Functiones et Approximatio
XXXV (2006), 339–357

A MEASURE FOR THE LINEAR INDEPENDENCE OF VALUES OF A CERTAIN GENERALIZATION OF THE EXPONENTIAL FUNCTION

Rolf Wallisser

Dedicated to Professor Eduard Wirsing
on the occasion of his 75th birthday

Abstract: In 1873 Hermite proved the transcendence of the basis e of the natural logarithm. In the more than 140 years since this paper was published, several quite different proofs were given (see Transcendental Numbers by N.I. Fel'dman and Yu.V. Nesterenko, Springer 1998). One of the best known versions goes back to Hurwitz, who used divisibility properties to show that a certain linear form doesn't vanish. Recently I generalized this method (see [20]), using some results on higher congruences going back to Dedekind and Hasse. We regained a former result of Carlson on linear independence of certain values of the function

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{Q(1)\cdots Q(n)},$$

Q a certain polynomial. Here I change the method used in [20] (see Chap. 5) to give a new proof of Galochkin's measure for the $\mathbb{Q}$–linear independence of these values.
Keywords: irrationality and Q–linear independence.

1. Introduction

Let $Q \in \mathbb{Z}[x]$ be a polynomial of degree $q \geqslant 1$ with integer coefficients and without zeros at the non–negative integers. Carlson [3] investigated already in 1935 the arithmetical nature of values of the function

$$G(x) = \sum_{n=0}^{\infty} \frac{x^n}{Q(1)Q(2)\cdots Q(n)}. \tag{1}$$

Throughout the paper empty products (e.g. corresponding to $n = 0$ in the latter series) and sums are interpreted as 1 and 0, respectively.

Carlson showed by means of a certain Padé–approximation to G that for any distinct nonzero rationals $\alpha_1, \dots, \alpha_h$ the numbers 1, $G(\alpha_1), \dots, G^{(q-1)}(\alpha_1), \dots,$

2000 Mathematics Subject Classification: 11J72, 11J82, 11K60, 11J25, 11A07.

$G(\alpha_h), \ldots, G^{(q-1)}(\alpha_h)$ are linearly independent over $\mathbb{Q}$. In 1979 and 1988 much more was proved for this and more general functions by Galochkin (see [6],[7]) and other mathematicians. One can get a first impression of the extensive research during the last century on the arithmetic nature of the values of special functions like E–functions, G–functions, hypergeometric functions in the monograph of Fel'dman and Nesterenko [5] and one can find there a broad overview of the literature in this area.

In this paper my aim is to show that aside the lines which Carlson or Galochkin used it is possible to get their result with the method Hurwitz [11] applied to prove the transcendence of e. Here we have to replace the divisibility properties in the paper of Hurwitz by theorems on higher congruences going back to Dedekind [4] and Hasse [9]. Gerst and Brillhart [8] have written a very readable paper on this using only relatively simple algebraic considerations. In addition to the result of Carlson, which I gained already in [20], we can get here the same measure which Galochkin obtained for the linear independence of the values in question of G.

The theorem of Carlson with the measure of Galochkin is the following:

Theorem 1. *Let $\alpha_1, \ldots, \alpha_h$ be distinct non–zero rational numbers. For any $\varepsilon > 0$ there exists an integer $H_0(\varepsilon)$ such that for any non–zero vector*

$$\left(s_0, s_1^{(0)}, \ldots, s_h^{(0)}, \ldots, s_1^{(q-1)}, \ldots, s_h^{(q-1)}\right) \in \mathbb{Z}^{h \cdot q+1}$$

of height $H := \max\limits_{\substack{1 \leqslant j \leqslant h \\ 0 \leqslant \nu \leqslant q-1}} \left\{|s_j^{(\nu)}|\right\} > H_0(\varepsilon)$, *one has*

$$\left|s_0 + \sum_{j=1}^{h} \sum_{\nu=0}^{q-1} s_j^{(\nu)} G^{(\nu)}(\alpha_j)\right| \geqslant H^{-\left(\frac{hq+1}{\kappa} q - 1\right) - \varepsilon}, \tag{2}$$

where κ denotes the number of different irreducible factors of Q.

Remarks

i) For a linear polynomial Q one has in comparision to Dirichlet's approximation theorem the sharp measure $H^{-h-\varepsilon}$. As in Baker [1] Chap. 10, our method enables us to replace ε by a function $\varepsilon(H)$ tending to zero if H goes to infinity like $c(\log\log H)^{-1}$; c a constant independent of H.

ii) G satisfies a linear differential equation of order q with coefficients which are polynomials over $\mathbb{Q}$. Therefore it is not possible to replace the order $(q-1)$ of differentation in the theorem by a higher one.

iii) One can assume that the numbers $\alpha_j (1 \leqslant j \leqslant h)$ in the theorem are integers. To show this, let us denote

$$\alpha_j := \frac{s_j}{t_j}(1 \leqslant j \leqslant h),\ s_j, t_j \in \mathbb{Z}^*,\ a := t_1 \cdot t_2 \cdots t_h,\ \alpha_j^* \in \mathbb{Z},$$
$$t_j \alpha_j^* := s_j a,\ Q^*(x) := a \cdot Q(x).$$

If the theorem is proved for integers α_j we apply this result to the function

$$G^*(x) = \sum_{n=0}^{\infty} \frac{1}{Q(1)\cdots Q(n)} \left(\frac{x}{a}\right)^n = \sum_{n=0}^{\infty} \frac{x^n}{Q^*(1)\cdots Q^*(n)};$$

for $(s_0, s_1, \ldots, s_n) \in \mathbb{Z}^{h+1}\backslash\vec{0}$ it follows

$$s_0 + s_1 G^*(\alpha_1^*) + \ldots + s_h G^*(\alpha_h^*) \neq 0.$$

The term on the left in this relation is by definition of G^* and $\alpha_j^* := \alpha_j \cdot a$ equal to

$$s_0 + s_1 G(\alpha_1) + \ldots + s_h\, G(\alpha_h).$$

Acknowledgement. The author thanks the referee for many valuable suggestions and for the correction of misprints.

2. The method of Hilbert–Perron–Skolem

Let

$$f(x) = \sum_{n=0}^{\infty} k_n x^n,\ k_0 = 1. \tag{3}$$

One chooses a "starting–polynomial"

$$P(x) := P_0(x) := \sum_{n=0}^{m} \gamma_n k_n x^n \tag{4}$$

and derives m other polynomials from P:

$$P_\mu(x) = \sum_{n=\mu}^{m} \gamma_n\, k_{n-\mu}\, x^{n-\mu},\ 1 \leqslant \mu \leqslant m. \tag{5}$$

The approximating–polynomial P^* belonging to P is the following:

$$P^*(x) := \sum_{\mu=0}^{m} P_\mu(x). \tag{6}$$

Using the relations (4) and (5) one obtains

$$P^*(x) = \sum_{n=0}^{m} \gamma_n \sum_{\nu=0}^{n} k_\nu x^\nu \tag{7}$$

and as $f(0) = k_0 = 1$

$$P^*(0) = \sum_{n=0}^{m} \gamma_n. \tag{8}$$

Finally it follows for the "remainder–term"

$$\Delta(x) := P^*(x) - P^*(0) f(x) \tag{9}$$

with (7) and (8)

$$\begin{aligned} \Delta(x) &= \sum_{n=0}^{m} \gamma_n \sum_{\nu=0}^{n} k_\nu x^\nu - \sum_{n=0}^{m} \gamma_n \sum_{\nu=0}^{\infty} k_\nu x^\nu \\ &= -\sum_{n=0}^{m} \gamma_n \sum_{\nu=n+1}^{\infty} k_\nu x^\nu. \end{aligned} \tag{10}$$

To prove Theorem 1 one has to show that for every non–zero vector

$$(s_0, s_1^{(0)}, \dots, s_h^{(0)}, \dots, s_1^{(q-1)}, \dots, s_h^{(q-1)}) \in \mathbb{Z}^{hq+1} \tag{11}$$

the linear form

$$\Lambda = s_0 \cdot 1 + \sum_{j=1}^{h} \sum_{\nu=0}^{q-1} s_j^{(\nu)} G^{(\nu)}(\alpha_j) \tag{12}$$

does not vanish.

To show this one takes the "approximating–polynomial" P^* (see (6)) to introduce the linear form

$$\Lambda^* = s_0 P^*(0) + \sum_{j=1}^{h} \sum_{\nu=0}^{q-1} s_j^{(\nu)} P^{*(\nu)}(\alpha_j). \tag{13}$$

Using the "remainder term" $\Delta(x)$ (see (9)) one gets the following connection between the linear forms Λ and Λ^*

$$\Lambda^* = P^*(0)\Lambda + \sum_{\nu=0}^{q-1} \sum_{j=1}^{h} s_j^{(\nu)} \Delta^{(\nu)}(\alpha_j). \tag{14}$$

If one chooses an appropriate "starting–polynomial" P one can show firstly that the linear form Λ^* is different from zero and secondly that

$$\Big| \sum_{\nu=0}^{q-1} \sum_{j=1}^{h} s_j^{(\nu)} \Delta^{(\nu)}(\alpha_j) \Big| \leqslant \frac{1}{2} |\Lambda^*|. \tag{15}$$

In this way one gets from (14)

$$\big| P^*(0) \big| \cdot |\Lambda| \geqslant \frac{1}{2} |\Lambda^*| > 0. \tag{16}$$

From this inequaliy one concludes $\Lambda \neq 0$ which means the linear independence over $\mathbb{Q}$ of the values $1, G(\alpha_1) \dots, G(\alpha_h), \dots, G^{(q-1)}(\alpha_h)$. To give a measure for the linear independence of this values one has to find quantitative versions, depending on the height H of the vector (11), for the inequalities (15) and (16).

Remark. In the paper of Skolem [17] and in [2] one finds further applications of the method of Hilbert–Perron–Skolem.

3. The form of the polynomials P_μ in the special case of the function G defined in 1

We apply now the method of Hilbert–Perron–Skolem described in Chap. 2 to the function G defined in Chap. 1. In this case the values k_n in (3) are given by

$$k_n := \frac{1}{Q(1)Q(2)\cdots Q(n)}. \tag{17}$$

Let δ denote the operator $x\frac{d}{dx}$. The function G satisfies the linear differential equation of order q

$$Q(\delta)G(x) = Q(0) + x\,G(x). \tag{18}$$

If one applies $Q(\delta)$ to P_μ one gets

$$\begin{aligned} Q(\delta)P_\mu(x) &= \sum_{n=\mu}^{m} \gamma_n\, k_{n-\mu}\, Q(\delta)x^{n-\mu} \qquad (19)\\ &= \sum_{n=\mu}^{m} \gamma_n\, k_{n-\mu}\, Q(n-\mu)x^{n-\mu}\\ &= \gamma_\mu + x \sum_{n=\mu+1}^{m} \gamma_n\, k_{n-(\mu+1)}\, x^{n-(\mu+1)}\\ &= P_\mu(0) + xP_{\mu+1}(x), \quad 0 \leqslant \mu \leqslant m. \end{aligned}$$

If the "starting–polynomial" $P_0(x) := P(x)$ vanishes at zero with multiplicity m_0, it follows

$$P_\mu(0) = \gamma_\mu = 0 \quad \text{for} \quad \mu = 0, \dots, m_0 - 1 \tag{20}$$

and from (19) we have

$$P_\mu(x) = \Big(\frac{1}{x}\, Q(\delta)\Big)^{\mu} \circ P(x) \quad \text{for} \quad 0 \leqslant \mu \leqslant m_0. \tag{21}$$

At the end of this chapter we will citate in form of two lemmata known properties of the operators δ and $D = \frac{d}{dx}$ giving with (21) the following representation of the polynomials P_μ,

$$P_\mu(x) = \sum_{j=0}^{\mu\cdot q} \alpha_{j,\mu}\, x^{j-\mu}\, P^{(j)}(x), \quad \mu \leqslant m_0. \tag{22}$$

The coefficients $\alpha_{j,\mu}$ are integers and if $Q \in \mathbb{Z}[x]$ has the form

$$Q(x) = a_0 + a_1 x + \cdots + a_q x^q, \quad a_q \neq 0,\ q \geqslant 1 \tag{23}$$

one gets for the highest coefficient of P_μ

$$\alpha_{\mu q,\mu} = a_q^\mu. \tag{24}$$

Lemma 1. *For* $\delta = x\frac{d}{dx}$ *and* $D = \frac{d}{dx}$ *we have*

$$\delta^j = \sum_{i=0}^{j} \rho_{i,j}\, x^i\, D^i, \quad \rho_{ij} \in \mathbb{Z},$$
$$\rho_{0,0} := 1;\ \rho_{0,j} := 0,\ \rho_{i,j} := i\rho_{i,j-1} + \rho_{i-1,j-1}\ (0 < i < j),$$
$$\rho_{j,j} := \rho_{j-1,j-1}\ (0 < j),\ \rho_{i,j} := 0\ \ (i = -1 \text{or } i > j).$$

Lemma 2. *For* $Q(x) = \sum_{\ell=0}^{q} a_\ell\, x^\ell$ *one has*

i) $Q(\delta) = \sum_{j=0}^{q} \alpha_j\, x^j\, D^i,\ \alpha_j := \sum_{\ell=j}^{q} \rho_{j,\ell}\, a_\ell\ (0 \leqslant j \leqslant q)$

ii) $\Big(\frac{1}{x}\, Q(\delta)\Big)^\mu = \sum_{j=0}^{\mu\cdot q} \alpha_{j,\mu} x^{j-\mu}\, D^j,\ \alpha_{0,0} := 1,\ \alpha_{j,1} := \alpha_j$

$\alpha_{\mu q,\mu} = \alpha_{q,1}^\mu = \alpha_q^\mu = (\rho_{q,q}\, a_q)^\mu = a_q^\mu\,.$

4. A survey of some results on higher congruences

Let $\mathbb{P}$ denote the set of prime numbers. For the proof of Theorem 1, one needs some results on those primes p for which the congruence $f(x) \equiv 0(\operatorname{mod} p)$ is solvable. Here $f(x)$ denotes a polynomial with rational integer coefficients which is not identically zero (mod p). Moreover results on those primes p are needed for which the given polynomial splits completely in $\deg f$ linear factors ($\operatorname{mod} p$). In algebraic–number–theory one can find many papers in this direction going back to Dedekind [4] and Hasse [9]. Gerst and Brillhart [8] have given an excellent more elementary introduction to these problems. For better readability of this work I mention here some notations, definitions, conclusions and theorems. All proofs can be found in [8].

Definition 1. A prime p for which $f(x)$ is not identically zero ($\operatorname{mod} p$) and for which the congruence $f(x) \equiv 0(\operatorname{mod} p)$ is solvable is called a prime divisor of f.

Proposition 1. Schur [16] has shown that every nonconstant polynomial f has an infinite number of prime divisors.

Definition 2. Let $f(x) = a\prod f_i(x)^{\alpha_i}(a, \alpha_i, \in \mathbb{Z}, \alpha_i > 0, f_i(x) \in \mathbb{Z}[x]$ dinstinct, primitive and irreducible) be the unique factorisation of $f(x)$ into irreducible polynomials. $f(x)$ is said to "split completely"$(\bmod p)$, p a prime, iff each $f_i(x)$ is congruent $(\bmod p)$ to a product of $\deg f_i$ distinct linear factors and p doesn't divide the discriminant of $g(x) = \prod f_i(x)$.

If $f(x)$ "splits completely" $(\bmod p)$ then $f(x)$ is congruent $(\bmod p)$ to a product of $\deg f$ many linear factors of the form $ax + b$, $a, b \in \mathbb{Z}$, $p \nmid a$. It is this property which is needed to prove Theorem 1.

Theorem 5 of the work of Gerst and Brillhart [8] gives a general information on the prime divisors of a polynomial. The following Proposition 2 is proved there as a corollary (see [8], page 258).

Proposition 2. Every nonconstant polynomial $f(x)$ has an infinite number of prime divisors p for which it "splits completely" (mod p).

Proposition 3. If $n = \deg f$, γ_i $(1 \leqslant i \leqslant m)$ are the roots of $f(x)$, $K = Q(\gamma_1, \ldots, \gamma_m)$ and if $h(x) \in \mathbb{Z}[x]$ is an irreducible polynomial with a primitive element of K as a root then each f_i (see Def. 2) will "split completely"(mod p) for almost all prime divisors p of $h(x)$ The prime divisors in proposition 2 are the prime divisors of $h(x)$ which do not divide the discriminant of $g(x) = \prod f_i(x)$ (see [8], proof of the corollary on page 258).

To give the measure in our Theorem 1 we use Čebotarev's theorem (see Tschebotareff [19] or Narkiewicz [13]). The polynomial $h(x)$ in the remark above is irreducible and normal. Therefore after Theorem 4 of [8] the primes p for which f "splits completely" $(\bmod p)$ belong to the trivial group $\{e\}$ of the Galois group $G(h)$ of $h(x)$. This primes have, if $|G(h)|$ denotes the order of $G(h)$, the Dirichlet–density $\frac{1}{|G(h)|}$. So there is (e.g. [13] pp 344) the "prime–number–theorem":

Proposition 4. Let $\pi_f(x)$ denote the number of primes p for which f "splits completely" $(\bmod p)$ and which do not exceed x. Then

$$\pi_f(x) = \Big(\frac{1}{|G(h)|} + \mathrm{o}(1)\Big)\ \frac{x}{\log x}.$$

5. The choice of the "starting–polynomial" P and some divisibility properties of the values $P_\mu^{(\rho)}(\alpha_j)$

To prove the quantitative part of Theorem 1 it is necessary that the prime p, which rules the "starting–polynomial", can be chosen independently of the coefficients $s_j^{(\nu)}$ of the linear form Λ. This is in contrast to my proof of the qualitative part in [20]. Here we choose for each sufficiently large prime p, for which Q "splits completely" (mod p), a new "starting polynomial". Then it is possible to use the prime number theorem of Čebotarev to get the sharp measure of Galochkin.

Without loss of generality we can assume that

$$\gcd\left(s_0, s_1^{(0)}, \dots, s_h^{(0)}, \dots, s_1^{(q-1)}, \dots, s_h^{(q-1)}\right) = 1. \tag{25}$$

If the numbers $s_j^{(\nu)}$ have no common divisor then for every prime $p \in \mathbb{P}$ there exists in the set $S = \left\{s_0, \dots, s_h^{(q-1)}\right\}$ an integer which is not divisible by p.

Let for a prime $p \in \mathbb{P}$ s_0 or $s_{j_0}^{(\nu_0)}$ be the "highest" term in S which is not divisible by p. We distinguish the cases:

$$p \nmid s_0 \quad \text{and} \quad p \mid s_j^{(\nu)} \ (1 \leqslant j \leqslant h, 0 \leqslant \nu \leqslant q-1) \tag{26α}$$

$$p \nmid s_{j_0}^{(\nu_0)}, \tag{26β}$$

$$p \mid s_j^{(\nu)}, \qquad (1 \leqslant j \leqslant h), \ (\nu_0 < \nu \leqslant q-1)$$

$$p \mid s_j^{(\nu_0)}, \quad (j_0 < j \leqslant h).$$

In case α or β we have for the linear form Λ^* (see (13)) the relations:

$$\Lambda^* = s_0\, P^*(0) + pA_1, \ A_1 \in \mathbb{Z} \tag{27α}$$

$$\Lambda^* = s_0\, P^*(0) + \sum_{j=1}^{h} \sum_{\nu=0}^{\nu_0-1} s_j^{(\nu)} P^{*(\nu)}(\alpha_j) \tag{27β}$$

$$+ \sum_{j=1}^{j_0} s_j^{(\nu_0)}\, P^{*(\nu_0)}(\alpha_j) + p\, A_2, A_2 \in \mathbb{Z}.$$

Let now $p \in \mathbb{P}$ be a prime number which satisfies the conditions:

i) Q "splits completely" (mod p)
(see Definition 2 and Proposition 2 in Chap. 4) (28)

ii) $p > q (=: \deg Q)$

Assume in case α respectively β in addition:

$$p \nmid s_0 \text{ and } p \nmid a_q \prod_{j=1}^{h} (\alpha_j) \tag{29α}$$

$$p \nmid s_{j_0}^{(\nu_0)} \text{ and } p \nmid a_q \alpha_{j_0} \prod_{\substack{j=1 \\ j \neq j_0}}^{h} (\alpha_{j_0} - \alpha_j) \tag{29β}$$

Define now the "starting–polynomial" P in case α respectively β in the following way:

$$P(x) := x^{p-1} \quad \prod_{j=1}^{h} (x - \alpha_j)^{p \cdot q} \tag{30α}$$

$$P(x) := x^p (x - \alpha_{j_0})^{(p-1)q + \nu_0} \quad \prod_{\substack{j=1 \\ j \neq j_0}}^{h} (x - \alpha_j)^{pq} \tag{30β}$$

The derivatives of P satisfy in case α resp. β the relations:

$$P^{(\ell)}(0) = \begin{cases} 0 & \ell < p-1 \\ (p-1)! \prod_{j=1}^{h} (-\alpha_j)^{pq} & \ell = p-1 \\ pA_3, A_3 \in \mathbb{Z} & \ell \geqslant p \end{cases} \tag{31 α}$$

i) $$P^{(\ell)}(\alpha_{j_0}) = \begin{cases} 0, & \ell < (p-1)q + \nu_0, \\ \alpha_{j_0}^p \big((p-1)q+\nu_0\big)! \prod_{\substack{j=1 \\ j \neq j_0}}^{h} (\alpha_{j_0} - \alpha_j)^{pq}, & \ell = (p-1)q + \nu_0 \end{cases}$$

ii) $1 \leqslant j \leqslant h,\ j \neq j_0,$ (31 β)

$$P^{(\ell)}(\alpha_j) = \begin{cases} 0 & \ell < qp \\ (qp)! \alpha_j^p (\alpha_j - \alpha_{j_0})^{(p-1)q+\nu_0} \prod_{\substack{r=1 \\ r \neq j_0, j}}^{h} (\alpha_j - \alpha_r)^{pq}, & \ell = pq \end{cases}$$

iii) $$P^{(\ell)}(0) = \begin{cases} 0 & \ell < p \\ p!(-\alpha_{j_0})^{(p-1)q+\nu_0} \prod_{\substack{j=1 \\ j \neq j_0}}^{h} (-\alpha_j)^{pq}, & \ell = p. \end{cases}$$

The Taylor development at $x = 0$ of the "starting–polynomial" P begins in case α resp. β with the term cx^{p-1} resp. $c \cdot x^p, c \neq 0$. Therefore one concludes from the relation (22)

$$P_\mu^{(\rho)}(x) = \sum_{j=0}^{\mu \cdot q} \alpha_{j\mu} \big(x^{j-\mu} P^{(j)}(x)\big)^{(\rho)}, \mu \leqslant m_0, \tag{32}$$

with $m_0 = p-1$ in case α and $m_0 = p$ in case β.

From the definition of the "weights" γ_n in (4) we have

$$\gamma_{p-1}\, k_{p-1} = \frac{P^{(p-1)}(0)}{(p-1)!}. \tag{33}$$

Because of the form of the k_n in case of the function G, given in Chap. 3 (17), and from the definition of the "starting polynomial" P in (30), we find with (31α) and (33) in case α,

$$P_\mu(0) = 0 \quad \text{for} \quad \mu = 0, \ldots, p-2 \tag{34α}$$

$$P_{p-1}(0) = \gamma_{p-1} = Q(1) \cdots Q(p-1) \prod_{j=1}^{h} (-\alpha_j)^{pq}.$$

In case β, one gets from (31β) and (32)

$$P_\mu^{(\rho)}(\alpha_{j_0}) = \begin{cases} 0 & 0 \leqslant \mu < p-1,\ 0 \leqslant \rho \leqslant q-1 \\ 0 & \text{for } \mu = p-1,\ 0 \leqslant \rho < \nu_0 \\ A_{\nu_0} & \mu = p-1,\ \rho = \nu_0 \end{cases} \tag{34β}$$

$$A_{\nu_0} := \big((p-1)q+\nu_0\big)!\,\alpha_{j_0}^p \Big(\prod_{\substack{j=1\\ j\neq j_0}}^{h} (\alpha_{j_0}-\alpha_j)\Big)^{pq} \alpha_{j_0}^{(p-1)(q-1)} \cdot \alpha_{(p-1)q,p-1} \tag{35}$$

$$\alpha_{(p-1)q,p-1} = a_q^{p-1}.$$

To find such properties for $\mu \geqslant p$, one takes the representation

$$P_\mu^{(\rho)}(\alpha_j) = \sum_{n=\mu+\rho}^{m} \gamma_n\, k_{n-\mu}\rho! \binom{n-\mu}{\rho} \alpha_j^{n-(\mu+\rho)} \tag{36}$$

and one expresses $\gamma_n \cdot k_{n-\mu}$ in the following way:

$$\gamma_n \cdot k_{n-\mu} = \gamma_n \cdot k_n\, \frac{k_{n-\mu}}{k_n} = \gamma_n \cdot k_n \cdot Q(n-\mu+1)\cdots Q(n). \tag{37}$$

For $\mu \geqslant p$ the product $Q(n-\mu+1)\cdots Q(n)$ has at least p factors. By the assumption (28) that the polynomial Q "splits completely"(mod p) in q linear factors of the form $ax+b,\ p\nmid a$, one gets

$$p^q \,\big|\, Q(n-\mu+1)\cdots Q(n),\ \mu \geqslant p. \tag{38}$$

If one remembers that the $\alpha_j, 1 \leqslant j \leqslant h$, could be chosen as integers one recognizes that $\gamma_n \cdot k_n$ is also an integer. Therefore we have from (37) and (38)

$$p^q \,\big|\, \gamma_n \cdot k_{n-\mu},\ p \leqslant \mu \leqslant n \leqslant m. \tag{39}$$

From this we conclude with (36)

$$p^q \,\big|\, P_\mu^{(\rho)}(\alpha_j),\ p \leqslant \mu \leqslant n \leqslant m,\ 0 \leqslant \rho < q,\ 1 \leqslant q,\ 0 \leqslant j \leqslant h \ \ (\alpha_0 := 0). \tag{40}$$

To show that the linear form Λ^* doesn't vanish one needs also the following facts:

In case α: Let $\overline{q}$ be the highest exponent with the property

$$p^{\overline{q}} \big\| \big(Q(1)\, Q(2) \cdots Q(p-1) Q(p)\big) \tag{41}$$

$\big(p^q \| a$ denotes $p^q | a$ but $p^{q+1} \nmid a\big)$.

Because Q "splits completely" (mod p) we have $\overline{q} \geqslant q$. From relation (37) follows that we have too for $\mu \geqslant p$

$$p^{\overline{q}} \big| \gamma_n\, k_{n-\mu}, \quad p \leqslant \mu \leqslant n \leqslant m. \tag{42α}$$

Therefore for $\alpha_0 := 0$ and $1 \leqslant j \leqslant h$

$$p^{\overline{q}} \big| P_\mu^{(\rho)}(\alpha_j),\ p \leqslant \mu \leqslant m,\ 0 \leqslant \rho < q. \tag{43α}$$

We see too from (29α), $(34\,\alpha)$ and (41) that we have for the highest power of p dividing $P_{p-1}(0)$

$$p^{\hat{q}} \big\| P_{p-1}(0);\ 0 \leqslant \hat{q} \leqslant \overline{q}. \tag{44α}$$

In case β, we have from (29β), $(34\,\beta)$ and (35)

$$p^q \nmid P_{p-1}^{(\nu_0)}(\alpha_{j_0})$$

and from (30β) and (32)

$$P_\mu(0) = 0,\ \mu = 0, \ldots, p-1 \tag{45}$$

$$P_\mu^{(\rho)}(\alpha_j) = 0,\ j \neq j_0,\ 1 \leqslant j \leqslant h,\ \mu < p,\ \rho < q.$$

6. Non–vanishing of the linear form Λ^*

P^* (see (6)) is defined by

$$P^*(x) = \sum_{\mu=0}^{m} P_\mu(x)$$

where P is the "starting–polynomial" (30α resp. 30β). We have for the degree m of P in the cases α resp. β:

$$m = (p-1) + hpq \tag{46α}$$

$$m = (p-1)q + \nu_0 + (h-1)pq + p \leqslant pqh + p - 1. \tag{46β}$$

From the definition of P in (30α) follows with (32) in case α

$$P^*(0) = \sum_{\mu=p-1}^{m} P_\mu(0) = \sum_{\mu=p-1}^{m} \gamma_\mu, \tag{47}$$

$$P^{*(\rho)}(\alpha_j) = \sum_{\mu=p}^{m} P_\mu^{(\rho)}(\alpha_j),\ 1 \leqslant j \leqslant h,\ 0 \leqslant \rho < q,$$

and with (30β), (34β), (35) and (45) in case β

$$\begin{cases} P^{*(\rho)}(\alpha_j) = \displaystyle\sum_{\mu=p}^{m} P_\mu^{(\rho)}(\alpha_j),\ j \neq j_0,\ \rho \leqslant q-1, \\ P^{*(\rho)}(\alpha_{j_0}) = \displaystyle\sum_{\mu=p}^{m} P_\mu^{(\rho)}(\alpha_{j_0}),\ \rho < \nu_0, \\ P^{*(\rho)}(\alpha_{j_0}) = \displaystyle\sum_{\mu=p-1}^{m} P_\mu^{(\rho)}(\alpha_{j_0}),\ \nu_0 < \rho \leqslant q-1, \\ P^{*(\nu_0)}(\alpha_{j_0}) = P_{p-1}^{(\nu_0)}(\alpha_{j_0}) + \displaystyle\sum_{\mu=p}^{m} P_\mu^{(\nu_0)}(\alpha_{j_0}). \end{cases} \tag{48}$$

We assumed $p > q$ and $0 \leqslant \nu_0 < q$. Therefore one has

$$p^{q-1} \| \big((p-1)q + \nu_0\big)! \tag{49}$$

From (49), ($30\,\beta$) and (32) one gets

$$p^q \mid P^{(\rho)}_{p-1}(\alpha_{j_0}), \quad \nu_0 < \rho < q. \tag{$50\,\beta$}$$

If one takes the results of (43α), (44α) and (47) one gets in case α

$$p^{\hat{q}} \| P^*(0),\ 0 \leqslant \hat{q} \leqslant \overline{q};\ \ p^{\hat{q}} \big| P^{*(\rho)}(\alpha_j),\ 1 \leqslant j \leqslant h,\ 0 \leqslant \rho < q, \tag{$51\,\alpha$}$$

and in case β with (40), (45) and (48)

$$\begin{cases} p^q \nmid P^{*(\nu_0)}(\alpha_{j_0}) \\ p^q \mid P^{*(\rho)}(\alpha_j) \qquad \text{for} \quad (j,\rho) \neq (j_0, \nu_0) \\ p^q \mid P^*(0). \end{cases} \tag{$51\,\beta$}$$

Case α is defined such that every $s_j^{(\nu)}$ in $\Lambda^*(1 \leqslant j \leqslant h,\ 0 \leqslant \nu < q)$ is divisible by p. Therefore we have from ($51\,\alpha$) that

$$p^{\hat{q}+1} \big| \big(\Lambda^* - s_0 P^*(0)\big). \tag{$52\,\alpha$}$$

But in case α we have by definition $p \nmid s_0$, therefore one gets from ($51\,\alpha$) that

$$p^{\hat{q}+1} \nmid s_0 P^*(0). \tag{$53\,\alpha$}$$

From ($52\,\alpha$) and ($53\,\alpha$) together we conclude

$$p^{\hat{q}+1} \nmid \Lambda^* \text{ or } \Lambda^* \neq 0. \tag{$54\,\alpha$}$$

In case β we see from ($51\,\beta$) that

$$p^q \big| \big(\Lambda^* - s_j^{(\nu_0)} P^{*(\nu_0)}(\alpha_{j_0})\big) \tag{$55\,\beta$}$$

and

$$p^q \nmid s_j^{(\nu_0)} P^{*(\nu_0)}(\alpha_{j_0}). \tag{$56\,\beta$}$$

From ($55\,\beta$) and ($56\,\beta$) we conclude

$$p^q \nmid \Lambda^* \text{ or } \Lambda^* \neq 0. \tag{$57\,\beta$}$$

7. A lower bound for Λ^*

Let t be a prime number and let μ_t denote the number

$$\mu_t := \big| \big\{ x \in \mathbb{Z},\ 0 \leqslant x < t,\ Q(x) \equiv 0 (\operatorname{mod} t) \big\} \big|. \tag{58}$$

$\big(|\{\ \}|$ denotes the cardinality of the set $|\{\ \}|\big)$. After Nagel [12] one has the estimate

$$\sum_{t\leqslant x}\mu_t\,\frac{\log t}{t}=\kappa\log x+\mathrm{O}(1) \tag{59}$$

(κ denotes the number of irreducible factors of Q).

In (37) we have seen that for $\mu\geqslant p$ every coefficient $\gamma_n\,k_{n-\mu}$ in the representation (36) of $P_\mu^{(\rho)}(\alpha_j)$ contains the factor

$$Q(n-\mu+1)\cdots Q(n)\ (p\leqslant\mu\leqslant n\leqslant m).$$

In all these products the argument of Q goes at least through p consecutive positive integers. Therefore, by the definition of μ_t in (58) in each one of these products the prime number t occurs at least in order $\mu_t\left[\frac{p}{t}\right]$. In consequence for $\mu\geqslant p$, every number $P_\mu(0)$ or $P_\mu^{(\rho)}(\alpha_j)$ $(1\leqslant j\leqslant h, 0\leqslant\rho\leqslant q-1)$ is divisible by a positive integer B_p with

$$B_p\geqslant\prod_{t\in\mathbb{P}}t^{\mu_t\left[\frac{p}{t}\right]}. \tag{60}$$

For $\mu<p-1$ we have $P_\mu(0)=0$ and $P_\mu^{(\rho)}(\alpha_j)=0(1\leqslant j\leqslant h, 0\leqslant\rho\leqslant q-1)$. Further we have in the cases α resp. β,

$$\prod_{t\in P}t^{\mu_t\left[\frac{p-1}{t}\right]}\big|P_{p-1}(0),\ P_\mu^{(\rho)}(\alpha_j)=0(\mu<p,1\leqslant j\leqslant h,0\leqslant\rho<q). \tag{61 α}$$

$$\begin{cases}\big((p-1)q+\nu_0\big)!\big|\,P_{p-1}^{(\nu_0)}(\alpha_{j_0}),\ P_{p-1}^{(\nu_0)}(\alpha_{j_0})\neq0\\ P_{p-1}^{(\rho)}(\alpha_j)=0\text{ for }1\leqslant j\leqslant h,\ j\neq j_0,\ 0\leqslant\rho\leqslant q-1,\ P_{p-1}(0)=0\\ P_{p-1}^{(\rho)}(\alpha_{j_0})=0\text{ for }0\leqslant\rho<\nu_0\\ \big((p-1)q+\nu_0\big)!p\big|P_{p-1}^{(\rho)}(\alpha_{j_0})\text{ for }\nu_0<\rho\leqslant q-1\end{cases} \tag{61 β}$$

Let D_α denote the number

$$D_\alpha:=\prod_{t\in\mathbb{P}}t^{\mu_t\left[\frac{p-1}{t}\right]}.$$

From (60) and (61 α) we get in case α

$$D_\alpha\big|\,P^*(0)\text{ and }D_\alpha\big|\,P^{*(\rho)}(\alpha_j)\ \ (1\leqslant j\leqslant h,\ 0\leqslant\rho<q). \tag{62 α}$$

In the same way one concludes from (61 β) in case β that we have for

$$\begin{aligned}&D_\beta:=\gcd\big(\big((p-1)q+\nu_0\big)!,D_\alpha\big)\\ &D_\beta\big|\,P^*(0)\text{ and }D_\beta\big|\,P^{*(\rho)}(\alpha_j)\ \ (1\leqslant j\leqslant h,\ 0\leqslant\rho<q).\end{aligned} \tag{62 β}$$

From this we get for

$$D := \gcd\big((p-1)!q, D_\alpha\big)$$

that in both cases α and β

$$D|\Lambda^*. \tag{63}$$

From the definition of μ_t in (58) follows $\mu_t \leqslant q, q := \deg Q$. We find in this way

$$D \geqslant \exp\Big(\sum_{\substack{t\leqslant p\\ t\in\mathbb{P}}} \mu_t \frac{\log t}{t}\, t\cdot \Big[\frac{p-1}{t}\Big]\Big). \tag{64}$$

Finally one gets with (59), (63) and (64), because of $\Lambda^* \neq 0$ (see $(54\,\alpha)$ or $(57\,\beta)$), the lower bound

$$|\Lambda^*| \geqslant \exp\big(\kappa\, p\log p + \mathrm{O}(p)\big). \tag{65}$$

8. An estimate of the remainder terms $\Delta^{(\nu)}(\alpha_j)$

$\Delta(x)$ was defined in (9). Since we have $\gamma_n = 0$ for $0 \leqslant n < p-1$, one gets for $0 \leqslant \nu \leqslant q-1$

$$\begin{aligned}\Delta^{(\nu)}(x) &= -\sum_{n=p-1}^{m} \gamma_n \cdot \sum_{\rho=n+1}^{\infty} k_\rho\, \nu! \binom{\rho}{\nu} x^{\rho-\nu} \qquad (66)\\ &= -\sum_{n=p-1}^{m} \gamma_n \cdot k_n\, \nu! \binom{n}{\nu} x^{n-\nu} \cdot \sum_{\rho=n+1}^{\infty} \frac{\binom{\rho}{\nu}}{\binom{n}{\nu}}\, \frac{k_\rho}{k_n} \cdot x^{\rho-n}.\end{aligned}$$

Because of $Q \in \mathbb{Z}[x]$ and since Q is of degree q there is an integer $n_0(Q)$ so that

$$\text{for all } \ n > n_0(Q),\ \big|Q(n)\big| \geqslant \frac{n^q}{2}. \tag{67}$$

In (66) we have $\rho > n \geqslant p-1$. It follows from (67) that if p ist sufficiently large we have

$$\Big|\frac{k_\rho}{k_n}\Big| = \frac{1}{|Q(n+1)\cdots Q(\rho)|} \leqslant \Big(\frac{2}{n^q}\Big)^{(\rho-n)}.$$

Considering only numbers x with the property

$$\frac{4|x|}{p} < \frac{1}{2}$$

one can get the estimate

$$\begin{aligned}\Big|\sum_{\rho=n+1}^{\infty} \frac{\binom{\rho}{\nu}}{\binom{n}{\nu}}\, \frac{k_\rho}{k_n}\, x^{\rho-n}\Big| &\leqslant \sum_{\rho=n+1}^{\infty} 2^\rho \Big(\Big(\frac{2}{n^q}\Big)|x|\Big)^{\rho-n} \qquad (68)\\ &= \sum_{\rho=n+1}^{\infty} 2^n \Big(\frac{4}{n^q}\ |x|\Big)^{\rho-n}\\ &\leqslant 2^{n+3}\cdot \frac{|x|}{n^q} \leqslant 2^{n+2}|x|.\end{aligned}$$

Let P_* denote the polynomial

$$P_*(x) := \sum_{n=p-1}^{m} \left|\gamma_n\, k_n\right| x^n. \tag{69}$$

With (66) and (68) one gets

$$\left|\Delta^{(\nu)}(x)\right| \leqslant 4 \cdot 2^m |x|\, P_*^{(\nu)}\big(|x|\big), \quad 0 \leqslant \nu \leqslant q-1. \tag{70}$$

If $\tilde{p}$ and $\tilde{q} \in \mathbb{C}[x]$ are polynomials,

$$\tilde{p}(x) := \sum_{\ell=0}^{L} a_\ell\, x^\ell, \;\; \tilde{q}(x) := \sum_{\ell=0}^{L} b_\ell\, x^\ell,$$

one writes

$$\tilde{p}(x) \ll \tilde{q}(x) \Leftrightarrow \quad |a_\ell| \leqslant |b_\ell| \text{ for } \ell = 0, 1, \ldots, L.$$

From the definition of P in (4), $(30\,\alpha)$ and $(30\,\beta)$ it follows that there are constants $c_1 > 0$ and $c_2 > 0$, which depend only on $\alpha_1, \ldots, \alpha_h$ so that we have

$$P_*(x) \ll \big(c_1(1+x)\big)^m$$

and

$$P_*^{(\nu)}(x) \ll c_2^m (1+x)^m, \quad 1 \leqslant \nu \leqslant q-1. \tag{71}$$

For a $p \in \mathbb{P}$ that satisfies

$$8|\alpha_j| \leqslant p, \quad 1 \leqslant j \leqslant h,$$

we get from (70) and (71)

$$\left|\Delta^{(\nu)}(\alpha_j)\right| \leqslant c_3^p, \quad 1 \leqslant j \leqslant h. \tag{72}$$

Here $c_3 > 0$ is a constant, which depends on $\alpha_1, \ldots, \alpha_h$ and Q, but not on $p \in \mathbb{P}$.

9. A measure for the linear independence

We have to estimate for every non–zero vector

$$\big(s_0, s_1^{(0)}, \ldots, s_h^{(q-1)}\big) \tag{73}$$

with the property (25), the linear form

$$\Lambda := s_0\, G(0) + \sum_{\nu=0}^{q-1} \sum_{j=1}^{h} s_j^{(\nu)}\; G^{(\nu)}(\alpha_j)$$

from below.

The linear forms Λ and Λ^* (see (13)) are connected (see (14)) in the following way:

$$\Lambda^* = P^*(0)\Lambda + \sum_{\nu=0}^{q-1} \sum_{j=1}^{h} s_j^{(\nu)} \Delta^{(\nu)}(\alpha_j) \tag{74}$$

If H denotes the height of the vector in (73),

$$H := \max_{\nu,j} \big(|s_j^{(\nu)}|\big),$$

it follows from (74) with (65) and (72)

$$\begin{aligned} \big|P^*(0)\Lambda\big| &\geqslant |\Lambda^*| - \sum_{\nu=0}^{q-1} \sum_{j=1}^{h} \big|s_j^{(\nu)} \Delta^{(\nu)}(\alpha_j)\big| \\ &\geqslant \exp\big(\kappa\, p \log p + \mathrm{O}(p)\big) - H \exp(p \cdot c_4) \end{aligned} \tag{75}$$

where $c_4 > 0$ is a constant independent of p and H.

We have (see (5) – (8))

$$\big|P^*(0)\big| \leqslant \sum_{n=p-1}^{m} |\gamma_n| \leqslant \sum_{n=p-1}^{m} \frac{1}{|k_n|} |\gamma_n\, k_n| \leqslant \frac{1}{|k_m|} \cdot P_*(1) \tag{76}$$

with

$$\frac{1}{|k_m|} = \big|Q(1) \cdots Q(m)\big|. \tag{77}$$

There is a constant $c_5(Q) > 0$ such that

$$Q(x) \ll c_5(1+x)^q. \tag{78}$$

For the degree m of the "starting–polynomial" P (see (30)) we have in both cases α and β (see (46))

$$m \leqslant hpq + p - 1. \tag{79}$$

With (77) – (79) one gets

$$\frac{1}{|k_m|} \leqslant \big(c_6(1+m)^q\big)^m \leqslant \exp\big((hq^2+q)p \log p + \mathrm{O}(p)\big). \tag{80}$$

By definition of P_* in (69) and its connection with P one has

$$\big|P_*(1)\big| \leqslant c_7^p \tag{81}$$

and with (76), (80), (81) one derives

$$\big|P^*(0)\big| \leqslant \exp\big((hq^2+q)p \log p + \mathrm{O}(p)\big). \tag{82}$$

Let ${}_QP_n$, $n=1,2,\dots,$ be an enumeration of those prime numbers p, for which the polynomial Q "splits completely" (mod p). Given H, let n_0 be the smallest positive integer such that

$$2H\exp(c_4 \cdot {}_Q P_{n_0}) \leqslant \exp(\kappa\ {}_QP_{n_0} \log\ {}_QP_{n_0}). \tag{83}$$

From (75) and (83) follows for sufficiently large H

$$\bigl|P^*(0)\Lambda\bigr| \geqslant \exp\bigl(\kappa\ {}_QP_{n_0} \log {}_QP_{n_0}(1+\mathrm{o}(1))\bigr)$$

and together with (82)

$$|\Lambda| \geqslant \exp\Bigl(-(hq^2+q)+\kappa+\mathrm{o}(1)\Bigr)\, {}_QP_{n_0}\ \log\ {}_QP_{n_0}. \tag{84}$$

By the prime–number–theorem of Čebotarev (see [14] or [19]) one has for the number ${}_Q\pi(x)$ of primes ${}_QP_n \leqslant x$

$${}_Q\pi(x) = \bigl(c_Q+\mathrm{o}(1)\bigr)\ \frac{x}{\log x}, \tag{85}$$

$c_Q :=$ Čebotarev density of the sequence $\{{}_QP_n\}_N$. From (85) one concludes as in the case of the prime number theorem (see Schwarz [18], pp 106)

$${}_QP_n = \frac{1}{c_Q}\bigl(1+\mathrm{o}(1)\bigr)\ n\cdot\log n. \tag{86}$$

The definition of n_0 in (83) gives

$$2H\exp\bigl(c_4\, {}_QP_{n_0-1}\bigr) > \exp\Bigl(\kappa\ {}_QP_{n_0-1}\log\ {}_QP_{n_0-1}\Bigr) \tag{87}$$

or

$$H^{-1} \leqslant \exp\bigl(-\kappa+\mathrm{o}(1)\bigr)\ {}_QP_{n_0-1}\log {}_QP_{n_0-1}.$$

From (86) one gets: For all $\varepsilon>0$ there exists $n_0(\varepsilon)$ such that for all $n>n_0(\varepsilon)$

$${}_QP_{n-1} > \frac{1}{1+\varepsilon}\ {}_QP_n. \tag{88}$$

With (87) and (88) one has for sufficiently large H

$$H^{-1} \leqslant \exp\bigl(-\kappa+\mathrm{o}(1)\bigr)\ {}_QP_{n_0}\cdot\log_Q P_{n_0}. \tag{89}$$

If one writes (84) in the form

$$|\Lambda| \geqslant \exp\Bigl(-\Bigl(\frac{hq^2+q}{\kappa}-1\Bigr)+\mathrm{o}(1)\Bigr)\ \kappa\ {}_QP_{n_0}\cdot\log\ {}_QP_{n_0}$$

one derives from (87) and (89) for $H>H_0(\varepsilon)$

$$|\Lambda| \geqslant H^{-\left(\frac{hq+1}{\kappa}q-1\right)-\varepsilon}.$$

References

[1] A. Baker, *Transcendental Number Theory.* Cambridge University Press 1975.

[2] P. Bundschuh and R. Wallisser, *Maße für die lineare Unabhängigkeit von Werten ganz transzendenter Lösungen gewisser Funktionalgleichungen.* Abh. Math Sem. Univ. Hamburg. I Band 69(1999), 103–122; II Band 73 (2003), 1–12.

[3] F. Carlson, *Sur une propriété arithmétique de quelques functions entières.* Arkiv för Mathematik, Astronomi och Fysik. Bd 25A. (1935), N: 07. 1–13.

[4] R. Dedekind, *Über den Zusammenhang zwischen der Theorie der Ideale und der höheren Kongruenzen*, Abh. Akad. Wiss. Göttingen, 23 (1878), 3–37.

[5] N.I. Fel'dman and Yu.V. Nesterenko, *Transcendental Numbers, Number Theory IV.* Encycl. of Math. Sc. 44, Springer 1998.

[6] A.I. Galochkin, *On Diophantine approximations of the values of some entire functions with algebraic coefficients*, Vestn. Mosk. Univ. Ser. 1, 34 No. 1 (1979), 26–30.

[7] A.I. Galochkin, *On effective bounds for certain linear forms*, New Advances in Transcendence Theory. Cambridge Univ. Press, Cambridge (1988), 207–214.

[8] I. Gerst and J. Brillhart, *On the prime divisors of polynomials.* Amer. Math. Monthly 78 (1971), 250–266.

[9] H. Hasse, *Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper*, 2nd ed., Physica, Würzburg 1965.

[10] D. Hilbert, *Über die Transzendenz der Zahlen e und* π. Math. Ann. 43 (1893), 216–219.

[11] A. Hurwitz, *Beweis der Transzendenz der Zahl e.* Math. Ann. 43 (1893), 220–221.

[12] T. Nagel, *Généralisation d'un théorème de Tchebycheff.* Journ. de Math. Pures et Appliquées (8^e série), tome IV (1921), 343–356.

[13] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers.* PWN Tom 57. Warszawa 1974.

[14] J. Neukirch, *Algebraische Zahlentheorie*, Kap VII, § 13 Dichtigkeitssätze, Springer–Verlag 1992.

[15] O. Perron, *Irrationalzahlen*, 2nd Edition, § 48. Chelsea Publishing Company 1951.

[16] I. Schur, *Über die Existenz unendlich vieler Primzahlen in einigen speziellen arithmetischen Progressionen*, S.–B. Berlin. Math. Ges., 11 (1912), 40–50.

[17] Th. Skolem, *Some theorems on irrationality and linear independence.* Skand. Mat. Kongr. 11, Trondheim (1949), 77–98.

[18] W. Schwarz, *Einführung in Methoden und Ergebnisse der Primzahltheorie.* BI Hochschultaschenbücher Bibliographisches Institut AG Mannheim 278/278a*, 1969.

[19] N.G. Tschebotareff (Čebotarev), *Die Bestimmung der Dichtigkeit einer Menge von Primzahlen, welche zu einer gegebenen Substitutionsklasse gehören.* Math. Ann. 95 (1926), 191–228.

[20] R. Wallisser, *Linear independence of values of a certain generalization of the exponential function – a new proof of a theorem of Carlson.* Journal de Théorie des Nombres de Bordeaux 17 (2005), 381–396.

Address: Fakultät für Mathematik und Physik, der Albert–Ludwigs–Universität Freiburg, Abteilung für Reine Mathematik, Eckerstr. 1, D–79104 Freiburg

Received: 25 October 2005; **revised:** 26 May 2006