d *in* $\{33 \to 64, 69 \to 76, 85 \to 100, 117 \to 124, 129 \to 136, 153 \to 160, 169 \to 172, 181 \to 184, 201 \to 208, 241 \to 244, 265 \to 268\}$ *are mixed strict sums of* 11 *biquadrates, (resp. strict sums of* 12 *biquadrates).*

b) All other polynomials with degree $d \in \{29 \to 271\}$ *are mixed strict sums of* 10 *biquadrates, (resp. strict sums of* 11 *biquadrates).*

Proof. By direct computations one proves that the process works in the case b) and that the process need one step more in the case a).

The method fails for polynomials of degree $\leqslant 28$. For instance if P has degree 4, the recursive process applied to $n_0 = 1$ gives $n_i = 1$ for all integer i, if P has degree 28, the recursive process applied to $n_0 = 7$ gives $n_1 = 6, n_2 = 5$, $n_3 = 4, n_4 = 3$ and $n_i = 3$ for all integer $i \geqslant 4$. ■

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