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Some classes of operators associated with generalized Aluthge transformation

Masatoshi Ito

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Abstract. In this paper, firstly we shall give simplified proofs of the results on generalized Aluthge transformation in [11][12][14] and [16]. Secondly we shall discuss a generalization of both classes of class $A(k)$ operators defined in [9] and w-hyponormal operators defined in [2].

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§1. Introduction

We shall consider bounded linear operators on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible.

An operator T is said to be *p-hyponormal* for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$ and an operator T is said to be *log-hyponormal* if T is invertible and $\log T^*T \geq \log TT^*$. p-hyponormal and log-hyponormal operators are defined as extensions of hyponormal one, i.e., $T^*T \geq TT^*$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem "*$A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for any $\alpha \in [0,1]$*", and every p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function.

Let T be a p-hyponormal operator whose polar decomposition is $T = U|T|$. Aluthge [1] introduced the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which is called Aluthge transformation, and also showed the following result.

Theorem A.1 ([1]). *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $0 < p < 1$ and U be unitary. Then the following assertions hold:*

(1) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is $(p + \frac{1}{2})$-hyponormal if $0 < p < \frac{1}{2}$.*

(2) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is hyponormal if $\frac{1}{2} \le p < 1$.*

As a natural generalization of Aluthge transformation, the operator $\tilde{T}_{s,t} = |T|^s U|T|^t$ for $s > 0$ and $t > 0$ can be considered. The following Theorem A.2 on $\tilde{T}_{s,t}$ is a generalization of Theorem A.1 on $\tilde{T}$.

Theorem A.2 ([11][12][16]). *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $p > 0$. Then the following assertions hold:*

(1) $\tilde{T}_{s,t} = |T|^s U|T|^t$ *is $\frac{p+\min\{s,t\}}{s+t}$-hyponormal for $s > 0$ and $t > 0$ such that $\max\{s,t\} \ge p$.*

(2) $\tilde{T}_{s,t} = |T|^s U|T|^t$ *is hyponormal for $s > 0$ and $t > 0$ such that $\max\{s,t\} \le p$.*

We remark that Theorem A.2 yields Theorem A.1 when putting $s = t = \frac{1}{2}$ and the proof of [11] is cited under the condition $N(T) = N(T^*)$. As a parallel result to Theorem A.2 for log-hyponormal operators, the following Theorem A.3 is given in [14].

Theorem A.3 ([14]). *Let $T = U|T|$ be the polar decomposition of a log-hyponormal operator. Then $\tilde{T}_{s,t} = |T|^s U|T|^t$ is $\frac{\min\{s,t\}}{s+t}$-hyponormal for $s > 0$ and $t > 0$.*

We remark that Theorem A.3 is a parallel result to Theorem A.2. In fact, Theorem A.3 corresponds to Theorem A.2 in the case $p \to +0$ since p-hyponormality of T (i.e., $(T^*T)^p \ge (TT^*)^p$) approaches log-hyponormality of T (i.e., $\log T^*T \ge \log TT^*$) as $p \to +0$.

On the other hand, an operator T belongs to *class A* if $|T^2| \ge |T|^2$ and *class A(k)* for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \ge |T|^2$. We call an operator T class $A(k)$ operator briefly if T belongs to class $A(k)$. An operator T is class A if and only if T is class $A(1)$. On class $A(k)$ operators, we have the following Theorem A.4 in [9].

Theorem A.4 ([9]).

(1) *For each $k > 0$, every k-hyponormal operator is a class $A(k)$ operator.*

(2) *Every log-hyponormal operator is a class $A(k)$ operator for $k > 0$.*

(3) *For each $k > 0$, every invertible class $A(k)$ operator is a class $A(l)$ operator for $l \geq k$.*

An operator T is said to be *w-hyponormal* if $|\tilde{T}| \geq |T| \geq |\tilde{T}^*|$. We remark that w-hyponormal operator is defined by using Aluthge transformation $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$. w-hyponormal operator was defined by Aluthge and Wang [2] and the following Theorem A.5 is shown in [2].

Theorem A.5 ([2]).

(1) *If T is a p-hyponormal operator for $p > 0$, then T is w-hyponormal.*

(2) *If T is a log-hyponormal operator, then T is w-hyponormal.*

(3) *If T is a w-hyponormal operator, then $|T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{*^2}|$ hold.*

Theorem A.5 states that the class of w-hyponormal operators includes the classes of p-hyponormal operators and log-hyponormal operators, and also the class of w-hyponormal operators is included in the class of class A operators.

In this paper, firstly we shall give simplified proofs of Theorem A.2 and Theorem A.3 in section 2.

Secondly we shall discuss a generalization of both classes of class A(k) operators and w-hyponormal operators in section 3.

§2. Simplified proofs of Theorem A.2 and Theorem A.3

We need the following theorems and lemmas in order to give proofs of the results in this paper.

Theorem F (Furuta inequality [6]).

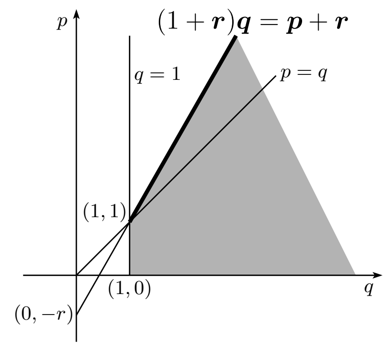
If $A \geq B \geq 0$, then for each $r \geq 0$,

(i) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.



FIGURE

It is shown in [13] that the domain drawn for p, q and r in the Figure is the best possible one for Theorem F.

On the other hand, chaotic order is defined by $\log A \geq \log B$ for positive and invertible operators A and B. Chaotic order is weaker than usual order $A \geq B$ since $\log t$ is an operator monotone function. Ando [3] shows that $\log A \geq \log B$ is equivalent to that $A^p \geq (A^{\frac{p}{2}}B^pA^{\frac{p}{2}})^{\frac{1}{2}}$ holds for all $p \geq 0$. By using Theorem F, a generalization of Ando's characterization is given as follows.

Theorem B.1 ([4][5][7][15]). *Let A and B be positive invertible operators. Then the following properties are mutually equivalent:*

(i) $\log A \geq \log B$.

(ii) $A^r \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for all $p \geq 0$ and $r \geq 0$.*

Very recently, by his ingenious technique, Uchiyama [15] obtains a simplified proof of Theorem B.1 by only using Theorem F.

Lemma F ([8]). *Let $A > 0$ and B be an invertible operator. Then*

$$(BAB^*)^{\lambda} = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{\lambda-1}A^{\frac{1}{2}}B^*$$

holds for any real number λ.

We remark that Lemma F holds without invertibility of A and B in case $\lambda \geq 1$.

Lemma 2.1. *Let $A \geq 0$ and $T = U|T|$ be the polar decomposition of T. Then for each $\alpha > 0$ and $\beta > 0$, the following statements hold:*

(1) $U^*U(|T|^\beta A|T|^\beta)^\alpha = (|T|^\beta A|T|^\beta)^\alpha$.

(2) $UU^*(|T^*|^\beta A|T^*|^\beta)^\alpha = (|T^*|^\beta A|T^*|^\beta)^\alpha$.

(3) $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$.

(4) $(U^*|T^*|^\beta A|T^*|^\beta U)^\alpha = U^*(|T^*|^\beta A|T^*|^\beta)^\alpha U$.

Proof of Lemma 2.1.
Proof of (1). We remark that

$$N(|T|) = N(|T|^\beta) \subset N(|T|^\beta A|T|^\beta) = N((|T|^\beta A|T|^\beta)^\alpha),$$

i.e., $\overline{R((|T|^\beta A|T|^\beta)^\alpha)} \subset \overline{R(|T|)}$. Since U^*U is the initial projection onto $\overline{R(|T|)}$, we have $U^*U(|T|^\beta A|T|^\beta)^\alpha = (|T|^\beta A|T|^\beta)^\alpha$ for $\alpha > 0$.

Proof of (2). Since $T^* = U^*|T^*|$ is the polar decomposition of T^*, we have (2) by applying (1).

Proof of (3). Firstly we have

$$\begin{aligned}(U|T|^\beta A|T|^\beta U^*)^2 &= U|T|^\beta A|T|^\beta U^* \cdot U|T|^\beta A|T|^\beta U^* \\ &= U(|T|^\beta A|T|^\beta)^2 U^*\end{aligned}$$

since U^*U is the initial projection onto $\overline{R(|T|^\beta)}$. Similarly, by induction,

$$(U|T|^\beta A|T|^\beta U^*)^{\frac{n}{m}} = U(|T|^\beta A|T|^\beta)^{\frac{n}{m}} U^*$$

holds for any natural number n and m by using (1), so that the continuity of an operator yields $(U|T|^\beta A|T|^\beta U^*)^\alpha = U(|T|^\beta A|T|^\beta)^\alpha U^*$ by attending $\frac{n}{m} \to \alpha$, so the proof is complete.

Proof of (4). Since $T^* = U^*|T^*|$ is the polar decomposition of T^*, we have (4) by applying (3).

Hence the proof of Lemma 2.1 is complete. □

Proof of Theorem A.2.
Proof of (1). Let $A = |T|^{2p}$ and $B = |T^*|^{2p}$. p-hyponormality of T ensures $A \geq B \geq 0$. Applying Theorem F to $A \geq B \geq 0$ since $(1+\frac{t}{p})\frac{s+t}{p+\min\{s,t\}} \geq \frac{s}{p}+\frac{t}{p}$

and $\frac{s+t}{p+\min\{s,t\}} \geq 1$, we have

$$
\begin{aligned}
(\tilde{T}^*_{s,t}\tilde{T}_{s,t})^{\frac{p+\min\{s,t\}}{s+t}} &= (|T|^t U^*|T|^{2s}U|T|^t)^{\frac{p+\min\{s,t\}}{s+t}} \\
&= (U^*U|T|^t U^*|T|^{2s}U|T|^t U^*U)^{\frac{p+\min\{s,t\}}{s+t}} \\
&= (U^*|T^*|^t|T|^{2s}|T^*|^t U)^{\frac{p+\min\{s,t\}}{s+t}} \\
&= U^*(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{p+\min\{s,t\}}{s+t}}U \quad \text{by (4) of Lemma 2.1} \qquad (2.1)\\
&= U^*(B^{\frac{t}{2p}}A^{\frac{s}{p}}B^{\frac{t}{2p}})^{\frac{p+\min\{s,t\}}{s+t}}U \\
&\geq U^*B^{\frac{p+\min\{s,t\}}{p}}U \\
&= U^*|T^*|^{2(p+\min\{s,t\})}U \\
&= |T|^{2(p+\min\{s,t\})}.
\end{aligned}
$$

Again applying Theorem F to $A \geq B \geq 0$ since $(1+\frac{s}{p})\frac{s+t}{p+\min\{s,t\}} \geq \frac{t}{p}+\frac{s}{p}$ and $\frac{s+t}{p+\min\{s,t\}} \geq 1$, we have

$$
\begin{aligned}
(\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{p+\min\{s,t\}}{s+t}} &= (|T|^s U|T|^{2t}U^*|T|^s)^{\frac{p+\min\{s,t\}}{s+t}} \\
&= (|T|^s|T^*|^{2t}|T|^s)^{\frac{p+\min\{s,t\}}{s+t}} \\
&= (A^{\frac{s}{2p}}B^{\frac{t}{p}}A^{\frac{s}{2p}})^{\frac{p+\min\{s,t\}}{s+t}} \qquad (2.2)\\
&\leq A^{\frac{p+\min\{s,t\}}{p}} \\
&= |T|^{2(p+\min\{s,t\})}.
\end{aligned}
$$

Hence (2.1) and (2.2) ensure

$$(\tilde{T}^*_{s,t}\tilde{T}_{s,t})^{\frac{p+\min\{s,t\}}{s+t}} \geq |T|^{2(p+\min\{s,t\})} \geq (\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{p+\min\{s,t\}}{s+t}},$$

that is, $\tilde{T}_{s,t}$ is $\frac{p+\min\{s,t\}}{s+t}$-hyponormal.

Proof of (2). p-hyponormality of T ensures

$$|T|^{2s} \geq |T^*|^{2s} \tag{2.3}$$

and

$$|T|^{2t} \geq |T^*|^{2t} \tag{2.4}$$

for $p \geq \max\{s,t\}$ by Löwner-Heinz theorem. By (2.3) and (2.4), we have

$$\tilde{T}^*_{s,t}\tilde{T}_{s,t} = |T|^t U^*|T|^{2s}U|T|^t \geq |T|^t U^*|T^*|^{2s}U|T|^t = |T|^{2(s+t)} \tag{2.5}$$

and

$$\tilde{T}_{s,t}\tilde{T}^*_{s,t} = |T|^s U|T|^{2t}U^*|T|^s = |T|^s|T^*|^{2t}|T|^s \leq |T|^{2(s+t)}. \tag{2.6}$$

Hence (2.5) and (2.6) ensure

$$\tilde{T}^*_{s,t}\tilde{T}_{s,t} \geq |T|^{2(s+t)} \geq \tilde{T}_{s,t}\tilde{T}^*_{s,t},$$

that is, $\tilde{T}_{s,t}$ is hyponormal.

Whence the proof of Theorem A.2 is complete. □

Proof of Theorem A.3. Suppose T is log-hyponormal, i.e.,

$$\log|T|^2 \geq \log|T^*|^2. \tag{2.7}$$

By Theorem B.1, (2.7) is equivalent to

$$|T|^{2p} \geq (|T|^p|T^*|^{2r}|T|^p)^{\frac{p}{p+r}} \quad \text{for all } p \geq 0 \text{ and } r \geq 0. \tag{2.8}$$

By Lemma F, (2.8) is equivalent to the following (2.9).

$$(|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r} \quad \text{for all } p \geq 0 \text{ and } r \geq 0. \tag{2.9}$$

Then

$$\begin{aligned}
(\tilde{T}^*_{s,t}\tilde{T}_{s,t})^{\frac{\min\{s,t\}}{s+t}} &= (|T|^tU^*|T|^{2s}U|T|^t)^{\frac{\min\{s,t\}}{s+t}} \\
&= (U^*U|T|^tU^*|T|^{2s}U|T|^tU^*U)^{\frac{\min\{s,t\}}{s+t}} \\
&= (U^*|T^*|^t|T|^{2s}|T^*|^tU)^{\frac{\min\{s,t\}}{s+t}} \\
&= U^*(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{\min\{s,t\}}{s+t}}U \quad \text{by (4) of Lemma 2.1} \\
&\geq U^*|T^*|^{2\min\{s,t\}}U \\
&= |T|^{2\min\{s,t\}}
\end{aligned} \tag{2.10}$$

and the last inequality holds by (2.9) and Löwner-Heinz theorem.

On the other hand,

$$\begin{aligned}
(\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{\min\{s,t\}}{s+t}} &= (|T|^sU|T|^{2t}U^*|T|^s)^{\frac{\min\{s,t\}}{s+t}} \\
&= (|T|^s|T^*|^{2t}|T|^s)^{\frac{\min\{s,t\}}{s+t}} \\
&\leq |T|^{2\min\{s,t\}}
\end{aligned} \tag{2.11}$$

and the last inequality holds by (2.8) and Löwner-Heinz theorem.

Therefore (2.10) and (2.11) ensure

$$(\tilde{T}^*_{s,t}\tilde{T}_{s,t})^{\frac{\min\{s,t\}}{s+t}} \geq |T|^{2\min\{s,t\}} \geq (\tilde{T}_{s,t}\tilde{T}^*_{s,t})^{\frac{\min\{s,t\}}{s+t}},$$

that is, $\tilde{T}_{s,t}$ is $\frac{\min\{s,t\}}{s+t}$-hyponormal.

Hence the proof of Theorem A.3 is complete. □

§3. A generalization of w-hyponormal and class $A(k)$

As a generalization of both class $A(k)$ operators and w-hyponormal operators, we shall introduce a new class of operators as follows:

Definition 3.1. *For each $s > 0$ and $t > 0$, an operator T belongs to class $wA(s,t)$ if an operator T satisfies*

$$(|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \tag{3.1}$$

and

$$|T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}. \tag{3.2}$$

We remark that (3.1) is equivalent to (3.2) by Lemma F if T is invertible.

Firstly we have the following two propositions.

Proposition 3.2. *Let $T = U|T|$ be the polar decomposition of T and $\tilde{T}_{s,t} = |T|^sU|T|^t$ for $s > 0$ and $t > 0$. Then T is a class $wA(s,t)$ operator if and only if T satisfies*

$$|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} \geq |T|^{2t} \tag{3.3}$$

and

$$|T|^{2s} \geq |\tilde{T}^*_{s,t}|^{\frac{2s}{s+t}}. \tag{3.4}$$

We would like to cite the following result by Proposition 3.2 or scrutinizing the proof of Theorem A.3.

Remark 3.3. *Let $T = U|T|$ be the polar decomposition of an operator T which belongs to class $wA(s,t)$ for $s > 0$ and $t > 0$. Then $\tilde{T}_{s,t} = |T|^sU|T|^t$ is $\frac{\min\{s,t\}}{s+t}$-hyponormal, that is, Theorem A.3 on log-hyponormal remains valid for T in $wA(s,t)$.*

Proposition 3.4. *Let $T = U|T|$ be the polar decomposition of T and $\tilde{T}_{s,t} = |T|^sU|T|^t$ for $s > 0$ and $t > 0$. Then the following assertions hold;*

(1) *T is a class $wA(1,1)$ operator if and only if $|T^2| \geq |T|^2$ and $|T^*|^2 \geq |T^{*2}|$ hold.*

(2) *If T is a class $wA(s,1)$ operator, then T is class $A(s)$. Especially an invertible operator T is a class $wA(s,1)$ operator if and only if T is a class $A(s)$ operator.*

(3) *T is a class $wA(s,s)$ operator if and only if* $|\tilde{T}_{s,s}| \geq |T|^{2s} \geq |\tilde{T}^*_{s,s}|$.

(4) *T is a class $wA(\frac{1}{2},\frac{1}{2})$ operator if and only if T is a w-hyponormal operator.*

Proposition 3.2 states that (3.1) (resp. (3.2)) can be rewritten in (3.3) (resp. (3.4)) using generalized Aluthge transformation $\tilde{T}_{s,t} = |T|^s U |T|^t$. And also Proposition 3.4 asserts that class $wA(s,t)$ is a generalization of both class $A(k)$ operators and w-hyponormal operators.

Proof of Proposition 3.2.
(a). Proof of the result that (3.1) *is equivalent to* (3.3). Suppose that

$$|\tilde{T}_{s,t}|^{\frac{2t}{s+t}} = (|T|^t U^* |T|^{2s} U |T|^t)^{\frac{t}{s+t}} \geq |T|^{2t}. \tag{3.3}$$

(3.3) ensures the following (3.5).

$$U(|T|^t U^* |T|^{2s} U |T|^t)^{\frac{t}{s+t}} U^* \geq U |T|^{2t} U^*. \tag{3.5}$$

And also (3.3) follows from (3.5) by (1) of Lemma 2.1. Hence (3.3) is equivalent to (3.5), and (3.5) holds if and only if

$$(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t} \tag{3.1}$$

by (3) of Lemma 2.1, so that (3.1) is equivalent to (3.3).

(b). Proof of the result that (3.2) *is equivalent to* (3.4).

Since $(|T|^s |T^*|^{2t} |T|^s)^{\frac{s}{s+t}} = (|T|^s U |T|^{2t} U^* |T|^s)^{\frac{s}{s+t}} = |\tilde{T}^*_{s,t}|^{\frac{2s}{s+t}}$, it is easily obtained.

Hence the proof of Proposition 3.2 is complete. □

Proof of Proposition 3.4.
Proof of (1).
(a). Proof of the result that $|T^2| \geq |T|^2$ is equivalent to $|\tilde{T}_{1,1}| \geq |T|^2$. We easily obtain $|T^2| = (T^*T^*TT)^{\frac{1}{2}} = (|T|U^*|T|^2U|T|)^{\frac{1}{2}} = |\tilde{T}_{1,1}|$, so that the proof is complete.

(b). Proof of the result that $|T^*|^2 \geq |T^{*^2}|$ is equivalent to $|T|^2 \geq |\tilde{T}^*_{1,1}|$. Suppose that

$$|T^*|^2 \geq |T^{*^2}| = (TTT^*T^*)^{\frac{1}{2}} = (U|T||T^*|^2|T|U^*)^{\frac{1}{2}}. \tag{3.6}$$

By (3) of Lemma 2.1, (3.6) holds if and only if

$$U|T|^2U^* \geq U(|T|U|T|^2U^*|T|)^{\frac{1}{2}}U^*. \tag{3.7}$$

(3.7) ensures the following (3.8) by (1) of Lemma 2.1.

$$|T|^2 \geq (|T|U|T|^2U^*|T|)^{\frac{1}{2}} = |\tilde{T}^*_{1,1}|. \tag{3.8}$$

And also (3.7) follows from (3.8), so that the proof is complete.

Finally $|\tilde{T}_{1,1}| \geq |T|^2$ and $|T|^2 \geq |\tilde{T}^*_{1,1}|$ hold if and only if T is class $wA(1,1)$ by Proposition 3.2. Hence the proof of (1) is complete by (a) and (b).

Proof of (2). If T is class $wA(s,1)$, then the following (3.9) holds.

$$(|T^*||T|^{2s}|T^*|)^{\frac{1}{s+1}} \geq |T^*|^2. \tag{3.9}$$

(3.9) ensures the following (3.10).

$$U^*(|T^*||T|^{2s}|T^*|)^{\frac{1}{s+1}}U \geq U^*|T^*|^2U. \tag{3.10}$$

And also (3.9) follows from (3.10) by (2) of Lemma 2.1. Hence (3.9) is equivalent to (3.10), and (3.10) holds if and only if

$$(T^*|T|^{2s}T)^{\frac{1}{s+1}} = (U^*|T^*||T|^{2s}|T^*|U)^{\frac{1}{s+1}} \geq |T|^2,$$

by (4) of Lemma 2.1, that is, (3.9) holds if and only if T is class $A(s)$. Therefore T is class $A(s)$ if T is class $wA(s,1)$.

Moreover assume that T is invertible. Then (3.9) is equivalent to the following (3.11) by Lemma F.

$$|T|^{2s} \geq (|T|^s|T^*|^2|T|^s)^{\frac{s}{s+1}}. \tag{3.11}$$

Consequently, if T is invertible and class $A(s)$, then (3.9) and (3.11) holds, that is, T is class $wA(s,1)$. Hence the proof of (2) is complete.

Proof of (3). We have only to put $t = s$ in Proposition 3.2.

Proof of (4). We have only to put $s = \frac{1}{2}$ in (3).

Whence the proof of Theorem 3.4 is complete. □

We obtain the following Theorem 3.5 as an extension of Theorem A.4 and Theorem A.5.

Theorem 3.5.

(1) *For each $p > 0$, every p-hyponormal operator is a class $wA(s,t)$ operator for $s > 0$ and $t > 0$.*

(2) *Every log-hyponormal operator is a class $wA(s,t)$ operator for $s > 0$ and $t > 0$.*

(3) *For each $s > 0$ and $t > 0$, every class $wA(s,t)$ operator is a class $wA(\alpha,\beta)$ operator for any $\alpha \geq s$ and $\beta \geq t$.*

In fact Theorem 3.5 implies Theorem A.4 by putting $p = s$ and $t = \beta = 1$ in Theorem 3.5 and (2) of Proposition 3.4, and also Theorem 3.5 implies Theorem A.5 by putting $s = t = \frac{1}{2}$ and $\alpha = \beta = 1$ in Theorem 3.5 and (1) and (4) of Proposition 3.4.

In order to give a proof of Theorem 3.5, we need the following Theorem 3.6.

Theorem 3.6. *Let A and B be positive operators such that*

$$A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} \tag{3.12}$$

and

$$(B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0} \tag{3.13}$$

hold for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then the following inequalities hold:

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} \tag{3.14}$$

and

$$(B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \geq B^{\beta} \tag{3.15}$$

for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

We remark that Theorem 3.6 does not require invertibility of A and B. Theorem 3.6 implies the following Theorem C.1 since (3.12) (resp. (3.14)) is equivalent to (3.13) (resp. (3.15)) by Lemma F if A and B are invertible.

Theorem C.1 ([10]). *Let A and B be positive invertible operators such that*

$$A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} \tag{3.12}$$

holds for fixed $\alpha_0 > 0$ and $\beta_0 > 0$. Then the following inequality holds:

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} \tag{3.14}$$

for all $\alpha \geq \alpha_0$ and $\beta \geq \beta_0$.

Proof of Theorem 3.6.
(a). Proof of (3.14). Applying Theorem F to (3.13), we have

$$\{B^{\frac{\beta_0 r_1}{2}} (B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0 p_1}{\alpha_0+\beta_0}} B^{\frac{\beta_0 r_1}{2}}\}^{\frac{1+r_1}{p_1+r_1}} \geq B^{\beta_0(1+r_1)} \tag{3.16}$$

for any $p_1 \geq 1$ and $r_1 \geq 0$. Putting $p_1 = \frac{\alpha_0+\beta_0}{\beta_0} \geq 1$ in (3.16), we have

$$(B^{\frac{\beta_0(1+r_1)}{2}} A^{\alpha_0} B^{\frac{\beta_0(1+r_1)}{2}})^{\frac{\beta_0(1+r_1)}{\alpha_0+\beta_0+\beta_0 r_1}} \geq B^{\beta_0(1+r_1)} \tag{3.17}$$

for any $r_1 \geq 0$. Put $\beta = \beta_0(1+r_1) \geq \beta_0$ in (3.17). Then we have

$$(B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha_0+\beta}} \geq B^{\beta} \quad \text{for } \beta \geq \beta_0. \tag{3.18}$$

Next we show $f(\beta) = (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta}}$ is decreasing for $\beta \geq \beta_0$. By Löwner-Heinz theorem, (3.18) ensures the following (3.19).

$$(B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{w}{\alpha_0+\beta}} \geq B^{w} \quad \text{for } 0 \leq w \leq \beta. \tag{3.19}$$

Then we have

$$\begin{aligned} f(\beta) &= (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta}} \\ &= \{(A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0+\beta+w}{\alpha_0+\beta}}\}^{\frac{\alpha_0}{\alpha_0+\beta+w}} \\ &= \{A^{\frac{\alpha_0}{2}} B^{\frac{\beta}{2}} (B^{\frac{\beta}{2}} A^{\alpha_0} B^{\frac{\beta}{2}})^{\frac{w}{\alpha_0+\beta}} B^{\frac{\beta}{2}} A^{\frac{\alpha_0}{2}}\}^{\frac{\alpha_0}{\alpha_0+\beta+w}} \quad \text{by Lemma F} \\ &\geq (A^{\frac{\alpha_0}{2}} B^{\beta+w} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta+w}} \quad \text{by (3.19)} \\ &= f(\beta+w). \end{aligned}$$

Hence $f(\beta)$ is decreasing for $\beta \geq \beta_0$. Therefore

$$A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta}} \quad \text{for } \beta \geq \beta_0 \tag{3.20}$$

holds since $A^{\alpha_0} \geq (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta_0}} = f(\beta_0) \geq f(\beta) = (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0}{\alpha_0+\beta}}$.

Again applying Theorem F to (3.20), we have

$$A^{\alpha_0(1+r_2)} \geq \{A^{\frac{\alpha_0 r_2}{2}} (A^{\frac{\alpha_0}{2}} B^{\beta} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 p_2}{\alpha_0+\beta}} A^{\frac{\alpha_0 r_2}{2}}\}^{\frac{1+r_2}{p_2+r_2}} \tag{3.21}$$

for any $p_2 \geq 1$ and $r_2 \geq 0$. Putting $p_2 = \frac{\alpha_0+\beta}{\alpha_0} \geq 1$ in (3.21), we have

$$A^{\alpha_0(1+r_2)} \geq (A^{\frac{\alpha_0(1+r_2)}{2}} B^{\beta} A^{\frac{\alpha_0(1+r_2)}{2}})^{\frac{\alpha_0(1+r_2)}{\alpha_0+\beta+\alpha_0 r_2}} \tag{3.22}$$

for any $r_2 \geq 0$. Put $\alpha = \alpha_0(1+r_2) \geq \alpha_0$ in (3.22). Then we have

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} \quad \text{for all } \alpha \geq \alpha_0 \text{ and } \beta \geq \beta_0, \tag{3.23}$$

so that the proof of (a) is complete.

(b). Proof of (3.15). Applying Theorem F to (3.12), we have

$$A^{\alpha_0(1+r_3)} \geq \{A^{\frac{\alpha_0 r_3}{2}} (A^{\frac{\alpha_0}{2}} B^{\beta_0} A^{\frac{\alpha_0}{2}})^{\frac{\alpha_0 p_3}{\alpha_0+\beta_0}} A^{\frac{\alpha_0 r_3}{2}}\}^{\frac{1+r_3}{p_3+r_3}} \tag{3.24}$$

for any $p_3 \geq 1$ and $r_3 \geq 0$. Putting $p_3 = \frac{\alpha_0+\beta_0}{\alpha_0} \geq 1$ in (3.24), we have

$$A^{\alpha_0(1+r_3)} \geq (A^{\frac{\alpha_0(1+r_3)}{2}} B^{\beta_0} A^{\frac{\alpha_0(1+r_3)}{2}})^{\frac{\alpha_0(1+r_3)}{\alpha_0+\beta_0+\alpha_0 r_3}} \tag{3.25}$$

for any $r_3 \geq 0$. Put $\alpha = \alpha_0(1+r_3) \geq \alpha_0$ in (3.25). Then we have

$$A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta_0}} \quad \text{for } \alpha \geq \alpha_0. \tag{3.26}$$

Next we show $g(\alpha) = (B^{\frac{\beta_0}{2}} A^{\alpha} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha+\beta_0}}$ is increasing for $\alpha \geq \alpha_0$. By Löwner-Heinz theorem, (3.26) ensures the following (3.27).

$$A^{u} \geq (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{u}{\alpha+\beta_0}} \quad \text{for } 0 \leq u \leq \alpha. \tag{3.27}$$

Then we have

$$\begin{aligned}
g(\alpha) &= (B^{\frac{\beta_0}{2}} A^{\alpha} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha+\beta_0}} \\
&= \{(B^{\frac{\beta_0}{2}} A^{\alpha} B^{\frac{\beta_0}{2}})^{\frac{\alpha+\beta_0+u}{\alpha+\beta_0}}\}^{\frac{\beta_0}{\alpha+\beta_0+u}} \\
&= \{B^{\frac{\beta_0}{2}} A^{\frac{\alpha}{2}} (A^{\frac{\alpha}{2}} B^{\beta_0} A^{\frac{\alpha}{2}})^{\frac{u}{\alpha+\beta_0}} A^{\frac{\alpha}{2}} B^{\frac{\beta_0}{2}}\}^{\frac{\beta_0}{\alpha+\beta_0+u}} \quad \text{by Lemma F} \\
&\leq (B^{\frac{\beta_0}{2}} A^{\alpha+u} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha+u+\beta_0}} \quad \text{by (3.27)} \\
&= g(\alpha+u).
\end{aligned}$$

Hence $g(\alpha)$ is increasing for $\alpha \geq \alpha_0$. Therefore

$$(B^{\frac{\beta_0}{2}} A^\alpha B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha+\beta_0}} \geq B^{\beta_0} \quad \text{for } \alpha \geq \alpha_0 \tag{3.28}$$

holds since $(B^{\frac{\beta_0}{2}} A^\alpha B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha+\beta_0}} = g(\alpha) \geq g(\alpha_0) = (B^{\frac{\beta_0}{2}} A^{\alpha_0} B^{\frac{\beta_0}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geq B^{\beta_0}$.

Again applying Theorem F to (3.28), we have

$$\{B^{\frac{\beta_0 r_4}{2}} (B^{\frac{\beta_0}{2}} A^\alpha B^{\frac{\beta_0}{2}})^{\frac{\beta_0 p_4}{\alpha+\beta_0}} B^{\frac{\beta_0 r_4}{2}}\}^{\frac{1+r_4}{p_4+r_4}} \geq B^{\beta_0(1+r_4)} \tag{3.29}$$

for any $p_4 \geq 1$ and $r_4 \geq 0$. Putting $p_4 = \frac{\alpha+\beta_0}{\beta_0} \geq 1$ in (3.29), we have

$$(B^{\frac{\beta_0(1+r_4)}{2}} A^\alpha B^{\frac{\beta_0(1+r_4)}{2}})^{\frac{\beta_0(1+r_4)}{\alpha+\beta_0+\beta_0 r_4}} \geq B^{\beta_0(1+r_4)} \tag{3.30}$$

for any $r_4 \geq 0$. Put $\beta = \beta_0(1+r_4) \geq \beta_0$ in (3.30). Then we have

$$(B^{\frac{\beta}{2}} A^\alpha B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \geq B^\beta \quad \text{for all } \alpha \geq \alpha_0 \text{ and } \beta \geq \beta_0, \tag{3.31}$$

so that the proof of (b) is complete.

Whence the proof of Theorem 3.6 is complete. □

Proof of Theorem 3.5.
Proof of (1). Suppose that T is p-hyponormal for $p > 0$, i.e., $|T|^{2p} \geq |T^*|^{2p}$, and also let $A = |T|^{2p}$ and $B = |T^*|^{2p}$. Applying Theorem F to $A \geq B \geq 0$ since $(1+\frac{t}{p})\frac{s+t}{t} \geq \frac{s}{p} + \frac{t}{p}$ and $\frac{s+t}{t} \geq 1$, we have

$$(|T^*|^t |T|^{2s} |T^*|^t)^{\frac{t}{s+t}} = (B^{\frac{t}{2p}} A^{\frac{s}{p}} B^{\frac{t}{2p}})^{\frac{t}{s+t}} \geq B^{\frac{t}{p}} = |T^*|^{2t} \tag{3.1}$$

for $s > 0$ and $t > 0$. Again applying Theorem F to $A \geq B \geq 0$ since $(1+\frac{s}{p})\frac{s+t}{s} \geq \frac{t}{p} + \frac{s}{p}$ and $\frac{s+t}{s} \geq 1$, we have

$$(|T|^s |T^*|^{2t} |T|^s)^{\frac{s}{s+t}} = (A^{\frac{s}{2p}} B^{\frac{t}{p}} A^{\frac{s}{2p}})^{\frac{s}{s+t}} \leq A^{\frac{s}{p}} = |T|^{2s} \tag{3.2}$$

for $s > 0$ and $t > 0$. Therefore T is class $wA(s,t)$ for $s > 0$ and $t > 0$.

Proof of (2). Suppose that T is log-hyponormal, i.e.,

$$\log |T|^2 \geq \log |T^*|^2. \tag{2.7}$$

By Theorem B.1, (2.7) is equivalent to

$$|T|^{2p} \geq (|T|^p |T^*|^{2r} |T|^p)^{\frac{p}{p+r}} \quad \text{for all } p \geq 0 \text{ and } r \geq 0. \tag{2.8}$$

By Lemma F, (2.8) is equivalent to the following (2.9).

$$(2.9) \qquad (|T^*|^r|T|^{2p}|T^*|^r)^{\frac{r}{p+r}} \geq |T^*|^{2r} \quad \text{for all } p \geq 0 \text{ and } r \geq 0.$$

Putting $p = s$ and $r = t$ in (2.9) and (2.8), we have

$$(3.1) \qquad (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}$$

and

$$(3.2) \qquad |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}.$$

Therefore T is class $wA(s,t)$ for $s > 0$ and $t > 0$.

Proof of (3). Suppose that T is class $wA(s,t)$ for $s > 0$ and $t > 0$, i.e., the following (3.1) and (3.2) hold.

$$(3.1) \qquad (|T^*|^t|T|^{2s}|T^*|^t)^{\frac{t}{s+t}} \geq |T^*|^{2t}.$$

$$(3.2) \qquad |T|^{2s} \geq (|T|^s|T^*|^{2t}|T|^s)^{\frac{s}{s+t}}.$$

By Theorem 3.6, we have

$$(|T^*|^\beta|T|^{2\alpha}|T^*|^\beta)^{\frac{\beta}{\alpha+\beta}} \geq |T^*|^{2\beta} \quad \text{and} \quad |T|^{2\alpha} \geq (|T|^\alpha|T^*|^{2\beta}|T|^\alpha)^{\frac{\alpha}{\alpha+\beta}}$$

for any $\alpha \geq s$ and $\beta \geq t$. Therefore T is class $wA(\alpha,\beta)$ for any $\alpha \geq s$ and $\beta \geq t$.

Hence the proof of Theorem 3.5 is complete. □

§4. Concluding remark

In Theorem 3.6, for $\alpha > 0$ and $\beta > 0$, we might expect that $A^\alpha \geq (A^{\frac{\alpha}{2}}B^\beta A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}}$ is equivalent to $(B^{\frac{\beta}{2}}A^\alpha B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \geq B^\beta$ even if A and B are not invertible. But it is not true by the following Example 4.1.

Example 4.1. *There exists positive operators A and B such that $A^\alpha \geq (A^{\frac{\alpha}{2}}B^\beta A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}}$ and $(B^{\frac{\beta}{2}}A^\alpha B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \ngeq B^\beta$ for any $\alpha > 0$ and $\beta > 0$.*

Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$A^{\alpha} - (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

and

$$(B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} - B^{\beta} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \ngeq 0$$

for $\alpha > 0$ and $\beta > 0$. Therefore $A^{\alpha} \geq (A^{\frac{\alpha}{2}} B^{\beta} A^{\frac{\alpha}{2}})^{\frac{\alpha}{\alpha+\beta}}$ and $(B^{\frac{\beta}{2}} A^{\alpha} B^{\frac{\beta}{2}})^{\frac{\beta}{\alpha+\beta}} \ngeq B^{\beta}$ for any $\alpha > 0$ and $\beta > 0$.

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Masatoshi Ito
Department of Applied Mathematics, Faculty of Science, Science University of Tokyo,
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan
E-mail address: `m-ito@am.kagu.sut.ac.jp`