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Examples of 4-dimensional symplectic-Haantjes manifolds

Keiichi Kikuchi and Tsukasa Takeuchi

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Abstract. Symplectic-Haantjes manifolds are constructed for several Hamiltonian systems following P. Tempesta and G. Tondo [14], which yield the complete integrability of systems.

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§1. Introduction

Recently, certain ways of characterizing integrable systems with $(1,1)$-tensors have been investigated (cf.[2], [4], [5], [11], [12], [13], [16], etc.). On the other hand, according to [10], [14] and [15] established new method of using $(1,1)$-tensor field for the integrable system. P. Tempesta and G. Tondo [14] introduce a concept of symplectic-Haantjes manifolds or $\omega\mathcal{H}$ manifolds and Lenard-Haantjes chain to treat completely integrable Hamiltonian system by means of the Haantjes tensor [3]. For a (1,2)-tensor field L, the Haantjes tensor $\mathcal{H}_L$ is given by Definition 2.1 below. If $\mathcal{H}_L$ vanishes, the tensor is called a *Haantjes operator*. In [14], [15], Tempesta and Tondo showed that the existence of an $\omega\mathcal{H}$ manifold is a necessary and sufficient condition for a non-degenerate Hamiltonian system to be completely integrable. They showed an algorithm for solving the inverse problem, that is, for a given set of involutive functions, a Haantjes structure of the involutive functions is constructed by using Lenard-Haantjes chains.

We consider the system with respect to the hydrodynamic type on the orbit space with Hsiang-Lawson metric in the case of the Berger sphere (as in Section 3), via S^1-equivariant CMC-H (constant mean curvature H) immersion and the corresponding Lagrangian formalism [7], [8], [9]. In Section 4, we construct of symplectic-Haantjes manifold for a Hamiltonian system of Section 3.

In Section 5, we construct $\omega\mathcal{H}$ manifolds of Hamiltonian systems of geodesic flow of two-dimensional Minkowski space. [13] showed complete integrability of the geodesic flow of the Minkowski metric using a $(1,1)$ tensor, however we construct the complete integrability of two-dimensional case by obtaining different $(1,1)$ tensors from [13] in this paper. In this Section, we revise [6] and add some results.

In Sections 3 and 4, we construct a geometrical example of 4-dimensional symplectic Haantjes manifold.

§2. Haantjes operator, symplectic-Haantjes manifold, Lenard-Haantjes chain

In this Section, we recall basic concepts of Haantjes operators, symplectic-Haantjes manifolds and Lenard-Haantjes chains (see for details, e.g., [14]).

Let M be a differentiable manifold and $L : TM \to TM$ be a $(1,1)$ tensor field, i.e., a field of linear operators on the tangent space at each point of M.

Definition 2.1. *The Nijenhuis torsion of L is the skew-symmetric* $(1,2)$ *tensor field defined by*

$$\mathcal{N}_L(X,Y) = L^2[X,Y] + [LX,LY] - L\left([X,LY] + [LX,Y]\right)$$

and the Haantjes tensor associated with L is the $(1,2)$ *tensor field defined by*

$$\mathcal{H}_L(X,Y) = L^2\mathcal{N}_L(X,Y) + \mathcal{N}_L(LX,LY) - L\left(\mathcal{N}_L(X,LY) + \mathcal{N}_L(LX,Y)\right)$$

where $X, Y \in TM$ *and* $[\ ,\ ]$ *denotes the commutator of two vector fields.*

In local coordinates $x = (x_1, \cdots, x_n)$, the Nijenhuis torsion and the Haantjes tensor can be written in the form

$$(\mathcal{N}_L)^i_{jk} = \sum_{\alpha=1}^{n} \left(\frac{\partial L^i_k}{\partial x^\alpha} L^\alpha_j - \frac{\partial L^i_j}{\partial x^\alpha} L^\alpha_k + \left(\frac{\partial L^\alpha_j}{\partial x^k} - \frac{\partial L^\alpha_k}{\partial x^j} \right) L^i_\alpha \right)$$

and

(2.1)

$$(\mathcal{H}_L)^i_{jk} = \sum_{\alpha,\beta=1}^{n} \left\{ L^i_\alpha (L^\alpha_\beta (\mathcal{N}_L)^\beta_{jk} - (\mathcal{N}_L)^\alpha_{\beta k} L^\beta_j) + ((\mathcal{N}_L)^i_{\alpha\beta} L^\alpha_j - L^i_\alpha (\mathcal{N}_L)^\alpha_{j\beta}) L^\beta_k \right\},$$

respectively.

Tempesta and Tondo [14] remarks that the skew-symmetry of the Nijenhuis torsion implies that of the Haantjes tensor.

Definition 2.2. *A (1,1)-tensor is called Haantjes operator when its Haantjes tensor vanishes.*

Now they introduce a concept of symplectic-Haantjes manifold ($\omega\mathcal{H}$ manifold) in [14]. They can formulate the theory of Hamiltonian integrable systems naturally by means of symplectic-Haantjes manifolds.

Definition 2.3. *An $\omega\mathcal{H}$ manifold $(M, \omega, K_0, K_1, \cdots, K_{n-1})$ is a $2n$-dimension symplectic manifold M, endowed with n endomorphisms of TM*

$$K_\alpha : TM \mapsto TM, \quad \alpha = 0, \cdots, n-1,$$

which satisfy the following conditions :

- $K_0 = I$.
- *Their Haantjes tensor vanishes identically, i.e.*

$$\mathcal{H}_{K_\alpha}(X, Y) = 0, \quad \forall X, Y \in TM, \quad \alpha = 0, \cdots, n-1.$$

- *The endomorphisms are compatible with ω (or equivalently, with the corresponding symplectic operator $\Omega := \omega^\flat$) :*

$$K_\alpha^T \Omega = \Omega K_\alpha, \quad \alpha = 0, \cdots, n-1,$$

  that is, the operators ΩK_α are skew symmetric.

- *The endomorphisms are commuting each other, i.e. they form a commutative ring $\mathcal{K}$:*

$$K_\alpha K_\beta = K_\beta K_\alpha, \quad \alpha = 0, \cdots, n-1,$$

  and also generate a module over the ring of smooth functions on M :

$$\mathcal{H}_{\left(\sum_{\alpha=0}^{n-1} a_\alpha(\boldsymbol{x}) K_\alpha\right)}(X, Y) = 0, \quad \forall X, Y \in TM,$$

  where $a_\alpha(x)$ are arbitrary smooth functions on M.

The $(n+1)$-ple $(\omega, K_0, K_1, \cdots, K_{n-1})$ will be called the $\omega\mathcal{H}$ structure associated with the $\omega\mathcal{H}$ manifold and $\mathcal{K}$ the Haantjes module (ring).

By using the Haantjes operators, we can generalize the notion of integrability, which is called a Lenard-Haantjes chain.

Definition 2.4. *Let $(M, \omega, K_0, K_1, \cdots, K_{n-1})$ be a $2n$-dimensional $\omega\mathcal{H}$ manifold and $\{\mathcal{H}_j\}_{1\le j\le n}$ be n independent functions which satisfy the following relations :*

$$d\mathcal{H}_j = K_{j-1}^T d\mathcal{H},\ \ j = 1, \cdots, n,\ \ \mathcal{H} := \mathcal{H}_1.$$

The functions $\{\mathcal{H}_j\}_{1\le j\le n}$ are called a Lenard-Haantjes chain generated by the function $\mathcal{H}$.

Let us consider Hamiltonian systems with two degrees of freedom. In [14], Tempesta and Tondo proposed a general procedure to compute a Haantjes operator adapted to the Lenard-Haantjes chain formed by two integrals of motion in involution. Let (M, ω) be a four dimentional symplectic manifold. They searched for a Haantjes operator K whose minimal polynomial should be of degree two, namely, the maximum degree allowed by their assumptions:

$$m_K(x) = x^2 - c_1 x - c_2,$$

where c_1 and c_2 are functions on M.

We construct the Haantjes operator K according to the conditions in [14].

$$K^T\Omega = \Omega K, \tag{2.2}$$

$$K^T d\mathcal{H} = d\mathcal{H}_2, \tag{2.3}$$

$$(K^T)^2 d\mathcal{H} = (c_1 K^T + c_2 I) d\mathcal{H}, \tag{2.4}$$

$$\mathcal{H}_K(X, Y) = 0, \quad \forall X, Y \in TM, \tag{2.5}$$

where $\Omega = \omega^\flat$ and I denotes the identity operator.

§3. Berger sphere, Hsiang-Lawson metric, Lagrangian

Let $S^3 \subset \mathbb{C}\times\mathbb{C}$ be the unit 3-sphere. The following metric g_β $(\beta > -1)$ on S^3 is called to be the Berger metric :

$$(g_\beta)_z(v, w) = \langle v, w\rangle + \beta \langle v, iz\rangle \langle w, iz\rangle\,,$$

where $v = (v_1, v_2), w = (w_1, w_2) \in T_zS^3$ and $\langle v, w\rangle = \mathrm{Re}(v\overline{w})$.

Then $S^3_\beta := (S^3, g_\beta)$ is called to be the Berger sphere $(\beta > -1)$. X denotes the orbit space via g_β-isometric S^1-action $r_t : S^3 \to S^3$ as follows :

$$r_t(z) = (z_1, e^{it}z_2),\ \ z = (z_1, z_2) \in S^3.$$

As the parameterization of X we use the following map :

$$(\theta, \phi) \to (e^{i\phi}\cos\theta, \sin\theta),\ \ 0 \le \phi \le 2\pi,\ \ 0 \le \theta \le \frac{\pi}{2}.$$

Let $X^\circ := X\backslash(\partial X \cup \{pole\})$. For a curve $\gamma : J \subset \mathbb{R} \to X^\circ$, we consider S^1-equivariant map $\mu : N \to S^3_\beta$ such that $\gamma \circ \pi = \sigma \circ \mu$, where $N = \gamma^{-1}(S^3_\beta) := \{(z, y) \in J \times S^3_\beta \,|\, \gamma(z) = \sigma(y)\}$ is the induced bundle, not an inverse map, $\pi : N \to J$ and $\sigma : S^3_\beta \to X^\circ$ are Riemannian submersions. This commutative diagram is called the pull-back construction [1], [7]. Throughout the paper, we assume that μ is an S^1-equivariant CMC-H (constant mean curvature H) immersion. Then the orbital metric h_β on X° is given by

$$h_\beta : ds^2 = d\theta^2 + \frac{(1+\beta)\cos^2\theta}{1+\beta\sin^2\theta} d\phi^2. \tag{3.1}$$

The volume function V of orbits and the Hsiang-Lawson metric $\hat{h}_\beta = V^2 h_\beta$ on X° are given as follows:

$$V = 2\pi \sin\theta\sqrt{1+\beta\sin^2\theta}, \quad \hat{h}_\beta = (\hat{h}_\beta)_1 d\theta^2 + (\hat{h}_\beta)_2 d\phi^2 \tag{3.2}$$

where $(\hat{h}_\beta)_1 = 4\pi^2\sin^2\theta(1+\beta\sin^2\theta)$, $(\hat{h}_\beta)_2 = 4\pi^2(1+\beta)\sin^2\theta\cos^2\theta$.

$\tau(\gamma) = \nabla_{\gamma'}\gamma'$ and $\hat{\tau}(\gamma) = \hat{\nabla}_{\gamma'}\gamma'$ stand for the tension fields of $\gamma = \gamma(s)$ with respect to the metric h_β and $\hat{h}_\beta$, respectively, where s is the arc-length parameter with respect to the orbital metric h_β.

On the orbit space (X°, h_β), the velocity vector field of a curve $\gamma(s) = (\theta(s), \phi(s))$ is given by the following component functions :

$$\theta'(s) = \cos\lambda(s), \quad \phi'(s) = \frac{\sqrt{1+\beta\sin^2\theta(s)}\sin\lambda(s)}{\sqrt{1+\beta}\cos\theta(s)}, \tag{3.3}$$

where $\lambda(s)$ stands for an auxiliary function with variable s, then (3.3) can be obtained by using $\sin^2\lambda(s) + \cos^2\lambda(s) = 1$ and (3.1). Then, using the conformal transformation of the metric, we have the following formula :

$$h_\beta(\tau(\gamma), \eta) - \eta(\log V) = h_\beta(\hat{\tau}(\gamma), \eta) = 2H, \tag{3.4}$$

where η denotes the unit normal vector field to γ :

$$\eta(s) = -\sin\lambda(s)\frac{\partial}{\partial\theta} + \frac{\sqrt{1+\beta\sin^2\theta(s)}\cos\lambda(s)}{\sqrt{1+\beta}\cos\theta(s)}\frac{\partial}{\partial\phi}.$$

In the following, we consider the motion of a particle as time parameter s on the orbit space X° with the Hsiang-Lawson metric $\hat{h}_\beta$. In general, this motion has the Lagrangian

$$\mathcal{L} = \frac{1}{2}\left\{(\hat{h}_\beta)_1(\theta')^2 + (\hat{h}_\beta)_2(\phi')^2\right\} - G(\theta, \phi),$$

where $G = G(\theta, \phi)$ called as a potential function is a smooth function on the configuration space and θ', ϕ' stand for the derivatives of $\theta = \theta(s)$, $\phi = \phi(s)$ by s, respectively.

Then we can regard the Lagrangian $\mathcal{L}$ as a smooth function on the tangent bundle TX° of X°.

Via the canonical momenta p_1, p_2 conjugate to θ, ϕ :

$$p_1 = \frac{\partial \mathcal{L}}{\partial \theta'} = (\hat{h}_\beta)_1 \theta' = (\hat{h}_\beta)_1 \cos\lambda, \tag{3.5}$$

$$p_2 = \frac{\partial \mathcal{L}}{\partial \phi'} = (\hat{h}_\beta)_2 \phi' = (\hat{h}_\beta)_2 \frac{\sqrt{1+\beta\sin^2\theta}}{\sqrt{1+\beta}\cos\theta}\sin\lambda, \tag{3.6}$$

we have the Hamiltonian

$$\mathcal{H} = \theta' p_1 + \phi' p_2 - \mathcal{L} = \frac{1}{2}\left(\hat{h}^1_\beta p_1^2 + \hat{h}^2_\beta p_2^2\right) + G(\theta, \phi),$$

where $\hat{h}^1_\beta$ and $\hat{h}^2_\beta$ are the inverse of $(\hat{h}_\beta)_1$ and $(\hat{h}_\beta)_2$, respectively.

On the orbit space $(X^\circ, \hat{h}_\beta)$, we consider the system of hydrodynamic type as follows. Assume that smooth functions ν^1 and ν^2 on X° do not depend on ϕ. Moreover, let either $\nu^1 = \nu^1(\theta)$ or $\nu^2 = \nu^2(\theta)$ be nonzero-valued as a function of θ only. Then we can consider the functions $\mathcal{H}$ and $\mathcal{H}_2$ on the phase space :

$$\mathcal{H} = \frac{1}{2}\left(\hat{h}^1_\beta p_1^2 + \hat{h}^2_\beta p_2^2\right) + G(\theta, \phi),$$
$$\mathcal{H}_2 = \frac{1}{2}\left(\nu^1 \hat{h}^1_\beta p_1^2 + \nu^2 \hat{h}^2_\beta p_2^2\right),$$

where we assume that $\mathcal{H}$ and $\mathcal{H}_2$ are in involution.

The Poisson bracket of $\mathcal{H}$ and $\mathcal{H}_2$ is calculated as follows :

$$\begin{aligned}\{\mathcal{H}_2, \mathcal{H}\} = &\left\{\frac{1}{2}\left(\left(\frac{\partial}{\partial\theta}\hat{h}^1_\beta\right)p_1^2 + \left(\frac{\partial}{\partial\theta}\hat{h}^2_\beta\right)p_2^2\right) + \frac{\partial G}{\partial \theta}\right\}\nu^1 \hat{h}^1_\beta p_1 \\ &+ \frac{\partial G}{\partial\phi}\nu^2\hat{h}^2_\beta p_2 - \frac{1}{2}\hat{h}^1_\beta p_1\left\{\frac{\partial}{\partial\theta}(\nu^1\hat{h}^1_\beta)p_1^2 + \frac{\partial}{\partial\theta}(\nu^2\hat{h}^2_\beta)p_2^2\right\},\end{aligned}$$

since $\hat{h}^1_\beta$ and $\hat{h}^2_\beta$ depend only on the variable θ. Hereafter, we consider the case $\nu^1 = 0$. Then we have

$$\{\mathcal{H}_2, \mathcal{H}\} = \frac{\partial G}{\partial\phi}\nu^2\hat{h}^2_\beta p_2 - \frac{1}{2}\hat{h}^1_\beta\frac{\partial}{\partial\theta}(\nu^2\hat{h}^2_\beta)p_1 p_2^2.$$

Thus $\{\mathcal{H}_2, \mathcal{H}\} = 0$ implies that

$$\frac{\partial G}{\partial\phi}\nu^2\hat{h}^2_\beta = 0, \ \ \hat{h}^1_\beta\frac{\partial}{\partial\theta}(\nu^2\hat{h}^2_\beta) = 0,$$

from which, we obtain that $\nu^2 = k(\hat{h}_\beta)_2$ (k is nonzero constant). Then the potential function G depends only on the variable θ.

§4. Construction of symplectic-Haantjes manifold for a Hamiltonian system

Assume that $\nu^1 = 0$, $\nu^2 = k(\hat{h}_\beta)_2$ (k is nonzero constant) and the potential function G depends only on θ. Then we can consider the functions

$$\mathcal{H} = \frac{1}{2}\left(\hat{h}^1_\beta p_1^2 + \hat{h}^2_\beta p_2^2\right) + G(\theta),$$
$$\mathcal{H}_2 = \frac{1}{2} k p_2^2,$$

where we assume that λ is a suitable function of θ as in (3.5) and (3.6), $\mathcal{H}$ and $\mathcal{H}_2$ are in involution, k is nonzero constant. Using above functions, we construct a Haantjes operator.

We put a 4-dimensional square matrix

$$K = (K^i_j)$$
$$= \begin{pmatrix} a(\theta) & b(\theta) & 0 & b(\theta) \\ c(\theta, p_1, p_2) & a(\theta) & -b(\theta) & 0 \\ 0 & c(\theta, p_1, p_2) & a(\theta) & c(\theta, p_1, p_2) \\ -c(\theta, p_1, p_2) & 0 & b(\theta) & a(\theta) \end{pmatrix} = \begin{pmatrix} a & b & 0 & b \\ c & a & -b & 0 \\ 0 & c & a & c \\ -c & 0 & b & a \end{pmatrix},$$

where $a = k(\hat{h}_\beta)_2$, $b = b(\theta)$, $c = -k\hat{h}^1_\beta(\hat{h}_\beta)^2_2 p_1 p_2^{-1}$.

This function matrix K satisfies conditions (2.2), (2.3), and the potential function G is given such that the following formula is satisfied:

$$\frac{\partial G}{\partial \theta} = -\frac{1}{2}\left(\frac{\partial \hat{h}^1_\beta}{\partial \theta} p_1^2 + \frac{\partial \hat{h}^2_\beta}{\partial \theta} p_2^2\right) + \frac{1}{k} b(\hat{h}^2_\beta)^2 p_2,$$

where, from (3.5) and (3.6), G can be regarded as a function of θ, under the assumption that λ is a suitable function of θ. Then, minimal polynomial of K is

$$m_K(x) = x^2 - 2ax + a^2. \tag{4.1}$$

From (4.1), we put $c_1 = 2a$ and $c_2 = -a^2$, the condition (2.4) is satisfied. We set $x_1 = \theta$, $x_2 = \phi$, $x_3 = p_1$, $x_4 = p_2$ and $K\left(\frac{\partial}{\partial x_j}\right) = \sum_{i=1}^{4} K^i_j \frac{\partial}{\partial x_i}$ ($j =$

$1, 2, 3, 4)$. For example, we have, using the definition of the Nijenhuis torsion:

$$
\begin{aligned}
(\mathcal{N}_K)^1_{12} &= \sum_{\alpha=1}^{4} \frac{\partial b}{\partial x_\alpha} K^\alpha_1 - \sum_{\alpha=1}^{4} \frac{\partial a}{\partial x_\alpha} K^\alpha_2 \\
&\qquad + a\left(\frac{\partial a}{\partial x_2} - \frac{\partial b}{\partial x_1}\right) + b\left(\frac{\partial c}{\partial x_2} - \frac{\partial a}{\partial x_1}\right) - b\frac{\partial c}{\partial x_2} \\
&= b'a - a'b - ab' - ba' \\
&= -2a'b, \\
(\mathcal{N}_K)^2_{14} &= -\sum_{\alpha=1}^{4} \frac{\partial c}{\partial x_\alpha} K^\alpha_4 + c\left(\frac{\partial a}{\partial x_4} - \frac{\partial b}{\partial x_1}\right) + a\frac{\partial c}{\partial x_4} - b\left(-\frac{\partial c}{\partial x_1}\right) \\
&= -\frac{\partial c}{\partial \theta} b - \frac{\partial c}{\partial p_1} c - \frac{\partial c}{\partial p_2} a - b'c + a\frac{\partial c}{\partial p_2} + b\frac{\partial c}{\partial \theta} \\
&= -b'c - c\frac{\partial c}{\partial p_1},
\end{aligned}
$$

where $a' := \dfrac{\partial a}{\partial \theta}$, $b' := \dfrac{\partial b}{\partial \theta}$.

Then we get the following equations which calculate the components of a Nijenhuis torsion $\mathcal{N}_K = \mathcal{N}$:

$$
\begin{aligned}
&\mathcal{N}^1_{12} = -2a'b,\ \mathcal{N}^1_{14} = -2a'b,\ \mathcal{N}^2_{12} = -b'c - c\frac{\partial c}{\partial p_1},\ \mathcal{N}^2_{13} = a'b - b\frac{\partial c}{\partial p_2}, \\
&\mathcal{N}^2_{14} = -b'c - c\frac{\partial c}{\partial p_1},\ \mathcal{N}^2_{23} = -bb' - b\frac{\partial c}{\partial p_1},\ \mathcal{N}^2_{24} = -a'b - b\frac{\partial c}{\partial p_2}, \\
&\mathcal{N}^2_{34} = bb' + b\frac{\partial c}{\partial p_1},\ \mathcal{N}^3_{12} = -a'c - c\frac{\partial c}{\partial p_2},\ \mathcal{N}^3_{14} = -a'c - c\frac{\partial c}{\partial p_2}, \\
&\mathcal{N}^3_{23} = a'b - b\frac{\partial c}{\partial p_2},\ \mathcal{N}^3_{34} = -a'b + b\frac{\partial c}{\partial p_2},\ \mathcal{N}^4_{12} = b'c + c\frac{\partial c}{\partial p_1}, \\
&\mathcal{N}^4_{13} = -a'b + b\frac{\partial c}{\partial p_2},\ \mathcal{N}^4_{14} = b'c + c\frac{\partial c}{\partial p_1},\ \mathcal{N}^4_{23} = b'b + b\frac{\partial c}{\partial p_1}, \\
&\mathcal{N}^4_{24} = a'b + b\frac{\partial c}{\partial p_2},\ \mathcal{N}^4_{34} = -bb' - b\frac{\partial c}{\partial p_1},
\end{aligned}
$$

where $\mathcal{N}^i_{jk} = 0$ (otherwise, except skew-symmetric ones with the above formulas). From (2.1), we have

$$
\begin{aligned}
(4.2)\qquad (\mathcal{H}_K)^i_{jk} = \sum_{\alpha,\beta=1}^{n} \Big\{ & K^i_\alpha K^\alpha_\beta (\mathcal{N}_K)^\beta_{jk} + K^\alpha_j K^\beta_k (\mathcal{N}_K)^i_{\alpha\beta} \\
& + K^i_\alpha K^\beta_j (\mathcal{N}_K)^\alpha_{k\beta} + K^i_\alpha K^\beta_k (\mathcal{N}_K)^\alpha_{\beta j} \Big\}.
\end{aligned}
$$

For example, we have, from the components of K and Nijenhuis torsion :

$$
\begin{aligned}
&\sum_{\alpha,\beta=1}^{4} K^1_\alpha K^\alpha_\beta (\mathcal{N}_K)^\beta_{12} \\
&\quad = \sum_{\alpha=1}^{4} K^1_\alpha \left(K^\alpha_1 (\mathcal{N}_K)^1_{12} + K^\alpha_2 (\mathcal{N}_K)^2_{12} + K^\alpha_3 (\mathcal{N}_K)^3_{12} + K^\alpha_4 (\mathcal{N}_K)^4_{12} \right) \\
&\quad = K^1_1 \left(K^1_1 (\mathcal{N}_K)^1_{12} + K^1_2 (\mathcal{N}_K)^2_{12} + K^1_4 (\mathcal{N}_K)^4_{12} \right) \\
&\qquad + K^1_2 \left(K^2_1 (\mathcal{N}_K)^1_{12} + K^2_2 (\mathcal{N}_K)^2_{12} + K^2_3 (\mathcal{N}_K)^3_{12} \right) \\
&\qquad + K^1_4 \left(K^4_1 (\mathcal{N}_K)^1_{12} + K^4_3 (\mathcal{N}_K)^3_{12} + K^4_4 (\mathcal{N}_K)^4_{12} \right) \\
&\quad = a^2(-2a'b) + ab \left(-b'c - c\frac{\partial c}{\partial p_1} \right) + ab \left(b'c + c\frac{\partial c}{\partial p_1} \right) \\
&\qquad + bc(-2a'b) + ba \left(-b'c - c\frac{\partial c}{\partial p_1} \right) + b(-b) \left(-a'c - c\frac{\partial c}{\partial p_2} \right) \\
&\qquad + b(-c)(-2a'b) + b^2 \left(-a'c - c\frac{\partial c}{\partial p_2} \right) + ba \left(b'c + c\frac{\partial c}{\partial p_1} \right) \\
&\quad = -2a^2 a' b.
\end{aligned}
$$

Similarly, we have

$$\sum_{\alpha,\beta=1}^{4} K^\alpha_1 K^\beta_2 (\mathcal{N}_K)^1_{\alpha\beta} = -2a^2 a' b.$$

Also, we have

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{4} K^1_\alpha K^\beta_1 (\mathcal{N}_K)^\alpha_{2\beta} &= K^1_1 K^1_1 (\mathcal{N}_K)^1_{21} + K^1_2 \left(K^1_1 (\mathcal{N}_K)^2_{21} + K^4_1 (\mathcal{N}_K)^2_{24} \right) \\
&\quad + K^1_4 \left(K^1_1 (\mathcal{N}_K)^4_{21} + K^4_1 (\mathcal{N}_K)^4_{24} \right) \\
&= 2a^2 a' b + ba \left(b'c + c\frac{\partial c}{\partial p_1} \right) + b(-c) \left(-a'b - b\frac{\partial c}{\partial p_2} \right) \\
&\quad + ba \left(-b'c - c\frac{\partial c}{\partial p_1} \right) + b(-c) \left(a'b + b\frac{\partial c}{\partial p_2} \right) \\
&= 2a^2 a' b.
\end{aligned}
$$

Similarly, we have

$$\sum_{\alpha,\beta=1}^{4} K^1_\alpha K^\beta_2 (\mathcal{N}_K)^\alpha_{\beta 1} = 2a^2 a' b.$$

Then we have

$$
\begin{aligned}
(\mathcal{H}_K)^1_{12} = \sum_{\alpha,\beta=1}^{n} \Big\{ & K^1_\alpha K^\alpha_\beta (\mathcal{N}_K)^\beta_{12} + K^\alpha_1 K^\beta_2 (\mathcal{N}_K)^1_{\alpha\beta} \\
& + K^1_\alpha K^\beta_1 (\mathcal{N}_K)^\alpha_{2\beta} + K^1_\alpha K^\beta_2 (\mathcal{N}_K)^\alpha_{\beta 1} \Big\} \\
= 0. &
\end{aligned}
$$

Secondly, we calculate $(\mathcal{H}_K)^2_{34}$.

$$
\begin{aligned}
& \sum_{\alpha,\beta=1}^{4} K^2_\alpha K^\alpha_\beta (\mathcal{N}_K)^\beta_{34} \\
&= K^2_1 \left(K^1_2 (\mathcal{N}_K)^2_{34} + K^1_4 (\mathcal{N}_K)^4_{34} \right) + K^2_2 \left(K^2_2 (\mathcal{N}_K)^2_{34} + K^2_3 (\mathcal{N}_K)^3_{34} \right) \\
&\quad + K^2_3 \left(K^3_2 (\mathcal{N}_K)^2_{34} + K^3_3 (\mathcal{N}_K)^3_{34} + K^3_4 (\mathcal{N}_K)^4_{34} \right) \\
&= cb \left(bb' + b\frac{\partial c}{\partial p_1} \right) + cb \left(-bb' - b\frac{\partial c}{\partial p_1} \right) + a^2 \left(bb' + b\frac{\partial c}{\partial p_1} \right) \\
&\quad + a(-b) \left(-a'b + b\frac{\partial c}{\partial p_2} \right) - bc \left(bb' + b\frac{\partial c}{\partial p_1} \right) - ba \left(-a'b + b\frac{\partial c}{\partial p_2} \right) \\
&\quad - bc \left(-bb' - b\frac{\partial c}{\partial p_1} \right) \\
&= a^2bb' + 2aa'b^2 + a^2b\frac{\partial c}{\partial p_1} - 2ab^2\frac{\partial c}{\partial p_2}.
\end{aligned}
$$

Similarly, we have

$$
\sum_{\alpha,\beta=1}^{4} K^\alpha_3 K^\beta_4 (\mathcal{N}_K)^2_{\alpha\beta} = a^2bb' + a^2b\frac{\partial c}{\partial p_1} + 2ab^2\frac{\partial c}{\partial p_2}.
$$

Also we have

$$
\begin{aligned}
\sum_{\alpha,\beta=1}^{4} K^2_\alpha K^\beta_3 (\mathcal{N}_K)^\alpha_{4\beta} &= K^2_2 \left(K^2_3 (\mathcal{N}_K)^2_{42} + K^3_3 (\mathcal{N}_K)^2_{43} \right) + K^2_3 K^3_3 (\mathcal{N}_K)^3_{43} \\
&= a(-b) \left(a'b + b\frac{\partial c}{\partial p_2} \right) + a^2 \left(-bb' - b\frac{\partial c}{\partial p_1} \right) \\
&\quad - ba \left(a'b - b\frac{\partial c}{\partial p_2} \right) \\
&= -2aa'b^2 - a^2bb' - a^2b\frac{\partial c}{\partial p_1}.
\end{aligned}
$$

Similarly, we have

$$\sum_{\alpha,\beta=1}^{4} K^2_\alpha K^\beta_4 (\mathcal{N}_K)^\alpha_{\beta 3} = -a^2 bb' - a^2 b \frac{\partial c}{\partial p_1}.$$

Consequently, we have

$$\begin{aligned}(\mathcal{H}_K)^2_{34} &= \sum_{\alpha,\beta=1}^{n} \Big\{ K^2_\alpha K^\alpha_\beta (\mathcal{N}_K)^\beta_{34} + K^\alpha_3 K^\beta_4 (\mathcal{N}_K)^2_{\alpha\beta} \\ &\qquad + K^2_\alpha K^\beta_3 (\mathcal{N}_K)^\alpha_{4\beta} + K^2_\alpha K^\beta_4 (\mathcal{N}_K)^\alpha_{\beta 3} \Big\} \\ &= 0.\end{aligned}$$

Similarly, we can prove that all components of Haantjes tensor vanish.

Thus the function matrix K satisfies the condition (2.5). Hence, we get a Haantjes operator K. Thus, we construct a symplectic Haantjes manifold $(T^*X^\circ, \omega, I, K)$.

§5. Other examples

In this Section, we construct $\omega\mathcal{H}$ manifolds in three cases. Let us consider the Hamiltonian system of the geodesic flow of 2-dimensional Minkowski space (cf. [13])

$$\mathcal{H} = \frac{1}{2}(-p_1^2 + p_2^2) \tag{5.1}$$

with an independent integral of motion

$$\mathcal{H}_2 = p_1. \tag{5.2}$$

Thus, $\mathcal{H}$ has Haantjes operator K in the following way.

We consider $\mathcal{G} = \mathcal{G}(q, p)$ which is functionally independent of $\mathcal{H}$. We assume the Poisson bracket $\{\mathcal{H}, \mathcal{G}\}$ vanishes, that is

$$\{\mathcal{H}, \mathcal{G}\} = \sum_{i=1}^{2} \left(\frac{\partial \mathcal{H}}{\partial p_i} \frac{\partial \mathcal{G}}{\partial q_i} - \frac{\partial \mathcal{H}}{\partial q_i} \frac{\partial \mathcal{G}}{\partial p_i} \right) = -p_1 \frac{\partial \mathcal{G}}{\partial q_1} + p_2 \frac{\partial \mathcal{G}}{\partial q_2} = 0.$$

Then we get the following condition:

$$p_1 \frac{\partial \mathcal{G}}{\partial q_1} = p_2 \frac{\partial \mathcal{G}}{\partial q_2}. \tag{5.3}$$

The function $\mathcal{H}_2$ in (5.2) as $\mathcal{G}$ satisfies the condition (5.3). Under the condition (2.2), we put a 4-dimensional square matrix

$$K = \begin{pmatrix} a_{11} & a_{12} & 0 & b_{12} \\ a_{21} & a_{22} & -b_{12} & 0 \\ 0 & c_{12} & a_{11} & a_{21} \\ -c_{12} & 0 & a_{12} & a_{22} \end{pmatrix}.$$

Then from the condition (2.3), we get the following relationship

$$\begin{aligned} a_{21} &= \frac{a_{11}p_1 + 1}{p_2}, \\ a_{22} &= \frac{a_{12}p_1}{p_2}, \\ b_{12} &= 0. \end{aligned}$$

Further, we put $c_1 = (a_{12}p_1 + a_{11}p_2)p_2^{-1}$ and $c_2 = a_{12}p_2^{-1}$, the condition (2.4) is satisfied. We set $x_1 = p_1$, $x_2 = p_2$, $x_3 = q_1$, $x_4 = q_2$. For example, we have, using the definition of the Nijenhuis torsion :

$$\begin{aligned} (\mathcal{N}_K)^1_{jk} &= \sum_{\alpha=1}^{4} \left(\frac{\partial K^\alpha_j}{\partial x_k} - \frac{\partial K^\alpha_k}{\partial x_j} \right) K^1_\alpha = \left(\frac{\partial K^2_j}{\partial x_k} - \frac{\partial K^2_k}{\partial x_j} \right) K^1_2, \\ (\mathcal{N}_K)^1_{12} &= \left(\frac{\partial K^2_1}{\partial x_2} - \frac{\partial K^2_2}{\partial x_1} \right) K^1_2 \\ &= \left\{ \frac{\partial}{\partial p_2} \left(\frac{a_{11}p_1 + 1}{p_2} \right) - \frac{\partial}{\partial p_1} \left(\frac{a_{12}p_1}{p_2} \right) \right\} a_{12} \\ &= a_{12} \left(-\frac{a_{11}p_1 + 1}{p_2^2} - \frac{a_{12}}{p_2} \right) \\ &= -\frac{a_{12}(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^2}, \\ (\mathcal{N}_K)^1_{23} &= \left(\frac{\partial K^2_2}{\partial x_3} - \frac{\partial K^2_3}{\partial x_2} \right) K^1_2 = \left\{ \frac{\partial}{\partial q_1} \left(\frac{a_{12}p_1}{p_2} \right) - 0 \right\} a_{12} = 0. \end{aligned}$$

Then we get the following equations which calculate the components of the

Nijenhuis torsion $\mathcal{N}_K = \mathcal{N}$:

$$\mathcal{N}^1_{12} = -\frac{a_{12}(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^2}, \ \mathcal{N}^2_{12} = -\frac{a_{12}p_1(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^3},$$
$$\mathcal{N}^3_{12} = -\frac{c_{12}(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^2}, \ \mathcal{N}^3_{14} = -\frac{(a_{11}p_1 + 1)(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^3},$$
$$\mathcal{N}^3_{24} = \frac{a_{11}(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^2}, \ \mathcal{N}^4_{14} = -\frac{a_{12}p_1(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^3},$$
$$\mathcal{N}^4_{24} = \frac{a_{12}(a_{11}p_1 + a_{12}p_2 + 1)}{p_2^2},$$

where $\mathcal{N}^i_{jk} = 0$ (otherwise, except skew-symmetric ones with the above formulas). From (4.2), for example, we have, from the components of K and Nijenhuis torsion :

$$\begin{aligned}
&\sum_{\alpha,\beta=1}^{4} K^3_\alpha K^\alpha_\beta (\mathcal{N}_K)^\beta_{12} \\
&\qquad = K^3_2 K^2_2 (\mathcal{N}_K)^2_{12} + K^1_1 \left(K^3_2 (\mathcal{N}_K)^2_{12} + K^1_1 (\mathcal{N}_K)^3_{12}\right) + K^2_1 K^1_2 (\mathcal{N}_K)^3_{12}, \\
&\sum_{\alpha,\beta=1}^{4} K^\alpha_1 K^\beta_2 (\mathcal{N}_K)^3_{\alpha\beta} \\
&\qquad = K^1_1 K^2_2 (\mathcal{N}_K)^3_{12} - K^2_1 K^1_2 (\mathcal{N}_K)^3_{12} + K^3_2 \left(K^1_2 (\mathcal{N}_K)^3_{14} + K^2_2 (\mathcal{N}_K)^3_{24}\right), \\
&\sum_{\alpha,\beta=1}^{4} K^3_\alpha K^\beta_1 (\mathcal{N}_K)^\alpha_{2\beta} \\
&\qquad = -K^3_2 \left(K^1_1 (\mathcal{N}_K)^2_{12} + K^3_2 (\mathcal{N}_K)^2_{24}\right) - K^1_1 \left(K^1_1 (\mathcal{N}_K)^3_{12} + K^3_2 (\mathcal{N}_K)^3_{24}\right) \\
&\qquad\quad - K^2_1 \left(K^1_1 (\mathcal{N}_K)^4_{12} + K^3_2 (\mathcal{N}_K)^4_{24}\right), \\
&\sum_{\alpha,\beta=1}^{4} K^3_\alpha K^\beta_2 (\mathcal{N}_K)^\alpha_{\beta 1} = -K^3_2 K^2_2 (\mathcal{N}_K)^2_{12} - K^1_1 K^2_2 (\mathcal{N}_K)^3_{12}.
\end{aligned}$$

From above equations, we get

$$\begin{aligned}
(\mathcal{H}_K)^3_{12} &= K^3_2 \left(K^1_2 (\mathcal{N}_K)^3_{14} + K^2_2 (\mathcal{N}_K)^3_{24}\right) - K^3_2 K^3_2 (\mathcal{N}_K)^2_{24} - K^1_1 K^3_2 (\mathcal{N}_K)^3_{24} \\
&\quad - K^2_1 \left(K^1_1 (\mathcal{N}_K)^4_{12} + K^3_2 (\mathcal{N}_K)^4_{24}\right) \\
&= -\frac{c_{12}(a_{11}p_1 + a_{12}p_2 + 1)(a_{11}^2 p_2 + a_{11}a_{12}p_1 + 2a_{12})}{p_2^3}.
\end{aligned}$$

If $(\mathcal{H}_K)^3_{12} = 0$, we can choose $a_{11} = 0$, $a_{12} = 0$ as one of the solutions.

Consequently, we have

$$(\mathcal{H}_K)^3_{12} = 0,$$

where $a_{11} = 0$, $a_{12} = 0$ and $\mathcal{N} \neq 0$. Similarly, we can prove that all components of Haantjes tensor vanish. Thus we get the Haantjes operator K :

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ p_2^{-1} & 0 & 0 & 0 \\ 0 & c_{12} & 0 & p_2^{-1} \\ -c_{12} & 0 & 0 & 0 \end{pmatrix},$$

where $\mathcal{N} \neq 0$ and $c_{12} \neq 0$. Hence we construct a $\omega\mathcal{H}$ manifold $(T^*\mathbb{R}^2, \omega, I, K)$.

Now, we put the function

$$\mathcal{H}_3 = k_1 p_1 + k_2 p_2. \tag{5.4}$$

This function (5.4) satisfies the condition (5.3). Then, by the same calculation as above, we construct the Haantjes operator K' :

$$K' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 5p_2^{-1} & k_2 p_2^{-1} & 0 & 0 \\ 0 & c_{12} & 0 & 5p_2^{-1} \\ -c_{12} & 0 & 0 & k_2 p_2^{-1} \end{pmatrix},$$

where $(\mathcal{N}_{K'})^2_{12} = (\mathcal{N}_{K'})^4_{14} = -5k_2 p_2^{-3}$, $(\mathcal{N}_{K'})^3_{12} = -5c_{12} p_2^{-2}$, $(\mathcal{N}_{K'})^3_{14} = -25 p_2^{-3}$ and otherwise except skew-symmetric ones with the above formulas. Hence, we get a different $\omega\mathcal{H}$ manifold $(T^*\mathbb{R}^2, \omega, I, K')$ from the one above.

Moreover, we consider

$$\mathcal{H}_4 = k p_1 p_2. \tag{5.5}$$

Similar to the above, this function (5.5) satisfies the condition (5.3). Then, if we put

$$K'' = \begin{pmatrix} 0 & -5 & 0 & 0 \\ 5 & 0 & 0 & 0 \\ 0 & c_{12} & 0 & 5 \\ -c_{12} & 0 & -5 & 0 \end{pmatrix},$$

K'' is a Haantjes operator. In fact, (5.1), (5.5) and K'' satisfy the conditions (2.2), (2.3), (2.4) and (2.5). Therefore, we construct an $\omega\mathcal{H}$ manifold $(T^*\mathbb{R}^2, \omega, I, K'')$.

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Keiichi Kikuchi
Department of Mathematics, Tokai University
Takanawa 2-3-23, Minatoku, Tokyo 108-8619, Japan
E-mail: kikuchi6@ozzio.jp

Tsukasa Takeuchi
Department of Mathematics, Tokyo University of Science
Kagurazaka 1-3, Shinjuku, Tokyo 162-8601, Japan
E-mail: takeuchi.tsukasa@rs.tus.ac.jp