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Quantization of simply-laced isomonodromy systems by the quantum spectral curve method

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Abstract. We quantize the simply-laced isomonodromy systems using the theory of Manin matrices and Talalaev's quantum spectral curve method.

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§1. Introduction

It is an interesting problem to quantize isomonodromic deformation equations. In [11] Reshetikhin showed that the Knizhnik–Zamolodchikov system is a quantization of the Schlesinger equations, which govern the isomonodromic deformations of linear differential systems of the form

$$\frac{du}{dx} = \sum_{i=1}^{m} \frac{R_i}{x - t_i} u,$$

where the matrices R_i do not depend on x (a similar result has been also obtained by Harnad, see [7]). Reshetikhin's result was generalized in [1, 5], where the authors constructed a quantization of the equations of Jimbo–Miwa–Môri–Sato [8], which govern the isomonodromic deformations of systems of the form

$$\frac{du}{dx} = T + \sum_{i=1}^{m} \frac{R_i}{x - t_i} u$$

where the matrices T, R_i do not depend on x and T is diagonal with distinct eigenvalues. In [9] Nagoya–Sun further generalized the above results. They quantized the Hamiltonian system governing the isomonodromic deformations of systems of the form

$$\frac{du}{dx} = Ax + B + \sum_{i=1}^{m} \frac{R_i}{x - t_i} u,$$

where the matrices A, B, R_i do not depend on x and A is diagonal with distinct eigenvalues.

On the other hand, in [2] Boalch introduced an interesting class of Hamiltonian systems of isomonodromy type, called the *simply-laced isomonodromy systems*. They *partially* govern the isomonodromic deformations of systems of the form

$$\frac{du}{dx} = Ax + T + [A, Y] + \sum_{i=1}^{m} \frac{R_i}{x - t_i} u,$$

where the matrices A, T, Y, R_i do not depend on x and A, T are diagonal. Since A is not assumed to have distinct eigenvalues, such systems contain the systems considered by Nagoya–Sun. Boalch showed that the simply-laced isomonodromy systems have a beautiful $\mathrm{SL}_2(\mathbb{C})$-symmetry, which specializes to the well-known Harnad duality (see [6]) when $A = 0$.

Recently, Rembado [10] quantized the simply-laced isomonodromy systems. In this note, we give a different way to quantize the simply-laced isomonodromy systems. Our approach is to use the theory of Manin matrices and Talalaev's quantum spectral curve method (see [3, 13]). As mentioned in [10], our result has been announced in 2015.

This note is organized as follows. Section 2 is the classical theory. The first three subsections are devoted to a brief review on Boalch's simply-laced isomonodromy systems and their remarkable properties. In Section 2.4, we give some useful expressions of the Hamiltonians of the simply-laced isomonodromy systems. For instance, we express the Hamiltonians in terms of the spectral curve (see Theorem 2.10, which we call the *determinant formula*). They are interesting in their own right and seem to be new. Section 3 is the quantum theory. In Section 3.1 we first construct the deformation quantization of the phase space and some commutative subalgebra $\mathcal{H}$ in which our quantized Hamiltonians live. For the construction of $\mathcal{H}$ and the proof of commutativity we use Talalaev's quantum spectral curve method. In Section 3.2, we show that $\mathcal{H}$ is invariant under some $\mathrm{SL}_2(\mathbb{C})$-symmetry using the theory of Manin matrices. In Section 3.3, we finally construct the quantized Hamiltonians and prove that our quantized systems satisfy the integrability condition (Theorem 3.10).

§2. Simply-laced isomonodromy systems

In this section we recall the definition of simply-laced isomonodromy systems and their basic properties.

2.1. The Poisson structure

Throughout this note we fix the following data:

- non-empty finite sets Σ, I and a surjective map $\pi\colon \Sigma \to I$;
- a finite dimensional $\mathbb{C}$-vector space V_λ for each $\lambda \in \Sigma$.

Put $\Sigma_i = \pi^{-1}(i)$ for each $i \in I$ (so $\Sigma = \bigsqcup_{i\in I} \Sigma_i$) and define

$$W_i = \bigoplus_{\lambda\in\Sigma_i} V_\lambda \quad (i \in I),$$
$$V = \bigoplus_{i\in I} W_i = \bigoplus_{\lambda\in\Sigma} V_\lambda.$$

For $\Gamma \in \mathrm{End}(V)$ and $i, j \in I$, let $\Gamma_{ij} \in \mathrm{Hom}(W_j, W_i)$ be the (i,j)-block of Γ with respect to the decomposition $V = \bigoplus_{i\in I} W_i$. We often write $\Gamma = \Theta + \Xi$, where $\Theta = \bigoplus_{i\in I} \Theta_i \in \bigoplus_{i\in I} \mathrm{End}(W_i)$ is the block diagonal part of Γ and $\Xi = (\Xi_{ij})$ is the block off-diagonal part.

Let $\mathfrak{Z}$ be the center of the closed subgroup $\prod_{i\in I} \mathrm{GL}(W_i) \subset \mathrm{GL}(V)$ and $\mathfrak{z}$ be its Lie algebra. By the definition $\mathfrak{Z}$ consists of all $C \in \mathrm{GL}(V)$ of the form

$$C = \bigoplus_{i\in I} c_i\, 1_{W_i} \quad (c_i \in \mathbb{C}^\times).$$

Let $\mathcal{W} = \mathbb{C}\langle x, \partial\rangle$ be the first Weyl algebra. Consider elements $M = M(\partial, x)$ of $\mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{W}$ of the form

$$M(\partial, x) = A_1\partial - A_0 x - \Gamma \quad (A_0, A_1 \in \mathfrak{z},\ \Gamma = \Theta + \Xi \in \mathrm{End}(V)).$$

Since $A_0, A_1 \in \mathfrak{z}$, they have the form

$$A_0 = \bigoplus_{i\in I} a_{0i}\, 1_{W_i}, \quad A_1 = \bigoplus_{i\in I} a_{1i}\, 1_{W_i} \quad (a_{0i}, a_{1i} \in \mathbb{C}).$$

Let $\mathcal{M} \subset \mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{W}$ be the set consisting of all such M satisfying the following conditions:

1. $(a_{0i}, a_{1i}) \neq (0, 0)$ for any $i \in I$.

2. The map $\boldsymbol{a}\colon I \to \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, $i \mapsto a_{0i}/a_{1i}$ (which we call the *spectral map*) is injective.

3. For any $i \in I$, the i-th diagonal block Θ_i of Γ is semisimple with eigenspaces V_λ, $\lambda \in \Sigma_i$. Namely, it has the form

$$\Theta_i = \bigoplus_{\lambda \in \Sigma_i} \theta_\lambda \, 1_{V_\lambda},$$

where θ_λ, $\lambda \in \Sigma_i$ are distinct complex numbers.

We may identify $\mathcal{M}$ with the direct product $\mathbb{A} \times \mathbb{T} \times \mathbb{M}$, where

$$\mathbb{A} := \left\{ (a_{0i}, a_{1i})_{i \in I} \in \left(\mathbb{C}^2 \setminus \{(0,0)\}\right)^I \,\middle|\, \begin{vmatrix} a_{0i} & a_{0j} \\ a_{1i} & a_{1j} \end{vmatrix} \neq 0 \ (i \neq j) \right\},$$

$$\mathbb{T} := \left\{ \bigoplus_{\lambda \in \Sigma} \theta_\lambda \, 1_{V_\lambda} \in \bigoplus_{\lambda \in \Sigma} \mathbb{C} 1_{V_\lambda} \,\middle|\, \theta_\lambda \neq \theta_\mu \text{ if } \pi(\lambda) = \pi(\mu), \lambda \neq \mu \right\},$$

$$\mathbb{M} := \{\, \Xi \in \operatorname{End}(V) \mid \Xi_{ii} = 0 \ (i \in I) \,\} = \bigoplus_{i,j \in I; i \neq j} \operatorname{Hom}(W_j, W_i).$$

In this way we regard $\mathcal{M}$ as a non-singular affine variety. Observe that the complex algebraic torus $\mathfrak{Z}$ freely acts on $\mathcal{M}$ by the left multiplication and the spectral map is $\mathfrak{Z}$-invariant.

Let us introduce a Poisson structure on $\mathcal{M}$. For convenience, fix a basis of V which respects the decomposition $V = \bigoplus_{\lambda \in \Sigma} V_\lambda$. Define a bivector Π on $\mathcal{M} = \mathbb{A} \times \mathbb{T} \times \mathbb{M}$ by

$$\Pi = -\frac{1}{2} \sum_{i,j \in I,\, i \neq j} \sum_{p,q} \begin{vmatrix} a_{0i} & a_{0j} \\ a_{1i} & a_{1j} \end{vmatrix} \frac{\partial}{\partial (\Xi_{ij})_{pq}} \wedge \frac{\partial}{\partial (\Xi_{ji})_{qp}},$$

where $(\Xi_{ij})_{pq}$ are the matrix entries of the (i,j)-block of $\Xi \in \mathbb{M}$ with respect to the fixed basis. Obviously it defines a $\mathfrak{Z}$-invariant Poisson structure on $\mathcal{M}$.

Recall that $\mathrm{SL}_2(\mathbb{C})$ acts on the Weyl algebra $\mathcal{W}$ by

$$\mathrm{SL}_2(\mathbb{C}) \ni g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} \partial \\ x \end{pmatrix} \longmapsto g \begin{pmatrix} \partial \\ x \end{pmatrix} = \begin{pmatrix} a\partial + bx \\ c\partial + dx \end{pmatrix}.$$

This action induces a right $\mathrm{SL}_2(\mathbb{C})$-action on $\mathcal{M}$ commuting with the $\mathfrak{Z}$-action as follows:

$$M = \begin{pmatrix} A_1 & -A_0 \end{pmatrix} \begin{pmatrix} \partial \\ x \end{pmatrix} - \Gamma \overset{g}{\longmapsto} M^g = \begin{pmatrix} A_1 & -A_0 \end{pmatrix} g \begin{pmatrix} \partial \\ x \end{pmatrix} - \Gamma.$$

By a direct calculation one can check that if $\boldsymbol{a}$ is the spectral map of M, then the spectral map of M^g is $g^{-1}\boldsymbol{a}$, where $g^{-1}\colon \mathbb{P}^1 \to \mathbb{P}^1$ is the Möbius transformation defined by g^{-1}. This action preserves the Poisson structure since the determinant

$$\begin{vmatrix} a_{0i} & a_{0j} \\ a_{1i} & a_{1j} \end{vmatrix}$$

is invariant under the $\mathrm{SL}_2(\mathbb{C})$-action.

2.2. Symplectic fiber bundles

Fix an injective map $\boldsymbol{a}\colon I \to \mathbb{P}^1$, $i \mapsto a_i$ and let us describe the closed Poisson subvariety $\mathcal{M}_{\boldsymbol{a}} \subset \mathcal{M}$ consisting of all $M \in \mathcal{M}$ whose spectral map is $\boldsymbol{a}$.

Put

$$I_{\mathrm{fin}} = \{\, i \in I \mid \boldsymbol{a}(i) \neq \infty \,\}, \quad U = \bigoplus_{i \in I_{\mathrm{fin}}} W_i, \quad W_\infty = \bigoplus_{i \in I \setminus I_{\mathrm{fin}}} W_i.$$

Then $V = W_\infty \oplus U$, and $W_\infty = W_{\boldsymbol{a}^{-1}(\infty)}$ if $\infty \in \boldsymbol{a}(I)$ (otherwise $W_\infty = 0$). For $M = A_1\partial - A_0 x - \Gamma \in \mathcal{M}_{\boldsymbol{a}}$, define $C = \bigoplus_{i \in i} c_i\, 1_{W_i} \in \mathfrak{Z}$ by

$$c_i = \begin{cases} -a_{0i} & (a_i = \infty), \\ a_{1i} & (a_i \neq \infty). \end{cases}$$

In terms of the decomposition $V = W_\infty \oplus U$, the matrix $C^{-1}M$ is expressed as

$$C^{-1}M = \begin{pmatrix} 0 & 0 \\ 0 & 1_U \end{pmatrix} \partial - \begin{pmatrix} -1_{W_\infty} & 0 \\ 0 & A \end{pmatrix} x - C^{-1}\Gamma,$$

where

$$A = \bigoplus_{i \in I_{\mathrm{fin}}} a_i\, 1_{W_i} \in \operatorname{End}(U).$$

Put $T = C^{-1}\Theta \in \mathbb{T}$ and decompose it as

$$T = \bigoplus_{i \in I} T_i, \quad T_i \in \operatorname{End}(W_i).$$

Each T_i has the form

$$T_i = \bigoplus_{\lambda \in \Sigma_i} t_\lambda\, 1_{V_\lambda},$$

where t_λ, $\lambda \in \Sigma_i$ are given by

$$t_\lambda = \begin{cases} -a_{0i}^{-1}\theta_\lambda & (a_i = \infty), \\ a_{1i}^{-1}\theta_\lambda & (a_i \neq \infty). \end{cases}$$

Proposition 2.1. *The map*

$$\mathcal{M}_{\boldsymbol{a}} \to \mathfrak{Z} \times \mathbb{M} \times \mathbb{T}, \quad M \mapsto (C, C^{-1}\Xi, T)$$

is a $\mathfrak{Z}$-equivariant isomorphism, where $\mathfrak{Z}$ acts on $\mathfrak{Z} \times \mathbb{M} \times \mathbb{T}$ by

$$Z \ni \gamma\colon (C, X, T) \mapsto (\gamma C, X, T).$$

In particular, $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$ is isomorphic to $\mathbb{M} \times \mathbb{T}$.

Proof. The map $\mathfrak{Z} \times \mathbb{M} \times \mathbb{T} \to \mathcal{M}_{\boldsymbol{a}}$ defined by

$$(C, X, T) \mapsto C \begin{pmatrix} 0 & 0 \\ 0 & 1_U \end{pmatrix} \partial - C \begin{pmatrix} -1_{W_\infty} & 0 \\ 0 & A \end{pmatrix} x - C(T + X)$$

gives an inverse. □

The Poisson structure on $\mathcal{M}_{\boldsymbol{a}}$ descends to a Poisson structure on the quotient $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$, whose symplectic leaves are exactly the fibers of the projection $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z} \to \mathbb{T}$, $[M] \mapsto T$. Thus $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$ has a structure of symplectic fiber bundle over $\mathbb{T}$. On the other hand, the two-form on $\mathbb{M}$ defined by

$$\begin{aligned} \omega_{\boldsymbol{a}} &= -\frac{1}{2} \sum_{i,j \in I,\, i \neq j} c_i c_j \begin{vmatrix} a_{0i} & a_{0j} \\ a_{1i} & a_{1j} \end{vmatrix}^{-1} \operatorname{tr}(dX_{ij} \wedge dX_{ji}) \\ &= -\sum_{i,j \in I_{\mathrm{fin}},\, i \neq j} \frac{\operatorname{tr}(dX_{ij} \wedge dX_{ji})}{2(a_i - a_j)} - \sum_{i \in I_{\mathrm{fin}}} \operatorname{tr}(dX_{i\infty} \wedge dX_{\infty i}), \end{aligned}$$

where $X_{i\infty}, X_{\infty i}$ are the blocks of X for $\operatorname{Hom}(W_\infty, W_i)$, $\operatorname{Hom}(W_i, W_\infty)$, makes $\mathbb{M}$ into a symplectic manifold, which we denote by $\mathbb{M}_{\boldsymbol{a}}$. It is easy to see that the above isomorphism $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z} \xrightarrow{\simeq} \mathbb{M}_{\boldsymbol{a}} \times \mathbb{T}$ is an isomorphism of symplectic fiber bundles. We regard $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$ as the trivial symplectic fiber bundle in this way.

2.3. Simply-laced isomonodromy systems

Fix an injective map $\boldsymbol{a}\colon I \to \mathbb{P}^1$. Take any $M = A_1\partial - A_0 x - \Gamma \in \mathcal{M}_{\boldsymbol{a}}$ and consider the differential equation $Mv = 0$ for (locally defined) V-valued analytic function $v(x)$. Clearly this equation is invariant under the $\mathfrak{Z}$-action. Using the decomposition $V = W_\infty \oplus U$, we write

$$T = T_\infty \oplus T_{\mathrm{fin}}, \quad C^{-1}\Gamma = \begin{pmatrix} T_\infty & P \\ Q & B \end{pmatrix}.$$

Note that the block diagonal part of B with respect to the decomposition $U = \bigoplus_{i \in I_{\mathrm{fin}}} W_i$ is equal to T_{fin}. Define

$$L(x) = Ax + B + Q(x - T_\infty)^{-1}P \in \mathrm{End}(U) \otimes_{\mathbb{C}} \mathbb{C}(x).$$

Then $C^{-1}M$ is decomposed as

$$\begin{aligned} C^{-1}M &= \begin{pmatrix} x - T_\infty & -P \\ -Q & \partial - Ax - B \end{pmatrix} \\ &= \begin{pmatrix} x - T_\infty & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -(x - T_\infty)^{-1}P \\ -Q & \partial - Ax - B \end{pmatrix} \\ &= \begin{pmatrix} x - T_\infty & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -Q & 1 \end{pmatrix} \begin{pmatrix} 1 & -(x - T_\infty)^{-1}P \\ 0 & \partial - L(x) \end{pmatrix} \end{aligned} \tag{2.1}$$

in $\mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{W} \otimes_{\mathbb{C}[x]} \otimes \mathbb{C}(x)$. Thus generically the equation $Mv = 0$ for $v = w \oplus u$ is equivalent to the system of equations

$$w = (x - T_\infty)^{-1}Pu, \quad \frac{du}{dx} = L(x)u,$$

which reduces to the second equation $du/dx = L(x)u$ for u as the first equation uniquely determines w from u.

If $\infty \in \boldsymbol{a}(I)$, then $T_\infty = T_{\boldsymbol{a}^{-1}(\infty)}$ and

$$Q(x - T_\infty)^{-1}P = \sum_{\lambda \in \Sigma_{\boldsymbol{a}^{-1}(\infty)}} \frac{Q\mathrm{Id}_\lambda P}{x - t_\lambda},$$

where Id_λ denotes the idempotent of $\mathrm{End}(W_\infty)$ for V_λ. In particular, $L(x)$ has an at most simple pole at each eigenvalue of T_∞. If $\infty \notin \boldsymbol{a}(I)$, then $W_\infty = 0$ and

$$L(x) = Ax + B, \quad A = A_1^{-1}A_0, \; B = A_1^{-1}\Gamma.$$

The map

$$\mathcal{L}_{\boldsymbol{a}} \colon \mathcal{M}_{\boldsymbol{a}} \to \mathrm{End}(U) \otimes_{\mathbb{C}} \mathbb{C}(x), \quad M \mapsto L(x)$$

is $\mathfrak{Z}$-invariant as so is the map $M \mapsto C^{-1}M$. Thus it descends to a map $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z} \simeq \mathbb{M}_{\boldsymbol{a}} \times \mathbb{T} \to \mathrm{End}(U) \otimes_{\mathbb{C}} \mathbb{C}(x)$, which is explicitly given by

$$\mathbb{M} \times \mathbb{T} \ni (X, T) \mapsto Ax + T_{\mathrm{fin}} + B^\circ + Q(x - T_\infty)^{-1}P,$$

where we write

$$X = \begin{pmatrix} 0 & P \\ Q & B^\circ \end{pmatrix}.$$

The following fact is well-known in the formal reduction theory of linear ordinary differential equations (see [2, Lemma C.4]).

Proposition 2.2. *For any $M \in \mathcal{M}_{\boldsymbol{a}}$, there exists a formal series*

$$\widehat{F} = 1_U + F_1/x + F_2/x^2 + \cdots, \quad F_i \in \operatorname{End}(U)$$

such that

$$\widehat{F} L \widehat{F}^{-1} + \frac{d\widehat{F}}{dx}\widehat{F}^{-1} = Ax + T_{\mathrm{fin}} + \widehat{R}(x), \quad \widehat{R}(x) = \frac{R + R_1/x + R_2/x^2 + \cdots}{x}$$

with $R, R_i \in \operatorname{End}(U)$ commuting with A, T_{fin} and $[R^s, R_i] = -iR_i$, where R^s is the semisimple part of R.

Using the above $\widehat{F}$, let us define our Hamiltonian systems.

Definition 2.3 ([2, Theorem 5.9])**.** The *simply-laced isomonodromy system* is the non-autonomous Hamiltonian system on the symplectic fiber bundle $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z} = \mathbb{M}_{\boldsymbol{a}} \times \mathbb{T} \to \mathbb{T}$ with the Hamiltonian one-form $\varpi_{\boldsymbol{a}} = \sum_{\lambda \in \Sigma} H^{\boldsymbol{a}}_\lambda \, dt_\lambda$ defined by

$$H^{\boldsymbol{a}}_\lambda(M) := \begin{cases} \dfrac{1}{2} \operatorname*{Res}_{x=t_\lambda} \left(\operatorname{tr}(L(x)^2)\, dx\right) & (a_{\pi(\lambda)} = \infty), \\ \operatorname*{Res}_{x=\infty} \operatorname{tr}\left(\dfrac{\partial \widehat{F}}{\partial x}\widehat{F}^{-1} \mathrm{Id}^U_\lambda x \, dx\right) & (a_{\pi(\lambda)} \neq \infty), \end{cases}$$

where Id^U_λ denotes the idempotent of $\operatorname{End}(U)$ for V_λ.

Remark 2.4. Our symplectic form on $\mathbb{M}_{\boldsymbol{a}}$ is minus Boalch's original one, while the definition of Hamiltonians is the same. This is because our sign convention for the associated Hamiltonian equation is different to Boalch's: if m_i are local coordinates on $\mathbb{M}_{\boldsymbol{a}}$ then we consider the system of differential equations $\partial m_i/\partial t_\lambda = \{H^{\boldsymbol{a}}_\lambda, m_i\}$, while Boalch considers $\partial m_i/\partial t_\lambda = \{m_i, H^{\boldsymbol{a}}_\lambda\}$.

The simply-laced isomonodromy system is completely integrable and governs the isomonodromic deformations of the linear differential system $du/dx = L(x)u$ along t_λ's; see [2, Theorems 5.7, 6.1]. Furthermore, the systems for various $\boldsymbol{a}$ have the following beautiful symmetry. Recall that each $g \in \mathrm{SL}_2(\mathbb{C})$ gives a $\mathfrak{Z}$-equivariant Poisson automorphism of $\mathcal{M}$. It induces a Poisson isomorphism

$$\Phi_g \colon \mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z} \to \mathcal{M}_{g^{-1}\boldsymbol{a}}/\mathfrak{Z},$$

covering some automorphism $T \mapsto T^g = \bigoplus t^g_\lambda 1_{V_\lambda}$ of the base space $\mathbb{T}$ as a bundle map. It follows from [2, Theorem 5.4] that for any $g \in \mathrm{SL}_2(\mathbb{C})$, there exists $\Lambda \in \mathfrak{z}$ such that for any (local) solution $T \mapsto X(T) \in \mathbb{M}_{\boldsymbol{a}}$ of the Hamiltonian system with Hamiltonian one-form $\Phi^*_g \varpi_{g^{-1}\boldsymbol{a}}$, the map

$$T \mapsto e^{\Lambda T^2} X(T) e^{-\Lambda T^2}$$

is a solution of the simply-laced isomonodromy system $\varpi_{\boldsymbol{a}}$. In particular, the two Hamiltonian systems $\Phi_g^*\varpi_{g^{-1}\boldsymbol{a}}$, $\varpi_{\boldsymbol{a}}$ are gauge equivalent. Thus the difference $\Phi_g^*\varpi_{g^{-1}\boldsymbol{a}} - \varpi_{\boldsymbol{a}}$ may be non-zero but comes from some gauge transformation of the symplectic fiber bundle $\mathbb{M}_{\boldsymbol{a}} \times \mathbb{T}$.

For instance, take any $i \in I_{\text{fin}}$ and put

$$g_i = \begin{pmatrix} a_i & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}). \tag{2.2}$$

Then $g_i^{-1}(a_i) = \infty$, and a direct calculation shows

$$t_\lambda^{g_i} = \begin{cases} -t_\lambda & (\lambda \in \Sigma_i), \\ t_\lambda & (a_{\pi(\lambda)} = \infty), \\ \dfrac{t_\lambda}{a_i - a_{\pi(\lambda)}} & (\text{otherwise}). \end{cases}$$

Thus for any $\lambda \in \Sigma_i$, we have

$$\Phi_{g_i}^*(H_\lambda^{g_i^{-1}\boldsymbol{a}}\, dt_\lambda) = H_\lambda^{g_i^{-1}\boldsymbol{a}}(M^{g_i})\frac{dt_\lambda^{g_i}}{dt_\lambda}\, dt_\lambda = -H_\lambda^{g_i^{-1}\boldsymbol{a}}(M^{g_i})\, dt_\lambda.$$

In this case, we can show the following:

Proposition 2.5. *For any $\lambda \in \Sigma_i$ and $M \in \mathcal{M}_{\boldsymbol{a}}$, we have*

$$H_\lambda^{\boldsymbol{a}}(M) = -H_\lambda^{g_i^{-1}\boldsymbol{a}}(M^{g_i}).$$

Note that if we put $L_i(x) = \mathcal{L}_{g_i^{-1}\boldsymbol{a}}(M^{g_i})$, then

$$H_\lambda^{g_i^{-1}\boldsymbol{a}}(M^{g_i}) = \frac{1}{2}\operatorname*{Res}_{x=-t_\lambda}\left(\mathrm{tr}(L_i(x)^2)\, dx\right).$$

Thus for any $\lambda \in \Sigma$, the Hamiltonian $H_\lambda^{\boldsymbol{a}}$ can be described as the residue of the trace of the square of some matrix-valued rational function. The proof of Proposition 2.5 will be given in the next subsection.

2.4. Trace and determinant formulae for Hamiltonians

Fix an injective map $\boldsymbol{a}\colon I \to \mathbb{P}^1$. In this section we introduce some useful formulae for the Hamiltonians $H_\lambda^{\boldsymbol{a}}$ and use them to prove Proposition 2.5. The results in this section are based on our earlier work [14].

For $M = A_1\partial - A_0x - \Theta - \Xi \in \mathcal{M}_{\boldsymbol{a}}$, let $M_0 = M_0(\partial, x) \in \mathcal{M}_{\boldsymbol{a}}$ be its block diagonal part:

$$M_0(\partial, x) = M - \Xi = A_1\partial - A_0x - \Theta.$$

Theorem 2.6 (Trace formula). *For $i \in I_{\mathrm{fin}}$, $\lambda \in \Sigma_i$ and $M \in \mathcal{M}_{\boldsymbol{a}}$, the following equality holds:*

$$H_\lambda^{\boldsymbol{a}}(M) = -\sum_{k=1}^\infty \frac{1}{k} \operatorname*{Res}_{x=\infty} \left(\operatorname*{Res}_{y=a_i x + t_\lambda} x \operatorname{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k \right] dy \right) dx.$$

Let us prove the theorem. Fix $i \in I_{\mathrm{fin}}$, $\lambda \in \Sigma_i$ and $M \in \mathcal{M}_{\boldsymbol{a}}$. Using the fixed basis of V, we identify the coordinate ring of the complex affine variety $\mathfrak{gl}(V)$ with the polynomial ring $\mathbb{C}[z_{pq}; p,q = 1,2,\dots,\dim V]$, and put $Z = (z_{pq})$. Let $\mathbb{C}[\![\mathfrak{gl}(V)]\!]$ be the formal completion of the local ring of $\mathfrak{gl}(V)$ at 0, which is identified with the ring of formal power series $\mathbb{C}[\![z_{pq}; p,q = 1,2,\dots,\dim V]\!]$. The adjoint action of $\mathrm{GL}(V)$ on $\mathfrak{gl}(V)$ induces an action on $\mathbb{C}[\![\mathfrak{gl}(V)]\!]$.

Put $\overline{y} = y - a_i x - t_\lambda$ and embed $\mathbb{C}(x,y)$ in $\mathbb{C}(\!(\overline{y})\!)(\!(x^{-1})\!)$ in the obvious manner.

Lemma 2.7. *The substitution $Z = \Xi M_0(y,x)^{-1}$ gives a well-defined map*

$$\mathbb{C}[\![\mathfrak{gl}(V)]\!]^{\mathrm{GL}(V)} \to \mathbb{C}(\!(\overline{y})\!)(\!(x^{-1})\!).$$

Proof. Since any element of $\mathbb{C}[\![\mathfrak{gl}(V)]\!]^{\mathrm{GL}(V)}$ is uniquely expressed as a formal series $\sum_{k=0}^\infty c_k \operatorname{tr}(Z^k)$, it is sufficient to show

$$\lim_{k\to\infty} \operatorname{ord}_{1/x} \left(\operatorname{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k \right] \right) = \infty,$$

where $\operatorname{ord}_{1/x}$ denotes the order of a formal Laurent series in x^{-1} with coefficients in $\mathbb{C}(\!(\overline{y})\!)$. For $\mu, \nu \in \Sigma$, let $\Xi_{\mu\nu}$ be the (μ,ν)-block of Ξ with respect to the decomposition $V = \bigoplus_{\mu\in\Sigma} V_\mu$. Then we have

$$\operatorname{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k \right] = \sum_{\mu_1,\dots,\mu_k \in \Sigma} \frac{\operatorname{tr}\left(\Xi_{\mu_1\mu_2}\Xi_{\mu_2\mu_3}\cdots\Xi_{\mu_k\mu_1}\right)}{\prod_{l=1}^k f_{\mu_l}(y,x)},$$

where

$$f_\mu(y,x) = a_{1\pi(\mu)} y - a_{0\pi(\mu)} x - \theta_\mu \quad (\mu \in \Sigma).$$

For $\mu \in \Sigma$ with $a_{\pi(\mu)} = \infty$, we have

$$\frac{1}{f_\mu(y,x)} = \frac{1}{-a_{0\pi(\mu)}(x - t_\mu)},$$

while for $\mu \in \Sigma$ with $\pi(\mu) \in I_{\mathrm{fin}}$, we have

$$\frac{1}{f_\mu(y,x)} = \frac{1}{a_{1\pi(\mu)}(\overline{y} - (a_{\pi(\mu)} - a_i)x - (t_\mu - t_\lambda))}.$$

Hence

$$\operatorname{ord}_{1/x}\left(\frac{1}{f_\mu(y,x)}\right) \geq \begin{cases} 0 & (\mu \in \Sigma_i), \\ 1 & (\mu \in \Sigma \setminus \Sigma_i), \end{cases}$$

which implies

$$\operatorname{ord}_{1/x}\left(\prod_{l=1}^{k} f_{\mu_l}(y,x)^{-1}\right) \geq \#\{\, l \in \{1,2,\ldots,k\} \mid \pi(\mu_l) \neq i \,\}$$

for $\mu_1, \mu_2, \ldots, \mu_l \in \Sigma$. On the other hand, $\Xi_{\mu\nu} = 0$ if $\pi(\mu) = \pi(\nu)$ (recall that Ξ is block off-diagonal). It follows that if

$$\#\{\, l \in \{1,2,\ldots,k\} \mid \pi(\mu_l) = i \,\} > \frac{k}{2},$$

then

$$\operatorname{tr}(\Xi_{\mu_1\mu_2}\Xi_{\mu_2\mu_3}\cdots\Xi_{\mu_k\mu_1}) = 0.$$

Thus we obtain

$$\operatorname{ord}_{1/x}\left(\operatorname{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k\right]\right) \geq \frac{k}{2} \to \infty \quad (k \to \infty).$$

□

We apply Lemma 2.7 to the formal series

$$\operatorname{tr}\log(1-Z) = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr} Z^k,$$

which is equal to

$$\log\det(1-Z) = \sum_{k=1}^{\infty} \frac{1}{k}(1-\det(1-Z))^k.$$

Substituting $\Xi M_0(y,x)^{-1}$ for Z, we obtain

$$1 - Z = 1 - \Xi M_0(y,x)^{-1} = (M_0(y,x) - \Xi)M_0(y,x)^{-1} = M(y,x)M_0(y,x)^{-1},$$

and hence

$$\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k\right] = \sum_{k=1}^{\infty} \frac{1}{k}\left(1 - \frac{\det M(y,x)}{\det M_0(y,x)}\right)^k$$

as elements of $\mathbb{C}(\!(\overline{y})\!)(\!(x^{-1})\!)$. On the other hand, the decomposition (2.1) yields

$$\frac{\det M(y,x)}{\det M_0(y,x)} = \frac{\det(y - L(x))}{\det(y - Ax - T_{\mathrm{fin}})}.$$

Taking the formal series $\widehat{F}$ shown in Proposition 2.2, we have

$$\begin{aligned}\det(y - L(x)) &= \det(\widehat{F}(y - L)\widehat{F}^{-1}) \\ &= \det(y - Ax - T_{\mathrm{fin}} - \widehat{R} + \widehat{F}'\widehat{F}^{-1}).\end{aligned}$$

Thus

$$\frac{\det M(y,x)}{\det M_0(y,x)} = \frac{\det(y - Ax - T_{\mathrm{fin}} - \widehat{R} + \widehat{F}'\widehat{F}^{-1})}{\det(y - Ax - T_{\mathrm{fin}})}.$$

Lemma 2.8. *The substitution* $Z = (\widehat{R} - \widehat{F}'\widehat{F}^{-1})(y - Ax - T_{\mathrm{fin}})^{-1}$ *gives a well-defined map*

$$\mathbb{C}[\![\mathfrak{gl}(V)]\!]^{\mathrm{GL}(V)} \to \mathbb{C}(\!(\overline{y})\!)(\!(x^{-1})\!).$$

Proof. For each $\mu \in \Sigma$ with $a_{\pi(\mu)} \neq \infty$, we have

$$\mathrm{ord}_{1/x}\left(\frac{1}{y - a_{\pi(\mu)}x - t_\mu}\right) \geq 0,$$

which together with the inequality $\mathrm{ord}_{1/x}(\widehat{R} - \widehat{F}'\widehat{F}^{-1}) \geq 1$ shows

$$\mathrm{ord}_{1/x}\left(\mathrm{tr}\left[\left((\widehat{R} - \widehat{F}'\widehat{F}^{-1})(y - Ax - T_{\mathrm{fin}})^{-1}\right)^k\right]\right) \geq k.$$

This completes the proof. □

Applying the above lemma to the formal series $\mathrm{tr}\log(1-Z) = \log\det(1-Z)$, we obtain

$$\begin{aligned}&\sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}\left[\left((\widehat{R} - \widehat{F}'\widehat{F}^{-1})(y - Ax - T_{\mathrm{fin}})^{-1}\right)^k\right] \\ &\qquad = \sum_{k=1}^{\infty}\frac{1}{k}\left(1 - \frac{\det(y - Ax - T_{\mathrm{fin}} - \widehat{R} + \widehat{F}'\widehat{F}^{-1})}{\det(y - Ax - T_{\mathrm{fin}})}\right)^k \\ &\qquad = \sum_{k=1}^{\infty}\frac{1}{k}\left(1 - \frac{\det M(y,x)}{\det M_0(y,x)}\right)^k = \sum_{k=1}^{\infty}\frac{1}{k}\,\mathrm{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k\right].\end{aligned}$$

Thus Theorem 2.6 follows from the lemma below.

Lemma 2.9. *The following equality holds:*

$$\sum_{k=1}^{\infty}\frac{1}{k}\operatorname*{Res}_{x=\infty}\operatorname*{Res}_{\overline{y}=0} x\,\mathrm{tr}\left[\left((\widehat{R} - \widehat{F}'\widehat{F}^{-1})(y - Ax - T_{\mathrm{fin}})^{-1}\right)^k\right] d\overline{y}\,dx = -H_\lambda^{\boldsymbol{a}}(M).$$

Proof. From the inequalities shown in the proof of the previous lemma we easily deduce

$$\operatorname*{Res}_{x=\infty}\operatorname*{Res}_{\overline{y}=0} x\,\mathrm{tr}\left[\left((\widehat{R}-\widehat{F}'\widehat{F}^{-1})(y-Ax-T_{\mathrm{fin}})^{-1}\right)^k\right]d\overline{y}\,dx=0 \quad (k\geq 3).$$

Furthermore, since $\widehat{R}_{\mu\nu}=0$ $(\mu\neq\nu)$ we have

$$\begin{aligned}
&\operatorname*{Res}_{x=\infty}\operatorname*{Res}_{\overline{y}=0} x\,\mathrm{tr}\left[\left((\widehat{R}-\widehat{F}'\widehat{F}^{-1})(y-Ax-T_{\mathrm{fin}})^{-1}\right)^2\right]d\overline{y}\,dx\\
&\quad=\sum_{\substack{\mu\in\Sigma\\ a_{\pi(\mu)}\neq\infty,\ \mu\neq\lambda}}\operatorname*{Res}_{x=\infty} x\frac{\mathrm{tr}\left((\widehat{R}-\widehat{F}'\widehat{F}^{-1})_{\lambda\mu}(\widehat{R}-\widehat{F}'\widehat{F}^{-1})_{\mu\lambda}\right)}{(a_i-a_{\pi(\mu)})x+(t_\lambda-t_\mu)}\,dx\\
&\quad=\sum_{\substack{\mu\in\Sigma\\ a_{\pi(\mu)}\neq\infty,\ \mu\neq\lambda}}\operatorname*{Res}_{x=\infty} x\frac{\mathrm{tr}\left((\widehat{F}'\widehat{F}^{-1})_{\lambda\mu}(\widehat{F}'\widehat{F}^{-1})_{\mu\lambda}\right)}{(a_i-a_{\pi(\mu)})x+(t_\lambda-t_\mu)}\,dx,
\end{aligned}$$

which is zero because $\mathrm{ord}_{1/x}(\widehat{F}'\widehat{F}^{-1})\geq 2$. Finally, a direct calculation shows

$$\begin{aligned}
&\operatorname*{Res}_{x=\infty}\operatorname*{Res}_{\overline{y}=0} x\,\mathrm{tr}\left((\widehat{R}-\widehat{F}'\widehat{F}^{-1})(y-Ax-T_{\mathrm{fin}})^{-1}\right)d\overline{y}\,dx\\
&\quad=\operatorname*{Res}_{x=\infty} x\,\mathrm{tr}\left((\widehat{R}-\widehat{F}'\widehat{F}^{-1})_{\lambda\lambda}\right)dx=\mathrm{tr}(R_1)_{\lambda\lambda}-H^{\boldsymbol{a}}_\lambda(M).
\end{aligned}$$

Since $[R^s,R_1]=R_1$ we have $\mathrm{tr}(R_1)_{\lambda\lambda}=0$. Thus we obtain the desired formula. □

The above arguments also yield the following formula:

Theorem 2.10 (Determinant formula)**.** *For $i\in I_{\mathrm{fin}}$, $\lambda\in\Sigma_i$ and $M\in\mathcal{M}_{\boldsymbol{a}}$, the following equality holds:*

$$H^{\boldsymbol{a}}_\lambda(M)=-\sum_{k=1}^\infty\frac{1}{k}\operatorname*{Res}_{x=\infty}\left(\operatorname*{Res}_{y=a_ix+t_\lambda} x\left(1-\frac{\det M(y,x)}{\det M_0(y,x)}\right)^k dy\right)dx.$$

For $\lambda\in\Sigma$ with $a_{\pi(\lambda)}=\infty$, we can also describe the Hamiltonian $H^{\boldsymbol{a}}_\lambda$ in a similar form.

Proposition 2.11. *For $\lambda\in\Sigma$ with $a_{\pi(\lambda)}=\infty$ and $M\in\mathcal{M}_{\boldsymbol{a}}$, the following equalities hold:*

$$\begin{aligned}
H^{\boldsymbol{a}}_\lambda(M)&=-\sum_{k=1}^\infty\frac{1}{k}\operatorname*{Res}_{y=\infty}\left(\operatorname*{Res}_{x=t_\lambda} y\,\mathrm{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k\right]dx\right)dy\\
&=-\sum_{k=1}^\infty\frac{1}{k}\operatorname*{Res}_{y=\infty}\left(\operatorname*{Res}_{x=t_\lambda} y\left(1-\frac{\det M(y,x)}{\det M_0(y,x)}\right)^k dx\right)dy.
\end{aligned}$$

Proof. We embed $\mathbb{C}(x,y)$ in $\mathbb{C}(\!(x-t_\lambda)\!)(\!(y^{-1})\!)$. Then a direct calculation shows

$$\operatorname{ord}_{1/y}\left(\frac{1}{a_{0\pi(\mu)}y - a_{1\pi(\mu)}x - \theta_\mu}\right) \geq \begin{cases} 0 & (a_{\pi(\mu)} = \infty), \\ 1 & (a_{\pi(\mu)} \neq \infty) \end{cases}$$

for every $\mu \in \Sigma$. Thus arguments similar to the proofs of Lemmas 2.7, 2.9 yield the equalities among the infinite sums

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}\left[\left(\Xi M_0(y,x)^{-1}\right)^k\right] \\
&\qquad = \sum_{k=1}^{\infty} \frac{1}{k}\left(1 - \frac{\det M(y,x)}{\det M_0(y,x)}\right)^k \\
&\qquad = \sum_{k=1}^{\infty} \frac{1}{k}\left(1 - \frac{\det(y - L(x))}{\det(y - Ax - T_{\mathrm{fin}})}\right)^k \\
&\qquad = \sum_{k=1}^{\infty} \frac{1}{k} \operatorname{tr}\left[\left((L(x) - Ax - T_{\mathrm{fin}})(y - Ax - T_{\mathrm{fin}})^{-1}\right)^k\right]
\end{aligned}$$

in $\mathbb{C}(\!(x-t_\lambda)\!)(\!(y^{-1})\!)$. Since

$$(y - Ax - T_{\mathrm{fin}})^{-1} = y^{-1} \sum_{l \geq 0} (Ax + T_{\mathrm{fin}})^l y^{-l},$$

the order counting shows

$$\begin{aligned}
&\operatorname*{Res}_{y=\infty} y \operatorname{tr}\left[\left((L(x) - Ax - T_{\mathrm{fin}})(y - Ax - T_{\mathrm{fin}})^{-1}\right)^k\right] dy \\
&\qquad\qquad = \begin{cases} 0 & (k \geq 3), \\ -\operatorname{tr}\left[(L - Ax - T_{\mathrm{fin}})^2\right] & (k = 2), \\ -\operatorname{tr}\left[(L - Ax - T_{\mathrm{fin}})(Ax + T_{\mathrm{fin}})\right] & (k = 1). \end{cases}
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{1}{k} \operatorname*{Res}_{y=\infty} y \operatorname{tr}\left[\left((L(x) - Ax - T_{\mathrm{fin}})(y - Ax - T_{\mathrm{fin}})^{-1}\right)^k\right] dy \\
&\qquad = -\frac{1}{2} \operatorname{tr}\left[(L - Ax - T_{\mathrm{fin}})^2\right] - \operatorname{tr}\left[(L - Ax - T_{\mathrm{fin}})(Ax + T_{\mathrm{fin}})\right] \\
&\qquad = -\frac{1}{2} \operatorname{tr}\left[(L - Ax - T_{\mathrm{fin}})(L + Ax + T_{\mathrm{fin}})\right] \\
&\qquad = -\frac{1}{2} \operatorname{tr}\left[L^2 - (Ax + T_{\mathrm{fin}})^2\right],
\end{aligned}$$

whose residue at $x = t_\lambda$ is equal to that of $-\operatorname{tr}(L^2)/2$. □

As an application of Theorem 2.6 and Proposition 2.11, we will give a proof of Proposition 2.5.

Proof of Proposition 2.5. Define variables x_i, y_i by

$$\begin{pmatrix} y_i \\ x_i \end{pmatrix} = g_i \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} a_i y - x \\ y \end{pmatrix}.$$

Then

$$M^{g_i}(y,x) = M(y_i, x_i), \quad M_0^{g_i}(y,x) = M_0(y_i, x_i).$$

Also, for $F \in \mathbb{C}(x,y) = \mathbb{C}(x_i, y_i)$ we have

$$\operatorname*{Res}_{y=\infty} \left(\operatorname*{Res}_{x=-t_\lambda} F\, dx \right) dy = - \operatorname*{Res}_{x_i=\infty} \left(\operatorname*{Res}_{y_i=a_i x_i + t_\lambda} F\, dy_i \right) dx_i.$$

Thus Theorem 2.6 and Proposition 2.11 yield

$$\begin{aligned} -H_\lambda^{g_i^{-1}\boldsymbol{a}}(M^{g_i}) &= \sum_{k=1}^\infty \frac{1}{k} \operatorname*{Res}_{y=\infty} \left(\operatorname*{Res}_{x=-t_\lambda} y \operatorname{tr} \left[\left(\Xi M_0^{g_i}(y,x)^{-1} \right)^k \right] dx \right) dy \\ &= - \sum_{k=1}^\infty \frac{1}{k} \operatorname*{Res}_{x_i=\infty} \left(\operatorname*{Res}_{y_i=a_i x_i + t_\lambda} x_i \operatorname{tr} \left[\left(\Xi M_0(y_i,x_i)^{-1} \right)^k \right] dy_i \right) dx_i \\ &= H_\lambda^{\boldsymbol{a}}(M). \end{aligned}$$

□

§3. Quantization

Fix an injective map $\boldsymbol{a}\colon I \to \mathbb{P}^1$. This section is devoted to quantize the simply-laced Hamiltonian system on $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$.

We denote the coordinate ring of a complex affine variety S by $\mathbb{C}[S]$.

3.1. Formal deformation quantization and Lax matrices

We first construct a formal deformation quantization of the affine Poisson variety $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$. Recall that for each $M = A_1\partial - A_0 x - \Theta - \Xi \in \mathcal{M}_{\boldsymbol{a}}$ we have defined

$$C = \bigoplus_{i \in I} c_i\, 1_{W_i}, \quad T = C^{-1}\Theta = \bigoplus_{\lambda \in \Sigma} t_\lambda\, 1_{V_\lambda}.$$

Varying M we thus obtain functions c_i, t_λ, $(\Xi_{ij})_{pq}$ on $\mathcal{M}_{\boldsymbol{a}}$, which satisfy

$$\{(\Xi_{ij})_{pq}, (\Xi_{kl})_{rs}\} = -\delta_{il}\delta_{jk}\delta_{ps}\delta_{qr} \begin{vmatrix} a_{0i} & a_{0j} \\ a_{1i} & a_{1j} \end{vmatrix}, \quad \{c_i, \cdot\} = \{t_\lambda, \cdot\} = 0,$$

where a_{0i}, a_{1i} $(i \in I)$ are defined by

$$(a_{1i}, -a_{0i}) = \begin{cases} (0, c_i) & (a_i = \infty), \\ (c_i, -c_i a_i) & (a_i \neq \infty). \end{cases}$$

We also regard c_i, t_λ as coordinate functions on $\mathfrak{Z} \times \mathbb{T}$. Then $\mathbb{C}[\mathcal{M}_{\boldsymbol{a}}]$ is a $\mathbb{C}[\mathfrak{Z} \times \mathbb{T}]$-algebra and every element of $\mathbb{C}[\mathfrak{Z} \times \mathbb{T}]$ is Casimir.

Let $\mathcal{A}_{\boldsymbol{a}}$ be the $\mathbb{C}[\mathfrak{Z} \times \mathbb{T}][\![\hbar]\!]$-algebra with generators $(\widehat{\Xi}_{ij})_{pq}$ $(i \neq j \in I$, $p = 1, 2, \ldots, \dim W_i$, $q = 1, 2, \ldots, \dim W_j)$ and fundamental relations

$$\left[(\widehat{\Xi}_{ij})_{pq}, (\widehat{\Xi}_{kl})_{rs}\right] = -\delta_{il}\delta_{jk}\delta_{ps}\delta_{qr}\hbar \begin{vmatrix} a_{0i} & a_{0j} \\ a_{1i} & a_{1j} \end{vmatrix}.$$

This is obviously a formal deformation quantization of the Poisson algebra $\mathbb{C}[\mathcal{M}_{\boldsymbol{a}}]$.

The matrices C, Θ, T may now be regarded as elements of $\mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{A}_{\boldsymbol{a}}$. Let $\widehat{\Xi} = (\widehat{\Xi}_{ij}) \in \mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{A}_{\boldsymbol{a}}$ be the block off-diagonal matrix with each $\widehat{\Xi}_{ij}$ having matrix entries $(\widehat{\Xi}_{ij})_{pq}$. Define

$$\widehat{M}(\partial, x) = A_1\partial - A_0 x - \Theta - \widehat{\Xi} \in \mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{A}_{\boldsymbol{a}} \otimes_{\mathbb{C}} \mathcal{W},$$

where

$$A_0 = \bigoplus_{i \in I} a_{0i}\, 1_{W_i}, \ \ A_1 = \bigoplus_{i \in I} a_{1i}\, 1_{W_i} \in \mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{A}_{\boldsymbol{a}}.$$

For $i, j \in I$, let $\widehat{M}_{ij}$ be the (i, j)-block of $\widehat{M}$ and $(\widehat{M}_{ij})_{pq}$ be its matrix entries.

Proposition 3.1. *The equality*

$$\left[(\widehat{M}_{ij})_{pq}, (\widehat{M}_{kl})_{rs}\right] = (\delta_{ij}\delta_{kl}\delta_{pq}\delta_{rs} - \delta_{il}\delta_{jk}\delta_{ps}\delta_{qr})\hbar \begin{vmatrix} a_{0i} & a_{0k} \\ a_{1i} & a_{1k} \end{vmatrix}$$

holds for any $i, j, k, l,\ p, q, r, s$.

Proof. By the definition we have

$$(\widehat{M}_{ij})_{pq} = \delta_{ij}\delta_{pq}(a_{1i}\partial - a_{0i}x - \theta_{i,p}) - (\widehat{\Xi}_{ij})_{pq},$$

where $\theta_{i,p}$ is the p-th diagonal entry of the i-th block Θ_i of Θ. Since the matrix entries of $\widehat{\Xi}$ commute with x, ∂ and the elements of $\mathbb{C}[\mathfrak{Z} \times \mathbb{T}]$, we obtain the desired formula as follows:

$$\begin{aligned} \left[(\widehat{M}_{ij})_{pq}, (\widehat{M}_{kl})_{rs}\right] &= \delta_{ij}\delta_{pq}\delta_{kl}\delta_{rs}[a_{1i}\partial - a_{0i}x - \theta_{i,p}, a_{1k}\partial - a_{0k}x - \theta_{k,r}] \\ &\quad + \left[(\widehat{\Xi}_{ij})_{pq}, (\widehat{\Xi}_{kl})_{rs}\right] \end{aligned}$$

$$= \delta_{ij}\delta_{pq}\delta_{kl}\delta_{rs}\hbar(-a_{1i}a_{0k} + a_{0i}a_{1k}) + \left[(\widehat{\Xi}_{ij})_{pq}, (\widehat{\Xi}_{kl})_{rs}\right]$$
$$= (\delta_{ij}\delta_{kl}\delta_{pq}\delta_{rs} - \delta_{il}\delta_{jk}\delta_{ps}\delta_{qr})\hbar \begin{vmatrix} a_{0i} & a_{0k} \\ a_{1i} & a_{1k} \end{vmatrix}.$$

□

We let $\mathfrak{Z}$ act on $\mathcal{A}_{\boldsymbol{a}}$ by

$$\mathfrak{Z} \ni \gamma = \bigoplus_{i\in I} \gamma_i \, 1_{W_i} \colon (c_i, t_\lambda, (\widehat{\Xi}_{ij})_{pq}) \mapsto (\gamma_i^{-1}c_i, t_\lambda, \gamma_i^{-1}(\widehat{\Xi}_{ij})_{pq}),$$

so that $\gamma\colon \widehat{M} \mapsto \gamma^{-1}\widehat{M}$. This action induces an action on the quasi-classical limit $\mathbb{C}[\mathcal{M}_{\boldsymbol{a}}]$, which coincides with the one induced from the $\mathfrak{Z}$-action on $\mathcal{M}_{\boldsymbol{a}}$. Hence the invariant part $\mathcal{A}_{\boldsymbol{a}}^{\mathfrak{Z}} \subset \mathcal{A}_{\boldsymbol{a}}$ is a formal deformation quantization of the quotient space $\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}$.

Using the decomposition $V = W_\infty \oplus U$ we write

$$C^{-1}\widehat{\Xi} = \widehat{X} = \begin{pmatrix} 0 & \widehat{P} \\ \widehat{Q} & \widehat{B}^\circ \end{pmatrix}, \quad T = T_\infty \oplus T_{\mathrm{fin}}.$$

Let $\widehat{B}^\circ_{ij}$, $\widehat{Q}_i$, $\widehat{P}_i$ be the blocks of $\widehat{B}^\circ, \widehat{Q}, \widehat{P}$ with respect to the decomposition $U = \bigoplus_{i\in I_{\mathrm{fin}}} W_i$ (so they are the blocks of $\widehat{X}$). Then their matrix entries generate $\mathcal{A}_{\boldsymbol{a}}^{\mathfrak{Z}}$ as a $\mathbb{C}[\mathbb{T}][\![\hbar]\!]$-algebra and satisfy the following commutation relation:

$$\left[(\widehat{B}^\circ_{ij})_{pq}, (\widehat{B}^\circ_{kl})_{rs}\right] = -\delta_{il}\delta_{jk}\delta_{pr}\delta_{qs}\hbar(a_i - a_j), \quad \left[(\widehat{P}_i)_{pq}, (\widehat{Q}_j)_{rs}\right] = \delta_{ij}\delta_{ps}\delta_{qr}\hbar,$$

$$\left[(\widehat{B}^\circ_{ij})_{pq}, (\widehat{Q}_k)_{rs}\right] = \left[(\widehat{B}^\circ_{ij})_{pq}, (\widehat{P}_k)_{rs}\right] = \left[(\widehat{Q}_i)_{pq}, (\widehat{Q}_j)_{rs}\right] = \left[(\widehat{P}_i)_{pq}, (\widehat{P}_j)_{rs}\right] = 0.$$

Define

$$\widehat{L}(x) = Ax + T_{\mathrm{fin}} + \widehat{B}^\circ + \widehat{Q}(x - T_\infty)^{-1}\widehat{P} \in \mathrm{End}(U) \otimes_{\mathbb{C}} \mathcal{A}_{\boldsymbol{a}}^{\mathfrak{Z}} \otimes_{\mathbb{C}} \mathbb{C}(x).$$

Observe that the quasi-classical limit of $\widehat{L}(x)$ is the map

$$\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z} \to \mathrm{End}(U) \otimes_{\mathbb{C}} \mathbb{C}(x), \quad [M] \mapsto L(x)$$

regarded as an element of $\mathrm{End}(U) \otimes_{\mathbb{C}} \mathbb{C}[\mathcal{M}_{\boldsymbol{a}}/\mathfrak{Z}] \otimes_{\mathbb{C}} \mathbb{C}(x)$ (which we also denote by $L(x)$).

Proposition 3.2. *$\widehat{L}(x)$ is a Lax matrix of Gaudin type, i.e., it satisfies the following "RLL = LLR" relation:*

$$\left[(\widehat{L}_{ij})_{pq}(x), (\widehat{L}_{kl})_{rs}(y)\right] = \frac{\delta_{jk}\delta_{qr}\hbar}{x-y}\left((\widehat{L}_{il})_{ps}(y) - (\widehat{L}_{il})_{ps}(x)\right) - \frac{\delta_{li}\delta_{sp}\hbar}{x-y}\left((\widehat{L}_{kj})_{qr}(y) - (\widehat{L}_{kj})_{qr}(x)\right).$$

Proof. Put

$$\widehat{L}^+(x) = Ax + T_{\text{fin}} + \widehat{B}^\circ, \quad \widehat{L}^-(x) = \widehat{Q}(x - T_\infty)^{-1}\widehat{P},$$

so that $\widehat{L}(x) = \widehat{L}^+(x) + \widehat{L}^-(x)$. Denoting the diagonal entries of T_i by $t_{i,p}$, we have

$$(\widehat{L}^+_{ij})_{pq}(x) = \delta_{ij}\delta_{pq}(a_i x + t_{i,p}) + (\widehat{B}^\circ_{ij})_{pq}, \quad (\widehat{L}^-_{ij})_{pq}(x) = \sum_r \frac{(\widehat{Q}_i)_{pr}(\widehat{P}_j)_{rq}}{x - t_{\infty,r}},$$

and obviously

$$\Big[(\widehat{L}^+_{ij})_{pq}(x), (\widehat{L}^-_{kl})_{rs}(y)\Big] = 0.$$

Thus it is sufficient to show that both $\widehat{L}^+, \widehat{L}^-$ satisfy the $RLL = LLR$ relation. First, we have

$$\Big[(\widehat{L}^+_{ij})_{pq}(x), (\widehat{L}^+_{kl})_{rs}(y)\Big] = \Big[(\widehat{B}^\circ_{ij})_{pq}, (\widehat{B}^\circ_{kl})_{rs}\Big] = -\delta_{il}\delta_{jk}\delta_{ps}\delta_{qr}\hbar(a_i - a_j).$$

On the other hand,

$$(\widehat{L}^+_{il})_{ps}(y) - (\widehat{L}^+_{il})_{ps}(x) = \delta_{il}\delta_{ps}a_i(y - x),$$

and hence

$$\begin{aligned}\frac{\delta_{jk}\delta_{qr}}{x-y}\Big((\widehat{L}^+_{il})_{ps}(y) - (\widehat{L}^+_{il})_{ps}(x)\Big) - \frac{\delta_{li}\delta_{sp}}{x-y}\Big((\widehat{L}^+_{kj})_{qr}(y) - (\widehat{L}^+_{kj})_{qr}(x)\Big) \\ = -\delta_{il}\delta_{ps}\delta_{jk}\delta_{qr}a_i + \delta_{kj}\delta_{rq}\delta_{li}\delta_{sp}a_k = -\delta_{il}\delta_{ps}\delta_{jk}\delta_{qr}(a_i - a_j).\end{aligned}$$

Thus $\widehat{L}^+$ satisfy the $RLL = LLR$ relation. Next we have

$$\Big[(\widehat{L}^-_{ij})_{pq}(x), (\widehat{L}^-_{kl})_{rs}(y)\Big] = \sum_{u,v} \frac{\Big[(\widehat{Q}_i)_{pu}(\widehat{P}_j)_{uq}, (\widehat{Q}_k)_{rv}(\widehat{P}_l)_{vs}\Big]}{(x - t_{\infty,u})(y - t_{\infty,v})}.$$

The commutation relation for $(\widehat{Q}_i)_{pq}$, $(\widehat{P}_j)_{rs}$ implies

$$\Big[(\widehat{Q}_i)_{pu}(\widehat{P}_j)_{uq}, (\widehat{Q}_k)_{rv}(\widehat{P}_l)_{vs}\Big] = \delta_{jk}\delta_{qr}\delta_{uv}\hbar(\widehat{Q}_i)_{pu}(\widehat{P}_l)_{vs} - \delta_{li}\delta_{sp}\delta_{vu}\hbar(\widehat{Q}_k)_{rv}(\widehat{P}_j)_{uq}.$$

Hence

$$\begin{aligned}\sum_{u,v} \frac{\Big[(\widehat{Q}_i)_{pu}(\widehat{P}_j)_{uq}, (\widehat{Q}_k)_{rv}(\widehat{P}_l)_{vs}\Big]}{(x - t_{\infty,u})(y - t_{\infty,v})} \\ = \sum_u \frac{\delta_{jk}\delta_{qr}(\widehat{Q}_i)_{pu}(\widehat{P}_l)_{us} - \delta_{li}\delta_{sp}(\widehat{Q}_k)_{ru}(\widehat{P}_j)_{uq}}{(x - t_{\infty,u})(y - t_{\infty,u})}\end{aligned}$$

$$= \frac{\hbar}{x-y}\left(\sum_u \frac{\delta_{jk}\delta_{qr}(\widehat{Q}_i)_{pu}(\widehat{P}_l)_{us} - \delta_{li}\delta_{sp}(\widehat{Q}_k)_{ru}(\widehat{P}_j)_{uq}}{y - t_{\infty,u}} \right.$$
$$\left. - \sum_u \frac{\delta_{jk}\delta_{qr}(\widehat{Q}_i)_{pu}(\widehat{P}_l)_{us} - \delta_{li}\delta_{sp}(\widehat{Q}_k)_{ru}(\widehat{P}_j)_{uq}}{x - t_{\infty,u}}\right)$$
$$= \frac{\hbar}{x-y}\Big(\delta_{jk}\delta_{qr}(\widehat{L}^-_{il})_{ps}(y) - \delta_{il}\delta_{sp}(\widehat{L}^-_{kj})_{rq}(y)$$
$$- \delta_{jk}\delta_{qr}(\widehat{L}^-_{il})_{ps}(x) - \delta_{li}\delta_{sp}(\widehat{L}^-_{kj})_{rq}(x)\Big),$$

which shows the $RLL = LLR$ relation for $\widehat{L}^-$. □

For a square matrix $N = (N_{pq})$ with entries in a possibly non-commutative ring, let $\det^{\mathrm{col}} N$ be the column determinant of N:

$$\det{}^{\mathrm{col}} N := \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}(\sigma) N_{\sigma(1)1} \cdots N_{\sigma(n)n}.$$

Corollary 3.3. *Define* $\mathrm{qch}_p(\widehat{L})(x) \in \mathcal{A}^{\mathfrak{Z}}_{\boldsymbol{a}} \otimes_{\mathbb{C}} \mathbb{C}(x)$, $p = 0, 1, \ldots, \dim U$ *by*

$$\det{}^{\mathrm{col}}(\partial - \widehat{L}(x)) = \sum_{p=0}^{\dim U} \mathrm{qch}_p(\widehat{L})(x)\, \partial^{\dim U - p}.$$

1. *We have* $\mathrm{qch}_p(\widehat{L})(x)|_{\hbar=0} = \mathrm{ch}_p(L)(x)$, *where*

$$\det(y - L(x)) = \sum_{p=0}^{\dim U} \mathrm{ch}_p(L)(x)\, y^{\dim U - p}.$$

2. *We have*

$$\Big[\mathrm{qch}_p(\widehat{L})(x), \mathrm{qch}_q(\widehat{L})(y)\Big] = 0$$

as rational functions of x, y *for all* p, q.

Proof. This follows from Proposition 3.2 and Talalaev's result [12, Theorem 1] (see also [3, p. 3]). □

Take the Laurent expansion of each $\mathrm{qch}_p(\widehat{L})(x)$ at $x = \infty$:

$$\mathrm{qch}_p(\widehat{L})(x) = \sum_{m \in \mathbb{Z}} \mathrm{qch}_{p,m}(\widehat{L})\, x^m \in \mathcal{A}^{\mathfrak{Z}}_{\boldsymbol{a}} \otimes_{\mathbb{C}} \mathbb{C}(\!(x^{-1})\!).$$

Let $\mathcal{H}$ be the $\mathbb{C}[\mathbb{T}][\![\hbar]\!]$-subalgebra of $\mathcal{A}^{\mathfrak{Z}}_{\boldsymbol{a}}$ generated by $\mathrm{qch}_{p,m}(\widehat{L})$, $p = 1, 2, \ldots, \dim U$, $m \in \mathbb{Z}$. Then the above corollary implies:

Corollary 3.4. *The algebra* $\mathcal{H}$ *is commutative.*

3.2. $\mathrm{SL}_2(\mathbb{C})$-invariance of $\mathcal{H}$

Take any $g \in \mathrm{SL}_2(\mathbb{C})$. One can define a $\mathfrak{Z}$-equivariant $\mathbb{C}$-algebra isomorphism $g_* \colon \mathcal{A}_{g^{-1}\boldsymbol{a}} \to \mathcal{A}_{\boldsymbol{a}}$ by

$$(c_i, t_\lambda, (\widehat{\Xi}_{ij})_{pq}) \mapsto (c_i^g, t_\lambda^g, (\widehat{\Xi}_{ij})_{pq}),$$

where c_i^g $(i \in I)$, t_λ^g $(\lambda \in \Sigma)$ are defined so that

$$(a_{1i}, -a_{0i})g = \begin{cases} (0, c_i^g) & (g^{-1}(a_i) = \infty), \\ (c_i^g, -c_i^g g^{-1}(a_i)) & (g^{-1}(a_i) \neq \infty), \end{cases}$$

and

$$c^g_{\pi(\lambda)} t^g_\lambda = c_{\pi(\lambda)} t_\lambda.$$

Note that $c_i^g \in \mathbb{C}^\times c_i$, $t_\lambda^g \in \mathbb{C}^\times t_\lambda$, and the isomorphism between the quasi-classical limits induced from g_* coincides with the pull-back by the action $\mathcal{M}_{\boldsymbol{a}} \to \mathcal{M}_{g^{-1}\boldsymbol{a}}$, $M \mapsto M^g$.

Let $\widehat{M}^g, \widehat{L}^g$ be the transforms of the matrices $\widehat{M}, \widehat{L}$ associated to $g^{-1}\boldsymbol{a}$ by g_*. Then

$$\widehat{M}^g(\partial, x) = \begin{pmatrix} A_1 & -A_0 \end{pmatrix} g \begin{pmatrix} \partial \\ x \end{pmatrix} - \Theta - \widehat{\Xi} \in \mathrm{End}(V) \otimes_{\mathbb{C}} \mathcal{A}_{\boldsymbol{a}} \otimes_{\mathbb{C}} \mathcal{W}.$$

Theorem 3.5. *We have* $\mathrm{qch}_{p,m}(\widehat{L}^g) \in \mathcal{H}$ *for all* p, m.

In our proof of Theorem 3.5 we will use the general theory of *Manin matrices*. A square matrix $N = (N_{pq})$ with entries in a possibly non-commutative ring is called a Manin matrix if the equality

$$[N_{pq}, N_{rs}] = [N_{rq}, N_{ps}]$$

holds for any p, q, r, s. It is known that the column determinants of Manin matrices have the following nice properties (see [4]):

1. The column determinant of a Manin matrix is anti-symmetric with respect to columns/rows.

2. If two Manin matrices $N = (N_{pq})$, $N' = (N'_{pq})$ of the same size satisfy $[N_{pq}, N'_{rs}] = 0$ for all p, q, r, s, then NN' is also a Manin matrix and

$$\det^{\mathrm{col}}(NN') = \det^{\mathrm{col}}(N) \det^{\mathrm{col}}(N').$$

3. Let N be a Manin matrix expressed in a block form

$$N = \begin{pmatrix} N_{l\times l} & N_{l\times m} \\ N_{m\times l} & N_{m\times m} \end{pmatrix},$$

and assume that $N_{l\times l}$ has a two-sided inverse. Then the following *Schur's formula* holds:

$$\det^{\mathrm{col}} N = \det^{\mathrm{col}}(N_{l\times l}) \det^{\mathrm{col}}(N_{m\times m} - N_{m\times l} N_{l\times l}^{-1} N_{l\times m}).$$

Proposition 3.1 implies:

Corollary 3.6. *$\widehat{M}$ is a Manin matrix, i.e., the equality*

$$\left[(\widehat{M}_{kj})_{rq}, (\widehat{M}_{il})_{ps}\right] = \left[(\widehat{M}_{ij})_{pq}, (\widehat{M}_{kl})_{rs}\right]$$

holds for any $i, j, k, l,\ p, q, r, s$.

Proof.

$$\begin{aligned} \left[(\widehat{M}_{kj})_{rq}, (\widehat{M}_{il})_{ps}\right] &= (\delta_{kj}\delta_{il}\delta_{rq}\delta_{ps} - \delta_{kl}\delta_{ji}\delta_{rs}\delta_{qp})\hbar \begin{vmatrix} a_{0k} & a_{0i} \\ a_{1k} & a_{1i} \end{vmatrix} \\ &= -(\delta_{jk}\delta_{il}\delta_{rq}\delta_{ps} - \delta_{kl}\delta_{ij}\delta_{rs}\delta_{qp})\hbar \begin{vmatrix} a_{0i} & a_{0k} \\ a_{1i} & a_{1k} \end{vmatrix} \\ &= \left[(\widehat{M}_{ij})_{pq}, (\widehat{M}_{kl})_{rs}\right]. \end{aligned}$$

□

Proof of Theorem 3.5. Since the entries of C are central, the product $C^{-1}\widehat{M}$ is a Manin matrix and

$$\det^{\mathrm{col}}(C^{-1}\widehat{M}) = \det(C)^{-1} \det^{\mathrm{col}}(\widehat{M}).$$

On the other hand, we have

$$C^{-1}\widehat{M} = \begin{pmatrix} x - T_\infty & -\widehat{P} \\ -\widehat{Q} & \partial - Ax - T_{\mathrm{fin}} - \widehat{B}^\circ \end{pmatrix}$$

up to conjugation by a permutation matrix, and hence

$$\det^{\mathrm{col}}(C^{-1}\widehat{M}) = \det(x - T_\infty) \det^{\mathrm{col}}(\partial - \widehat{L}(x)) \tag{3.1}$$

by Schur's formula. Since $\det^{\mathrm{col}}(\partial - \widehat{L}(x)) \in \mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}(\!(x^{-1})\!) \otimes_{\mathbb{C}} \mathcal{W}$, the column determinant of $\widehat{M}$ may be expressed as

$$\det^{\mathrm{col}} \widehat{M}(\partial, x) = \det(C) \sum_{m,n\geq 0} h_{mn} x^m \partial^n, \quad h_{mn} \in \mathcal{H}.$$

Now we write
$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
and put
$$\begin{pmatrix} \tilde{\partial} \\ \tilde{x} \end{pmatrix} = g \begin{pmatrix} \partial \\ x \end{pmatrix} = \begin{pmatrix} a\partial + bx \\ c\partial + dx \end{pmatrix}.$$
Since $\widehat{M}^g(\partial, x) = \widehat{M}(\tilde{\partial}, \tilde{x})$, we have
$$\begin{aligned} \det{}^{\mathrm{col}} \widehat{M}^g(\partial, x) &= \det(C) \sum_{m,n\geq 0} h_{mn} \tilde{x}^m \tilde{\partial}^n \\ &= \det(C) \sum_{m,n\geq 0} h_{mn} (a\partial + bx)^m (c\partial + dx)^n. \end{aligned}$$
The right hand side lives in $\det(C)\mathcal{H}\otimes_{\mathbb{C}}\mathcal{W}$. Thus the equality (3.1) for $\widehat{M}^g, \widehat{L}^g$ shows
$$\det{}^{\mathrm{col}}(\partial - \widehat{L}^g(x)) \in \frac{\det(C)}{\det(C^g)} \mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}(\!(x^{-1})\!) \otimes_{\mathbb{C}} \mathcal{W},$$
where $C^g := \bigoplus_{i\in I} c_i^g 1_{W_i}$. Since $\det(C)/\det(C^g) \in \mathbb{C}^\times$, we obtain the assertion. □

3.3. Quantized simply-laced isomonodromy systems

For $i \in I$ and $\lambda \in \Sigma_i$, we define
$$\widehat{h}^{\boldsymbol{a}}_\lambda = \begin{cases} \dfrac{1}{2} \operatorname*{Res}_{x=t_\lambda} \operatorname{tr}\left(\widehat{L}(x)^2\right) dx & (i \notin I_{\mathrm{fin}}), \\ -\dfrac{1}{2} \operatorname*{Res}_{x=-t_\lambda} \operatorname{tr}\left(\widehat{L}^{g_i}(x)^2\right) dx & (i \in I_{\mathrm{fin}}), \end{cases}$$
where $g_i \in \mathrm{SL}_2(\mathbb{C})$ is defined in (2.2). Proposition 2.5 shows that the quasi-classical limit of each $\widehat{h}^{\boldsymbol{a}}_\lambda$ is equal to $H^{\boldsymbol{a}}_\lambda$.

The following lemma implies that $\widehat{h}^{\boldsymbol{a}}_\lambda \in \mathcal{H}$ for all $\lambda \in \Sigma$ (note that the residue of any exact meromorphic one-form is zero):

Lemma 3.7. *Let $\mathcal{R}$ be a possibly non-commutative ring and suppose that a matrix $N(x) = (N_{pq}(x)) \in M_n(\mathcal{R} \otimes_{\mathbb{C}} \mathbb{C}(x))$ satisfies the RLL = LLR relation:*
$$[N_{pq}(x), N_{rs}(y)] = \frac{\delta_{qr}(N_{ps}(y) - N_{ps}(x)) - \delta_{sp}(N_{qr}(y) - N_{qr}(y))}{x - y}.$$
Then
$$\operatorname{tr}(N(x)^2) = \mathrm{qch}_1(N)(x)^2 - 2\,\mathrm{qch}_2(N)(x) - (n-1)\operatorname{tr}(N'(x)),$$
where $N'(x) = dN/dx$.

Proof. A direct calculation shows

$$\mathrm{qch}_1(N) = \mathrm{tr}(N), \quad \mathrm{qch}_2(N) = \sum_{p<q}(N_{pp}N_{qq} - N_{qp}N_{pq}) - \sum_{p=1}^{n}(p-1)N'_{pp}.$$

On the other hand, the $RLL = LLR$ relation implies

$$N_{pp}N_{qq} = N_{qq}N_{pp}, \quad N_{qp}N_{pq} = N_{pq}N_{qp} - N'_{qq} + N'_{pp}.$$

Using the above we have

$$\begin{aligned}
\mathrm{tr}(N^2) - (\mathrm{tr}\, N)^2 &= \sum_{p<q}(-N_{pp}N_{qq} - N_{qq}N_{pp} + N_{pq}N_{qp} + N_{qp}N_{pq}) \\
&= \sum_{p<q}(-2N_{pp}N_{qq} + 2N_{qp}N_{pq} + N'_{qq} - N'_{pp}) \\
&= -2\,\mathrm{qch}_2(N) - 2\sum_{p=1}^{n}(p-1)N'_{pp} + \sum_{p<q}(N'_{qq} - N'_{pp}) \\
&= -2\,\mathrm{qch}_2(N) - (n-1)\sum_{p=1}^{n} N'_{pp},
\end{aligned}$$

which gives the desired equality. □

Take any $\lambda, \mu \in \Sigma$ and put $i = \pi(\lambda)$, $j = \pi(\mu)$. We calculate $\partial\widehat{h}^{\boldsymbol{a}}_{\lambda}/\partial t_{\mu} - \partial\widehat{h}^{\boldsymbol{a}}_{\mu}/\partial t_{\lambda}$.

Lemma 3.8. *If* $a_i = \infty$ *then*

$$\frac{\partial\widehat{h}^{\boldsymbol{a}}_{\lambda}}{\partial t_{\mu}} = \begin{cases} \dfrac{1}{2(t_\lambda - t_\mu)^2}\,\mathrm{tr}\left(\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\widehat{Q}\mathrm{Id}_\mu\widehat{P} + \widehat{Q}\mathrm{Id}_\mu\widehat{P}\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right) & (a_j = \infty), \\ \dfrac{1}{c_i c_j}\,\mathrm{tr}\left(\mathrm{Id}^V_\mu\widehat{\Xi}\mathrm{Id}^V_\lambda\widehat{\Xi}\right) & (a_j \neq \infty), \end{cases}$$

where $\mathrm{Id}^V_\lambda, \mathrm{Id}^V_\mu$ *denote the idempotents of* $\mathrm{End}(V)$ *for* V_λ, V_μ*, respectively.*

Proof. Decompose $\widehat{L} = \widehat{L}^+ + \widehat{L}^-$ as in the proof of Proposition 3.2. Since the matrix entries of $\widehat{L}^+(x)$ commute with those of $\widehat{L}^-(x)$ and are holomorphic at $x = t_\lambda$, we have

$$\operatorname*{Res}_{x=t_\lambda}\mathrm{tr}\left(\widehat{L}(x)^2\right)dx = 2\operatorname*{Res}_{x=t_\lambda}\mathrm{tr}\left(\widehat{L}^+(x)\widehat{L}^-(x)\right)dx + \operatorname*{Res}_{x=t_\lambda}\mathrm{tr}\left(\widehat{L}^-(x)^2\right)dx.$$

The two terms on the right hand side may be calculated as

$$\operatorname*{Res}_{x=t_\lambda}\mathrm{tr}\left(\widehat{L}^+(x)\widehat{L}^-(x)\right)dx = \operatorname*{Res}_{x=t_\lambda}\mathrm{tr}\left((Ax + T_{\mathrm{fin}} + \widehat{B}^\circ)\widehat{Q}(x - T_\infty)^{-1}\widehat{P}\right)dx$$

$$
\begin{aligned}
&= \operatorname{tr}\left((At_\lambda + T_{\mathrm{fin}} + \widehat{B}^\circ)\widehat{Q}\mathrm{Id}_\lambda \widehat{P}\right), \\
\operatorname*{Res}_{x=t_\lambda} \operatorname{tr}\left(\widehat{L}^-(x)^2\right) dx &= \operatorname*{Res}_{x=t_\lambda} \operatorname{tr}\left(\widehat{Q}(x-T_\infty)^{-1}\widehat{P}\widehat{Q}(x-T_\infty)^{-1}\widehat{P}\right) dx \\
&= \sum_{\substack{\nu\in\Sigma_i\\ \nu\neq\lambda}} \frac{\operatorname{tr}\left(\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\widehat{Q}\mathrm{Id}_\nu\widehat{P} + \widehat{Q}\mathrm{Id}_\nu\widehat{P}\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right)}{t_\lambda - t_\nu}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\widehat{h}^{\boldsymbol{a}}_\lambda = \operatorname{tr}&\left((At_\lambda + T_{\mathrm{fin}} + \widehat{B}^\circ)\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right) \\
&+ \frac{1}{2}\sum_{\substack{\nu\in\Sigma_i\\ \nu\neq\lambda}} \frac{\operatorname{tr}\left(\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\widehat{Q}\mathrm{Id}_\nu\widehat{P} + \widehat{Q}\mathrm{Id}_\nu\widehat{P}\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right)}{t_\lambda - t_\nu},
\end{aligned}
$$

and hence

$$
\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} = \begin{cases} \dfrac{\operatorname{tr}\left(\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\widehat{Q}\mathrm{Id}_\mu\widehat{P} + \widehat{Q}\mathrm{Id}_\mu\widehat{P}\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right)}{2(t_\lambda - t_\mu)^2} & (a_j = \infty), \\ \operatorname{tr}\left(\mathrm{Id}^U_\mu \widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right) & (a_j \neq \infty), \end{cases}
$$

where recall that Id^U_μ denotes the idempotent of $\mathrm{End}(U)$ for V_μ. Note that $\widehat{Q}, \widehat{P}$ are blocks of $C^{-1}\widehat{\Xi}$. Thus if $a_j \neq \infty$ then

$$
\operatorname{tr}\left(\mathrm{Id}^U_\mu\widehat{Q}\mathrm{Id}_\lambda\widehat{P}\right) = \frac{1}{c_i c_j}\operatorname{tr}\left(\mathrm{Id}^V_\mu\widehat{\Xi}\mathrm{Id}^V_\lambda\widehat{\Xi}\right).
$$

□

Define $\kappa_{ij} \in \mathbb{C}$ by

$$
\kappa_{ij} = \begin{cases} 0 & (i = j), \\ -1 & (a_i = \infty,\ a_j \neq \infty), \\ 1 & (a_i \neq \infty,\ a_j = \infty), \\ \dfrac{1}{a_i - a_j} & (\text{otherwise}). \end{cases}
$$

Proposition 3.9. *The following equality holds:*

$$
\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} - \frac{\partial \widehat{h}^{\boldsymbol{a}}_\mu}{\partial t_\lambda} = \hbar(\dim V_\lambda)(\dim V_\mu)\kappa_{ij}.
$$

Proof. First, suppose $i = j$. If $a_i = a_j = \infty$, then Lemma 3.8 shows

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} = \frac{\operatorname{tr}\left(\widehat{Q}\mathrm{Id}_\lambda \widehat{P}\widehat{Q}\mathrm{Id}_\mu \widehat{P} + \widehat{Q}\mathrm{Id}_\mu \widehat{P}\widehat{Q}\mathrm{Id}_\lambda \widehat{P}\right)}{2(t_\lambda - t_\mu)^2},$$

which is a symmetric function of (λ, μ). Hence

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} = \frac{\partial \widehat{h}^{\boldsymbol{a}}_\mu}{\partial t_\lambda}.$$

If $a_i = a_j \neq \infty$, then $\widehat{h}^{\boldsymbol{a}}_\lambda = -(g_i)_*(\widehat{h}^{g_i^{-1}\boldsymbol{a}}_\lambda)$ and hence

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} = -\frac{dt^{g_i}_\mu}{dt_\mu}\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t^{g_i}_\mu} = -\frac{dt^{g_i}_\mu}{dt_\mu}(g_i)_*\left(\frac{\partial \widehat{h}^{g_i^{-1}\boldsymbol{a}}_\lambda}{\partial t_\mu}\right) = (g_i)_*\left(\frac{\partial \widehat{h}^{g_i^{-1}\boldsymbol{a}}_\lambda}{\partial t_\mu}\right)$$

because $t^{g_i}_\mu = -t_\mu$. Since $\lambda, \mu \in \Sigma_i$ and $g_i^{-1}(a_i) = \infty$, $\partial \widehat{h}^{g_i^{-1}\boldsymbol{a}}_\lambda / \partial t_\mu$ is a symmetric function of (λ, μ). Hence

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} = \frac{\partial \widehat{h}^{\boldsymbol{a}}_\mu}{\partial t_\lambda}.$$

Next, suppose $i \neq j$. If $a_i = \infty$, then Lemma 3.8 shows

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} = \frac{1}{c_i c_j}\operatorname{tr}\left(\mathrm{Id}^V_\mu \widehat{\Xi}\mathrm{Id}^V_\lambda \Xi\right),$$

and

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\mu}{\partial t_\lambda} = -\frac{dt^{g_j}_\lambda}{dt_\lambda} g_j^*\left(\frac{\partial \widehat{h}^{g_j^{-1}\boldsymbol{a}}_\mu}{\partial t_\lambda}\right) = -\frac{dt^{g_j}_\lambda}{dt_\lambda}\frac{1}{c^{g_j}_j c^{g_j}_i}\operatorname{tr}\left(\mathrm{Id}^V_\lambda \widehat{\Xi}\mathrm{Id}^V_\mu \Xi\right).$$

A direct calculation shows

$$c^{g_j}_k = \begin{cases} -c_j & (k = j), \\ c_k & (a_k = \infty), \\ (a_j - a_k)c_k & (\text{otherwise}). \end{cases}$$

Hence

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\mu}{\partial t_\lambda} = \frac{1}{c_j c_i}\operatorname{tr}\left(\mathrm{Id}^V_\lambda \widehat{\Xi}\mathrm{Id}^V_\mu \Xi\right).$$

Thus we obtain

$$\frac{\partial \widehat{h}^{\boldsymbol{a}}_\lambda}{\partial t_\mu} - \frac{\partial \widehat{h}^{\boldsymbol{a}}_\mu}{\partial t_\lambda} = \frac{1}{c_i c_j}\operatorname{tr}\left(\mathrm{Id}^V_\mu \widehat{\Xi}\mathrm{Id}^V_\lambda \Xi - \mathrm{Id}^V_\lambda \widehat{\Xi}\mathrm{Id}^V_\mu \Xi\right).$$

If $a_i, a_j \neq \infty$, then

$$\frac{\partial \widehat{h}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} = -\frac{dt_\mu^{g_i}}{dt_\mu}(g_i)_* \left(\frac{\partial \widehat{h}_\lambda^{g_i^{-1}\boldsymbol{a}}}{\partial t_\mu} \right) = \frac{\operatorname{tr}\left(\mathrm{Id}_\mu^V \widehat{\Xi} \mathrm{Id}_\lambda^V \widehat{\Xi}\right)}{(a_i - a_j)^2 c_i c_j},$$

and hence

$$\frac{\partial \widehat{h}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} - \frac{\partial \widehat{h}_\mu^{\boldsymbol{a}}}{\partial t_\lambda} = \frac{\operatorname{tr}\left(\mathrm{Id}_\mu^V \widehat{\Xi} \mathrm{Id}_\lambda^V \widehat{\Xi} - \mathrm{Id}_\lambda^V \widehat{\Xi} \mathrm{Id}_\mu^V \widehat{\Xi}\right)}{(a_i - a_j)^2 c_i c_j}.$$

On the other hand, the commutation relations for the entries of $\widehat{\Xi}$ yield

$$\operatorname{tr}\left(\mathrm{Id}_\mu^V \widehat{\Xi} \mathrm{Id}_\lambda^V \widehat{\Xi} - \mathrm{Id}_\lambda^V \widehat{\Xi} \mathrm{Id}_\mu^V \widehat{\Xi}\right) = -\hbar (\dim V_\lambda)(\dim V_\mu) \begin{vmatrix} a_{0j} & a_{0i} \\ a_{1j} & a_{1i} \end{vmatrix}.$$

Also, by the definition we have

$$\begin{vmatrix} a_{0j} & a_{0i} \\ a_{1j} & a_{1i} \end{vmatrix} = \begin{cases} c_i c_j & (a_i = \infty,\ a_j \neq \infty), \\ -c_i c_j & (a_i \neq \infty,\ a_j = \infty), \\ -c_i c_j (a_i - a_j) & (\text{otherwise}). \end{cases}$$

Now the assertion immediately follows. □

For $\lambda \in \Sigma$, we define

$$\begin{aligned} \widehat{H}_\lambda^{\boldsymbol{a}} &= \widehat{h}_\lambda^{\boldsymbol{a}} - \frac{\hbar \dim V_\lambda}{2} \sum_{\mu \neq \lambda} (\dim V_\mu) \kappa_{\pi(\lambda)\pi(\mu)} t_\mu \\ &= \widehat{h}_\lambda^{\boldsymbol{a}} - \frac{\hbar \dim V_\lambda}{2} \sum_{j \neq \pi(\lambda)} \kappa_{\pi(\lambda) j} \operatorname{tr} T_j. \end{aligned}$$

The quasi-classical limit of each $\widehat{H}_\lambda^{\boldsymbol{a}}$ is equal to $H_\lambda^{\boldsymbol{a}}$.

Theorem 3.10. *For any $\lambda, \mu \in \Sigma$, the following equalities hold:*

$$\left[\widehat{H}_\lambda^{\boldsymbol{a}}, \widehat{H}_\mu^{\boldsymbol{a}}\right] = 0, \quad \frac{\partial \widehat{H}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} = \frac{\partial \widehat{H}_\mu^{\boldsymbol{a}}}{\partial t_\lambda}.$$

Proof. All the $\widehat{H}_\lambda^{\boldsymbol{a}}$ live in $\mathcal{H}$, and hence pairwise commute. Also, for $\lambda \neq \mu \in \Sigma$ we have

$$\frac{\partial \widehat{H}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} = \frac{\partial \widehat{h}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} - \frac{\hbar}{2} (\dim V_\lambda)(\dim V_\mu) \kappa_{\pi(\lambda)\pi(\mu)}.$$

By Proposition 3.9, we thus obtain

$$\frac{\partial \widehat{H}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} - \frac{\partial \widehat{H}_\mu^{\boldsymbol{a}}}{\partial t_\lambda} = \frac{\partial \widehat{h}_\lambda^{\boldsymbol{a}}}{\partial t_\mu} - \frac{\partial \widehat{h}_\mu^{\boldsymbol{a}}}{\partial t_\lambda} - \hbar (\dim V_\lambda)(\dim V_\mu) \kappa_{\pi(\lambda)\pi(\mu)} = 0,$$

which completes the proof. □

Thus the family $\{\widehat{H}^{\boldsymbol{a}}_\lambda\}_{\lambda\in\Sigma}$ gives a quantization of the simply-laced isomonodromy system.

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