SUT Journal of Mathematics
Vol. 57, No. 1 (2021), 55–75

On the Riemann hypothesis for self-dual weight enumerators of genera three and four

Koji Chinen and Yuki Imamura

(Received November 5, 2020)

Abstract. Zeta functions for linear codes were defined by I. Duursma in 1999. In the cases of genera less than three, S. Nishimura gave equivalent conditions for their Riemann hypothesis. In this paper, using a new method, we give similar equivalent conditions for the cases of genera three and four. Our method can be applied to smaller genera and leads to an alternative simple proofs of Nishimura's theorems. Using these results, we examine the Riemann hypothesis of some invariant polynomials. We also discuss the cases of genera greater than four and propose some new problems.

AMS 2010 *Mathematics Subject Classification.* Primary 11T71; Secondary 13A50, 12D10.

Key words and phrases. Zeta function for codes, invariant polynomial ring, Riemann hypothesis.

§1. Introduction

Zeta functions for linear codes were introduced by Iwan Duursma [7] in 1999 and they have attracted attention of many mathematicians:

Definition 1.1. *Let C be an $[n,k,d]$-code over $\mathbf{F}_q$ ($q = p^r$, p is a prime) with the Hamming weight enumerator $W_C(x,y)$. Then there exists a unique polynomial $P(T) \in \mathbf{R}[T]$ of degree at most $n-d$ such that*

$$\frac{P(T)}{(1-T)(1-qT)}(y(1-T)+xT)^n = \cdots + \frac{W_C(x,y)-x^n}{q-1}T^{n-d} + \cdots . \tag{1.1}$$

We call $P(T)$ and $Z(T) = P(T)/(1-T)(1-qT)$ the zeta polynomial and the zeta function of $W_C(x,y)$, respectively.

Note that we always assume $d, d^{\perp} \geq 2$ ($d^{\perp}$ is the minimum distance of $C^{\perp}$). If C is self-dual, then $P(T)$ satisfies the functional equation (see [8, §2]):

Theorem 1.2. *If C is self-dual, then we have*

$$P(T) = P\Big(\frac{1}{qT}\Big) q^g T^{2g}, \tag{1.2}$$

where $g = n/2 + 1 - d$.

The number g is called the *genus* of C. It is appropriate to formulate the Riemann hypothesis as follows:

Definition 1.3. *The code C satisfies the Riemann hypothesis if all the zeros of $P(T)$ have the same absolute value $1/\sqrt{q}$.*

The reader is referred to [9] and [10] for other results by Duursma.

Remark. The definition of the zeta function can be extended straightforwardly to much wider classes of invariant polynomials: let $W(x, y)$ be a polynomial of the form

$$W(x,y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i \in \mathbf{C}[x,y] \quad (A_d \neq 0) \tag{1.3}$$

which satisfy $W^{\sigma_q}(x, y) = \pm W(x, y)$ for some $q \in \mathbf{R}$, $q > 0$, $q \neq 1$, where

$$\sigma_q = \frac{1}{\sqrt{q}} \begin{pmatrix} 1 & q-1 \\ 1 & -1 \end{pmatrix} \quad \text{(the MacWilliams transform)} \tag{1.4}$$

and the action of $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ on a polynomial $f(x, y) \in \mathbf{C}[x, y]$ is defined by $f^\sigma(x, y) = f(ax + by, cx + dy)$. Then we can formulate the zeta function and the Riemann hypothesis for $W(x, y)$ in the same way as Definitions 1.1 and 1.3. For the results in this direction, the reader is referred to [2]–[6], for example. We assume $d, d^\perp \geq 2$ also in this case, where $d^\perp$ is defined by $W^{\sigma_q}(x, y) = B_0 x^n + B_{d^\perp} x^{n-d^\perp} y^{d^\perp} + \cdots$ where $B_0, B_{d^\perp} \neq 0$, when considering the zeta function of $W(x, y)$.

We call $W(x, y)$ "σ_q-invariant" if we have $W^{\sigma_q}(x, y) = W(x, y)$. We do not know much about the Riemann hypothesis for σ_q-invariant polynomials, but one of the remarkable results is the following theorem by Nishimura [12, Theorem 1], an equivalent condition for a σ_q-invariant polynomial of genus one to satisfy the Riemann hypothesis:

Theorem 1.4 (Nishimura). *A σ_q-invariant polynomial*

$$W(x, y) = x^{2d} + A_d x^d y^d + \cdots$$

satisfies the Riemann hypothesis if and only if

$$\frac{\sqrt{q}-1}{\sqrt{q}+1}\binom{2d}{d} \leq A_d \leq \frac{\sqrt{q}+1}{\sqrt{q}-1}\binom{2d}{d}. \tag{1.5}$$

Nishimura also deduces the following, the case of genus two ([12, Theorem 2]):

Theorem 1.5 (Nishimura). *A σ_q-invariant polynomial $W(x,y) = x^{2d+2} + A_d x^{2d+2-d} y^d + \cdots$ satisfies the Riemann hypothesis if and only if the both roots of the quadratic polynomial*

$$(1.6)\quad A_d X^2 - \left((d-q)A_d + \frac{d+1}{d+2}A_{d+1}\right) X - (d+1)(q+1)\left(A_d + \frac{A_{d+1}}{d+2}\right) + (q-1)\binom{2d+2}{d}$$

are contained in $[-2\sqrt{q}, 2\sqrt{q}]$.

The purpose of this article is to establish analogous equivalent conditions for the cases of genera three and four. Our main results are Theorems 3.1 and 3.3. Nishimura [12] uses (1.1) directly as well as a property of the binomial moments (see [12, (3.1)]) to prove Theorems 1.4 and 1.5, which requires some complicated calculations. In this paper, we use the relation between the zeta polynomial and the MDS weight enumerators (see Theorem 2.3) instead of (1.1) and the binomial moments. Our method also applies to smaller genera and leads to alternative proofs of Nishimura's theorems. The proofs are considerably simplified and shortened.

By our theorems and the preceding results of Nishimura, we can verify the truth of the Riemann hypothesis of $W(x,y)$ by small number of coefficients A_i: the number of parameters which are needed coincides with the genus g (see [12]). For example, we need only three parameters A_d, A_{d+1} and A_{d+2} in the case of genus three. Moreover, in many cases, we have $A_{d+1} = 0$ (and $A_{d+2} = 0$ also holds in some cases, see Example 4.5) and the verification of the Riemann hypothesis is simplified.

Here we mention the famous problem by Duursma ([9, Open Problem 4.2]):

Problem 1.6 (Duursma). *Prove or disprove that all extremal weight enumerators satisfy the Riemann hypothesis.*

An extremal weight enumerator is that of an "extremal code", a self-dual code over the finite fields $\mathbf{F}_2$, $\mathbf{F}_3$ or $\mathbf{F}_4$ which has maximal possible minimum distance at a given code length n (see Definition 4.4 for detail). The extremal property is also defined for σ_q-invariant polynomials in the preceding remark.

As an application of our main results, we examine the Riemann hypothesis for some σ_q-invariant polynomials. First we consider some extremal ones. This case includes an example of so-called "Type III extremal weight enumerators", examples of extremal $\sigma_{4/3}$-invariant polynomials which were dealt with by the first named author [4], and an extremal σ_q-invariant polynomial for $q = 6+2\sqrt{5}$. The last one is another example of an extremal invariant polynomial not satisfying the Riemann hypothesis (see also [5]). Next we examine a certain sequence of σ_q-invariant polynomials, that is,

$$W_{m,q}(x,y) = (x^2 + (q-1)y^2)^m \quad (m \geq 2) \tag{1.7}$$

as the numbers q and m vary. As was mentioned in Remark before, q can take other numbers than prime powers. In this context, we can notice the tendency that the Riemann hypothesis becomes harder to hold if q or $\deg W(x,y)$ are larger. Some of the results in [4] and [5] also support it. Theorems 3.1 and 3.3 can illustrate this tendency by considering (1.7).

Lastly, in synthesis of the contents of this paper, we give some remarks and problems. It is desirable that the results will be generalized to genera greater than four. The first step will be an explicit expression of $P(T)$ via the coefficients A_i of $W(x,y)$. It is predicted in Conjecture 6.2. We propose some other problems related to the Riemann hypothesis of $W_{m,q}(x,y)$.

The rest of the paper is organized as follows: in Section 2, we give some notions and theorems which are needed in the later sections. In Section 3, we give statements and proofs of our main results. We also give alternative proofs of Nishimura's theorems. In Section 4, we examine the Riemann hypothesis for various invariant polynomials. Section 5 is devoted to the observation of the behavior of $W_{m,q}(x,y)$ of genus up to four. In the last section, we give some remarks based on numerical experiments and propose some future problems, including the case of $g \geq 5$.

§2. Preliminaries

We begin this section by giving the definition of the MDS codes (MacWilliams-Sloane [11, p. 317]) and some known results related to them.

Definition 2.1 (MDS code). *Let C be an $[n,k,d]$ code over $\mathbf{F}_q$. We call C an MDS code if*

$$d = n - k + 1 \tag{2.1}$$

is satisfied.

Because of the relation (2.1), the weight enumerator of an MDS code is determined by n and d, so it is often denoted by $M_{n,d}(x,y)$. The weight distribution is well known:

Theorem 2.2. *Let*

$$M_{n,d}(x,y) = x^n + \sum_{w=d}^{n} M_w^{(n,d)} x^{n-w} y^w.$$

Then we have

$$M_w^{(n,d)} = \binom{n}{w} \sum_{j=0}^{w-d} (-1)^j \binom{w}{j} (q^{w-d+1-j} - 1). \tag{2.2}$$

Proof. MacWilliams-Sloane [11, Ch. 11, Theorem 6]. □

We can generalize and use (2.2) to the case where $q \in \mathbf{R}$, $q > 0$ and $q \neq 1$. As to the zeta function for a polynomial of the form (1.3), the following theorem is fundamental for our later discussion:

Theorem 2.3. *Let $W(x,y)$ be a polynomial of the form (1.3) and*

$$P(T) = a_0 + a_1 T + \cdots + a_r T^r$$

be the zeta polynomial of $W(x,y)$. Then we have

$$W(x,y) = a_0 M_{n,d}(x,y) + a_1 M_{n,d+1}(x,y) + \cdots + a_r M_{n,d+r}(x,y). \tag{2.3}$$

Proof. Duursma [8, p.59, (5)]. □

Remark. As is shown in [8], Theorem 2.3 is also valid for $W(x,y)$ which is not invariant under σ_q.

If $W(x,y)$ is of the form (1.3), σ_q-invariant and of genus g, the zeta polynomial $P(T)$ of $W(x,y)$ can be written as

$$P(T) = a_0 + a_1 T + \cdots + a_{2g} T^{2g}$$

(see [8, p.59, (6)]). By the functional equation (1.2), we can deduce

$$\begin{aligned} P(T) = \ & a_0 + a_1 T + \cdots + a_g T^g \\ & + a_{g-1} q T^{g+1} + \cdots + a_1 q^{g-1} T^{2g-1} + a_0 q^g T^{2g}. \end{aligned} \tag{2.4}$$

Moreover, we have

$$a_g = 1 - (1+q^g)a_0 - (1+q^{g-1})a_1 - \cdots - (1+q)a_{g-1} \tag{2.5}$$

since $P(1) = 1$ ([8, p.59, (7)]). It follows that $P(T)$ is determined by g parameters $a_0, a_1 \cdots, a_{g-1}$. We also have a factorization

$$P(T) = a_0 q^g \prod_{i=1}^{g} (T^2 + b_i T + 1/q) \tag{2.6}$$

($b_i \in \mathbf{R}$) since $\deg P(T)$ is even (essentially the same but a little different form is used in [8, p.60, (10)]). The case $g = 2$ is used in [12, p.2356].

Lastly we introduce notations expressing the elementary symmetric polynomials. For $1 \le k \le n$, let

$$e_k(X_1, X_2, \cdots, X_n) = \sum_{1 \le j_1 < j_2 < \cdots < j_k \le n} X_{j_1} X_{j_2} \cdots X_{j_k}$$

(we usually define $e_0(X_1, X_2, \cdots, X_n) = 1$, but we do not need the case $k = 0$). Later we use the cases where $n = 3, 4$ and $X_j = b_j$ in connection to the expansion of the right hand side of (2.6), so we further define

$$e_k^{(n)} = e_k(b_1, b_2, \cdots, b_n).$$

For example,

$$\begin{aligned} e_1^{(3)} &= b_1 + b_2 + b_3, \\ e_2^{(3)} &= b_1 b_2 + b_2 b_3 + b_3 b_1, \\ e_3^{(3)} &= b_1 b_2 b_3. \end{aligned}$$

§3. Main theorems

First we consider the case of genus three. In this case, we have $n = 2d + 4$ by $g = n/2 + 1 - d = 3$ and $\deg P(T) = 2g = 6$. We prove the following:

Theorem 3.1. *A σ_q-invariant polynomial $W(x, y) = x^{2d+4} + A_d x^{d+4} y^d + \cdots$ satisfies the Riemann hypothesis if and only if all the roots of the cubic polynomial*

$$f_3 X^3 + f_2 X^2 + f_1 X + f_0 \tag{3.1}$$

are contained in $[-2\sqrt{q}, 2\sqrt{q}]$, where f_i is defined as follows.

$$\begin{aligned} f_3 &= A_d, \\ f_2 &= (q-d)A_d - \frac{d+1}{d+4} A_{d+1}, \\ f_1 &= \frac{1}{2}(d^2 - 2qd + d - 6q)A_d + (d - q + 1)\frac{d+1}{d+4} A_{d+1} + \frac{(d+1)(d+2)}{(d+3)(d+4)} A_{d+2}, \\ f_0 &= \frac{1}{2}(q+1)(d^2 + 3d - 4q + 2)A_d + (q+1)(d+1)(d+2)\frac{A_{d+1}}{d+4} \\ &\quad + (q+1)\frac{(d+1)(d+2)}{(d+3)(d+4)} A_{d+2} - (q-1)\binom{2d+4}{d+4}. \end{aligned}$$

Proof. We have

$$P(T) = a_0 + a_1T + a_2T^2 + a_3T^3 + a_2qT^4 + a_1q^2T^5 + a_0q^3T^6 \tag{3.2}$$

and

$$a_3 = 1 - (1+q^3)a_0 - (1+q^2)a_1 - (1+q)a_2 \tag{3.3}$$

(see (2.4) and (2.5)). The identity (2.6) becomes

$$P(T) = a_0q^3 \prod_{i=1}^{3}(T^2 + b_iT + 1/q). \tag{3.4}$$

Suppose we expand the right hand side of (3.4) and then compare the coefficients in (3.2) and (3.4). Then we get

$$\begin{aligned} e_1^{(3)} &= a_1/a_0q, \\ e_2^{(3)} &= (a_2 - 3a_0q)/a_0q^2, \\ e_3^{(3)} &= (a_3 - 2a_1q)/a_0q^3. \end{aligned} \tag{3.5}$$

We can see that b_i are the roots of the cubic polynomial

$$a_0q^3X^3 - a_1q^2X^2 + (a_2 - 3a_0q)qX - a_3 + 2a_1q. \tag{3.6}$$

Considering the distribution of the roots of each factor $T^2 + b_iT + 1/q$ in (3.4), we can see that a self-dual weight enumerator $W(x, y)$ of genus three satisfies the Riemann hypothesis if and only if b_1, b_2 and b_3 are contained in $[-2/\sqrt{q}, 2/\sqrt{q}]$. By change of variable in (3.6), we get the following:

Lemma 3.2. *$W(x, y)$ satisfies the Riemann hypothesis if and only if all the roots of the polynomial*

$$a_0X^3 - a_1X^2 + (a_2 - 3a_0q)X - a_3 + 2a_1q \tag{3.7}$$

are contained in $[-2\sqrt{q}, 2\sqrt{q}]$.

Our next task is to express the coefficients a_i in (3.7) by way of A_i in $W(x, y)$. Note that we need a_0, a_1, a_2 only (a_3 is determined by (3.3)). By

Theorem 2.3, we have

$$\begin{aligned}
W(x,y) &= a_0 M_{n,d}(x,y) + a_1 M_{n,d+1}(x,y) + a_2 M_{n,d+2}(x,y) + \cdots \\
&= a_0\Big(x^n + \sum_{w=d}^{n} M_w^{(n,d)} x^{n-w} y^w\Big) \\
&\quad + a_1\Big(x^n + \sum_{w=d+1}^{n} M_w^{(n,d+1)} x^{n-w} y^w\Big) \\
&\quad + a_2\Big(x^n + \sum_{w=d+2}^{n} M_w^{(n,d+2)} x^{n-w} y^w\Big) \\
&\quad + \cdots
\end{aligned}$$

$$\begin{aligned}
= x^n + a_0 M_d^{(n,d)} x^{n-d} y^d + a_0 M_{d+1}^{(n,d)} x^{n-d-1} y^{d+1} &+ a_0 M_{d+2}^{(n,d)} x^{n-d-2} y^{d+2} \\
&+ \cdots \\
+ a_1 M_{d+1}^{(n,d+1)} x^{n-d-1} y^{d+1} &+ a_1 M_{d+2}^{(n,d+1)} x^{n-d-2} y^{d+2} \\
&+ \cdots \\
&+ a_2 M_{d+2}^{(n,d+2)} x^{n-d-2} y^{d+2} \\
&+ \cdots
\end{aligned}$$

On the other hand we have

$$W(x,y) = x^n + A_d x^{n-d} y^d + A_{d+1} x^{n-d-1} y^{d+1} + A_{d+2} x^{n-d-2} y^{d+2} + \cdots .$$

Comparing the coefficients of $x^{n-d}y^d$, $x^{n-d-1}y^{d+1}$ and $x^{n-d-2}y^{d+2}$, we get the following system of linear equations of a_0, a_1, a_2:

$$\begin{aligned}
a_0 M_d^{(n,d)} &= A_d, \\
a_0 M_{d+1}^{(n,d)} + a_1 M_{d+1}^{(n,d+1)} &= A_{d+1}, \\
a_0 M_{d+2}^{(n,d)} + a_1 M_{d+2}^{(n,d+1)} + a_2 M_{d+2}^{(n,d+2)} &= A_{d+2}.
\end{aligned} \tag{3.8}$$

By Theorem 2.2, we have $M_{d+i}^{(n,d+i)} = \binom{n}{d+i}(q-1) \neq 0$, so the solution of the system (3.8) is given by

$$a_0 = \frac{1}{M_d^{(n,d)}} A_d, \tag{3.9}$$

$$a_1 = -\frac{M_{d+1}^{(n,d)}}{M_d^{(n,d)} M_{d+1}^{(n,d+1)}} A_d + \frac{1}{M_{d+1}^{(n,d+1)}} A_{d+1},$$

$$a_2 = \frac{1}{M_d^{(n,d)} M_{d+2}^{(n,d+2)}} \left(\frac{M_{d+1}^{(n,d)} M_{d+2}^{(n,d+1)}}{M_{d+1}^{(n,d+1)}} - M_{d+2}^{(n,d)} \right) A_d$$
$$- \frac{M_{d+2}^{(n,d+1)}}{M_{d+1}^{(n,d+1)} M_{d+2}^{(n,d+2)}} A_{d+1} + \frac{1}{M_{d+2}^{(n,d+2)}} A_{d+2}.$$

These expressions seem very complicated, but Theorem 2.2 simplifies them as follows:

$$a_0 = \frac{1}{q-1} \cdot \frac{1}{\binom{n}{d}} A_d,$$
$$a_1 = \frac{1}{q-1} \left(\frac{d-q}{\binom{n}{d}} A_d + \frac{1}{\binom{n}{d+1}} A_{d+1} \right), \tag{3.10}$$
$$a_2 = \frac{1}{q-1} \left(\frac{d(d+1-2q)}{2\binom{n}{d}} A_d + \frac{d+1-q}{\binom{n}{d+1}} A_{d+1} + \frac{1}{\binom{n}{d+2}} A_{d+2} \right).$$

We obtain the theorem by combining (3.10), (3.3) and Lemma 3.2. □

Now we proceed to the case of genus four. In this case, we have $n = 2d + 6$ by $g = n/2 + 1 - d = 4$ and $\deg P(T) = 2g = 8$. The result is the following:

Theorem 3.3. *A σ_q-invariant polynomial $W(x, y) = x^{2d+6} + A_d x^{d+6} y^d + \cdots$ satisfies the Riemann hypothesis if and only if all the roots of the polynomial*

$$a_0 X^4 - a_1 X^3 + (a_2 - 4a_0 q) X^2 - (a_3 - 3a_1 q) X + a_4 - 2a_2 q + 2a_0 q^2 \tag{3.11}$$

are contained in $[-2\sqrt{q}, 2\sqrt{q}]$, where a_0, a_1, a_2 are the same as (3.10), a_3 is given by

$$a_3 = \frac{1}{q-1} \left(\frac{d(d+1)(d+2-3q)}{6\binom{n}{d}} A_d \right. \tag{3.12}$$
$$\left. + \frac{(d+1)(d+2-2q)}{2\binom{n}{d+1}} A_{d+1} + \frac{d+2-q}{\binom{n}{d+2}} A_{d+2} + \frac{1}{\binom{n}{d+3}} A_{d+3} \right)$$

and a_4 is given by

$$a_4 = 1 - (1 + q^4) a_0 - (1 + q^3) a_1 - (1 + q^2) a_2 - (1 + q) a_3. \tag{3.13}$$

Proof. The proof is similar to that of Theorem 3.1. We give an outline. The zeta polynomial is

$$P(T) = a_0 + a_1 T + a_2 T^2 + a_3 T^3 + a_4 T^4 + a_3 q T^5 + a_2 q^2 T^6 + a_1 q^3 T^7 + a_0 q^4 T^8 \tag{3.14}$$

and we have (3.13) by (2.4) and (2.5). The identity (2.6) becomes

$$P(T) = a_0 q^4 \prod_{i=1}^{4} (T^2 + b_i T + 1/q). \tag{3.15}$$

Instead of (3.5) we get

$$\begin{aligned} e_1^{(4)} &= a_1/a_0 q, \\ e_2^{(4)} &= (a_2 - 4a_0 q)/a_0 q^2, \\ e_3^{(4)} &= (a_3 - 3a_1 q)/a_0 q^3. \\ e_4^{(4)} &= (a_4 - 2a_2 q + 2a_0 q^2)/a_0 q^4. \end{aligned}$$

Using these, we can verify in a similar manner to Lemma 3.2 that $W(x, y)$ satisfies the Riemann hypothesis if and only if all the roots of (3.11) are contained in $[-2\sqrt{q}, 2\sqrt{q}]$. The relation between a_0, a_1, a_2, a_3 and $A_d, A_{d+1}, A_{d+2}, A_{d+3}$ are given by four equations

$$\sum_{j=0}^{i} a_j M_{d+i}^{(n,d+j)} = A_{d+i} \qquad (i = 0, 1, 2, 3).$$

Note that the equations for $i = 0, 1, 2$ coincide with (3.8), so a_0, a_1, a_2 are given by (3.10). For the coefficient a_3, a direct calculation brings (3.12). □

As was mentioned in the first section, our method also applies to the cases of $g = 1, 2$ and simplifies the proof. Here we give an outline of it:

(Alternative proof of Theorem 1.4) Note that $n = 2d$ by $g = 1 = n/2 - d + 1$. We have $P(T) = a_0 + (1 - (1 + q)a_0)T + a_0 q T^2$ by (2.4) and (2.5). Thus, $P(T)$ itself is a quadratic polynomial, so we can omit the process which requires (2.6). It is easy to see that both roots of $P(T)$ are on the circle $|T| = 1/\sqrt{q}$ if and only if $(1 - (1 + q)a_0)^2 - 4{a_0}^2 q \leq 0$. It is equivalent to

$$\frac{(\sqrt{q} - 1)^2}{(q - 1)^2} \leq a_0 \leq \frac{(\sqrt{q} + 1)^2}{(q - 1)^2}.$$

Using Theorem 2.3 (instead of (1.1) and the binomial moment), we have

$$\begin{aligned} W(x,y) &= a_0 M_{n,d}(x,y) + a_1 M_{n,d+1}(x,y) + a_0 q M_{n,d+2}(x,y) \\ &= x^n + a_0 M_d^{(n,d)} x^{n-d} y^d + \cdots \\ &= x^n + A_d x^{n-d} y^d + \cdots . \end{aligned}$$

The parameter a_0 is determined by the coefficient of $x^{n-d}y^d$ as

$$a_0 = \frac{A_d}{M_d^{(n,d)}} = \frac{A_d}{(q-1)\binom{n}{d}} = \frac{A_d}{(q-1)\binom{2d}{d}}.$$

This immediately implies (1.5) under the condition $q > 1$.

(Alternative proof of Theorem 1.5) Note that $n = 2d+2$ by $g = 2 = n/2 - d + 1$. We have $P(T) = a_0 + a_1 T + a_2 T^2 + a_1 q T^3 + a_0 q^2 T^4$ by (2.4). We also have $a_2 = 1 - (1+q^2)a_0 - (1+q)a_1$ by (2.5). The identity (2.6) becomes

$$P(T) = a_0 q^2 (T^2 + b_1 T + 1/q)(T^2 + b_2 T + 1/q). \tag{3.16}$$

Similarly to our main theorems, we have

$$\begin{aligned} e_1^{(2)} &= b_1 + b_2 = a_1/a_0 q, \\ e_2^{(2)} &= b_1 b_2 = (a_2 - 2a_0 q)/a_0 q^2. \end{aligned}$$

Hence we can see that the Riemann hypothesis is true if and only if both roots of the polynomial

$$a_0 X^2 - a_1 X + a_2 - 2a_0 q \tag{3.17}$$

are contained in the interval $[-2\sqrt{q}, 2\sqrt{q}]$. We must express a_0 and a_1 by A_d and A_{d+1}. This can be done by using Theorem 2.3, rather than (1.1) and the binomial moment. We have

$$\begin{aligned} W(x,y) &= a_0 M_{n,d}(x,y) + a_1 M_{n,d+1}(x,y) + a_2 M_{n,d+2}(x,y) \\ &\quad + a_1 q M_{n,d+3}(x,y) + a_0 q^2 M_{n,d+4}(x,y) \\ &= x^n + a_0 M_d^{(n,d)} x^{n-d} y^d + (a_0 M_{d+1}^{(n,d)} + a_1 M_{d+1}^{(n,d+1)}) x^{n-d-1} y^{d+1} \\ &\quad + \cdots \\ &= x^n + A_d x^{n-d} y^d + A_{d+1} x^{n-d-1} y^{d+1} + \cdots . \end{aligned}$$

Comparing the coefficients, we get a system of linear equations

$$\begin{aligned} A_d &= a_0 M_d^{(n,d)}, \\ A_{d+1} &= a_0 M_{d+1}^{(n,d)} + a_1 M_{d+1}^{(n,d+1)} \end{aligned}$$

which gives (note that $n = 2d + 2$)

$$\begin{aligned} a_0 &= \frac{A_d}{M_d^{(n,d)}} = \frac{1}{q-1} \cdot \frac{1}{\binom{2d+2}{d}} A_d, \\ a_1 &= \frac{1}{M_{d+1}^{(n,d+1)}} \left(-\frac{M_{d+1}^{(n,d)}}{M_d^{(n,d)}} A_d + A_{d+1} \right) \\ &= \frac{1}{q-1} \left(\frac{d-q}{\binom{2d+2}{d}} A_d + \frac{1}{\binom{2d+2}{d+1}} A_{d+1} \right). \end{aligned}$$

These values enable us to describe the polynomial (3.17) explicitly and we can obtain Theorem 1.5.

§4. Extremal divisible polynomials

First we review some basic facts on the extremal divisible σ_q-invariant polynomials.

Definition 4.1. *A polynomial $W(x, y)$ in the form (1.3) is called "divisible by c" ($c \in \mathbf{N}$, $c > 1$) if*

$$A_i \neq 0 \Rightarrow c | i \quad (d \leq i \leq n)$$

is satisfied.

In the case of existing self-dual codes, the following theorem is well-known (see [15]):

Theorem 4.2 (Gleason-Pierce)**.** *Suppose a self-dual code over $\mathbf{F}_q$ is divisible by $c > 1$, that is, the weight of any codeword is divisible by c. Then, (q, c) must be one of the following:*

$$(q, c) = (2, 2), (2, 4), (3, 3), (4, 2)$$

or q is arbitrary and $c = 2$.

The cases $(q, c) = (2, 2), (2, 4), (3, 3)$ and $(4, 2)$ are called "Types I, II, III and IV", respectively. The Mallows-Sloane bound is a set of inequalities which bound the minimum distance d by the code length n for these types (see [10, Theorem 3]):

Theorem 4.3 (Mallows-Sloane bound)**.** *We have the following upper bounds for the minimum distance d by the code length n:*

$$\begin{aligned}
&\text{(Type I)} \qquad d \le 2\left[\frac{n}{8}\right] + 2,\\
&\text{(Type II)} \qquad d \le 4\left[\frac{n}{24}\right] + 4,\\
&\text{(Type III)} \qquad d \le 3\left[\frac{n}{12}\right] + 3,\\
&\text{(Type IV)} \qquad d \le 2\left[\frac{n}{6}\right] + 2,
\end{aligned}$$

where $[x]$ *means the largest integer not exceeding* x.

Now we can define the notion of the extremal codes and the extremal weight enumerators:

Definition 4.4. *Among the codes of Types I through IV, the codes which attain the equality in Theorem 4.3 are called extremal codes. Weight enumerators of extremal codes are called extremal weight enumerators.*

Problem 1.6 is solved affirmatively only for two sequences of Type IV extremal weight enumerators (see [10] for one sequence, the code length is of the form $6k$ ($k \in \mathbf{N}$), see [13] for the other sequence, that is of the form $6k-2$). It is not solved for other cases (Types I through III and the code length $6k+2$ of Type IV), but no counterexample is known so far.

Our first example is the following:

Example 4.5. The extremal Type III weight enumerator of degree 16:

$$W(x,y) = x^{16} + 224x^{10}y^6 + 2720x^7y^9 + 3360x^4y^{12} + 256xy^{15}.$$

It is obtained by calculating $(4W_4(x,y)W_{12}(x,y) - W_4(x,y)^4)/3$, where

$$\begin{aligned}
W_4(x,y) &= x^4 + 8xy^3,\\
W_{12}(x,y) &= x^{12} + 264x^6y^6 + 440x^3y^9 + 24y^{12}.
\end{aligned}$$

They are the generators of the ring of Type III weight enumerators (see [14, p.137]). The polynomial $W(x,y)$ is of genus three and we have $A_d = A_6 = 224$, $A_{d+1} = A_{d+2} = 0$, so the calculation is simplified a lot. By Theorem 3.1, we have $f_3 = 224$, $f_2 = -672$, $f_1 = -1344$, $f_0 = 3696$. Let

$$g(X) = 2X^3 - 6X^2 - 12X + 33.$$

Then the polynomial (3.1) becomes $112g(X)$. We can easily verify

$$\begin{aligned}
&g(-2\sqrt{3}) = -39 - 24\sqrt{3} < 0,\\
&g(0) = 33 > 0, \quad g(3) = -3 < 0,\\
&g(2\sqrt{3}) = -39 + 24\sqrt{3} > 0.
\end{aligned}$$

So, all the roots of $g(X)$ are in the interval $[-2\sqrt{3}, 2\sqrt{3}]$, and the Riemann hypothesis is true.

Divisibility and the extremal property are extended similarly to other σ_q-invariant polynomials. Among them, we next consider $\sigma_{4/3}$-invariant polynomials. This family of polynomials was discovered by [4] and an analog of the Mallows-Sloane bound is proved in [6, Theorem 4.2 (i)]. A $\sigma_{4/3}$-invariant polynomial of the form (1.3) is extremal if $d = 2[n/12] + 2$. There are two extremal $\sigma_{4/3}$-invariant polynomials of genus three: the cases $(n, d) = (8, 2)$ and $(12, 4)$.

Example 4.6. The $\sigma_{4/3}$-invariant polynomials can be constructed by two generators $W_{2,4/3}(x, y)$ and $\varphi_6(x, y)^2$ where

$$\begin{aligned} W_{2,4/3}(x, y) &= x^2 + \frac{1}{3}y^2, \\ \varphi_6(x, y) &= x^6 - 5x^4y^2 + \frac{5}{3}x^2y^4 - \frac{1}{27}y^6 \end{aligned}$$

(see [6, Section 4]).

(i) *The case* $(n, d) = (8, 2)$.

The extremal polynomial is given by $W_{2,4/3}(x, y)^4 = x^8 + (4/3)x^6y^2 + (2/3)x^4y^4 + \cdots$. We have $A_d = A_2 = 4/3$, $A_{d+1} = 0$, and $A_{d+2} = A_4 = 2/3$. Theorem 3.1 tells us that the polynomial (3.1) becomes $(4/3)g(X)$, where

$$g(X) = X^3 - \frac{2}{3}X^2 - \frac{52}{15}X + \frac{56}{45}.$$

We can see that

$$\begin{aligned} &g(-2) = -\frac{112}{45} < 0, \\ &g(0) = \frac{56}{45} > 0, \quad g(1) = -\frac{17}{9} < 0, \\ &g(2\sqrt{4/3}) = \frac{1}{45}(112\sqrt{3} - 104) > 0. \end{aligned}$$

So, all the roots of $g(X)$ are in the interval $[-2\sqrt{4/3}, 2\sqrt{4/3}]$, and the Riemann hypothesis is true.

(ii) *The case* $(n, d) = (12, 4)$.

The extremal polynomial is given by

$$\frac{1}{6}(5W_{2,4/3}(x, y)^6 + \varphi_6(x, y)^2) = x^{12} + \frac{55}{9}x^8y^4 - \frac{176}{81}x^6y^6 + \cdots$$

(see also [6, Example 4.7]). We have $A_d = A_4 = 55/9$, $A_{d+1} = 0$ and $A_{d+2} = A_6 = -176/81$. The polynomial (3.1) becomes $(55/9)g(X)$, where

$$g(X) = X^3 - \frac{8}{3}X^2 + \frac{10}{21}X + \frac{4}{3}.$$

We can see that

$$g(-1) = -\frac{59}{21} < 0, \quad g(0) = \frac{4}{3} > 0, \quad g(2) = -\frac{8}{21} < 0,$$
$$g(2\sqrt{4/3}) = \frac{4}{63}(122\sqrt{3} - 203) > 0.$$

So, all the roots of $g(X)$ are in the interval $[-2\sqrt{4/3}, 2\sqrt{4/3}]$, and the Riemann hypothesis is true.

Remark. By using the analog of Okuda's theorem (see [6, Theorem 4.6]) and the above example (ii), we can see that $W_{2,4/3}(x,y)^5$, the extremal $\sigma_{4/3}$-invariant polynomial of degree 10 also satisfies the Riemann hypothesis.

Lastly, we take an example from [5], an extremal σ_q-invariant polynomial for $q = 6 + 2\sqrt{5}$. An analog of the Mallows-Sloane bound is established for $\sigma_{6+2\sqrt{5}}$-invariant polynomials of even degree (see [5, Theorem 2.4]). A $\sigma_{6+2\sqrt{5}}$-invariant polynomial of the form (1.3) with even n is extremal if $d = 2[n/10]+2$. In [5], two extremal polynomials are considered and it proved that the Riemann hypothesis fails to hold for them (see [5, Section 3]). These are, an a wide sense, the first counterexamples of Problem 1.6. Here is another such example:

Example 4.7. The extremal $\sigma_{6+2\sqrt{5}}$-invariant polynomial of genus three exists for $(n,d) = (12,4)$, which is

$$\frac{1}{25}(19W_{6+2\sqrt{5},2}(x,y)^6 + 6W_{6+2\sqrt{5},2}(x,y)\psi_5(x,y)^2)$$
$$= x^{12} + (1485 + 660\sqrt{5})x^8y^4 + (7480 + 3344\sqrt{5})x^6y^6 + \cdots,$$

where

$$W_{6+2\sqrt{5},2}(x,y) = x^2 + (5 + 2\sqrt{5})y^2,$$
$$\psi_5(x,y) = x^5 - (50 + 20\sqrt{5})x^3y^2 + (225 + 100\sqrt{5})xy^4$$

(see [5, (2.1) and (2.2)]). We have $A_d = A_4 = 1485 + 660\sqrt{5}$, $A_{d+1} = 0$ and $A_{d+2} = A_6 = 7480 + 3344\sqrt{5}$. The polynomial (3.1) becomes $165g(X)$, where

$$g(X) = 9X^3 + (58 + 26\sqrt{5})X^2 - \frac{1}{7}(3806 + 1702\sqrt{5})X - \frac{1}{7}(2432 + 1088\sqrt{5}).$$

This polynomial has a root outside the interval $[-2(1+\sqrt{5}), 2(1+\sqrt{5})]$ (note that $(1+\sqrt{5})^2 = 6 + 2\sqrt{5}$) because $g(-20) < 0$ and $g(-19) > 0$. Thus the Riemann hypothesis does not hold.

§5. The sequence of the polynomials (1.7)

We examine the polynomials (1.7), which has essentially only one parameter q and is easy to see the phenomenon. Using Nishimura's results ($g = 1, 2$) and our theorem ($g = 3, 4$), we can see that the range of q for which the Riemann hypothesis is true are the following (we will mention the case $g = 4$ later):

$$g = 1: \quad 4 - 2\sqrt{3}\ (\approx 0.53590) \le q \le 4 + 2\sqrt{3}\ (\approx 7.46410) \quad (q \ne 1),$$
$$g = 2: \quad -4 + 2\sqrt{5}\ (\approx 0.47214) \le q \le \alpha^2\ (\approx 3.46812) \quad (q \ne 1),$$

where

$$\alpha = \frac{1}{6}\left(1 + \sqrt[3]{5(29 + 6\sqrt{6})} + \sqrt[3]{5(29 - 6\sqrt{6})}\right),$$

and

$$g = 3: \quad \beta_1\ (\approx 0.47448) \le q \le \beta_3^2\ (\approx 2.47607) \quad (q \ne 1), \tag{5.1}$$

where β_1 is the unique real root of the polynomial

$$100t^5 + 495t^4 + 2056t^3 - 2928t^2 + 1408t - 256$$

and β_3 is the positive root of the polynomial

$$13t^4 + 4t^3 - 20t^2 - 24t - 8.$$

The cases $g = 1$ and 2 are not very complicated, but the last case needs some explanation. The relevant coefficients of $W_{4,q}(x, y)$ are

$$A_d = A_2 = 4(q - 1),\ A_3 = 0,\ A_4 = 6(q - 1)^2.$$

Using these values, we get the explicit form of the polynomial (3.1) as follows:

$$g(X) := 5X^3 + 5(q - 2)X^2 - 2(11q - 6)X - 7q^2 + 20q - 8. \tag{5.2}$$

Let D_g be the discriminant of $g(X)$, X_1 and X_2 be the roots of $g'(X)$ (we assume X_1, X_2 are real and $X_1 \le X_2$). Then, by Theorem 3.1, $W_{4,q}(x, y)$ satisfies the Riemann hypothesis if and only if

$$D_g \ge 0,$$
$$-2\sqrt{q} \le X_1,\ X_2 \le 2\sqrt{q},$$
$$g(-2\sqrt{q}) \le 0,\ g(2\sqrt{q}) \ge 0.$$

We have

$$\frac{D_g}{35} = 100q^5 + 495q^4 + 2056q^3 - 2928q^2 + 1408q - 256,$$

so $D_g \geq 0$ is equivalent to

$$q \geq \beta_1 \tag{5.3}$$

with the above mentioned β_1. The roots X_i are given by

$$\begin{aligned} X_1 &= \frac{-5(q-2) - \sqrt{25q^2 + 230q - 80}}{15}, \\ X_2 &= \frac{-5(q-2) + \sqrt{25q^2 + 230q - 80}}{15}. \end{aligned}$$

The range of q satisfying $-2\sqrt{q} \leq X_1$ is (note that we also have $25q^2 + 230q - 80 \geq 0$)

$$\frac{\sqrt{609} - 23}{5} \leq q \leq \beta_2, \tag{5.4}$$

where β_2 is the square of the unique real root of the polynomial

$$10t^3 - 19t^2 - 20t - 6 \tag{5.5}$$

(this polynomial comes from the equation $-2\sqrt{q} = X_1$). The explicit value is

$$\begin{aligned} \beta_2 &= \frac{1}{300}\Bigg(761 + \sqrt[3]{386669681 + 396000\sqrt{17318}} \\ &\quad + \sqrt[3]{386669681 - 396000\sqrt{17318}}\Bigg) \end{aligned}$$

($\beta_2 \approx 7.38366$, this expression of β_2 can be obtained by constructing the cubic polynomial having the squares of roots of (5.5) as its roots: $100t^3 - 761t^2 + 172t - 36$). The inequality $X_2 \leq 2\sqrt{q}$ gives $(\sqrt{609} - 23)/5 \leq q$. Finally, putting $\sqrt{q} = t$, we have

$$\begin{aligned} g(-2\sqrt{q}) &= 13t^4 + 4t^3 - 20t^2 - 24t - 8, \\ g(2\sqrt{q}) &= 13t^4 - 4t^3 - 20t^2 + 24t - 8. \end{aligned}$$

The inequalities $g(-2\sqrt{q}) \leq 0$ and $g(2\sqrt{q}) \geq 0$ give

$$0 \leq q \leq \beta_3^2 \quad \text{and} \quad q \geq \beta_4^2 \approx 0.356397, \tag{5.6}$$

respectively. Gathering the inequalities (5.3), (5.4) and (5.6), we obtain the estimate (5.1).

The polynomial $W_{m,q}(x, y)$ is of genus four when $m = 5$:

$$\begin{aligned} W_{5,q}(x, y) &= (x^2 + (q-1)y^2)^5 \\ &= x^{10} + 5(q-1)x^8y^2 + 10(q-1)^2x^6y^4 + \cdots . \end{aligned}$$

The relevant coefficients are

$$\begin{aligned} &A_d = A_2 = 5(q-1), \\ &A_{d+1} = A_3 = 0, \quad A_{d+3} = A_5 = 0, \\ &A_{d+2} = A_4 = 10(q-1)^2. \end{aligned}$$

The polynomial (3.11), multiplied by 63, becomes

$$\begin{aligned} g(X) &= 7X^4 + 7(q-2)X^3 + 3(6-13q)X^2 - (18q^2 - 48q + 16)X \\ &\quad + 27q^2 - 28q + 8. \end{aligned} \tag{5.7}$$

It is difficult to obtain exact theoretical values of q for which the Riemann hypothesis of $W_{5,q}(x, y)$ is true, but some numerical experiments suggest that it should be true for $q_1 \le q \le q_2$ where $q_1 \approx 0.4929$ and $q_2 \approx 2.0436$. We can predict that the range of q is smaller than the case of $g = 3$.

Example 5.1. The case $g = 4$ and $q = 2$:

$$\begin{aligned} W_{5,2}(x, y) &= (x^2 + y^2)^5 \\ &= x^{10} + 5x^8y^2 + 10x^6y^4 + 10x^4y^6 + 5x^2y^8 + y^{10}. \end{aligned}$$

It expresses the weight distribution of an existing code: the direct sum code $C \oplus C \oplus C \oplus C \oplus C$ where $C = \{00, 11\}$. We have $A_d = A_2 = 5$, $A_{d+1} = A_3 = 0$, $A_{d+2} = A_4 = 10$, $A_{d+3} = A_5 = 0$ and $a_0 = 1/9$, $a_1 = 0$, $a_2 = -4/63$, $a_3 = -8/63$, $a_4 = -12/63$. The equivalent condition for the Riemann hypothesis of $W_{5,2}(x, y)$ is that all the roots of $g(X) = 7X^4 - 60X^2 - 8X + 60$ are contained in the interval $[-2\sqrt{2}, 2\sqrt{2}]$. We can easily see that

$$\begin{aligned} &g(-2\sqrt{2}) = 28 + 16\sqrt{2} > 0, \quad g(-2) = -52 < 0, \\ &g(0) = 60 > 0, \quad g(2) = -84 < 0, \\ &g(2\sqrt{2}) = 28 - 16\sqrt{2} > 0 \end{aligned}$$

and that $W_{5,2}(x, y)$ satisfies the Riemann hypothesis. Note that $W_{5,2}(x, y)$ is not extremal at this degree (the extremal σ_2-invariant polynomial of degree ten is $x^{10} + 15x^6y^4 + 15x^4y^6 + y^{10}$). The polynomial $W_{5,2}(x, y)$ is one of rare examples of weight enumerators which are not extremal but satisfy the Riemann hypothesis.

§6. Some remarks and problems

In this section, we give some observations and future problems.

We can see from the results of the previous sections that the range of q for which the Riemann hypothesis is true becomes smaller as $\deg W(x,y)$ becomes larger. We show some results of numerical experiment for $W_{m,q}(x,y)$. In the following table, "RH true" means the range of m where the Riemann hypothesis for $W_{m,q}(x,y)$ seems to be true:

q	RH true
2	$2 \le m \le 5$
$\frac{3}{2}$	$2 \le m \le 8$
$\frac{11}{10}$	$2 \le m \le 36$
$\frac{21}{20}$	$2 \le m \le 71$
$\frac{4}{5}$	$2 \le m \le 29$
$\frac{1}{2}$	$2 \le m \le 5$

These numerical examples also support the above observation. We propose the following problem:

Conjecture 6.1. *For any $m \ge 2$, there exists q ($q \approx 1$) and $W_{m,q}(x,y)$ satisfies the Riemann hypothesis.*

Another observation is on the formulas which express $P(T)$ via A_i such as (3.10) and (3.12). We can conjecture the following:

Conjecture 6.2. *Let $W(x,y)$ be a σ_q-invariant polynomial of the form (1.3) and of genus g ($g \in \mathbf{N}$). We suppose that the zeta polynomial of $W(x,y)$ is of the form (2.4). Then we have*

$$a_0 = \frac{1}{q-1} \cdot \frac{1}{\binom{n}{d}} A_d$$

and for $1 \le k \le g-1$, a_k is given by

$$\begin{aligned} a_k = \frac{1}{q-1} \Bigg(& \frac{(d)_{k-1}(d+k-1-kq)}{k!\binom{n}{d}} A_d \\ & + \frac{(d+1)_{k-2}(d+k-1-(k-1)q)}{(k-1)!\binom{n}{d+1}} A_{d+1} + \cdots \\ & + \frac{(d+k-1)_0(d+k-1-1\cdot q)}{1!\binom{n}{d+k-1}} A_{d+k-1} + \frac{1}{\binom{n}{d+k}} A_{d+k} \Bigg), \end{aligned} \tag{6.1}$$

where $(a)_j = a(a+1)(a+2)\cdots(a+j-1)$ if $j > 0$ and $(a)_0 = 1$.

If it is true, we will be able to prove that $W_{6,q}(x,y)$ $(g=5)$ satisfies the Riemann hypothesis if and only if all the roots of the polynomial

$$21X^5+21(q-2)X^4-7(20q-8)X^3-7(11q^2-28q+8)X^2 \\ +(187q^2-164q+40)X+33q^3-110q^2+72q-16$$

are contained in $[-2\sqrt{q}, 2\sqrt{q}]$. Some numerical experiments suggest that the equivalent condition will be $q_3 \le q \le q_4$, where $q_4 \approx 1.8045$ and q_3 exists in the interval $(0.515, 0.516)$. The range of q seems to be smaller than the case $g=4$.

In the case of $q=2$, We have long noticed, at least numerically, that $W_{m,2}(x,y)=(x^2+y^2)^m$ seems to satisfy the Riemann hypothesis only when $m=2,3,4,5$ (see [1]), but we have not known the reason. The calculation of this article may suggest a partial reason for this phenomenon. We conclude the article with the following problem:

Conjecture 6.3. *The σ_2-invariant polynomial $W_{m,2}(x,y)=(x^2+y^2)^m$ satisfies the Riemann hypothesis if and only if $m=2,3,4,5$.*

Acknowledgments

The first named author is supported by JSPS KAKENHI Grant Number JP20K03524. The authors would like to express their sincere gratitude to the referee and the editors for giving them pieces of valuable advice and for careful reading of the manuscript.

References

[1] K. Chinen: Zeta functions for linear codes and an analogue of the Riemann hypothesis — a survey of Iwan Duursma's work and a generalization (in Japanese), in Proceedings of Sendai Mini Symposium on Number Theory and Combinatorial Theory 2004, 31-44, published in March 2005, `http://staff.miyakyo-u.ac.jp/~taya/sendaiNC/2004/program.html`

[2] K. Chinen: Zeta functions for formal weight enumerators and the extremal property, Proc. Japan Acad. 81 Ser. A. (2005), 168-173.

[3] K. Chinen: An abundance of invariant polynomials satisfying the Riemann hypothesis, Discrete Math. 308 (2008), 6426-6440.

[4] K. Chinen: Divisible formal weight enumerators and extremal polynomials not satisfying the Riemann hypothesis, Discrete Math. 342 (2019), 111601.

[5] K. Chinen: Extremal invariant polynomials not satisfying the Riemann hypothesis, Appl. Algebra Engrg. Comm. Comput. 30 No.4 (2019), 275-284.

[6] K. Chinen: On some families of certain divisible polynomials and their zeta functions, Tokyo J. Math. 43 No. 1 (2020), 1-23.

[7] I. Duursma: Weight distribution of geometric Goppa codes, Trans. Amer. Math. Soc. 351, No.9 (1999), 3609-3639.

[8] I. Duursma: From weight enumerators to zeta functions, Discrete Appl. Math. 111 (2001), 55-73.

[9] I. Duursma: A Riemann hypothesis analogue for self-dual codes, DIMACS series in Discrete Math. and Theoretical Computer Science 56 (2001), 115-124.

[10] I. Duursma: Extremal weight enumerators and ultraspherical polynomials, Discrete Math. 268, No.1-3 (2003), 103-127.

[11] F. J. MacWilliams, N. J. A. Sloane: The Theory of Error-Correcting Codes, North-Holland, Amsterdam, 1977.

[12] S. Nishimura: On a Riemann hypothesis analogue for selfdual weight enumerators of genus less than 3, Discrete Appl. Math. **156** (2008), 2352-2358.

[13] T. Okuda, Relation between zeta polynomials and differential operators on some invariant rings (in Japanese), RIMS Kôkyûroku Bessatsu B20 (2010), 57-69. See also RIMS Kôkyûroku 1593 (2008), 145-153.

[14] V. S. Pless, Introduction to the Theory of Error-Correcting Codes, third ed., John Wiley & Sons, NewYork, 1998.

[15] N. J. A. Sloane, Self-dual codes and lattices, in: Relations Between Combinatorics and Other Parts of Mathematics, Proceedings of Symposium on Pure Mathematics 34, Ohio State University, Columbus, OH, 1978, Amer. Math. Soc., Providence, RI, 1979, pp. 273-308.

Koji Chinen
Department of Mathematics, School of Science and Engineering, Kindai University
3-4-1, Kowakae, Higashi-Osaka, 577-8502 Japan.
E-mail: `chinen@math.kindai.ac.jp`

Yuki Imamura
interprism Inc.
Chuo-ku, Tokyo Tsukiji 2-1-11 Galicia Ginza East, 104-0045 Japan.
E-mail: `yuki-i-xyz@outlook.jp`