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# NOTE ON THE SPECTRAL CONCENTRATION FOR THE SCHRÖDINGER OPERATOR WITH POINT INTERACTION

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**Abstract.** In this paper the family of the Schrödinger operator $H_\kappa = -d^2/dx^2 + q(x) + \delta(x-1)/\kappa$ in $L^2(0,\infty)$ is investigated. Roughly speaking, if $q(x) = o(x^{-2})$, then the spectral concentration occurs as $\kappa \to 0$.

*Key words and phrases.* Spectral concentration, Schrödinger operators, point interaction.

## 1. Introduction

We shall consider the Schrödinger operator family $\{H_\kappa\}$ $(\kappa \geq 0)$:

$$H_0 u(x) = -u''(x) + q(x)u(x) \quad \text{on } H_0^1(0,1) \cap H^2(0,1) \oplus H_0^1(1,\infty) \cap H^2(1,\infty),$$

$$H_\kappa u(x) = -u''(x) + q(x)u(x) + \frac{\delta(x-1)u(x)}{\kappa} \text{ in } L^2(0,\infty),\ \kappa > 0 \text{ (formally)},$$

where the domain of $H_\kappa$ is determined in Section 2 and $q$ is a real-valued continuous function on $[0,\infty)$ and $q(x) = o(1)$ as $x \to \infty$.

The various results for similar types of operators can be seen in [2], the references therein ([2, Chapter I.3]). Formally we see that $H_\kappa \to H_0$ as $\kappa \to 0$. In fact, we shall see that the convergence is the norm resolvent convergence (Lemma 4.2). We are interested in the embedded eigenvalues of $H_0$ and the positive spectrum of $H_\kappa$ as $\kappa \to 0$. We expect that the embedded eigenvalues of $H_0$ become resonances (cf. [7]). However, in this note we do not define resonances, and we call quasi-eigenstates instead of resonances. We shall investigate the spectral concentration (Theorem 1.1) and the exponential decay of the quasi-eigenstates of $H_\kappa$ (Theorem 4.9).

To describe our theorem we shall give some notations. We denote the resolvents of $H_0$ and $H_\kappa$ by $R_0(\zeta)$ and $R_\kappa(\zeta)$, for $\mathrm{Im}\zeta \neq 0$, respectively. That is , $R_0(\zeta) = (H_0 - \zeta)^{-1}$, $R_\kappa(\zeta) = (H_\kappa - \zeta)^{-1}$. Let $\tau$ be an isometry operator from $L^2(0,\infty)$ onto $L^2(1,\infty)$ such that $\tau u(x) = u(x-1)$ for $u \in L^2(0,\infty)$ and $U_0(t)$ a 1-parameter group on $L^2(0,\infty)$ such that $U_0(t)u(x) = e^{t/2}u(e^t x)$ $(t \in \mathbf{R})$. Since $\tau U_0(t)\tau^*$ is a 1-parameter unitary group on $L^2(1,\infty)$, there exists a unique self-adjoint operator $A$ in $L^2(1,\infty)$ such that $\tau U_0(t)\tau^* = e^{itA}$.

Remark that $e^{itA}u(x) = e^{t/2}u(e^t(x-1)+1)$ for $u \in L^2(1,\infty)$. Let $K_0$ be the Dirichlet Laplacian $-d^2/dx^2$ in $L^2(1,\infty)$ and $q(x) = q|_{(0,1)}(x) + q|_{[1,\infty)}(x) \equiv q_1(x) + q_2(x)$.

**Assumption (A).** We assume that $[q_2, A]$ is $K_0$-compact in the form sense and that $[[q_2, A], A]$ is $K_0$-bounded in the form sense.

**Theorem 1.1.** *Let $q(x)$ satisfy the Assumption (A), $\lambda_0 > 0$ an eigenvalue of $H_0$ and $P$ the orthogonal projection onto the eigenspace of $H_0$ corresponding to $\lambda_0$. Then there exists an closed interval $J(\kappa)$ such that $J(\kappa) \to \{\lambda_0\}$ $(\kappa \to 0)$ and*

$$E_\kappa(J(\kappa)) \longrightarrow P \ (\kappa \to 0) \text{ strongly,}$$

*where $E_\kappa$ is the spectral measure of $H_\kappa$.*

The organization of this note is as follows. In section 2 we shall define the formal operator $H_\kappa$ and determine the domain. In section 3 we describe the Livsic matrix and the Mourre theory. In section 4 we shall give a proof of Theorem 1.1 and the exponential time decay of quasi-eigenstates without the proof.

## 2. Definition of $H_\kappa$

We shall precisely define $H_\kappa$. We shall use the quadratic form (cf. [5, Chapter VI]).

**Lemma 2.1.** *Let $h_\kappa$ be a sesqui-linear form on $H_0^1(0,\infty)$ such that*

$$h_\kappa[u,v] = (u',v') + (qu,v) + \kappa^{-1}u(1)\overline{v(1)} \tag{2.1}$$

*for $u, v \in H_0^1(0,\infty)$. Then $h_\kappa$ is a symmetric closed form and $h_\kappa[u] := h_\kappa[u,u]$ is bounded from below: $h_\kappa[u] \ge m\|u\|^2$ $(u \in H_0^1(0,\infty))$, where $m := \inf_{x>0} q(x)$.*

*Proof.* Let $h_{max}$ be a sesqui-linear form on $H_0^1(0,\infty)$ such that

$$h_{max}[u,v] = (u',v') + (qu,v)$$

for $u, v \in H_0^1(0,\infty)$. Since $q(x)$ is continuous on $[0,\infty)$ and $q(x) \to 0$ as $x \to \infty$, we know that $h_{max}$ is a closed symmetric form and bounded from below ($h_{max}[u] \ge m\|u\|^2$) and that the operator associated with $h_{max}$ is $-d^2/dx^2 + q(x)$ with the Dirichlet condition at $x = 0$ in $L^2(0,\infty)$. (cf. [5, Chapter VI]). By $H_0^1(0,\infty) \subset C(0,\infty)$ the form $h_\kappa$ is well-defined on $H_0^1(0,\infty)$ and we can write (2.1) as

$$h_\kappa[u,v] = h_{max}[u,v] + \kappa^{-1}u(1)\overline{v(1)}$$

for $u, v \in H_0^1(0,\infty)$. Hence $D(h_{max}) = D(h_\kappa)$, $h_\kappa$ is symmetric and $h_\kappa[u] \geq h_{max}[u] \geq m\|u\|^2$. We shall prove that $h_\kappa$ is closed. It is easy to show that for $u \in H_0^1(0,\infty)$ and for $\varepsilon > 0$

$$|u(1)|^2 = \int_0^1 \frac{d}{dx}|u(x)|^2 dx = 2Re \int_0^1 u'(x)\overline{u(x)}dx$$
$$\leq \int_0^1 (\varepsilon|u'(x)|^2 + \varepsilon^{-1}|u(x)|^2)dx \leq \varepsilon\|u'\|^2 + \varepsilon^{-1}\|u\|^2.$$

Hence we see that

$$\kappa^{-1}|u'(1)|^2 \leq \varepsilon(h_{max} - m + 1)[u] + C\|u\|^2$$

for some $C \geq 0$. Since $h_{max} - m + 1$ is a closed symmetric positive form, we see by [5, Theorem VI.1.33] that $h_\kappa$ is a closed form. □

We shall define the operator $H_\kappa$ and determine its domain. By Lemma 2.1 and [5, Theorems VI.2.1 and 2.6] there exists a unique self-adjoint operator $H_\kappa$ associated with $h_\kappa$. And the domain of $H_\kappa$ is as follows:

**Proposition 2.2.** *$u \in D(H_\kappa)$ if and only if the following two conditions are satisfied:*

(*i*) $u \in H_0^1(0,\infty)$ *and* $u \in H^2(0,1) \oplus H^2(1,\infty)$,

(*ii*) $u(1) := u(1 \pm 0),\ \kappa(u'(1+0) - u'(1-0)) = u(1)$.

*Furthermore for* $u \in D(H_\kappa)$

$$H_\kappa u(x) = -u''(x) + q(x)u(x),\ x \neq 1.$$

*Proof.* Let $u \in D(H_\kappa)$. For $v \in C_0^\infty(0,1) \oplus C_0^\infty(1,\infty) (\subset H_0^1(0,\infty))$ by Lemma 2.1 and integration by parts we have

$$(H_\kappa u, v) = h_\kappa[u,v] = h_{max}[u,v] = (u, -v'') + (u, qv). \tag{2.2}$$

By using (2.2) we have $(H_\kappa u - qu, v) = (u, -v'')$. Hence we see that $u''|_{(0,1)}$ and $u''|_{(1,\infty)}$ exist in the distribution sense and that $v \in D(H_\kappa)$. Furthermore by integration by parts we have

$$(H_\kappa u, v) - (qu, v) = (-u'', v).$$

Since $C_0^\infty(0,1) \oplus C_0^\infty(1,\infty)$ is dense in $L^2(0,\infty)$, we see that $u'' \in L^2(0,\infty)$, that $u'|_{(0,1)}$ and $u'|_{(1,\infty)}$ are absolutely continuous and that $H_\kappa u(x) = -u''(x) + q(x)u(x)$ for $x \neq 1$. We shall prove that $\kappa(u'(1+0) - u'(1-0)) = u(1)$. By integration by parts we have for $u \in D(H_\kappa)$ and $v \in H_0^1(0,\infty)$

$$h_\kappa[u,v] = (-u'' + qu, v) + \kappa^{-1}u(1)\overline{v(1)} + u'(1-0)\overline{v(1-0)} - u'(1+0)\overline{v(1+0)}$$
$$= (H_\kappa u, v) = (-u'' + qu, v).$$

Since $v(1) := v(1+0) = v(1-0)$ attains any value, we have $\kappa(u'(1+0) - u'(1-0)) = u(1)$.

Conversely, if $u$ satisfies the conditions (i)-(ii), we can easily see that $h_\kappa[u,v] = (-u'' + qu, v)$ for any $v \in H_0^1(0,\infty)$. Thus this lemma has been proved. □

## 3. Livsic Matrix and Mourre Estimate

We shall introduce the Livsic matrix and the Mourre estimate. In general, let $\mathcal{H}$ be a Hilbert space with its inner product $(\cdot,\cdot)$ and $P$ an orthogonal projection onto a finite dimensional subspace of $\mathcal{H}$. And let $T$ be a closed operator in $\mathcal{H}$ with resolvent set $\rho(T)$. Then the Livsic matrix $B(z,T)$ in $P\mathcal{H}$ is determined by

$$P(T-z)^{-1}P = (B(z,T)-z)^{-1}$$

for $z \in \rho(T)$. In particular, if $T_0$ and $T_1$ are self-adjoint operators , $\lambda \in \sigma(T_0)$ is of finite multiplicity and $P$ is the orthogonal projection onto the eigenspace of $T_0$ corresponding to $\lambda$, then the Livsic matrix $B(z,T_1)$ is the following form:

$$B(z,T_1) = \lambda + PVP - PV\overline{P}(\overline{T_1}-z)^{-1}\overline{P}VP \tag{3.1}$$

where $V = T_1 - T_0$, $\overline{P} = I - P$ and $\overline{T_1} = \overline{P}T_1\overline{P}$.

**Assumption (G.1).** $g(z) = (az+b)/(cz+d)$, $a,b,c,d \in \mathbf{C}$ with $ad-bc \neq 0$. There exists $\kappa_0 > 0$ such that $c\lambda + d \neq 0$ for any $\lambda \in \overline{\cup_{0\le\kappa\le\kappa_0}\sigma(H_\kappa)}$.

**Assumption (AG).** There exists a neighborhood $I$ of $\lambda_0$ and a complex neighborhood $\Omega$ of $\lambda_0$ such that $B(g(z),g(H_\kappa))$ has a continuous extension from $\mathbf{C}\setminus\mathbf{R}$ to $I$ and the continuation satisfies

$$\|B(g(z),g(H_\kappa)) - B(g(w),g(H_\kappa))\| \le o(1)|g(z)-g(w)| \text{ as } \kappa \to 0$$

for any $z, w \in \Omega$.

**Theorem 3.1.** *([7, Theorem 4.1]) Suppose that Assumption (G.1) and (AG) are satisfied. Then there exists a closed interval $J(\kappa)$ such that*

$$\lim_{\kappa\to 0} J(\kappa) = \{\lambda_0\},\ s\text{-}\lim_{\kappa\to 0} E_\kappa(J(\kappa)) = P.$$

We shall describe the Mourre estimate [4]. Let $\mathcal{H}$ be the same as above and $H$ a self-adjoint operator in $\mathcal{H}$. Let $\mathcal{H}^s = \{u \in \mathcal{H}; \|(|H|+1)^{s/2}u\| < \infty\}$ for $s \ge 0$ and $\mathcal{H}^s$ the dual of $\mathcal{H}^{-s}$ for $s < 0$.

**Assumption (M).** (cf. [4, Definition 2.1]). Let $E \in \mathbf{R}$. Let $A$ be a self-adjoint operator (not necessesary the same $A$ as in Section 1) such that

(i) $D(A) \cap D(H)$ is a core for $H$,

(ii) $e^{itA}$ maps $D(H)$ to $D(H)$, and for each $u \in D(H)$

$$\sup_{|t|\le 1} \|He^{itA}u\| < \infty,$$

(iii) the form $i[H,A]$ defined on $D(A)\cap D(H)$ is bounded from below and closable. We shall denote the self-adjoint operator associated with $i[H,A]^a$

(form closure of $i[H,A]$) by the same symbol $i[H,A]^a$ and assume that $D(H) \subset D(i[H,A]^a)$.

(iv) The form $[[H,A]^a,A]$ defined on $D(A)\cap D(H)$ is closable and the self-adjoint operator associated with the form closure of $[[H,A]^a,A]$ is denoted by the same symbol $[[H,A]^a,A]^a$. And the operator $[[H,A]^a,A]^a$ is extended to a bounded operator from $\mathcal{H}^2$ to $\mathcal{H}^{-2}$.

(v) There exist $\alpha>0$, $\delta>0$ and a compact operator $K$ on $\mathcal{H}$ such that

$$E_H(J)i[H,A]^aE_H(J) \geq \alpha E_H(J) + E_H(J)KE_H(J),$$

where $E_H$ is the spectral measure of $H$ and $J=(E-\delta,E+\delta)$.

**Theorem 3.2.** *([4, Theorem 2.2 (iii), n=2]). Let $E\in\mathbf{R}$ and $A$ satisfy Assumption (M). Let $I\subset\sigma(H)\cap J$ be a relatively compact interval and $s>3/2$. Then for $\lambda\in I$ the following limit exists*

$$\lim_{\varepsilon\downarrow 0}(A^2+1)^{-s/2}(H-\lambda\pm i\varepsilon)^{-2}(A^2+1)^{-s/2}$$

*and equals*

$$\frac{d}{d\lambda}(A^2+1)^{-s/2}(H-\lambda\pm i0)^{-1}(A^2+1)^{-s/2}.$$

## 4. Proof of Theorem 1.1

To apply Theorem 3.1 to the proof of Theorem 1.1 we need some lemmas. Since the following fact is well-known, we omit the proof (cf. [3]).

**Lemma 4.1.** *Let $v_1(\zeta;x)$, $v_2(\zeta;x)$ be fundamental solutions of $-u''+qu=\zeta u$ on $[0,1]$ such that*

$$\begin{pmatrix} v_1(\zeta;0) & v_2(\zeta;0) \\ v_1'(\zeta;0) & v_2'(\zeta;0)\end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}$$

*and $w_1(\zeta;x)$, $w_2(\zeta;x)$ be fundamental solutions of $-u''+qu=\zeta u$ on $[1,\infty)$ such that*

$$w_1(\zeta;1)=0,\ w_1'(\zeta;1)=1,\ w_2(\zeta;1)=1,\ \lim_{x\to\infty}w_2(\zeta;x)=0.$$

*Then the kernel of $R_0(\zeta)$, $\mathrm{Im}\zeta\neq 0$, is*

$$k_0(\zeta;x,y)=\begin{cases} \dfrac{\begin{vmatrix} v_1(\zeta;1) & v_2(\zeta;1) \\ v_1(\zeta;x) & v_2(\zeta;x)\end{vmatrix}}{v_1(\zeta;1)}v_1(\zeta;y), & 0\leq y\leq x\leq 1, \\ \dfrac{\begin{vmatrix} v_1(\zeta;1) & v_2(\zeta;1) \\ v_1(\zeta;y) & v_2(\zeta;y)\end{vmatrix}}{v_1(\zeta;1)}v_1(\zeta;x), & 0\leq x< y\leq 1, \\ w_1(\zeta;y)w_2(\zeta;x), & 1\leq y\leq x, \\ w_1(\zeta;x)w_2(\zeta;y), & 1\leq x<y. \end{cases}$$

For functions $u,v$ we denote the Wronskian of $u,v$ by $W(u,v)(x)$.

**Lemma 4.2.** *Let $S(\kappa,\zeta)=R_\kappa(\zeta)-R_0(\zeta)$. Then for $\mathrm{Im}\zeta\neq 0$ we have*

$$S(\kappa,\zeta)u(x)=a(\kappa,\zeta)(u,\overline{\phi(\zeta;\cdot)})\phi(\zeta;x), \tag{4.1}$$

*where $a(\kappa,\zeta)=\kappa/(v_1(\zeta;1)-\kappa W(v_1,w_2)(\zeta;1))$ and*

$$\phi(\zeta;x)=\begin{cases}\dfrac{v_1(\zeta;x)}{\sqrt{v_1(\zeta;1)}}, & 0\le x\le 1,\\ \sqrt{v_1(\zeta;1)}w_2(\zeta;x), & 1\le x<\infty.\end{cases} \tag{4.2}$$

*where we take a branch of square-root such that $\mathrm{Im}\sqrt{z}>0$ for $\mathrm{Im}z\neq 0$.*

*Proof.* We put $f(x)=S(\kappa,\zeta)u(x)$ for $u\in L^2(0,\infty)$. Then we have, by Lemma 2.2,

$$-f''(x)+q(x)f(x)=\zeta f(x),\quad x\neq 1,$$

and $f\in L^2(0,\infty)$. So we can put

$$f(x)=\begin{cases} bv_1(\zeta;x), & 0\le x<1,\\ cw_2(\zeta;x), & 1<x<\infty.\end{cases}$$

Using the relations

$$R_0(\zeta)u(1\pm 0)=0,\ R_\kappa(\zeta)u(1):=R_\kappa(\zeta)u(1+0)=R_\kappa(\zeta)u(1-0),$$
$$(R_\kappa(\zeta)u)'(1+0)-(R_\kappa(\zeta)u)'(1-0)=\frac{1}{\kappa}R_\kappa(\zeta)u(1)$$

we have

$$\begin{aligned}
&f(1-0)=bv_1(\zeta;1)=cw_2(\zeta;1)=f(1+0),\\
&f'(1+0)-f'(1-0)=cw_2'(\zeta;1)-bv_1'(\zeta;1)\\
=&\frac{1}{\kappa}R_\kappa(\zeta)u(1)-\{(R_0(\zeta)u)'(1+0)-(R_0(\zeta)u)'(1-0)\}\\
=&\frac{1}{\kappa}(R_\kappa(\zeta)u(1)-R_0(\zeta)u(1))-\{(R_0(\zeta)u)'(1+0)-(R_0(\zeta)u)'(1-0)\}\\
=&\frac{1}{\kappa}f(1)-\{(R_0(\zeta)u)'(1+0)-(R_0(\zeta)u)'(1-0)\}\\
=&\frac{1}{\kappa}bv_1(\zeta;1)-\{(R_0(\zeta)u)'(1+0)-(R_0(\zeta)u)'(1-0)\}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
c&=b\frac{v_1(\zeta;1)}{w_2(\zeta;1)}=bv_1(\zeta;1),\\
b&=\frac{\kappa}{v_1(\zeta;1)-\kappa W(v_1,w_2)(\zeta;1)}\{(R_0(\zeta)u)'(1+0)-(R_0(\zeta)u)'(1-0)\}.
\end{aligned}$$

By Lemma 4.1 we know that

$$\begin{aligned}&(R_0(\zeta)u)'(1+0) - (R_0(\zeta)u)'(1-0)\\ &= \int_1^\infty w_2(\zeta;y)u(y)dy + \int_0^1 \frac{v_1(\zeta;y)}{v_1(\zeta;1)}u(y)dy.\end{aligned}$$

Hence we get

$$\begin{aligned}&f(x)\\ =&\begin{cases} a(\kappa,\zeta)(\int_0^1 v_1(\zeta;y)u(y)dy + \int_1^\infty w_2(\zeta;y)u(y)dy)v_1(\zeta;x), & 0\le x<1,\\ a(\kappa,\zeta)v_1(\zeta;1)(\int_0^1 \frac{v_1(\zeta;y)}{v_1(\zeta;1)}u(y)dy + \int_1^\infty w_2(\zeta;y)u(y)dy)w_2(\zeta;x), & 1<x<\infty.\end{cases}\end{aligned}$$

Therefore $\phi(\zeta;x)$ in (4.2) satisfies (4.1). □

*Proof of Theorem 1.1.* Since $q$ is continous and bounded on $[0,\infty)$, we may assume that $-1$ is not the eigenvalue of $H_0$ and $H_\kappa$ for $0<\kappa\le\kappa_0$. Hence we put $g(z) = 1/(z+1)$ and show that the Assumption (AG) is verified. Putting $S(\kappa)\equiv S(\kappa,-1)$, $a(\kappa)\equiv a(\kappa,-1)$ and $h\equiv\phi(x;-1)$ we see that $S(\kappa) = a(\kappa)(\cdot,\overline{h})h\equiv a(\kappa)V$. Hence by (3.1) the Livsic matrix $B(g(z),g(H_\kappa))$ of $g(H_\kappa)$ is

$$\begin{aligned}B(g(z),g(H_\kappa)) =& g(\lambda_0) + a(\kappa)PVP\\ &- a(\kappa)^2 PV\overline{P}(\overline{g(H_\kappa)} - g(z))^{-1}\overline{P}VP,\end{aligned} \tag{4.3}$$

where $\overline{g(H_\kappa)} = \overline{P}g(H_\kappa)\overline{P}$.

We see that the Lipschitz continuity of $B(g(\lambda\pm i0),g(H_\kappa))$ follows from that of the third term on the right-hand side of (4.3). Let $\psi$ be a normalized eigenfunction of $H_0$ corresponding to $\lambda_0$ and we shall investigate the Lipschitz continuity of

$$(PV\overline{P}(\overline{g(H_\kappa)} - g(\lambda\pm i0))^{-1}\overline{P}VP\psi,\psi)$$

with respect to $\lambda$ in some real neighborhood $I$ of $\lambda_0$. Note that the Livsic matrix $B(g(z),g(H_\kappa))$ is the operator in $PL^2(0,\infty) = \{\alpha\psi\}_{\alpha\in\mathbf{C}}$.

**Lemma 4.3.** *The following equation holds:*

$$(V\overline{P}(\overline{g(H_\kappa)} - g(z))^{-1}\overline{P}V\psi,\psi) = (\psi,\overline{h})(h,\psi)\frac{(A_0(z)\overline{P}h,\overline{h})}{1 + a(\kappa)(A_0(z)\overline{P}h,\overline{h})}, \tag{4.4}$$

*where* $A_0(z) = (g(H_0) - g(z))^{-1}$.

*Proof.* Since the left-hand side of (4.4) is

$$(\psi,\overline{h})(h,\psi)(\overline{P}(\overline{g(H_\kappa)} - g(z))^{-1}\overline{P}h,\overline{h}),$$

we shall calculate the third factor. Remark that $\overline{P}g(H_0) = g(H_0)\overline{P}$. Putting $(\overline{g(H_\kappa)} - g(z))^{-1}\overline{P}h = u$ we calculate $u$ in the part of $\overline{P}L^2(0,\infty)$.

$$\begin{aligned}\overline{P}h &= (\overline{g(H_\kappa)} - g(z))\overline{P}u = (\overline{g(H_\kappa) - g(H_0)} + \overline{g(H_0)} - g(z))\overline{P}u \\ &= a(\kappa)\overline{P}V\overline{P}u + (g(H_0) - g(z))\overline{P}u \\ &= a(\kappa)(\overline{P}u, \overline{h})\overline{P}h + (g(H_0) - g(z))\overline{P}u.\end{aligned}$$

Hence we have

$$\overline{P}u = (1 - a(\kappa)(\overline{P}u, \overline{h}))A_0(z)\overline{P}h.$$

Noting that $(\overline{P}u, \overline{h}) = (1 - a(\kappa)(\overline{P}u, \overline{h}))(A_0(z)\overline{P}h, \overline{h})$, we obtain

$$(\overline{P}u, \overline{h}) = \frac{(A_0(z)\overline{P}h, \overline{h})}{1 + a(\kappa)(A_0(z)\overline{P}h, \overline{h})}.$$

□

Let $P_1$ be the orthogonal projection on $L^2(0,\infty)$ into $L^2(0,1)$ and $P_2 = I - P_1$. Then by the resolvent equation, we can easily obtain

$$\begin{aligned}(A_0(z)\overline{P}h, \overline{h}) =& -(z+1)(\overline{P}h, \overline{h}) - (z+1)^2(R_0(z)P_1\overline{P}h, \overline{h}) \\ & -(z+1)^2(R_0(z)P_2\overline{P}h, \overline{h}).\end{aligned}$$

Note that $P_1R_0(z) = R_0(z)P_1$. Since the operator $R_0(z)P_1$ in the second term on the right-hand side is identical to the resolvent of $-d^2/dx^2 + q(x)$ with the domain $H_0^1(0,1) \cap H^2(0,1)$, the second term on the right-hand side is continuous in some real neighborhood $I$ of $\lambda_0$. So we shall investigate the third term on the right-hand side.

If we prove the following theorem, then Theorem 1.1 immediately follows.

**Theorem 4.4.** *If $\lambda_0 > 0$, then $(R_0(\lambda \pm i0)P_2\overline{P}h, \overline{h})$ is Lipschitz continuous in some real neighborhood $I$ of $\lambda_0$ and the Lipschitz constant is $O(1)$ $(\kappa \to 0)$.*

To prove this theorem we shall use the Mourre estimate. We need some lemmas.

**Lemma 4.5.** *Let $A$ be the generator of $\tau U_0(t)\tau^*$ in Section 1 with domain $D(A)$. Let*

$$\mathcal{D} = \{u \in H^1(0,\infty); xu'(x) \in L^2(0,\infty)\}.$$

*Then*

$$A|_{\tau\mathcal{D}} = \frac{1}{2i}\left(\frac{d}{dx}(x-1) + (x-1)\frac{d}{dx}\right) \tag{4.5}$$

*and $A|_{\tau\mathcal{D}}$ is essentially self-adjoint, that is, $\tau\mathcal{D}$ is a core for $A$.*

*Proof.* Let $A_0$ be the generator of $U_0(t)$. If we prove that $\mathcal{D}$ is a core for $A_0$, then we see that $\tau\mathcal{D}$ is a core for $A$. To prove that $\mathcal{D}$ is a core for $A_0$ we shall verify the conditions in [6, Theorem X. 49].

It is easy to see that: (i) $\mathcal{D}$ is dense in $L^2(0,\infty)$; (ii) $(d/dx)u(e^t x) = e^t u'(e^t x) \in L^2(0,\infty)$ for $u \in \mathcal{D}$; (iii) $\|x(d/dx)(U_0(t)u)(x)\| = \|xu'(x)\|$ for $u \in \mathcal{D}$. Hence $\mathcal{D}$ is a core for $A_0$, and by the direct computation of $(d/dt)\tau U_0(t)\tau^* u(x)|_{t=0}$ for $u \in \tau\mathcal{D}$ we have (4.5). □

The following lemma is found in [1, Chapter 5].

**Lemma 4.6.** ([1]) *Let $V(x)$ be a real-valued continuous function and $\lambda < 0$. Let $u$ satisfy $-\Delta u + V(x)u(x) = \lambda u(x)$ in $\{x; |x| > R\}$ and $u \in L^2(|x| > R) \cap C(|x| > R)$. For any $\varepsilon > 0$ there exist a constant $C > 0$ such that*

$$|u(x)| \le Ce^{(-\sqrt{-\lambda}+\varepsilon)|x|}$$

*for $|x| > R$.*

*Proof of Theorem 4.4.* Remark that $D(H_0P_2) = H_0^1(1,\infty) \cap H^2(1,\infty)$. First, we shall verify the conditions of Theorem 3.2 for $H_0P_2$ and $A$ in Section 1.

(i) $C_0^\infty[1,\infty)$ is core for $H_0P_2$, so is $D(A) \cap D(H_0P_2)$.

(ii) By $e^{itA}u(x) = e^{t/2}u(e^t(x-1)+1)$ for $u \in H_0^1(1,\infty) \cap H^2(1,\infty)$

$$\begin{aligned} H_0P_2e^{itA}u(x) &= (K_0+q_2)e^{itA}u(x) \\ &= -e^{2t}e^{t/2}u''(e^t(x-1)+1) + e^{t/2}q_2(x)u(e^t(x-1)+1). \end{aligned}$$

Since $q_2$ is bounded, we see that $\sup_{|t|\le 1}\|H_0P_2e^{itA}u\| < \infty$.

(iii) For $u, v \in C_0^\infty[1,\infty)$

$$\begin{aligned} (i[H_0,A]u,v) &= (iAu, H_0v) - (iH_0u, Av) \\ &= (iAu, K_0v) - (iK_0u, Av) + (i[q_2,A]u,v) \\ &= (2K_0u,v) + (i[q_2,A]u,v) \\ &= (2H_0u,v) - (2q_2u,v) + (i[q_2,A]u,v). \end{aligned} \tag{4.6}$$

By Assumption (A) $i[H_0,A]$ is closable and bounded from below, and $i[H_0P_2,A]^a$ is self-adjoint and $D(H_0) \subset D([H_0,A]^a)$.

(iv) By (4.6) we see that for $u, v \in C_0^\infty[1,\infty)$

$$\begin{aligned} (i[i[H_0,A],A]u,v) =& -i(i[H_0,A]u,Av) + i(Au, i[H_0,A]v) \\ =& -i(2H_0u,Av) + i(2q_2u,Av) - i(i[q_2,A]u,Av) \\ &+ i(Au,2H_0v) - i(Au,2q_2v) + i(Au,i[q_2,A]v) \\ =& 2(i[H_0,A]u,v) - 2i([q_2,A]u,v) + i([i[q_2,A],A]u,v) \\ =& 4(H_0u,v) - 4(q_2u,v) - 2(i[q_2,A]u,v) + i([i[q_2,A],A]u,v). \end{aligned}$$

By Assumption (A) we see that $i[i[H_0P_2, A], A]$ is extended to a bounded operator from $D(H_0P_2)$ to $D(H_0P_2)^*$.

(v) Let $\delta > 0$ (small) satisfy $[\lambda_0 - \delta, \lambda_0 + \delta] \subset (0, \infty)$. Notice that $\sigma(H_0P_2) \cap (0, \infty) = \sigma_{ac}(H_0P_2)$. Then by (4.6)

$$\begin{aligned}
&E_{H_0P_2}(J) i[H_0P_2, A] E_{H_0P_2}(J) \\
=&2E_{H_0P_2}(J) H_0 E_{H_0P_2}(J) + E_{H_0P_2}(J)(i[q_2, A] - 2q_2) E_{H_0P_2}(J) \\
\geq&2(\lambda_0 - \delta) E_{H_0P_2}(J) + E_{H_0P_2}(J)(i[q_2, A]^a - 2q_2) E_{H_0P_2}(J).
\end{aligned}$$

By Assumption (A) the condition (v) is satisfied. Hence we obtain

$$\sup_{\lambda \in I} \| \frac{d}{d\lambda} (A^2 + 1)^{-s/2} (H_0P_2 - \lambda \pm i0)^{-1} (A^2 + 1)^{-s/2} \| < \infty$$

where $s > 3/2$ and $I \subset J$ is a closed interval containing $\lambda_0$. □

**Lemma 4.7.** *(cf. [4]). Let $s \in \mathbf{R}$. Then $F_s \equiv (A^2 + 1)^{s/2} (1 + x^2)^{-s/2} (1 + K_0)^{-s/2}$ is a bounded operator on $L^2(1, \infty)$ and so is $F_s^*$.*

*Proof.* It is sufficient to prove that for $u \in H_0^1(1, \infty)$

$$\|A(1 + x^2)^{-1/2} u\| \leq C(\|u\| + \|u'\|).$$

Since for large $x$

$$\begin{aligned}
&2iA(1 + x^2)^{-1/2} u = (x - 1)((1 + x^2)^{-1/2} u)' + ((x - 1)(1 + x^2)^{-1/2} u)' \\
=&(x - 1) O(x^{-2}) u(x) + (x - 1) O(x^{-1}) u'(x) + O(x^{-1/2}) u(x) + O(1) u'(x) \\
=&O(1)(u(x) + u'(x)).
\end{aligned}$$

By the duality and interpolation we see that both $F_s$ and $F_s^*$ are bounded on $L^2(1, \infty)$. □

We continue the proof of Theorem 4.4. By (4.1) and Lemma 4.6 we see that for any $\varepsilon > 0$ there exist constants $C > 0$ and $C' > 0$ such that $|h(x)| \leq Ce^{(-1+\varepsilon)x}$ and $|h''(x)| \leq C' e^{(-1+\varepsilon)x}$ as $x \to \infty$. Hence we have $(1 + x^2)(1 + K_0) P_2 h \equiv u \in L^2(1, \infty)$. By Lemma 4.7 we have

$$\begin{aligned}
&\frac{d}{d\lambda} (R_0(\lambda \pm i0) P_2 \overline{P} h, P_2 \overline{h}) \\
=&\frac{d}{d\lambda} (F_2^* [(A^2 + 1)^{-1} R_0(\lambda \pm i0)(A^2 + 1)^{-1}] F_2 u, u)
\end{aligned}$$

is bounded, and $(R_0(\lambda + i0) P_2 \overline{P} h, \overline{h})$ is the Lipschitz continuous. Thus we have completed the proof of Theorem 4.4. □

**Corrollary 4.8.** *Let* $\lambda_0 > 0$. *If* $q \in C^2$, $q(x), q'(x) = o(x^{-1})$ *and* $q''(x) = O(x^{-2})$ *as* $x \to \infty$, *then* $(R_0(\lambda \pm i0)P_2\overline{P}h, \overline{h})$ *is Lipschitz's continuous in some real neighborhood* $I$ *of* $\lambda_0$.

*Proof.* It is sufficient to verify Assumption (A). We calculate the commutators: For $u \in H_0^1(1,\infty)$

$$\begin{aligned}
[q_2, iA]u(x) &= 2(x-1)q_2'(x)u(x) \\
[[q_2, iA]^a, iA]u(x) &= 2(x-1)(2(x-1)q_2'(x))'u(x) \\
&= 4((x-1)q_2'(x) + (x-1)^2 q_2''(x))u(x).
\end{aligned}$$

Hence we see that by Sobolev's theorem $[q_2, A]^a$ is a compact operator in $L^2(1,\infty)$ and that $[[q_2, A]^a, A]^a$ is a bounded operator in $L^2(1,\infty)$. □

We give a theorem for the time decay of the quasi-eigenstates, but we do not give a proof (cf. [7, Section 5]). Instead of $(B(g(z), g(H_\kappa))\psi, \psi)$, we shall simply write $B(g(z), g(H_\kappa))$.

**Theorem 4.9.** *Suppose the same assumption in Theorem 1.1 and use the same notations in the proof (Section 4). Assume further that*

$$\lim_{\kappa\downarrow 0}\lim_{\varepsilon\downarrow 0} \mathrm{Im}\frac{1}{\kappa^2} B(g(\lambda_0 + i\varepsilon), g(H_\kappa)) \neq 0.$$

*Then we have*

$$|(e^{-itH_\kappa}\psi, \psi)| = \exp(-\alpha\kappa^2 t) + o(1)$$

*uniformly in* $t \geq 0$ *as* $\kappa \to 0$, *where*

$$\begin{aligned}
\alpha &= \mathrm{Im} g^{-1}(B(g(\lambda_0 + i0), g(H_0))) \\
&= \mathrm{Im}(\psi, \overline{h})(h, \psi)\frac{-(\lambda_0+1)^2(R_0(\lambda_0 + i0)P_2\overline{P}h, \overline{h})}{v(-1;1)^2|B(g(\lambda_0+i0), g(H_0))|^2}.
\end{aligned}$$

Hence we see that $\psi$ is the quasi-eigenstate of $H_\kappa$.

## References


1. S. Agmon, *Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations*, Princeton Univ. Press, 1982.
2. S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, *Solvable Models in Quantum Mechanics*, Springer-Verlag, 1988.
3. E. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, McGraw-Hill, 1955.
4. A. Jensen, E. Mourre and P. A. Perry, *Multiple commutator estimates and resolvent smoothness in quantum scattering theory*, Ann. Inst. Henri Poincare Physique Theorique **41** (1984), 207–225,.
5. T. Kato, *Perturbation Theory for Linear Operators*, 2nd ed, Springer-Verlag, 1980.

6. M. Reed and B. Simon, *Methods of Modern Mathematical Physics II, Fourier Analysis, Self-Adjoint Operators*, Academic Press, 1975.
7. K. Watanabe, *Spectral concentration and resonances for unitary operators: Applications to self-adjoint Problems*, Rev. Math. Phys. **7** (1995), 979-1011.

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