SUT Journal of Mathematics
Vol. 58, No. 1 (2022), 77–89
DOI: 10.55937/sut/1654320039

On the generalized Ramanujan-Nagell equation $x^2 + b^m = c^n$ with $a^2 + b^r = c^2$

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(Received August 19, 2021)

Abstract. Let a, b, c be pairwise relatively prime positive integers such that $a^2+b^r = c^2$ with r positive integer. Then we show that the equation $x^2+b^m = c^n$ has the positive integer solution $(x, m, n) = (a, r, 2)$ only under some conditions. The proof is based on elementary methods and Zsigmondy's theorem.

AMS 2020 *Mathematics Subject Classification.* 11D61.

Key words and phrases. Generalized Ramanujan-Nagell equation, Zsigmondy's theorem, Jeśmanowicz' conjecture.

§1. Introduction.

In 1913, Ramanujan [R] conjectured that the equation $x^2+7 = 2^n$ has only five positive integer solutions $(x, n) = (1, 3), (3, 4), (5, 5), (11, 7), (181, 15)$. In 1960, Nagell [N] resolved Ramanujan's conjecture. Tanahashi [Ta] and Toyoizumi [To] independently extended Nagell's result by showing that the equation $x^2 + 7^m = 2^n$ has only six solutions $(x, m, n) = (1, 1, 3), (3, 1, 4), (5, 1, 5), (11, 1, 7), (181, 1, 15), (13, 3, 9)$. Tanahashi [Ta] also established that the equation $x^2 + 11^m = 3^n$ has only one solution $(x, m, n) = (4, 1, 3)$. Let b and c be relatively prime positive integers greater than one. Then the generalized Ramanujan-Nagell equation

$$x^2 + b^m = c^n \tag{1.1}$$

in positive integers x, m and n has been studied by a number of authors.

As an analogue of Jeśmanowicz' conjecture ([J]) concerning primitive Pythagorean triples, the first author [Te1] proposed the following:

Conjecture 1. Let a, b, c be pairwise relatively prime positive integers such that $a^2 + b^2 = c^2$ with b odd. Then (1.1) has only one positive integer solution $(x, m, n) = (a, 2, 2)$.

The first author [Te1] proved that if p and q are primes such that (i) $q^2+1=2p$ and (ii) $d=1$ or even if $q \equiv 1 \mod 4$, then the Diophantine equation $x^2+q^m=p^n$ has only one positive integer solution $(x,m,n)=(p-1,2,2)$, where d is the order of a prime divisor of (p) in the ideal class group of $\mathbb{Q}(\sqrt{-q})$. Conjecture 1 has been verified to be true in many special cases:

- (Le [Le1]) $b > 8 \cdot 10^6$, $b \equiv 5 \mod 8$, c is a prime power.
- (Chen-Le [CL]) $b^2+1=2c$, $b \not\equiv 1 \mod 16$, b and c are odd primes.
- (Le [Le2]) $b \equiv 7 \mod 8$, either b or c is a prime.
- (Cao-Dong [CD]) $c \equiv 5 \mod 8$, b or c is a prime power.
- (Yuan-Wang [YW]) $b \equiv \pm 5 \mod 8$, c is a prime.

However, Conjecture 1 remains unsolved.

The first author [Te2] showed that if $2c-1$ is a prime and $2c-1 \equiv 3, 5 \pmod 8$, then the equation $x^2+(2c-1)^m=c^n$ has only one positive integer solution $(x,m,n)=(c-1,1,2)$, and proposed the following:

Conjecture 2. Let c be a positive integer with $c \geq 2$. Then the equation

$$x^2+(2c-1)^m=c^n$$

has only one positive integer solution $(x,m,n)=(c-1,1,2)$.

Conjecture 2 also has been verified to be true in several cases:

- (Terai [Te2]) $2 \leq c \leq 30$ with $c \neq 12, 24$.
- (Deng [D], Bennett-Billerey [BeBi]) $c=12, 24$.
- (Deng-Guo-Xu [DGX]) $2c-1=3^{2s+1}p^{2t+1}$, $2c-1=5^{2s+1}p^{2t+1}$, or $3 \leq c \leq 499$ with $c \equiv 3 \pmod 4$ and p prime.
- (Fujita-Terai [FT]) $2c-1=3p^l$ or $2c-1=5p^l$ with p prime.

The first author [Te3] has recently proved that if a, b, c are pairwise relatively prime positive integers such that $a^2+b^4=c^2$ with b odd, then (1.1) has only one positive integer solution $(x,m,n)=(a,4,2)$ under certain conditions.

In this paper, when $a^2+b^r=c^2$ with r positive integer, we consider (1.1). It is well known that any primitive Pythagorean triple a, b, c such that $a^2+b^2=c^2$ with b even can be parametrized as follows:

$$a=u^2-v^2, \quad b=2uv, \quad c=u^2+v^2,$$

where u, v are positive integers such that $\gcd(u, v) = 1$, $u \not\equiv v \pmod 2$ and $u > v$. Similarly, any positive integers a, b, c satisfying $a^2 + b^r = c^2$ and $\gcd(a, b) = 1$ can be parametrized as follows, according to Case (B_1): b is even or Case (B_2): b is odd.

$$(1.2) \qquad (\mathrm{B}_1) \qquad a = |u^r - 2^{r-2}v^r|, \quad b = 2uv, \quad c = u^r + 2^{r-2}v^r,$$

where u, v are positive integers such that $\gcd(u, v) = 1$ and $u \equiv 1 \pmod 2$; or

$$(1.3) \qquad (\mathrm{B}_2) \qquad a = \frac{u^r - v^r}{2}, \quad b = uv, \quad c = \frac{u^r + v^r}{2},$$

where u, v are positive integers such that $\gcd(u, v) = 1$, $uv \equiv 1 \pmod 2$ and $u > v$. The purpose of this paper is to show that (1.1) has only a trivial solution $(x, m, n) = (a, r, 2)$ under some conditions for the following four cases:

(i) $u = p,\ v = 1,\ r = 3$ in (1.2),

(ii) $u = 1,\ v = p,\ r = 3$ in (1.2),

(iii) $u = p,\ v = 1,\ r \geq 1$ in (1.3),

(iv) $u = p,\ v = 3,\ r \equiv 1 \pmod 2$ in (1.3),

where p is an odd prime.

Theorem 1.1. *Let p be a prime with $p \equiv 3, 5 \pmod 8$. Then the equation*

$$(1.4) \qquad x^2 + (2p)^m = (p^3 + 2)^n$$

has only one positive integer solution $(x, m, n) = (p^3 - 2, 3, 2)$.

Theorem 1.2. *Let p be a prime with $p \equiv 3, 5 \pmod 8$. Then the equation*

$$(1.5) \qquad x^2 + (2p)^m = (2p^3 + 1)^n, \quad m \geq 2$$

has only one positive integer solution $(x, m, n) = (2p^3 - 1, 3, 2)$.

Theorem 1.3. *Let p be a prime with $p \equiv 3, 5 \pmod 8$, and let r be a positive integer. Then the equation*

$$(1.6) \qquad x^2 + p^m = \left(\frac{p^r + 1}{2}\right)^n$$

has only one positive integer solution $(x, m, n) = ((p^r - 1)/2, r, 2)$.

Theorem 1.4. *Let p be a prime with $p \equiv 7, 11, 17 \pmod{24}$, and let r be odd. Then the equation*

$$x^2 + (3p)^m = \left(\frac{p^r + 3^r}{2}\right)^n \tag{1.7}$$

has only one positive integer solution $(x, m, n) = ((p^r - 3^r)/2, r, 2)$.

The organization of this paper is as follows. In Section 2, we quote the theorem of Zsigmondy concerning primitive prime divisor and results on the Lebesgue-Nagell equations $x^2 + D^m = y^n$ ($D = 2, 6, 12, 20$). In Sections 3 to 6, we use elementary methods and Zsigmondy's theorem to show Theorems 1.1 to 1.4, respectively. In Section 7, we describe whether the equation $x^2 + b^m = c^n$ with $a^2 + b^3 = c^2$ has two solutions or not.

§2. Preliminaries

The following is a direct consequence of an old version of the Primitive Divisor Theorem due to Zsigmondy [Z]:

Lemma 2.1 (Zsigmondy [Z])**.** *Let A and B be relatively prime integers with $A > B \geq 1$. Let $\{a_k\}_{k\geq 1}$ be the sequence defined as*

$$a_k = A^k + B^k.$$

If $k > 1$, then a_k has a prime factor not dividing $a_1 a_2 \cdots a_{k-1}$, whenever $(A, B, k) \neq (2, 1, 3)$.

In the proof of Theorems 7.1, 7.2, we need the following lemma concerning the Lebesgue-Nagell equations, which is immediate from Cohn [C], Le [Le3], Luca [Lu], Luca-Togbe [LT].

Lemma 2.2. (1) *All positive integer solutions of the equation*

$$x^2 + 2^m = y^n, \quad \gcd(x, y) = 1, \quad n \geq 3$$

are $(x, y, m, n) = (5, 3, 1, 3), (7, 3, 5, 4), (11, 5, 2, 3)$.

(2) *All positive integer solutions of the equation*

$$x^2 + 6^m = y^n, \quad \gcd(x, y) = 1, \quad n \geq 3$$

are $(x, y, m, n) = (2681, 193, 4, 3), (39151, 1153, 5, 3)$.

(3) *The equation*

$$x^2 + 12^m = y^n, \quad \gcd(x,y) = 1, \quad n \geq 3$$

has no positive integer solutions (x, y, m, n).

(4) *The equation*

$$x^2 + 20^m = y^n, \quad \gcd(x,y) = 1, \quad n \geq 3$$

has no positive integer solutions (x, y, m, n).

§3. Proof of Theorem 1.1

Let (x, m, n) be a solution of (1.4). Suppose that our assumptions are all satisfied.

Put $c = p^3 + 2$. Since $p \equiv 3, 5 \pmod 8$, we have $\left(\dfrac{c}{p}\right) = \left(\dfrac{p^3+2}{p}\right) = \left(\dfrac{2}{p}\right) = -1$, where $\left(\dfrac{*}{*}\right)$ is the Jacobi symbol. Hence (1.4) implies that n is even, say $n = 2N$. From (1.4), we have

$$(2p)^m = (c^N + x)(c^N - x).$$

Since $\gcd(c^N + x, c^N - x) = 2$, we obtain the following two cases:

$$\text{(3.1)} \qquad \begin{cases} c^N + x &= 2p^m \\ c^N - x &= 2^{m-1} \end{cases}$$

or

$$\text{(3.2)} \qquad \begin{cases} c^N + x &= 2^{m-1}p^m \\ c^N - x &= 2. \end{cases}$$

First consider Case (3.1). Adding these two equations yields

$$\text{(3.3)} \qquad 2^{m-2} + p^m = (p^3 + 2)^N.$$

If $m \leq 2$, then it is clear that equation (3.3) has no solutions. If $m = 3$, then equation (3.3) has only one solution $N = 1$ and so $n = 2$, $x = p^3 - 2$. Thus we may suppose that $m \geq 4$.

From (3.3), we have $\left(\dfrac{2}{p}\right)^{m-2} = \left(\dfrac{2}{p}\right)^N$. Since $\left(\dfrac{2}{p}\right) = -1$, this implies that $m \equiv N \pmod 2$. The proof is divided into two cases:

(i) m and N are odd, (ii) m and N are even.

Case (i) m and N are odd. Then taking equation (3.3) modulo 4 implies that $p^m \equiv (p^3+2)^N \pmod 4$, so $p \equiv p+2 \pmod 4$, which is impossible.
Case (ii) m and N are even. By the parametrization of primitive Pythagorean triples, it follows that

$$2^{\frac{m-2}{2}} = 2uv, \qquad p^{\frac{m}{2}} = u^2 - v^2, \qquad (p^3+2)^{\frac{N}{2}} = u^2 + v^2,$$

where u, v are positive integers such that $\gcd(u,v) = 1$, $u \not\equiv v \pmod 2$ and $u > v$. This shows that

$$u = 2^{\frac{m-2}{2}-1}, \quad v = 1, \quad u + v = p^{\frac{m}{2}}, \quad u - v = 1,$$

so $u = 2$ and $m = 6$, which do not satisfy $u + v = p^{\frac{m}{2}}$.

Next consider Case (3.2). Adding these two equations yields

$$2^{m-2}p^m + 1 = (p^3+2)^N. \tag{3.4}$$

Since $\left(\dfrac{2}{p}\right) = -1$, the equation (3.4) implies that N is even, say $N = 2N_1$. Then

$$2^{m-2}p^m = (p^3+2)^{2N_1} - 1 = ((p^3+2)^2 - 1)C_1 = (p+1)(p^2-p+1)(p^3+3)C_1$$

with $C_1 = \dfrac{(p^3+2)^{2N_1} - 1}{(p^3+2)^2 - 1}$, which is impossible, since $\gcd(p^2-p+1, p) = 1$ and $p^2 - p + 1 > 1$ is odd. This completes the proof of Theorem 1.1.

§4. Proof of Theorem 1.2

Let (x, m, n) be a solution of equation (1.5). Suppose that our assumptions are all satisfied.

Put $c = 2p^3 + 1$. Then taking equation (1.5) modulo 4 implies that $1 \equiv 3^n \pmod 4$, so n is even, say $n = 2N$. From (1.6), we have

$$(2p)^m = (c^N + x)(c^N - x).$$

Since $\gcd(c^N + x, c^N - x) = 2$, we obtain the following two cases:

$$\begin{cases} c^N + x = 2p^m \\ c^N - x = 2^{m-1} \end{cases} \tag{4.1}$$

or

$$\begin{cases} c^N + x = 2^{m-1}p^m \\ c^N - x = 2. \end{cases} \tag{4.2}$$

First consider Case (4.1). Adding these two equations yields

$$2^{m-2} + p^m = (2p^3 + 1)^N. \tag{4.3}$$

Since it follows from (4.3) that $1 = \left(\frac{2}{p}\right)^{m-2} = (-1)^{m-2}$, we see that m is even. This leads to

$$\left(2^{(m-2)/2}\right)^2 + \left(p^{m/2}\right)^2 \equiv 0 \pmod{2p^3+1},$$

which is impossible, since $2p^3 + 1 \equiv 3 \pmod 4$.

Next consider Case (4.2). Adding these two equations yields

$$2^{m-2}p^m + 1 = (2p^3 + 1)^N. \tag{4.4}$$

If $m \le 2$, then it is clear that (4.4) has no solutions. If $m = 3$, then (4.4) has only one solution $N = 1$ and so $n = 2,\ x = 2p^3 - 1$.

If $m \ge 4$, then taking (4.4) modulo 4 implies that $1 \equiv 3^N \pmod 4$ and so N is even, say $N = 2N_1$. Then

$$2^{m-2}p^m = (2p^3+1)^{2N_1} - 1 = ((2p^3+1)^2 - 1)C_2 = 4(p+1)(p^2-p+1)p^3C_2$$

with $C_2 = \dfrac{(2p^3+1)^{2N_1} - 1}{(2p^3+1)^2 - 1}$, which is impossible, since $\gcd(p^2 - p + 1, p) = 1$ and $p^2 - p + 1 > 1$ is odd. This completes the proof of Theorem 1.2.

§5. Proof of Theorem 1.3

Let (x, m, n) be a solution of equation (1.6). Suppose that our assumptions are all satisfied.

Put $c = \dfrac{p^r+1}{2}$. Then $\left(\dfrac{c}{p}\right) = -1$. Indeed,

$$\left(\frac{c}{p}\right) = \left(\frac{4c}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{p^r+1}{p}\right) = \left(\frac{2}{p}\right) = -1,$$

since $p \equiv 3, 5 \pmod 8$. Hence (1.6) implies that n is even, say $n = 2N$. From (1.6), we have

$$p^m = (c^N + x)(c^N - x).$$

Since $\gcd(c^N + x, c^N - x) = 1$, we obtain the following:

$$\begin{cases} c^N + x &= p^m \\ c^N - x &= 1. \end{cases}$$

Adding these two equations yields

$$p^m + 1 = 2c^N. \tag{5.1}$$

From the definition of c, we have

$$p^r + 1 = 2c.$$

If $m > r$, then it follows from Lemma 2.1 that (5.1) has no solutions. If $m = r$, then (5.1) has only one solution $N = 1$ and so $n = 2, x = (p^r - 1)/2$. If $m < r$, then (5.1) has no solutions, since

$$p^m + 1 < p^r + 1 = 2c \leq 2c^N.$$

This completes the proof of Theorem 1.3.

§6. Proof of Theorem 1.4

Let (x, m, n) be a solution of (1.7). Suppose that our assumptions are all satisfied.

Put $c = \dfrac{p^r + 3^r}{2}$. Since $p \equiv 7, 11, 17 \pmod{24}$, we see that $\left(\dfrac{6}{p}\right) = -1$. Then $\left(\dfrac{c}{p}\right) = -1$. Indeed,

$$\left(\frac{c}{p}\right) = \left(\frac{4c}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{p^r + 3^r}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{3}{p}\right)^r = \left(\frac{6}{p}\right) = -1.$$

Hence (1.7) implies that n is even, say $n = 2N$. From (1.7), we have

$$(3p)^m = (c^N + x)(c^N - x).$$

Since $\gcd(c^N + x, c^N - x) = 1$, we obtain the following two cases:

$$\begin{cases} c^N + x &= (3p)^m \\ c^N - x &= 1 \end{cases} \tag{6.1}$$

or

$$\begin{cases} c^N + x &= p^m \\ c^N - x &= 3^m. \end{cases} \tag{6.2}$$

First consider Case (6.1). Adding these two equations yields

$$(3p)^m + 1 = 2c^N. \tag{6.3}$$

Taking (6.3) modulo 3 implies that $1 \equiv 2c^N \pmod 3$, so N is odd. We want to show that equation (6.3) has no solutions. The proof is divided into two cases: Case (i) $p \equiv 7, 17 \pmod{24}$ and Case (ii) $p \equiv 11 \pmod{24}$.

Case (i). Then (6.3) leads to

$$1 = \left(\frac{2c}{p}\right) = \left(\frac{p^r + 3^r}{p}\right) = \left(\frac{3}{p}\right)^r = \left(\frac{3}{p}\right) = -1,$$

which is impossible.

Case (ii). Then (6.3) leads to

$$1 = \left(\frac{2c}{3}\right) = \left(\frac{p^r + 3^r}{3}\right) = \left(\frac{p}{3}\right)^r = \left(\frac{p}{3}\right) = \left(\frac{2}{3}\right) = -1,$$

which is impossible.

Next consider Case (6.2). Adding these two equations yields

$$p^m + 3^m = 2c^N. \tag{6.4}$$

From the definition of c, we have

$$p^r + 3^r = 2c.$$

If $m > r$, then it follows from Lemma 2.1 that (6.4) has no solutions. As in Section 5, we easily see that if $m \leq r$, then (6.4) has one the solution $(m, N) = (r, 1)$ and so $n = 2$, $x = (p^r - 3^r)/2$. This completes the proof of Theorem 1.4.

§7. the equation $x^2 + b^m = c^n$ with two solutions

In Theorem 1.2, the equation (1.5) with $m = 1$, that is,

$$x^2 + 2p = (2p^3 + 1)^n \tag{7.1}$$

remains to be solved. In general, it is difficult to treat (7.1). For small values of odd primes p, we can easily solve (7.1) by using the function `IntegralPoints(`E`)` of an elliptic curve E in Magma [BoCa]. When $p = 3$, equation (7.1) can be solved as follows:

Theorem 7.1. *All positive integer solutions of the equation*

$$x^2 + 6^m = 55^n \tag{7.2}$$

are $(x, m, n) = (7, 1, 1),\ (53, 3, 2)$.

Proof. By Theorem 1.2, it suffices to solve (7.2) with $m = 1$, that is,

$$x^2 + 6 = 55^n.$$

Put $n = 3N + r$ with $r = 0, 1, 2$. Then the above equation becomes

$$Y^2 = X^3 - 6 \cdot 55^{2r}$$

with $X = 55^{r+N}$ and $Y = 55^r \cdot x$. When $r = 0, 1, 2$, we obtain the following three elliptic curves, which can be easily solved by Magma:

$$E_0 : Y^2 = X^3 - 6$$

with $\operatorname{rank} E_0(\mathbb{Q}) = 0$ and there are no integer points on E_0;

$$E_1 : Y^2 = X^3 - 6 \cdot 55^2$$

with $\operatorname{rank} E_1(\mathbb{Q}) = 2$ and all integer points on E_1 are $(X, Y) = (55, \pm 385)$, $(295, \pm 5065)$;

$$E_2 : Y^2 = X^3 - 6 \cdot 55^4$$

with $\operatorname{rank} E_2(\mathbb{Q}) = 0$ and there are no integer points on E_2. Consequently the equation $x^2 + 6 = 55^n$ has only one solution $(x, n) = (7, 1)$. □

Remark. We remark that (7.2) can also be solved by Lemma 2.2 (2).

Similarly, for odd primes p such that $3 < p < 100$ and $p \equiv 3, 5 \pmod 8$, we can solve (7.1) by Magma. In all these cases, we see that (7.1) has no solutions.

It is worth noting that (7.2) has two solutions. In other cases, it is natural to ask if there are two solutions (x, m, n) to the equation

(7.3) $$x^2 + b^m = c^n \quad \text{with} \quad a^2 + b^3 = c^2 \quad \text{and} \quad \gcd(a, b) = 1.$$

By Magma, in the range $2 \le a, b \le 1000$ and $1 \le m, n \le 20$, we verified that (7.3) has two solutions in only five cases

$$(a, b, c) = (15, 4, 17),\ (53, 6, 55),\ (127, 8, 129),\ (431, 12, 433),\ (109, 20, 141).$$

It follows from Lemma 2.2 that (7.3) has exactly two solutions in all these cases. In fact, the following theorem holds. (The case $(a, b, c) = (53, 6, 55)$ has been already treated in Theorem 7.1).

Theorem 7.2. (1) *All positive integer solutions of the equation*

$$x^2 + 4^m = 17^n$$

are $(x, m, n) = (1, 2, 1),\ (15, 3, 2)$.

(2) *All positive integer solutions of the equation*

$$x^2 + 8^m = 129^n$$

are $(x, m, n) = (11, 1, 1),\ (127, 3, 2)$.

(3) *All positive integer solutions of the equation*

$$x^2 + 12^m = 433^n$$

are $(x, m, n) = (17, 2, 1),\ (431, 3, 2)$.

(4) *All positive integer solutions of the equation*

$$x^2 + 20^m = 141^n$$

are $(x, m, n) = (11, 1, 1),\ (109, 3, 2)$.

In view of all Theorems 1.1 to 7.2, we propose the following :

Conjecture 3. (1) Equation (7.3) has at most two positive integer solutions (x, m, n).

(2) Another possible non-trivial solution is $(x, m, n) = (x_1, 1, 1)$ or $(x_2, 2, 1)$ with x_1, x_2 some positive integer.

By elliminating the condition $\gcd(a, b) = 1$ in (7.3), the equation $x^2 + b^m = c^n$ has three or five solutions in some cases. For example, the following equations have only the solutions below, respectively:

$$x^2 + 6^m = 15^n, \qquad (x, m, n) = (3, 1, 1), (3, 3, 2), (207, 5, 4), (63, 6, 4),$$
$$x^2 + 8^m = 24^n, \qquad (x, m, n) = (4, 1, 1), (8, 3, 2), (2816, 5, 5).$$

Acknowledgments

The authors would like to thank the referee for his valuable suggestions. The first author is supported by JSPS KAKENHI Grant Number 18K03247.

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