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A DECOMPOSITION THEOREM OF g-MARTINGALES *

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Abstract. In this paper, we first introduce the notions of nonlinear mathematical expectations and nonlinear martingales via backward stochastic differential equations(BSDEs) introduced by Duffie & Epstein and Skiadas. And then, we prove a general nonlinear decomposition theorem. Our decomposition theorem generalizes Doob-Meyer decomposition theorem.

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1. Introduction

The famous Doob-Meyer decomposition theorem is an important theorem in stochastic calculus. For many years, various versions of decomposition theorems and their applications have been discussed by many researchers, such as optional decomposition theorem (El.Karoui and Quneez(1995), D. Kramkov (1994)]); F–S decomposition theorem (Föllmer and Schweizer(1991); Jacka's martingale representation theorem (Jacka (1992)) and Ansel and Stricker 's theorem (Ansel and Stricker(1992)) etc.. In this paper, we first introduce the notions of general expectations (in short g-expectations), (which usually are non-additive) and the corresponding g-martingales via a class of backward stochastic differential equations (BSDEs) introduced by Duffie and Epstien and Skiadas, and then, we study a decomposition theorem for g-supermartingales. This result extends Doob-Meyer Theorem. Since g-supermartingale discussed in this paper usually is non-linear, thus the classical method is not valid. The method used in this paper is different from the classical method.

This paper is organized as follows: In Section 2, we present the notions of g-expectations, conditional g-expectations and g-martingales via a class of BSDEs introduced by Duffie & Epstein and Skiadas. In Section 3, we study a decomposition theorem of g-supermartingales.

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2. g-expectation and g-martingale introduced via BSDEs

In this section, we first present a BSDE first introduced by Duffie and Epstein [4], and then, we introduce the notions of g-expectations and g-martingales.

Let $(\Omega, \mathcal{F}, P)$ be probability space endowed with the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the "usual hypotheses". We suppose that $\mathcal{F}_0$ is trivial set and $\mathcal{F} = \mathcal{F}_\infty = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$. All processes mentioned in this paper are supposed to be $\{\mathcal{F}_t\}$-adapted. For any given $t \in [0, \infty]$, let us denote by $L^1(\Omega, \mathcal{F}_t, P)$ the set of all R-valued, $\mathcal{F}_t$-measurable random variables ξ such that

$$E|\xi| < +\infty.$$

Let $T > 0$ be fixed time horizon, we denote by $L^1(0, T, \mathcal{F}, P)$ the set of all R-valued, $\{\mathcal{F}_t\}$-adapted processes $\{\phi_t\}$ such that

$$E\int_0^T |\phi_s| ds < +\infty.$$

We identify two processes ϕ^1 and ϕ^2 in $L^1(0, T, \mathcal{F}, P)$, if

$$E\int_0^T |\phi_s^1 - \phi_s^2| ds = 0.$$

Duffie and Epstein [4] introduced the following BSDE:

$$y_t = E\big[(\xi + \int_t^T g_s(y_s) ds)|\mathcal{F}_t\big], \quad 0 \leq t \leq T \tag{2.1}$$

Here $\xi \in L^1(\Omega, \mathcal{F}, P)$ is given and $g : [0, T] \times \Omega \times R \to R$ is $\mathcal{B}([0, T]) \otimes \mathcal{F} \otimes \mathcal{B}(R)|\mathcal{B}(R)$ measurable function satisfying the following conditions:

$$\left\{\begin{array}{ll} \text{(i)} & g \text{ is uniformly Lipschitz with Lipschitz constant } \mu, \text{ i.e. there} \\ & \text{exists a constant } \mu > 0, \text{ such that } \forall y^i \in R, (i = 1, 2), \\ & |g_t(y^1) - g_t(y^2)| \leq \mu |y^1 - y^2|, \quad \forall t \in [0, T], \text{a.s.} \\ \text{(ii)} & \text{For any } \quad y \in R, \{g_t(y)\} \in L^1(0, T, \mathcal{F}, P). \end{array}\right\} \tag{H.1}$$

The following existence and uniqueness theorem is a special case of [2]:

Lemma 2.1. *Assume (H.1) holds on g, if $\xi \in L^1(\Omega, \mathcal{F}, P)$, then*

(1) *BSDE (2.1) has a unique RCLL adapted solution $\{y_t\}$ in $L^1(0, T, \mathcal{F}; P)$.*
(2) *Particularly, if we choose $g_t(y) := a_t y + H_t$, then the solution (y_t) of linear BSDE(2.1) can be given by*

$$y_t = E[\xi \exp(\int_t^T a_s ds) + \int_t^T H_s \exp(\int_t^s a_r dr) ds / \mathcal{F}_t], \quad 0 \leq t \leq T$$

Where $\{a_t\}_{t\geq 0}$ *is a bounded process and* $\{H_t\} \in L^1(0,T,\mathcal{F},P)$.

The following Lemma is called comparison theorem which plays an important role in our main results.

Lemma 2.2. *Under the assumption of Lemma 2.1, let* (y_t) *be the solution of BSDE (2.1) and* $(\overline{y_t})$ *be the solution of the following BSDE:*

$$\overline{y}_t = E[(\overline{\xi} + \int_t^T \overline{g}_s ds)|\mathcal{F}_t], \quad 0 \leq t \leq T, \tag{2.2}$$

where $\overline{\xi} \in L^1(\Omega,\mathcal{F},P)$ *and* $\overline{g} \in L^1(0,T,\mathcal{F},P)$.

(1) *If* $\xi \geq \overline{\xi}$, $g_t(\overline{y}_t) \geq \overline{g}_t$ *a.s.,* $\forall t \in [0,T)$, *then*

$$y_t \geq \overline{y}_t, \quad \forall t \in [0,T).$$

(2) **(Strict comparison theorem)** *If* $\xi > \overline{\xi}$ *(i.e.* $\xi \geq \overline{\xi}$ *a.s. and* $\xi \neq \overline{\xi}$*), then*

$$y_t > \overline{y_t}, \quad t \in [0,T].$$

Proof. Set

$$\widehat{y}_t := y_t - \overline{y}_t; \quad \widehat{\xi} := \xi - \overline{\xi}; \quad H_t := g_t(\overline{y}_t) - \overline{g}_t;$$
$$a_s := \begin{cases} \frac{g_s(y_s)-g_s(\overline{y}_s)}{y_s-\overline{y}_s}, & \text{if} \quad y_s \neq \overline{y}_s \ ; \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, for all $t \in [0,T]$, $|a_t| \leq \mu$ for the reason that g satisfies uniform Lipschitz condition. With the above notations, $(\widehat{y}_t)$ can be viewed as the solution of the following linear BSDE:

$$\widehat{y}_t = E\big[\widehat{\xi} + \int_t^T (a_s\widehat{y}_s + H_s)ds|\mathcal{F}_t\big] \tag{2.3}$$

Solving linear BSDE (2.3), by Lemma 2.1(2), we can obtain

$$\widehat{y}_t = E\big[(\widehat{\xi}\exp(\int_t^T a_s ds) + \int_t^T H_s \exp(\int_t^s a_r dr)ds)\big|\mathcal{F}_t\big]. \tag{2.4}$$

It then follows by $\widehat{\xi} \geq 0$ and $H_t \geq 0$ for all $t \in [0,T]$ that the proof of (1) is complete.

Note that for all $t \in [0,T]$, $|a_t| \leq \mu$, from Eq.(2.4), we can deduce that

$$\widehat{y}_t \geq e^{-\mu(T-t)}E[\widehat{\xi}|\mathcal{F}_t] > 0.$$

The proof of (2) is complete. □

We now introduce the generalized notions of g-expectation and g-martingale via BSDE(2.1).

Definition 2.3. Suppose $\xi \in L^1(\Omega, \mathcal{F}, P)$ and (H.1) holds on g. Let $\{y_t\}$ be the solution of BSDE (2.1), we call $E^g_{t,T}(\xi)$ denoted by

$$E^g_{t,T}(\xi) := y_t, \quad 0 \le t \le T$$

the *conditional g-expectation* of random variable ξ on time interval $[t, T]$ generated by g, in short *conditional g-expectation;* if $t = 0$, we call $E_{0,T}(\xi)$ the*g-expectation* of ξ on time interval $[0, T]$.

Remark.

(1) The above definition is based on the following observation: For any $\xi \in L^1(\Omega, \mathcal{F}, P)$, if we denote the conditional mathematical expectation of random variable ξ by $y_t := E[\xi|\mathcal{F}_t]$, then conditional mathematical expectation $E[\xi|\mathcal{F}_t]$ is the solution of BSDE(2.1) under $g \equiv 0$. Moreover, $E\xi = y_0$.

(2) If g is nonlinear, then (conditional) g-expectation is nonlinear too, for this reason, we sometimes call g-expectation nonlinear mathematical expectation.

If we further assume that g satisfies the following condition:

$$g_t(0) = 0, \quad \forall t \in [0, T], \tag{H.2}$$

then,by the above definitions, we have

Lemma 2.4. *Assume $\xi \in L^1(\Omega, \mathcal{F}, P)$, (H.1) and (H.2) hold on g, then for any $r \in [0, T]$, let $\eta := E^g_{r,T}(\xi)$, then $\eta \in L^1(\Omega, \mathcal{F}_r, P)$ is the unique random variable such that*

$$E^g_{0,T}(\mathbf{1}_A\xi) = E^g_{0,r}(\mathbf{1}_A\eta), \quad \forall A \in \mathcal{F}_r; \tag{2.5}$$

Proof. Let (y_t) be the solution of BSDE:

$$y_t = E\big[(\xi + \int_t^T g_s(y_s)ds)\big|\mathcal{F}_t\big].$$

For any $A \in \mathcal{F}_r$, multiplying $\mathbf{1}_A$ on both sides of the above equation and then observing $y_t\mathbf{1}_A$ on $[r, T]$. From the assumption (H.2), we can deduce the following relation

$$g_t(\mathbf{1}_A(\omega)y) = g_t(y)\mathbf{1}_A(\omega), \quad \forall (t, \omega, y) \in [0, T] \times \Omega \times R.$$

Note that $\forall t \in [r, T]$, $y_t\mathbf{1}_A$ is $\mathcal{F}_t$-adapted and $y_t\mathbf{1}_A$ solving the following BSDE on $[r, T]$:

$$\widetilde{y}_t = E\big[(\xi\mathbf{1}_A + \int_t^T g_s(\widetilde{y}_s)ds)|\mathcal{F}_t\big], \quad t \in [0, T].$$

Let $(\widetilde{y}_t)$ be the solution of the above BSDE, immediately, by Lemma 2.1 (the unicity of solution), we have

$$y_s \mathbf{1}_A = \widetilde{y}_s, \forall s \in [r,T], \tag{2.6}$$

Set $\eta := y_r$, obviously $\eta \in L^1(\Omega, \mathcal{F}_r, P)$. According to the definition of $E^g_{0,T}(\mathbf{1}_A\xi)$, applying equality (2.6), we have

$$E^g_{0,T}(\mathbf{1}_A\xi) = \widetilde{y}_0 = E^g_{0,r}(\widetilde{y}_r) = E^g_{0,r}(y_r\mathbf{1}_A) = E^g_{0,r}(\eta\mathbf{1}_A).$$

We now prove η is unique. Assume that there exists another $\overline{\eta} \in L^1(\Omega, \mathcal{F}_r, P)$ such that for any $A \in \mathcal{F}_r$,

$$E^g_{0,r}(\eta\mathbf{1}_A) = E^g_{0,r}(\overline{\eta}\mathbf{1}_A) \tag{2.7}$$

but $P(\eta \neq \overline{\eta}) > 0$.

We can choose $A := \{\eta \neq \overline{\eta}\}$, obviously $A \in \mathcal{F}_r$, it then follows by Strict Comparison Theorem (Lemma 2.2(2)) that

$$E^g_{0,r}(\eta\mathbf{1}_A) \neq E^g_{0,r}(\overline{\eta}\mathbf{1}_A)$$

which is in contradiction to (2.7). The proof is complete. □

The following counter-example is to show that equation (2.5) does not hold without the assumption of $g(0) \equiv 0$.

Example. Suppose $\xi \in L^1(\Omega, \mathcal{F}, P)$, let $g \equiv 1$, we choose $r = \frac{T}{2} > 0$. Then there is no $\eta \in L^1(\Omega, \mathcal{F}_{\frac{T}{2}}, P)$ satisfying(2.5).

In fact, if there exists such η satisfying (2.5),i.e.

$$E^1_{0,T}(\xi\mathbf{1}_A) = E^1_{0,\frac{T}{2}}(\eta\mathbf{1}_A), \quad \forall A \in \mathcal{F}_{\frac{T}{2}}.$$

then by Definition 2.3, $E^1_{0,T}(\xi\mathbf{1}_A) = E(\xi\mathbf{1}_A + T)$ and $E^1_{0,\frac{T}{2}}(\eta\mathbf{1}_A) = E(\eta\mathbf{1}_A + \frac{T}{2})$.

Thus

$$E(\xi\mathbf{1}_A + T) = E(\eta\mathbf{1}_A + \frac{T}{2}), \quad \forall A \in \mathcal{F}_{\frac{T}{2}}.$$

In particular, let $A := \emptyset$, then from the above equality, we have $T = \frac{T}{2}$, which is impossible.

Remark. Similar to the classical mathematical expectation, we can call η defined in (2.5) the condtional g-expectation of ξ in the interval $[r,T]$. However, motivated by Lemma 2.4 (ii), we can define the more general conditional g-expectation without the assumption (H.2).

Using conditional g-expectation, we can naturally define g-martingale just as in the classical case. To do this, we first introduce the solution of BSDEs on a random variable interval.

Assume that τ is a stopping time with value in $[0,T]$, in this paper, we denote the solution of the following BSDE on random variable interval $[0,\tau]$:

$$y_t = E\big[(\xi + \int_t^{\tau} g_s(y_s)ds)\big|\mathcal{F}_t\big], \quad t \in [0,\tau]$$

in the sense of

$$y_t = E\big[(\xi + \int_t^{T} 1_{[0,\tau]}(s) g_s(y_s)ds)\big|\mathcal{F}_t\big], \quad t \in [0,T]$$

Similar to Definition 2.5, we denote $E^g_{t,\tau}(\xi)$ by

$$E^g_{t,\tau}(\xi) := y_t.$$

Definition 2.5. A right-continuous adapted process $\{X_t\}$ is called *g-martingale on* $[0,T]$ *(resp. g-supermartingale, g-submartingale)*, if for any $t \in [0,T]$, $E|X_t| < \infty$ and for any stopping times σ and τ , if $0 \le \sigma \le \tau < T$, then

$$E^g_{\sigma,\tau}(X_\tau) = X_\sigma, \qquad (\text{ resp } \qquad \le X_\sigma \ , \ \ge X_\sigma).$$

A g-supermartingale $\{X_t\}_{t\ge 0}$ is said to be of class (DL), if $\{X_t\}$ is of class (DL).

Remark.

(1) For notational simplicity, we adopt the above (strong) definition of g-martingale, In fact, we can adopt the following (weak) definition similar to the definition of the classical martingale i.e.

$$E^g_{s,t}(X_t) = X_s, \quad (\text{resp. } \le X_s, \ge X_s); \quad \forall 0 \le s \le t \le T.$$

Chen and Peng have showed that the above definitions are equivalent under some assumptions on g (see [3]).

(2) Obviously, if $g \equiv 0$, then a g-martingale is a classical martingale.

(3) g-martingales usually are nonlinear, i.e. g-martingales usually are non-additive.

3. Nonlinear Decomposition Theorem for g-martingales

In this section, we assume that $\{X_t\}$ is a right continuous g-supermartingale such that $E\int_0^T |X_s|ds < \infty$. thus, for each $n \ge 1$, the following BSDE:

$$y^n_t = E\big[X_T + \int_t^T \big(g_s(y^n_s) + n(X_s - y^n_s)^+\big)ds\big|\mathcal{F}_t\big], \qquad t \in [0,T]. \tag{3.1}$$

has a unique solution (y^n_t). Moreover, by Comparison Theorem (Lemma 2.2), for each $t \in [0,T]$, $\{y^n_t\}$ is increasing as n increases. Where $X^+ := \max\{X, 0\}$, no loss of generacity, in this section we assume $g_t(0) = 0$.

The following Lemma shows that $\{y^n_t\}$ is bounded by $\{X_t\}$.

Lemma 3.1. *Let $\{X_t\}_{t\geq 0}$ be a right continuous g-supermartingale and $\{y_t^n\}$ be the solution of BSDE (3.1), then for any $n > 0$, we have*

$$y_t^n \leq X_t; \quad t \in [0, T]. \tag{3.2}$$

Moreover,

$$|y_t^n| \leq |X_t| + |E_{t,T}^g(X_T)|. \tag{3.3}$$

Proof. Obviously, (3.2) holds when $t = T$, we now prove (3.2) holds when $t \in [0, T)$. We argue by contradiction. If $\{X_t\}$ is not the case, since $\{X_t\}$ and y_t^n are right continuous, thus, there exist $n \geq 1$ and $\delta > 0$, such that the measure of

$$\{(\omega, t): \quad y_t^n - X_t \geq \delta\}$$

is a non-zero subset of $\Omega \times [0, T]$, we denote by the following stopping times:

$$\left| \begin{array}{l} \sigma := \inf\{t \geq 0; y_t^n - X_t \geq \delta\} \wedge T; \\ \tau := \inf\{t \geq \sigma; y_t^n \leq X_t\} \wedge T. \end{array} \right.$$

Since $y_t^n - X_t$ is right continuous, we have

$$\begin{array}{ll} (i) & y_\sigma^n \geq X_\sigma + \delta, \quad \text{on}\{\sigma < T\} \\ (ii) & X_\tau \geq y_\tau^n. \end{array}$$

Obviously, $0 \leq \sigma \leq \tau \leq T$ and $P(\tau > \sigma) > 0$, otherwise, if $P(\tau = \sigma) = 1$, then (i) is in contradiction to (ii).

Furthermore, according to Comparison Theorem, from (ii), we can deduce that

$$E_{\sigma,\tau}^g(X_\tau) \geq E_{\sigma,\tau}^g(y_\tau^n).$$

Noting that

$$y_t^n - X_t \geq 0 \qquad \text{on} \quad [\sigma, \tau).$$

Thus, from equation (3.1),

$$\begin{aligned} y_\sigma^n &= E[X_T + \int_\sigma^T (g_s(y_s^n) + n(X_s - y_s^n)^+) ds|\mathcal{F}_\sigma] \\ &= E[y_\tau^n + \int_\sigma^\tau (g_s(y_s^n) + n(X_s - y_s^n)^+) ds|\mathcal{F}_\sigma] \\ &= E[y_\tau^n + \int_\sigma^\tau g_s(y_s^n) ds|\mathcal{F}_\sigma] \\ &= E_{\sigma,\tau}^g(y_\tau^n) \\ &\leq E_{\sigma,\tau}^g(X_\tau). \end{aligned}$$

On the other hand, since $\{X_t\}$ is a g-supermartingale, thus

$$X_\sigma \geq E^g_{\sigma,\tau}(X_\tau).$$

Consequently,

$$X_\sigma \geq E^g_{\sigma,\tau}(y^n_\tau) \geq y^n_\sigma.$$

This is in contrary with (i). Hence, we obtain (3.2).

From (3.2) and Lemma 2.2, we have

$$X_t \geq y^n_t \geq E^g_{t,T}(X_T)$$

which implies (3.3). The proof is complete. $\square$

Set

$$A^n_t := n\int_0^t (X_s - y^n_s)^+ ds, \quad 0 \leq t \leq T, \tag{3.4}$$

then, BSDE (3.1) can be rewritten as:

$$y^n_t = E\big[X_T + A^n_T + \int_t^T g_s(y^n_s)ds|\mathcal{F}_t\big] - A^n_t,\ 0 \leq t \leq T. \tag{3.5}$$

We have the following Lemma:

Lemma 3.2. *If $\{X_t\}_{t\geq 0}$ is a right-continuous g-supermartingale of class (DL), then $\{A^n_T\}_{n>0}$ defined in (3.4) is uniformly integrable in $L^1(\Omega, \mathcal{F}, P)$.*

Proof. see Appendix.

Lemma 3.3. *For the above (y^n_t) and A^n_T, we have*

(1) *There exists a constant C, which is independent of n, such that*

$$EA^n_T \leq C;$$

(2) *For each $t \in [0,T]$, $\lim_{n\to\infty} y^n_t = X_t$.*

Proof. (1) is from Lemma 3.2. Now let us prove (2)

According to (3.2) and (3.4), applying the above result, we can obtain

$$\begin{aligned} E\int_0^T |X_s - y^n_s|ds &= E\int_0^T (X_s - y^n_s)ds \\ &= \frac{EA^n_T}{n} \leq \frac{C}{n} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

which implies

$$\lim_{n\to\infty} y^n = X$$

in $L^1(0, T, \mathcal{F}, P)$, it then follows by the fact that (X_t), (y^n_t) are right-continuous in $[0,T)$ that we can prove (2). $\square$

The following is adopted from [11].

Lemma 3.4. *Let $\{X^i(\cdot)\}$ be a family of RCLL adapted increasing on $[0,T]$ (i.e. for any $t \in [0,T]$, $X^i(t) \uparrow X(t)$) such that $X(t) = b(t) - A(t)$, here $b(\cdot)$ is a RCLL adapted process and $A(\cdot)$ is a increasing process with $A(0) = 0$ and $EA(T) < \infty$. then, $X(\cdot)$ and $A(\cdot)$ are also RCLL processes.*

Proof. See Appendix.

The following theorem is so-called nonlinear decomposition theorem.

Theorem 3.5. *If $\{X_t\}$ is a right continuous g-supermartingale of class (DL), then there exists a unique RCLL increasing $\{A_t\}$, such that $\{X_t\}$ satisfies the following BSDE:*

$$X_t = E\big[X_T + A_T + \int_t^T g_s(X_s)ds|\mathcal{F}_t\big] - A_t, \qquad t \in [0,T].$$

Proof. For each $n > 0$, let (y_t^n) be the solution of BSDE:

$$y_t^n = E\big[X_T + A_T^n + \int_t^T g_s(y_s^n)ds\big|\mathcal{F}_t\big] - A_t^n,$$

Where $A_t^n = n\int_0^t (X_s - y_s^n)^+ ds$.

Obviously, for each $n > 0$, $\{A_t^n\}$ is a continuous and increasing process. By Lemma 3.2 and Lemma 3.3(1) and the Dunford-Pettis compactness criterion (Dunford & Schwartz (1963), P. 294), the set $\{A_T^n\}_{n>0}$ is relatively compact in the weak topology of $L^1(\Omega, \mathcal{F}, P)$. Thus, there exists $A_T \in L^1(\Omega, \mathcal{F}_T, P)$ such that A_T^n weakly converges to A_T in $L^1(\Omega, \mathcal{F}, P)$. Moreover, it is easy to check that for each $t \in [0,T]$, $E[A_T^n|\mathcal{F}_t]$ weakly converges to $E[A_T|\mathcal{F}_t]$, (see Problem 4.11, P.27 [7]),

Denote A_t by

$$A_t := E\big[X_T + A_T + \int_t^T g_s(X_s)ds|\mathcal{F}_t\big] - X_t, \qquad t \in [0,T] \tag{3.6}$$

We only need to check that, for each $t \geq 0$, A_t^n also is weakly converges to A_t.

In fact, since $E\int_0^T |y_s^n - X_s|ds \to 0$ as $n \to \infty$, thus

$$E\big[\int_t^T g_s(y_s^n)ds|\mathcal{F}_t\big] \longrightarrow E\big[\int_t^T g_s(X_s)ds|\mathcal{F}_t\big], \quad \text{weakly in } L^1.$$

Applying Lemma 3.3(2), we have

$$A_t^n = E\big[X_T + A_T^n + \int_t^T g_s(y_s^n)ds|\mathcal{F}_t\big] - y_t^n$$

weakly converges to

$$E\big[X_T + A_T + \int_t^T g_s(X_s)ds|\mathcal{F}_t\big] - X_t = A_t$$

That is

$$X_t = E\big[X_T + A_T + \int_t^T g_s(X_s)ds|\mathcal{F}_t\big] - A_t.$$

Obviously (A_t) is an increasing process with $A_0 = 0$ such that $EA_T < \infty$. From Lemma 3.4, (A_t) is RCLL process. The proof is complete. □

Remark. Obviously, if $g \equiv 0$, let $M_t := E[(X_T + A_T)|\mathcal{F}_t]$, then $X_t = M_t + A_t$ which is Doob-Meyer decomposition theorem.

4. Appendix

The proof of Lemma 3.2 is similar to the classical case:

The proof of Lemma 3.2.

Let $c > 0$ be fixed, and set

$$\tau_c^n = \inf\big\{t \geq 0; A_t^n > c\big\} \wedge T$$
$$\tau_{\frac{c}{2}}^n = \inf\big\{t \geq 0; A_t^n > \frac{c}{2}\big\} \wedge T.$$

then $0 \leq A_{\tau_c^n}^n \leq c$; $\{\tau_c^n < T\} \subset \{\tau_{\frac{c}{2}}^n < T\}$ and $(A_T^n - A_{\tau_{\frac{c}{2}}^n}^n)1_{\{\tau_c^n < T\}} \geq \frac{c}{2}$.

Applying the classical optional stopping theorem to BSDE (3.5),

$$y_{\tau_c^n}^n = E\big[X_T + A_T^n + \int_{\tau_c^n}^T g_s(y_s^n)ds\big|\mathcal{F}_{\tau_c^n}\big] - A_{\tau_c^n}^n. \tag{A.1}$$

Noting that $\{A_T^n > c\} = \{\tau_c^n < T\}$, applying inequality (3.2) and (3.3), we can obtain, from (A.1)

$$\begin{aligned}
EA_T^n \mathbf{1}_{[A_T^n > c]} &\leq E\left(y_{\tau_c^n}^n - X_T - \int_{\tau_c^n}^T g_s(y_s^n)ds\right)\mathbf{1}_{[\tau_c^n < T]} + cP(\tau_c^n < T) \\
&\leq E\big(X_{\tau_c^n} - X_T - \int_{\tau_c^n}^T g_s(y_s^n)ds\big)\mathbf{1}_{[\tau_c^n < T]} + cP(\tau_c^n < T) \\
&\leq E\left(X_{\tau_c^n} - X_T + \mu\int_{\tau_c^n}^T |y_s^n|ds\right)\mathbf{1}_{[\tau_c^n < T]} + cP(\tau_c^n < T) \\
&\leq E\left[X_{\tau_c^n} - X_T + \mu\int_0^T (|X_s| + |E_{s,T}^g(X_T)|)ds\right]\mathbf{1}_{[\tau_c^n < T]} \\
&\qquad + cP(\tau_c^n < T)
\end{aligned} \tag{A.2}$$

On the other hand, from (A.2),

$$\begin{aligned} E[y^n_{\tau^n_{\frac{c}{2}}} - X_T - \int_{\tau^n_{\frac{c}{2}}}^T g_s(y^n_s)ds]\mathbf{1}_{\{\tau^n_{\frac{c}{2}}<T\}} &= E(A^n_T - A^n_{\tau^n_{\frac{c}{2}}})\mathbf{1}_{\{\tau^n_{\frac{c}{2}}<T\}} \\ &\geq E\left(A^n_T - A^n_{\tau^n_{\frac{c}{2}}}\right)\mathbf{1}_{[\tau^n_c<T]} \\ &\geq \frac{c}{2}P(\tau^n_c<T). \end{aligned}$$

Applying (3.2) and (3.3),

$$cP(\tau^n_c<T) \leq 2E[X_{\tau^n_{\frac{c}{2}}} - X_T + \mu\int_0^T (|X_s| + |E^g_{s,T}(X_T)|)ds]\mathbf{1}_{[\tau^n_{\frac{c}{2}}<T]}).$$

Consequently, from (A.2)

$$\begin{aligned} EA^n_T\mathbf{1}_{[A^n_T>c]} \leq{}& E[X_{\tau^n_c} - X_T + \mu\int_0^T (|X_s| + |E^g_{s,T}(X_T)|)ds]\mathbf{1}_{[\tau^n_c<T]} \\ &+ 2E[X_{\tau^n_{\frac{c}{2}}} - X_T + \mu\int_0^T (|X_s| + |E^g_{s,T}(X_T)|)ds]\mathbf{1}_{\{\tau^n_{\frac{c}{2}}<T\}}. \end{aligned} \tag{A.3}$$

Note that $\{X_t\}$ is a g-supermartingale of class(DL), and $\{X_t\}$ and $E^g_{t,T}(X_T)$ belong to $L^1(0,T,\mathcal{F},P)$.

Thus, from BSDE (3.5), we have

$$\begin{aligned} P(\tau^n_c<T) = P(A^n_T>c) &= \frac{EA^n_T}{c} \\ &= \frac{1}{c}E[y^n_0 - X_T - \int_0^T g_s(y^n_s)ds] \\ &\leq \frac{1}{c}E[X_0 - X_T + \mu\int_0^T (|X_s| + |E^g_{s,T}(X_T)|)ds] \\ &\to 0 \qquad \text{as} \quad c\uparrow +\infty \end{aligned} \tag{A.4}$$

Similarly,

$$P(\tau^n_{\frac{c}{2}}<T)\to 0 \quad \text{as} \quad c\uparrow\infty.$$

Combining (A.3) with (A.4), we can now conclude that $\{A^n_T\}_{n>0}$ is uniformly integrable. □

The proof of Lemma 3.4.

Since the processes $b(\cdot)$ and $A(\cdot)$ have paths with left limits, so is $X(\cdot)$, thus we only need to prove that $X(\cdot)$ is right-continuous.

Since for any $t \in [0,T)$, $A(t+) \geq A(t)$, thus

$$X(t+) = b(t) - A(t+) \leq X(t). \tag{A.5}$$

On the other hand, for any $\delta > 0$, there exists a positive integer $j = j(\delta, t)$ such that $X(t) \leq X^j(t) + \delta$. but $X^j(\cdot)$ is RCLL, therefore, there exists a positive integer $\epsilon_0 = \epsilon_0(j, t, \delta)$ such that $X^j(t) \leq X^j(t+\epsilon) + \delta$, $\forall \epsilon \in (0, \epsilon_0]$. thus, for any $\epsilon \in (0, \epsilon_0]$,

$$X(t) \leq X^j(t+\epsilon) + 2\delta \leq X(t+\epsilon) + 2\delta.$$

in particular, $X(t) \leq X(t+) + 2\delta$ and $X(t) \leq X(t+)$. It then follows by (A.5) that we can obtain that $X(\cdot)$ is right-continuous. □

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