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Estimates on modulation spaces for Schrödinger operators with first order magnetic fields

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Abstract. In this paper, we give the estimate of the solutions to Schrödinger equation with vector potential on modulation spaces. We assume that vector potential $\boldsymbol{a}(t,x)$ is first degree polynomial with respect to x, and it corresponds to constant magnetic field or time-dependent magnetic field for physics.

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§1. Introduction

In this paper, we consider the initial value problem for the Schrödinger equation with magnetic vector potential

$$\begin{cases} i\partial_t u(t,x) + \frac{1}{2}\left(\nabla - i\boldsymbol{a}(t,x)\right)^2 u(t,x) = 0, & (t,x) \in \mathbb{R}\times\mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

and give estimates for the solutions in the frame work of modulation spaces, where $i = \sqrt{-1}$, $u(t,x)$ is a complex valued unknown function of $(t,x) \in \mathbb{R}\times\mathbb{R}^n$, $u_0(x)$ is a complex valued given function of $x \in \mathbb{R}^n$, $\partial_t u = \partial u/\partial t$, $\partial_{x_j} u = \partial u/\partial x_j$ $(j = 1,\dots,n)$ and $\nabla = (\partial_{x_1},\dots,\partial_{x_n})$.

Throughout this paper, we assume the following Assumption 1.1 on the magnetic vector potential $\boldsymbol{a}(t,x) = (a_1(t,x),\dots,a_n(t,x))$.

Assumption 1.1. For $k = 1,\dots,n$, k-th component a_k of $\boldsymbol{a}$ has the form

$$a_k(t,x) = \sum_{l=1}^{n} a_{k,l}(t)x_l \tag{1.2}$$

with $a_{k,l} \in C^\infty(\mathbb{R})$ for $l,k = 1,\dots,n$.

Definition 1.2 (Wave packet transform)**.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n)\setminus\{0\}$ and $f \in \mathcal{S}'(\mathbb{R}^n)$. The wave packet transform $W_\varphi f$ of f with the basic wave packet φ is defined by

$$W_\varphi f(x,\xi) \;=\; \int_{\mathbb{R}^n} \overline{\varphi(y-x)} e^{-iy\cdot\xi} f(y)dy, \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Definition 1.3 (Modulation space)**.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $1 \le p, q \le \infty$. We define the modulation spaces $M_\varphi^{p,q}(\mathbb{R}^n)$ as follows.

$$M_\varphi^{p,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|f\|_{M_\varphi^{p,q}} = \left\| \|W_\varphi f(x,\xi)\|_{L_x^p} \right\|_{L_\xi^q} < \infty \right\}.$$

We remark the following properties of modulation spaces (For more details, see §4 and §6 in [3]).

Lemma 1.4. *Let* $1 \le p, q, p_1, q_1, p_2, q_2 \le \infty$. *Then*

1. $M_\varphi^{p_1,q_1}(\mathbb{R}^n) \hookrightarrow M_\varphi^{p_2,q_2}(\mathbb{R}^n)$, *for* $p_1 \le p_2$, $q_1 \le q_2$.
2. $M_\varphi^{p,q_1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow M_\varphi^{p,q_2}(\mathbb{R}^n)$ *for* $1 \le q_1 \le \min(p,p')$ *and* $q_2 \ge \max(p,p')$ *with* $1/p + 1/p' = 1$. *In particular,* $M_\varphi^{2,2}(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ *holds.*
3. $\mathcal{S}(\mathbb{R}^n)$ *is dense in* $M_\varphi^{p,q}(\mathbb{R}^n)$ *for* $1 \le p, q < \infty$.
4. $M_\varphi^{p,q}(\mathbb{R}^n)$ *is a Banach space with norm* $\|\cdot\|_{M_\varphi^{p,q}}$.
5. *The definition of* $M_\varphi^{p,q}(\mathbb{R}^n)$ *is independent of the choice of the basic wave packet* φ. *More precisely, for any* $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, *the norm* $\|\cdot\|_{M_\varphi^{p,q}}$ *is equivalent to the norm* $\|\cdot\|_{M_\psi^{p,q}}$.

We denote the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ by $\hat{f}$ or $\mathcal{F}f$. The inverse Fourier transform of $f \in \mathcal{S}(\mathbb{R}^n)$ is defined by $\check{f}$ or $\mathcal{F}^*f$, similarly. For our statement, we define the Schrödinger operator of a free particle $e^{it\frac{\Delta}{2}}$ by

$$\left(e^{it\frac{\Delta}{2}} f\right)(x) = \mathcal{F}^*_{\xi\to x}\left[e^{-it\frac{\xi^2}{2}} \hat{f}(\xi)\right](x), \quad f \in \mathcal{S}(\mathbb{R}^n).$$

Our purpose in this study is to estimate the solution of the initial value problem (1.1) on modulation spaces. The following theorem is the main result of this paper.

Theorem 1.5. *Let* $1 \le p \le \infty$, $\varphi_0 \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$, $T > 0$ *and* $u(t,x)$ *be the solution of* (1.1) *in* $C(\mathbb{R}; L^2(\mathbb{R}^n))$ *for* $u_0 \in \mathcal{S}(\mathbb{R}^n)$.

If $\boldsymbol{a}$ *satisfies the Assumption 1.1, then there exists* $C_T > 0$ *such that*

$$\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \le C_T \|u_0\|_{M^{p,p}_{\varphi_0}}, \quad u_0 \in \mathcal{S}(\mathbb{R}^n) \tag{1.3}$$

for all $t \in [-T, T]$ *where we denote* $\varphi(t,x)$ *by* $e^{it\frac{\Delta}{2}}\varphi_0(x)$.

We often use the wave packet transform for the function $u(t,x)$ on $\mathbb{R}\times\mathbb{R}^n$ with the basic wave packet $\varphi(t,x)$ on $\mathbb{R}\times\mathbb{R}^n$, so we use the following notations

$$W_{\varphi(t,\cdot)}u(t,x,\xi) = W_{\varphi(t,\cdot)}\left[u(t,\cdot)\right](x,\xi) = \int_{\mathbb{R}^n} \overline{\varphi(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy$$

and $\|u(t)\|_{M^{p,q}_{\varphi}} = \left\| \|W_{\varphi}u(t,x,\xi)\|_{L^p_x}\right\|_{L^q_\xi}$, for the sake of convenience.

Remark 1.6. We cannot expect that Theorem 1.5 is valid with $M^{p,q}$ for $p \neq q$. In fact, for the constant magnetic potential $\boldsymbol{a}(t,x) = (-x_2/2, x_1/2)$ with $n = 2$, the L^2-unitary operator $U = e^{ix_1x_2/2}e^{i\partial_{x_1}\partial_{x_2}}$ transforms (1.1) to Schrödinger equation with the harmonic oscillator (1.4).

$$\begin{cases} i\partial_t v(t,x) + \frac{1}{2}\partial_{x_2}^2 v(t,x) = \frac{1}{2}x_2^2 v(t,x), & (t,x)\in\mathbb{R}\times\mathbb{R}^2, \\ v(0,x) = u_0(x), \quad x = (x_1,x_2)\in\mathbb{R}^2, \end{cases} \tag{1.4}$$

where $v(t,x) = Uu(t,x)$. By the wave packet transform, the solution of (1.1) $u(t,x)$ can be represented by

$$\left|W_{\varphi(t,\cdot)}u(t,x,\xi)\right| = |W_{\varphi_0}u_0(x_1, x_2\cos t - \xi_2\sin t, \xi_1, x_2\sin t - \xi_2\cos t)|$$

and according to [5] and the above equality, $\|u(t)\|_{M^{p,q}} \le C\|u_0\|_{M^{p,q}}$ does not hold generally.

This short note is a first step for the estimate of the solutions to Schrödinger equations with magnetic potentials in modulation spaces, which is an extension of the result by K. Kato-M. Kobayashi-S. Ito [5]. In the forthcoming paper, we will discuss the case that $\boldsymbol{a}(t,x)\in C^\infty(\mathbb{R}\times\mathbb{R}^n)$ with $|\partial_x^\alpha \boldsymbol{a}(t,x)| \le C(1+|x|)^{\rho-|\alpha|}$ for $\rho < 1$.

Schrödinger equations with time-independent magnetic potential $\boldsymbol{a}(t,x) = \boldsymbol{a}(x)$ have been investigated by B. Simon [10], T. Kato [7], and so on. In [10], B. Simon showed the essentially self-adjointness of $H_0 = -(\nabla - i\boldsymbol{a}(x))^2$ on C_0^∞ when $\mathrm{div}\boldsymbol{a} = 0$, $\boldsymbol{a}\in L^q_{\mathrm{loc}}(\mathbb{R}^n)$ with $q > \max(n,4)$. In [7], T. Kato relaxed some conditions of $\boldsymbol{a}(x)$ for H_0 to be essentially self-adjoint on C_0^∞ stated in [10]. In H. Leinfelder and C. G. Simader's work [9], they proved the existence and uniqueness of the L^2-solution to the equation (1.1) with $\boldsymbol{a}(x)$ under the more general assumption which lets $\boldsymbol{a}(x)$ to be in $L^4_{\mathrm{loc}}(\mathbb{R}^n)$ and its derivative to be in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$.

When the magnetic potential depends on time, this problem becomes more difficult and delicate. K. Yajima, in the work of [13], proved the existence and uniqueness of the L^2-solution to the equation (1.1) and L^p-smoothing property of the unitary propagator $\{U(t,s)|t,s\in\mathbb{R}\}$ of (1.1) assuming that growth of $\boldsymbol{a}(t,x)$ and $\partial_t\boldsymbol{a}(t,x)$ are equal to first degree polynomial at infinity; i.e. $|\boldsymbol{a}(t,x)| + |\partial_t\boldsymbol{a}(t,x)| \sim |x|$.

From these results, the equation (1.1) can be solved on $L^2(\mathbb{R}^n)$, while we cannot expect to solve this equation on $L^p(\mathbb{R}^n)$ for $p \neq 2$. On the contrary, there are many works on existence of solutions to the following Schrödinger equations with scholar potentials $V \in C^\infty(\mathbb{R} \times \mathbb{R}^n)$ in modulation spaces.

$$\begin{cases} i\partial_t u(t,x) + \frac{1}{2}\Delta u(t,x) = V(t,x)u(t,x), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

In the work of A. Bényi, K. Gröchenig, K. A. Okoudjou and L. G. Rogers [1], it is shown that the Schrödinger group $e^{it|\nabla|^\alpha}$ for $0 \leq \alpha \leq 2$ is bounded on $M^{p,q}(\mathbb{R}^n)$ and this implies that the free Schrödinger equation (i.e. equation (1.5) with $V(t,x) \equiv 0$) can be solved in $M^{p,q}(\mathbb{R}^n)$ for $1 \leq p,q \leq \infty$. B. Wang and H. Hudzik showed that the global well-posedness of the nonlinear Schrödinger equation with power type nonlinearity by using the dispersive estimate for the free Schrödinger equation in $M^{p,q}(\mathbb{R}^n)$, $\|u(t,\cdot)\|_{M^{p,q}} \leq C(1+|t|)^{-n(1/2-1/p)}\|u_0\|_{M^{p',q}}$, see [11]. In the work of [6], the solutions to the free Schrödinger equation or Schrödinger equation with the harmonic oscillator preserve the norm of modulation spaces, $\|u(t,\cdot)\|_{M^{p,q}_{\varphi(t,\cdot)}} = \|u_0\|_{M^{p,q}_{\varphi_0}}$. In the case of time-dependent potential, the estimate of the solution to equation (1.5) with quadratic or sub-quadratic potential on $M^{p,q}(\mathbb{R}^n)$, $\|u(t,\cdot)\|_{M^{p,p}_{\varphi(t,\cdot)}} \leq C_T\|u_0\|_{M^{p,p}_{\varphi_0}}$ is obtained in [2] and [5].

However, it seems that there is no result about the Schrödinger equation with time dependent magnetic potential in modulation spaces so far.

This paper is organized as follows. In Section 2, we introduce terminology and preliminalies. We will give the representation of solution of (1.1) by wave packet transform and introduce some lemmas on characteristics corresponding to (1.1). In Section 3, we will prove Theorem 1.5.

§2. Preliminaries

In this section, we give proofs of lemmas to prove Theorem 1.5.

Definition 2.1 (Inverse wave packet transform)**.** Let $\varphi \in \mathcal{S}(\mathbb{R}^n) \setminus \{0\}$ and $F \in \mathcal{S}'(\mathbb{R}^{2n})$, we define the adjoint operator W_φ^* of W_φ by

$$W_\varphi^*[F(y,\xi)](x) \ = \ \iint_{\mathbb{R}^{2n}} \varphi(x-y)e^{ix\cdot\xi}F(y,\xi)dy\bar{d}\xi, \quad x \in \mathbb{R}^n,$$

where $\bar{d}\xi = (2\pi)^{-n}d\xi$.

Then, for $f \in \mathcal{S}'(\mathbb{R}^n)$, the following inversion formula holds (see [4, Corollary 11.2.7]);

$$f(x) = \frac{1}{\|\varphi\|_{L^2}^2} W_\varphi^*[W_\varphi f](x).$$

For the proof of Theorem 1.5, we reduce (1.1) to a first-order partial differential equation in $\mathbb{R}^{2n}$ by using the wave packet transform. Using formula

$$\frac{1}{2}(\nabla - i\boldsymbol{a})^2 u = \frac{1}{2}\Delta u - \frac{1}{2}\left(i(\nabla\cdot\boldsymbol{a}) + \boldsymbol{a}^2\right)u - i\boldsymbol{a}\cdot\nabla u$$

and substituting $V(t,x) = i(\nabla\cdot\boldsymbol{a})(t,x) + \boldsymbol{a}^2(t,x)$ into [5, (8)–(10)], we have

$$\begin{aligned}(2.1)\quad & W_{\varphi(t,\cdot)}\left[i\partial_t u + \frac{1}{2}\Delta u - \frac{1}{2}\left(i(\nabla\cdot\boldsymbol{a}) + \boldsymbol{a}^2\right)u\right](t,x,\xi)\\ &= \Big(i\partial_t + i\xi\cdot\nabla_x - \frac{|\xi|^2}{2} - \frac{i}{2}(\nabla_x\cdot\boldsymbol{a})(t,x)\\ &\quad - \frac{1}{2}\boldsymbol{a}^2(t,x) - \frac{i}{2}\nabla_x\boldsymbol{a}^2(t,x)\cdot\nabla_\xi + \frac{1}{2}\nabla_x\boldsymbol{a}^2(t,x)\cdot x\Big)W_{\varphi(t,\cdot)}u(t,x,\xi)\\ &\quad + W_{\left(i\partial_t+\frac{\Delta}{2}\right)\varphi(t,\cdot)}u(t,x,\xi) + R_1u(t,x,\xi),\end{aligned}$$

where

$$\begin{aligned}R_1u(t,x,\xi) &= -\frac{1}{2}\sum_{k,l,l'=1}^{n} a_{k,l}(t)a_{k,l'}(t)\int \overline{\varphi_{l,l'}(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy,\\ \varphi_{l,l'}(t,y-x) &= (y_l - x_l)(y_{l'} - x_{l'})\varphi(t,y-x).\end{aligned}$$

By integration by parts, we have

$$\begin{aligned}(2.2)\qquad & W_{\varphi(t,\cdot)}\left[-i\boldsymbol{a}\cdot\nabla u\right](t,x,\xi)\\ &= \Big(-i\boldsymbol{a}(t,x)\cdot\nabla_x + i(\nabla_x\cdot\boldsymbol{a}(t,x)) + \xi\cdot\boldsymbol{a}(t,x)\\ &\quad + \nabla_x(\xi\cdot\boldsymbol{a}(t,x))\cdot(i\nabla_\xi - x)\Big)W_{\varphi(t,\cdot)}u(t,x,\xi) + R_2u(t,x,\xi),\end{aligned}$$

where

$$\begin{aligned}R_2u(t,x,\xi) &= \sum_{k,l=1}^{n} i\partial_{x_l}a_k(t,x)\int \overline{\varphi_l^k(t,y-x)}u(t,y)e^{-iy\cdot\xi}dy,\\ \varphi_l^k(t,y-x) &= (y_l - x_l)\partial_{y_k}\varphi(t,y-x).\end{aligned}$$

Taking $\varphi(t,x) = e^{it\frac{\Delta}{2}}\varphi_0(x)$ for $\varphi_0(x)\in\mathcal{S}(\mathbb{R}^n)$ and combining (2.1) and (2.2), we transform (1.1) into

$$\begin{cases}(i\partial_t + i\nabla_\xi H(t,x,\xi)\cdot\nabla_x - i\nabla_x H(t,x,\xi)\cdot\nabla_\xi\\ \qquad\qquad +h(t,x,\xi))W_{\varphi(t,\cdot)}u(t,x,\xi) = Ru(t,x,\xi),\\ W_{\varphi(0,\cdot)}u(0,x,\xi) = W_{\varphi_0}u_0(x,\xi),\end{cases}$$

where

$$\begin{aligned} H(t,x,\xi) &= \frac{1}{2}\left|\xi - \boldsymbol{a}(t,x)\right|^2, \\ h(t,x,\xi) &= -H(t,x,\xi) + \nabla_x H(t,x,\xi)\cdot x + \frac{i}{2}\nabla_x \cdot \boldsymbol{a}(t,x), \\ Ru(t,x,\xi) &= -R_1 u(t,x,\xi) - R_2 u(t,x,\xi). \end{aligned}$$

By the method of characteristics, we obtain the following lemma.

Lemma 2.2. *For $t \in \mathbb{R}$ and $x, \xi \in \mathbb{R}^n$, we define $x(s) = x(s;t,x,\xi)$ and $\xi(s) = \xi(s;t,x,\xi)$ as the solutions of*

$$\begin{cases} \dot{x}(s) = \nabla_\xi H(s, x(s), \xi(s)), & x(t) = x, \\ \dot{\xi}(s) = -\nabla_x H(s, x(s), \xi(s)), & \xi(t) = \xi. \end{cases} \tag{2.3}$$

Then the solution $u(t,x)$ to (1.1) *satisfies the integral equation*

$$\begin{aligned} W_{\varphi(t,\cdot)}u(t,x,\xi) = e^{-i\int_0^t h(s,x(s),\xi(s))ds}\bigg(& W_{\varphi(0,\cdot)}u_0(x(0),\xi(0)) \\ & - i\int_0^t e^{i\int_t^\tau h(s,x(s),\xi(s))ds} Ru(\tau, x(\tau), \xi(\tau))d\tau\bigg). \end{aligned} \tag{2.4}$$

We will introduce some properties of the characteristics defined as the solutions of (2.3) and won't prove the following lemma because this is proved in [5, Appendix A.].

Lemma 2.3 (see [5])**.** *Let $w_j(s;t,x,\xi) = x_j(s)$, $w_{n+j}(s;t,x,\xi) = \xi_j(s)$ $(1 \leq j \leq n)$ and Jacobian matrix of $(w_1,\cdots,w_n,w_{n+1},\cdots,w_{2n})(s;t,x,\xi)$ with respect to variable $(v_1,\cdots,v_n,v_{n+1},\cdots,v_{2n}) = (x_1,\cdots,x_n,\xi_1,\cdots,\xi_n)$ as follows:*

$$J(s;t,x,\xi) = \left(\frac{\partial w_j(s;t,x,\xi)}{\partial v_k}\right)_{\substack{j\to 1,\cdots,2n \\ k\downarrow 1,\cdots,2n}}.$$

Then for any $s,t \in \mathbb{R}$ satisfying $|s-t| \leq T$ and $x,\xi \in \mathbb{R}^n$, $\det J(s;t,x,\xi) = 1$.

§3. Proof of Theorem1.5

In this section, we prove Theorem 1.5 by using the lemmas in Section 2.

Proof of Theorem 1.5. Taking $L^p(\mathbb{R}^n_x)$-$L^p(\mathbb{R}^n_\xi)$ norm of the both sides of (2.4), we have

$$\left\|W_{\varphi(t,\cdot)}u(t,x,\xi)\right\|_{L^p_x L^p_\xi} \tag{3.1}$$

$$\leq C_T \left(\left\| W_{\varphi(0,\cdot)} u_0(x(0), \xi(0)) \right\|_{L^p_x L^p_\xi} + \int_0^t \| Ru(\tau, x(\tau), \xi(\tau) \|_{L^p_x L^p_\xi} d\tau \right).$$

First, we estimate the first term on the right hand side (3.1). Taking new variables $X = x(0; t, x, \xi)$ and $\Xi = \xi(0; t, x, \xi)$ we have from Lemma 2.3.

$$\begin{aligned}
\left\| W_{\varphi(0,\cdot)} u_0(x(0), \xi(0)) \right\|_{L^p_x L^p_\xi} &= \left(\iint |W_{\varphi(0,\cdot)} u_0(x(0), \xi(0))|^p dx d\xi \right)^{\frac{1}{p}} \\
&= \left(\iint |W_{\varphi(0,\cdot)} u_0(X, \Xi)|^p \left| \frac{\partial(x,\xi)}{\partial(X,\Xi)} \right| dX d\Xi \right)^{\frac{1}{p}} \\
&= \| W_{\varphi_0} u_0(x, \xi) \|_{L^p_x L^p_\xi}.
\end{aligned}$$

Next let us estimate the second term of the right hand side of (3.1). Since $u(\tau, y) = \|\varphi_0\|_{L^2}^{-2} W^*_{\varphi(\tau,\cdot)} \left[W_{\varphi(\tau,\cdot)} u \right] (\tau, y)$,

$$\begin{aligned}
&\|\varphi_0\|_{L^2}^2 \left\| \int \overline{\phi(\tau, y - x(\tau))} u(\tau, y) e^{-iy\cdot\xi(\tau)} dy \right\|_{L^p_x L^p_\xi} \\
&= \left\| \int \overline{\phi(\tau, y - x(\tau))} W^*_{\varphi(\tau,\cdot)} \left[W_{\varphi(\tau,\cdot)} u \right] (\tau, y) e^{-iy\cdot\xi(\tau)} dy \right\|_{L^p_x L^p_\xi} \\
&= \left\| \iiint \overline{\phi(\tau, y - x(\tau))} \varphi(\tau, y - z) W_{\varphi(\tau,\cdot)} u(\tau, z, \eta) e^{iy\cdot(\eta - \xi(\tau))} dy dz \bar{d}\eta \right\|_{L^p_x L^p_\xi}
\end{aligned}$$

holds for $\phi \in C(\mathbb{R}; \mathcal{S}(\mathbb{R}^n))$. Using the equality $\langle \eta - \xi(\tau) \rangle^2 (1 - \Delta_y) e^{iy\cdot(\eta - \xi(\tau))} = e^{iy\cdot(\eta-\xi(\tau))}$ and integration by parts for $2N$ times, we have

$$\begin{aligned}
&\iiint \overline{\phi(\tau, y - x(\tau))} \varphi(\tau, y - z) W_{\varphi(\tau,\cdot)} u(\tau, z, \eta) e^{iy\cdot(\eta - \xi(\tau))} dy dz \bar{d}\eta \\
&= \sum_{|\beta_1| + |\beta_2| \leq 2N} \iiint \overline{\partial_y^{\beta_1} \phi(\tau, y - x(\tau))} \partial_y^{\beta_2} \varphi(\tau, y - z) \\
&\qquad\qquad \times W_{\varphi(\tau,\cdot)} u(\tau, z, \eta) \frac{e^{iy\cdot(\eta - \xi(\tau))}}{\langle \eta - \xi(\tau) \rangle^{2N}} dy dz \bar{d}\eta.
\end{aligned}$$

Taking $L^p(\mathbb{R}^n_x)$-$L^p(\mathbb{R}^n_\xi)$ norm of the both sides of the above equality and new variables $X = x(\tau; t, x, \xi)$ and $\Xi = \xi(\tau; t, x, \xi)$, we have

$$\begin{aligned}
&\left\| \iiint \overline{\phi(\tau, y - x(\tau))} \varphi(\tau, y - z) W_{\varphi(\tau,\cdot)} u(\tau, z, \eta) e^{iy\cdot(\eta - \xi(\tau))} dy dz \bar{d}\eta \right\|_{L^p_x L^p_\xi} \\
\leq & C_T \sum_{|\beta_1| + |\beta_2| \leq 2N}
\end{aligned}$$

$$\begin{aligned}
&\left\| \iiint \left|\partial_y^{\beta_2}\varphi(\tau, y-z)\right| \left|W_{\varphi(\tau,\cdot)}u(\tau,z,\eta)\right| \frac{\left|\partial_y^{\beta_1}\phi(\tau, y-x(\tau))\right|}{\langle \eta-\xi(\tau)\rangle^{2N}} dydz\bar{d}\eta \right\|_{L_x^pL_\xi^p} \\
=&C_T \sum_{|\beta_1|+|\beta_2|\le 2N} \left\| \iint F(\tau,y,\eta) \frac{\left|\partial_y^{\beta_1}\phi(\tau, y-x(\tau))\right|}{\langle \eta-\xi(\tau)\rangle^{2N}} dy\bar{d}\eta \right\|_{L_x^pL_\xi^p} \\
=&C \sum_{|\beta_1|+|\beta_2|\le 2N} \left\| \left[F(\tau) * \frac{\left|\partial_y^{\beta_1}\phi(\tau)\right|}{\langle\cdot\rangle^{2N}} \right](\tau) \right\|_{L_x^pL_\xi^p},
\end{aligned}$$

where $F(\tau,y,\eta) = \left[\left|\partial_y^{\beta_2}\varphi(\tau)\right| * \left|W_{\varphi(\tau,\cdot)}u(\tau)\right|\right](y,\eta)$. Taking $N\in\mathbb{N}$ large as $2N>n$ and using Hausdorff-Young's inequality, we get

$$\begin{aligned}
&\|Ru(\tau,x(\tau),\xi(\tau)\|_{L_x^pL_\xi^p} \\
&\le \sum_{k,l=1}^{n} C_T \left\| \int \overline{\varphi_l^k(t,y-x(\tau))} u(\tau,y)e^{-iy\cdot\xi(\tau)}dy \right\|_{L_x^pL_\xi^p} \\
&\quad + \frac{1}{2}\sum_{k,l,l'=1}^{n} C_T \left\| \int \overline{\varphi_{l,l'}(t,y-x(\tau))} u(\tau,y)e^{-iy\cdot\xi(\tau)}dy \right\|_{L_x^pL_\xi^p} \\
&\le C_nC_T \sum_{|\beta_1|+|\beta_2|\le 2N} \left(\sum_{k,l=1}^{n} \left\|\partial_y^{\beta_1}\varphi_l^k(\tau)\right\|_{L_y^1} + \frac{1}{2}\sum_{k,l,l'=1}^{n} \left\|\partial_y^{\beta_1}\varphi_{l,l'}(\tau)\right\|_{L_y^1} \right) \\
&\quad \times \left\|\partial_y^{\beta_2}\varphi(\tau)\right\|_{L_y^1} \left\|W_{\varphi(\tau,\cdot)}u(\tau)\right\|_{L_z^pL_\eta^p} \\
&\le C_nC'_T \left\|W_{\varphi(\tau,\cdot)}u(\tau)\right\|_{L_z^pL_\eta^p},
\end{aligned}$$

where $C_n = \frac{3}{2}C\left\|\langle\cdot\rangle^{-2N}\right\|_{L_\eta^1}$.

Thus, using the above inequality and Gronwall inequality for (3.1), we obtain (1.3). □

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