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On the Hochschild cohomology ring of integral cyclic algebras

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Abstract. We determine the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ of an integral cyclic algebra Γ by giving a projective bimodule resolution of Γ and calculating cup product by means of a diagonal approximation map.

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§1. Introduction

Let $\mathbb{Z}$ be the ring of rational integers, p a prime integer and ζ a primitive p-th root of unity. We set $R = \mathbb{Z}[\zeta]$, $\omega_n = 1 - \zeta^n$ for any $n \in \mathbb{Z}$ and we denote $\omega_1 = 1 - \zeta$ by ω. We note that $pR = \omega^{p-1}R$ and that ω_k/ω_l is a unit in R for any k, l with $k, l \not\equiv 0 \mod p$.

Let a and b any nonzero rational integers and d the greatest common divisor of a and b. We let Γ be the integral cyclic R-algebra

$$\Gamma = \bigoplus_{0 \le k,l \le p-1} Ri^k j^l \quad \text{such that} \quad i^p = a,\ j^p = b,\ ji = \zeta ij.$$

In particular, in the case $p = 2$, Γ is just the generalized quaternion algebra over the ring of rational integers $\mathbb{Z}$.

In this paper, we consider the Hochschild cohomology group $HH^m(\Gamma) = \mathrm{Ext}^m_{\Gamma^e}(\Gamma, \Gamma)$ and the Hochschild cohomology ring $HH^*(\Gamma) = \bigoplus_{m \ge 0} HH^m(\Gamma)$ of Γ, where Γ^e denotes the enveloping algebra $\Gamma \otimes_R \Gamma^{op}$ of Γ. Unless otherwise stated, $\otimes$ denotes $\otimes_R$.

Although there is basically a small number of studies about the Hochschild cohomology for algebras over a commutative ring, the Hochschild cohomology of quaternion algebras or cyclic algebras appearing as orders in semisimple algebras over a field are studied in, for example, Hayami's works [1], [2], [3], [4], and [6], [7], [8] etc. However, Hochschild cohomology is an important tool for investigating module categories of algebras. In fact it is known that the Hochschild cohomology ring of an algebra over a commutative ring is an invariant under the equivalence of bounded derived categories as triangulated categories (cf. [5, Chapter 6]).

Concerning the integral cyclic algebra Γ above, in the case a is any nonzero integer and $b = -1$, the module structure of $HH^m(\Gamma)$ was already given in [6] using spectral sequence. In the case $p = 2$, a is any nonzero integer and $b = -1$, the ring structure of the Hochschild cohomology $HH^*(\Gamma)$ was also calculated in [8] using spectral sequence. In the case $p = 2$, a and b are any nonzero integers, that is, Γ is a generalized quaternion algebra, the ring structure of $HH^*(\Gamma)$ was determined in [2]. In this paper, we will generalize these results to the case of any prime number p.

In Section 2, we give a projective bimodule resolution of Γ, and applying the functor $\mathrm{Hom}_{\Gamma^e}(-, \Gamma)$ to the resolution, we have a double complex which gives the Hochschild cohomology group $HH^m(\Gamma)$. In Section 3, we determine the R-module structure of $HH^m(\Gamma)$ (Theorem 2):

$$HH^m(\Gamma) \cong \begin{cases} R \quad \text{for } m = 0, \\ (R/dpR)^{(m-1)/2} \oplus (R/d\omega R)^{(m+1)/2} \oplus (R/\omega R)^{(p^2-2)(m+1)/2} \\ \qquad\qquad \text{for } m \text{ odd}, \\ (R/dpR)^{(m-2)/2} \oplus (R/d\omega R)^{m/2} \oplus (R/\omega R)^{(p^2-2)m/2} \oplus (R/apR) \\ \qquad\qquad \oplus (R/bpR) \quad \text{for } m (\neq 0) \text{ even}. \end{cases}$$

In Section 4, we determine the ring structure of $HH^*(\Gamma)$. First, in Section 4.1, we define a 'diagonal approximation map' for the projective bimodule resolution of Γ in order to calculate the cup product on $HH^*(\Gamma)$. In Section 4.2, by calculating the cup products of generators of the Hochoschild cohomology groups $HH^m(\Gamma)$ for $m \geq 0$, we give a system of generators of the Hochschild cohomology ring $HH^*(\Gamma)$ as an R-algebra in Theorem 3. As a result, if $p \geq 3$, then the Hochschild cohomology ring $HH^*(\Gamma)$ is generated by the elements of $HH^1(\Gamma)$, $HH^2(\Gamma)$ and $HH^3(\Gamma)$. Furthermore, in that section, we present the relations that the generators of $HH^*(\Gamma)$ satisfy. In addition, we study the special case $|a| = |b| = 1$. In Section 5, we consider the ring structure of $HH^*(\Gamma)$ in the case $p = 2$.

§2. Projective resolution of Γ

First, we will give a Γ^e-projective resolution $(P_m, \Delta_m, \varepsilon)$ of Γ referring to [2]:

$$P_m = (\Gamma \otimes \Gamma)^{m+1} := (\Gamma \otimes \Gamma) \oplus (\Gamma \otimes \Gamma) \oplus \cdots \oplus (\Gamma \otimes \Gamma),$$

$$\Delta_m = \sum_{s+t=m} (\partial_{s,t} + \delta_{s,t}) \text{ for every integer } m \geq 0, \ \varepsilon \text{ is the augmentation.}$$

Here, for $s, t \geq 0$ with $m = s + t$, we define an element $c_{s,t} \in P_m$ by

$$c_{s,t} \ = \begin{cases} (0, \dots, 0, 1 \overset{t}{\check{\otimes}} 1, 0, \dots, 0) \in (\Gamma \otimes \Gamma)^{m+1} & \text{if } 0 \leq t \leq m, s+t = m, \\ (0, \dots, 0) & \text{otherwise.} \end{cases}$$

Then $P_m = \bigoplus_{s+t=m} \Gamma_{s,t}$, where we set $\Gamma_{s,t} := \Gamma c_{s,t} \Gamma$. We define Γ^e-homomorphisms $\partial_{s,t} : \Gamma_{s,t} \longrightarrow \Gamma_{s-1,t}$ and $\delta_{s,t} : \Gamma_{s,t} \longrightarrow \Gamma_{s,t-1}$ by

$$\partial_{s,t} = \begin{cases} \left.\begin{array}{l} \partial_1 : c_{s,t} \mapsto i c_{s-1,t} - c_{s-1,t} i \text{ for } s \text{ odd} \\ \partial_2 : c_{s,t} \mapsto \displaystyle\sum_{k=0}^{p-1} i^{p-1-k} c_{s-1,t} i^k \text{ for } s \text{ even} \end{array}\right\} \text{for } t \text{ even,} \\ \left.\begin{array}{l} \partial'_1 : c_{s,t} \mapsto i c_{s-1,t} - \zeta^{-1} c_{s-1,t} i \text{ for } s \text{ odd} \\ \partial'_2 : c_{s,t} \mapsto \displaystyle\sum_{k=0}^{p-1} \zeta^{-k} i^{p-1-k} c_{s-1,t} i^k \text{ for } s \text{ even} \end{array}\right\} \text{for } t \text{ odd,} \end{cases}$$

$$\delta_{s,t} = \begin{cases} \left.\begin{array}{l} \delta_1 : c_{s,t} \mapsto j c_{s,t-1} - c_{s,t-1} j \text{ for } t \text{ odd} \\ \delta_2 : c_{s,t} \mapsto \displaystyle\sum_{k=0}^{p-1} j^{p-1-k} c_{s,t-1} j^k \text{ for } t \text{ even} \end{array}\right\} \text{for } s \text{ even,} \\ \left.\begin{array}{l} \delta'_1 : c_{s,t} \mapsto (-1)(\zeta^{-1} j c_{s,t-1} - c_{s,t-1} j) \text{ for } t \text{ odd} \\ \delta'_2 : c_{s,t} \mapsto (-1) \displaystyle\sum_{k=0}^{p-1} \zeta^{-(p-1-k)} j^{p-1-k} c_{s,t-1} j^k \text{ for } t \text{ even} \end{array}\right\} \text{for } s \text{ odd.} \end{cases}$$

It is easy to see that the following equations hold:

$$\delta_{s,t-1} \circ \delta_{s,t} = 0, \quad \partial_{s-1,t} \circ \partial_{s,t} = 0, \quad \partial_{s,t-1} \circ \delta_{s,t} + \delta_{s-1,t} \circ \partial_{s,t} = 0.$$

Hence, setting each $\Gamma_{s,t}$ on each lattice point on the first quadrant, we have the following double complex:

$$(\Gamma_{s,t}, \partial_{s,t}, \delta_{s,t}) : \quad \begin{array}{ccccccc} \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta'_1} & & \downarrow{\scriptstyle \delta_1} & & \\ \Gamma_{0,2} & \xleftarrow[\partial_1]{} & \Gamma_{1,2} & \xleftarrow[\partial_2]{} & \Gamma_{2,2} & \xleftarrow[\partial_1]{} & \\ \downarrow{\scriptstyle \delta_2} & & \downarrow{\scriptstyle \delta'_2} & & \downarrow{\scriptstyle \delta_2} & & \\ \Gamma_{0,1} & \xleftarrow[\partial'_1]{} & \Gamma_{1,1} & \xleftarrow[\partial'_2]{} & \Gamma_{2,1} & \xleftarrow[\partial'_1]{} & \\ \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta'_1} & & \downarrow{\scriptstyle \delta_1} & & \\ \Gamma_{0,0} & \xleftarrow[\partial_1]{} & \Gamma_{1,0} & \xleftarrow[\partial_2]{} & \Gamma_{2,0} & \xleftarrow[\partial_1]{} & . \end{array}$$

Then, we show the Γ^e-projective resolution of Γ in the following proposition.

Proposition 1. *By taking the total complex of the above complex, we have the* Γ^e*-projective resolution of* Γ*:*

$$\cdots \xrightarrow{\Delta_3} P_2 \xrightarrow{\Delta_2} P_1 \xrightarrow{\Delta_1} P_0 \xrightarrow{\varepsilon} \Gamma \longrightarrow 0,$$

where $\Delta_m = \sum_{s+t=m}(\partial_{s,t} + \delta_{s,t})$ *and* ε *is the multiplication map.*

Proof. The exactness of the sequence is verified by giving a contracting homotopy. We define the following maps $T_{-1} : \Gamma \longrightarrow P_0$ and $T_m : P_m \longrightarrow P_{m+1}$ for $m \geq 0$ by

$$T_{-1}(\gamma) = c_{0,0}\gamma \quad (\gamma \in \Gamma);$$

for any even m,

$$T_m(i^u j^v c_{m,0}) = \begin{cases} 0 & \text{for } u = 0 \text{ and } v = 0, \\ \displaystyle\sum_{k=0}^{v-1} j^{v-1-k} c_{m,1} j^k & \text{for } u = 0 \text{ and } v \neq 0, \\ \displaystyle\sum_{k=0}^{u-1} i^{u-1-k} c_{m+1,0} i^k & \text{for } u \neq 0 \text{ and } v = 0, \\ \displaystyle\sum_{k=0}^{u-1} i^{u-1-k} c_{m+1,0} i^k j^v + i^u \sum_{k=0}^{v-1} j^{v-1-k} c_{m,1} j^k & \text{for } u \neq 0 \text{ and } v \neq 0, \end{cases}$$

$$T_m(i^u j^v c_{s,t}) = \begin{cases} 0 & \text{for } v = 0 \text{ and } t\,(\neq 0) \text{ even}, \\ i^u \displaystyle\sum_{k=0}^{v-1} j^{v-1-k} c_{s,t+1} j^k & \text{for } v \neq 0 \text{ and } t\,(\neq 0) \text{ even}, \\ 0 & \text{for } v \neq p-1 \text{ and } t \text{ odd}, \\ -\zeta^{-1} i^u c_{s,t+1} & \text{for } v = p-1 \text{ and } t \text{ odd}; \end{cases}$$

and for any odd m,

$$
T_m(i^u j^v c_{m,0}) = \begin{cases} 0 & \text{for } u \neq p-1 \text{ and } v = 0, \\ -\zeta i^u \displaystyle\sum_{k=0}^{v-1} j^{v-1-k} c_{m,1} j^k & \text{for } u \neq p-1 \text{ and } v \neq 0, \\ c_{m+1,0} & \text{for } u = p-1 \text{ and } v = 0, \\ \zeta^v c_{m+1,0} j^v - \zeta i^{p-1} \displaystyle\sum_{k=0}^{v-1} j^{v-1-k} c_{m,1} j^k & \text{for } u = p-1 \text{ and } v \neq 0, \end{cases}
$$

$$
T_m(i^u j^v c_{s,t}) = \begin{cases} 0 & \text{for } v = 0 \text{ and } t\,(\neq 0) \text{ even}, \\ -\zeta i^u \displaystyle\sum_{k=0}^{v-1} \zeta^k j^{v-1-k} c_{s,t+1} j^k & \text{for } v \neq 0 \text{ and } t\,(\neq 0) \text{ even}, \\ 0 & \text{for } v \neq p-1 \text{ and } t \text{ odd}, \\ i^u c_{s,t+1} & \text{for } v = p-1 \text{ and } t \text{ odd}. \end{cases}
$$

Then T_m's satisfy the equalities

$$
\begin{aligned} &\Delta_1 \circ T_0 + T_{-1} \circ \varepsilon = id_{P_0}, \\ &\Delta_{m+1} \circ T_m + T_{m-1} \circ \Delta_m = id_{P_m} \text{ for } m \geq 0. \end{aligned}
$$

That is, $\{T_m\}$ is a contracting homotopy. □

We remark that the exactness above is also verified by using spectral sequence.

Next, we will define a complex giving the Hochschild cohomology of Γ. Applying the functor $\mathrm{Hom}_{\Gamma^e}(-, \Gamma)$ to the double complex above, we have the following double complex on the third quadrant:

$$
\left(\Gamma^{s,t}, \partial^{s,t}, \delta^{s,t}\right) : \quad
\begin{array}{ccccccc}
\xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{2,0} & \xleftarrow[\widetilde{\partial_2}]{} & \Gamma^{1,0} & \xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{0,0} \\
& \downarrow \widetilde{\delta_1} & & \downarrow \widetilde{\delta_1'} & & \downarrow \widetilde{\delta_1} \\
\xleftarrow[\widetilde{\partial_1'}]{} & \Gamma^{2,1} & \xleftarrow[\widetilde{\partial_2'}]{} & \Gamma^{1,1} & \xleftarrow[\widetilde{\partial_1'}]{} & \Gamma^{0,1} \\
& \downarrow \widetilde{\delta_2} & & \downarrow \widetilde{\delta_2'} & & \downarrow \widetilde{\delta_2} \\
\xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{2,2} & \xleftarrow[\widetilde{\partial_2}]{} & \Gamma^{1,2} & \xleftarrow[\widetilde{\partial_1}]{} & \Gamma^{0,2} \\
& \downarrow \widetilde{\delta_1} & & \downarrow \widetilde{\delta_1'} & & \downarrow \widetilde{\delta_1}
\end{array}
$$

where we set $\Gamma^{s,t} := \mathrm{Hom}_{\Gamma^e}(\Gamma_{s,t}, \Gamma) \cong \Gamma$ and we identify $\Gamma^{s,t}$ with Γ. So $\partial^{s,t} := \mathrm{Hom}(\partial_{s+1,t}, \iota) : \Gamma^{s,t} \longrightarrow \Gamma^{s+1,t}$ and $\delta^{s,t} := \mathrm{Hom}(\iota, \delta_{s,t+1}) : \Gamma^{s,t} \longrightarrow \Gamma^{s,t+1}$ are explicitly given by

$$\partial^{s,t} = \begin{cases} \left.\begin{array}{l} \widetilde{\partial_1} : x \mapsto ix - xi \quad \text{for } s \text{ even} \\ \widetilde{\partial_2} : x \mapsto \displaystyle\sum_{k=0}^{p-1} i^{p-1-k} x i^k \quad \text{for } s \text{ odd} \end{array}\right\} & \text{for } t \text{ even}, \\ \left.\begin{array}{l} \widetilde{\partial'_1} : x \mapsto ix - \zeta^{-1} xi \quad \text{for } s \text{ even} \\ \widetilde{\partial'_2} : x \mapsto \displaystyle\sum_{k=0}^{p-1} \zeta^{-k} i^{p-1-k} x i^k \quad \text{for } s \text{ odd} \end{array}\right\} & \text{for } t \text{ odd}, \end{cases}$$

$$\delta^{s,t} = \begin{cases} \left.\begin{array}{l} \widetilde{\delta_1} : x \mapsto jx - xj \quad \text{for } t \text{ even} \\ \widetilde{\delta_2} : x \mapsto \displaystyle\sum_{k=0}^{p-1} j^{p-1-k} x j^k \quad \text{for } t \text{ odd} \end{array}\right\} & \text{for } s \text{ even}, \\ \left.\begin{array}{l} \widetilde{\delta'_1} : x \mapsto (-1)(\zeta^{-1} jx - xj) \quad \text{for } t \text{ even} \\ \widetilde{\delta'_2} : x \mapsto (-1) \displaystyle\sum_{k=0}^{p-1} \zeta^{-(p-1-k)} j^{p-1-k} x j^k \quad \text{for } t \text{ odd} \end{array}\right\} & \text{for } s \text{ odd} \end{cases}$$

for $x \in \Gamma^{s,t}$. Therefore, putting $Q^m := \bigoplus_{s+t=m} \Gamma^{s,t} \cong \Gamma^{m+1}$ and $\Delta^m := \sum_{s+t=m} (\partial^{s,t} + \delta^{s,t})$, we have the total complex of the above complex:

$$\cdots \xleftarrow{\Delta^2} Q^2 \xleftarrow{\Delta^1} Q^1 \xleftarrow{\Delta^0} Q^0 \longleftarrow 0.$$

§3. Module structure of $HH^m(\Gamma)$

In this section, we determine the module structure of $HH^m(\Gamma) = \mathrm{Ext}^m_{\Gamma^e}(\Gamma, \Gamma)$. First, we present any element of Γ by a matrix in $M_p(R)$. If x is any element in $\Gamma^{s,t}$, then there uniquely exist $x_{kl} \in R$ $(k, l = 1, 2, \dots, p)$ such that

$$x = \begin{pmatrix} 1 & i & \cdots & i^{p-1} \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1} & x_{p2} & \cdots & x_{pp} \end{pmatrix} \begin{pmatrix} 1 \\ j \\ \vdots \\ j^{p-1} \end{pmatrix}.$$

By corresponding $x \in \Gamma^{s,t}$ to the matrix $X = (x_{kl}) \in M_p(R)$ above, $\partial^{s,t}(X)$ and $\delta^{s,t}(X)$ are given by

$$\widetilde{\partial_1}(X) = \begin{pmatrix} 0 & a\omega x_{p2} & \cdots & a\omega_{p-1} x_{pp} \\ 0 & \omega x_{12} & \cdots & \omega_{p-1} x_{1p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \omega x_{p-12} & \cdots & \omega_{p-1} x_{p-1p} \end{pmatrix}, \quad \widetilde{\partial_2}(X) = \begin{pmatrix} apx_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ apx_{p1} & 0 & \cdots & 0 \\ px_{11} & 0 & \cdots & 0 \end{pmatrix},$$

$$\widetilde{\partial}'_1(X) = \begin{pmatrix} a\omega_{p-1}x_{p1} & 0 & a\omega x_{p3} & \cdots & a\omega_{p-2}x_{pp} \\ \omega_{p-1}x_{11} & 0 & \omega x_{13} & \cdots & \omega_{p-2}x_{1p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_{p-1}x_{p-11} & 0 & \omega x_{p-13} & \cdots & \omega_{p-2}x_{p-1p} \end{pmatrix},$$

$$\widetilde{\partial}'_2(X) = \begin{pmatrix} 0 & apx_{22} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & apx_{p2} & 0 & \cdots & 0 \\ 0 & px_{12} & 0 & \cdots & 0 \end{pmatrix};$$

$$\widetilde{\delta}_1(X) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ -b\omega x_{2p} & -\omega x_{21} & \cdots & -\omega x_{2p-1} \\ \vdots & \vdots & \ddots & \vdots \\ -b\omega_{p-1}x_{pp} & -\omega_{p-1}x_{p1} & \cdots & -\omega_{p-1}x_{pp-1} \end{pmatrix},$$

$$\widetilde{\delta}_2(X) = \begin{pmatrix} bpx_{12} & \cdots & bpx_{1p} & px_{11} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$\widetilde{\delta}'_1(X) = \begin{pmatrix} b\omega_{p-1}x_{1p} & \omega_{p-1}x_{11} & \cdots & \omega_{p-1}x_{1p-1} \\ 0 & 0 & \cdots & 0 \\ b\omega x_{3p} & \omega x_{31} & \cdots & \omega x_{3p-1} \\ \vdots & \vdots & \ddots & \vdots \\ b\omega_{p-2}x_{pp} & \omega_{p-2}x_{p1} & \cdots & \omega_{p-2}x_{pp-1} \end{pmatrix},$$

$$\widetilde{\delta}'_2(X) = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ -bpx_{22} & \cdots & -bpx_{2p} & -px_{21} \\ 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

For $s+t=m\,(s,t\geq 0)$, we define $c^{s,t}\in Q^m$ by

$$c^{s,t} = \begin{cases} (0,\dots,0,\overset{t}{\check{1}},0,\dots,0)\in Q^m=\Gamma^{m+1} & \text{if } 0\leq t\leq m, s+t=m, \\ (0,\dots,0) & \text{otherwise.} \end{cases}$$

Using above expressions, we obtain the R-module structure of the Hochschild cohomology group $HH^m(\Gamma)$. In fact, we directly calculate $\operatorname{Ker}\Delta^m$ and $\operatorname{Im}\Delta^{m-1}$. We present those R-modules only in the case m is even.

$$\operatorname{Ker}\Delta^m = \bigoplus_{t=0}^{m} Rc^{m-t,t} \oplus \bigoplus_{\substack{1\leq t<\leq m-1,\text{ odd;}\\ 2\leq k,l\leq p-1}} Ri^k j^l c^{m-t,t}$$

$$\oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 2\le l\le p-1}} Rj^l c^{m-t,t} \oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 2\le k\le p-1}} Ri^k c^{m-t,t}$$

$$\oplus \bigoplus_{\substack{0\le t\le m-2,\text{ even};\\ 0\le k'(\ne 1)\le p-1}} R\left(\frac{p}{\omega_{p-1+k'}} i^{p-1+k'} c^{m-t,t} + i^{k'} j c^{m-t-1,t+1}\right)$$

$$\oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 0\le l'(\ne 1)\le p-1}} R\left(ij^{l'} c^{m-t,t} + \frac{p}{\omega_{p-1+l'}} j^{p-1+l'} c^{m-t-1,t+1}\right),$$

$$\begin{aligned}\operatorname{Im}\Delta^{m-1} =& apRc^{m,0} \oplus \bigoplus_{1\le t\le m-1,\text{ odd}} d\omega R c^{m-t,t} \oplus \bigoplus_{2\le t\le m-2,\text{ even}} dpRc^{m-t,t}\\ &\oplus bpRc^{0,m} \oplus \bigoplus_{\substack{1\le t\le m-2,\text{ odd};\\ 2\le k,l\le p-1}} \omega R i^k j^l c^{m-t-1,t-1}\\ &\oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 2\le l\le p-1}} \omega Rj^l c^{m-t,t} \oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 2\le k\le p-1}} \omega Ri^k c^{m-t,t}\\ &\oplus \bigoplus_{\substack{0\le t\le m-2,\text{ even};\\ 0\le k'(\ne 1)\le p-1}} \omega R\left(\frac{p}{\omega_{p-1+k'}} i^{p-1+k'} c^{m-t,t} + i^{k'} j c^{m-t-1,t+1}\right)\\ &\oplus \bigoplus_{\substack{1\le t\le m-1,\text{ odd};\\ 0\le l'(\ne 1)\le p-1}} \omega R\left(ij^{l'} c^{m-t,t} + \frac{p}{\omega_{p-1+l'}} j^{p-1+l'} c^{m-t-1,t+1}\right).\end{aligned}$$

In the above calculation, we note that $\omega R = \omega_{p-1+k'} R$ for $0 \le k'(\ne 1) \le p-1$.

Theorem 2. *Let $\mathbb{Z}$ be the ring of rational integers, a, b any nonzero rational integers and d the greatest common divisor of a and b. Let p be a prime and ζ a primitive p-th root of unity. We set $R = \mathbb{Z}[\zeta]$ and put $\omega = 1 - \zeta$. Then the R-module structure of the Hochschild cohomology group of Γ is as follows:*

$$HH^m(\Gamma) \cong \begin{cases} R \quad \text{for } m = 0, \\ (R/dpR)^{(m-1)/2} \oplus (R/d\omega R)^{(m+1)/2} \oplus (R/\omega R)^{(p^2-2)(m+1)/2} \\ \qquad\qquad \text{for } m \text{ odd}, \\ (R/dpR)^{(m-2)/2} \oplus (R/d\omega R)^{m/2} \oplus (R/\omega R)^{(p^2-2)m/2} \\ \qquad\qquad \oplus (R/apR) \oplus (R/bpR) \quad \text{for } m(\ne 0) \text{ even}. \end{cases}$$

For the later use, we list the system of generators of each $HH^m(\Gamma)$ as an R-module represented by elements in $Q^m = \Gamma^{m+1}$ as follows, where we set $a' = a/d, b' = b/d$:

For $m = 1$,

$ij^l c^{1,0}$ for $1 \le l \le p-1$,
$i^k j c^{0,1}$ for $1 \le k \le p-1$,
$i^{k+1} j^l c^{1,0} - \dfrac{\omega_k}{\omega_l} i^k j^{l+1} c^{0,1}$ for $1 \le k, l \le p-1$ with $(k,l) \ne (p-1, p-1)$,
$a' j^{p-1} c^{1,0} - b' i^{p-1} c^{0,1}$.

For $m \ge 2$ even,
$c^{m-t,t}$ for $0 \le t \le m$,
$i^k j^l c^{m-t,t}$ for $2 \le k, l \le p-1$ and t odd,
$i^k c^{m-t,t}$ for $2 \le k \le p-1$ and t odd,
$j^l c^{m-t,t}$ for $2 \le l \le p-1$ and t odd,
$\dfrac{p}{\omega_{p-1+k}} i^{p-1+k} c^{m-t,t} + i^k j c^{m-t-1,t+1}$ for $0 \le k (\ne 1) \le p-1$ and t even,
$ij^l c^{m-t,t} + \dfrac{p}{\omega_{p-1+l}} j^{p-1+l} c^{m-t-1,t+1}$ for $0 \le l (\ne 1) \le p-1$ and t odd.

For $m \ge 3$ odd,
$ij^l c^{m-t,t}$ for $1 \le l \le p-1$ and t even,
$i^k j c^{m-t,t}$ for $1 \le k \le p-1$ and t odd,
$i^{k+1} j^l c^{m-t,t} - \omega_k/\omega_l i^k j^{l+1} c^{m-t-1,t+1}$
for $1 \le k, l \le p-1$ with $(k,l) \ne (p-1,p-1)$ and t even,
$a' j^{p-1} c^{m-t,t} - b' i^{p-1} c^{m-t-1,t+1}$ for t even,
$a' j c^{m-t,t} - b' i c^{m-t-1,t+1}$ for $0 \le t < m$ odd.

§4. The ring structure of $HH^*(\Gamma)$

In this section, we will determine the ring structure of $HH^*(\Gamma) = \bigoplus_{m\geq 0} HH^m(\Gamma)$.

4.1. Diagonal approximation and cup product

First, we define a map $\Phi_{s,t;s',t'} : \Gamma_{s+t,s'+t'} \longrightarrow \Gamma_{s,t} \otimes_\Gamma \Gamma_{s',t'}$ of Γ^e-modules by the map sending $c_{s+t,s'+t'}$ to

$$\begin{cases} \displaystyle\sum_{\substack{u+v+w=p-2,\\ u'+v'+w'=p-2}} \zeta^{(v+1)(v'+1)+2-uw'} i^u j^{u'} c_{s,t} i^v j^{v'} \otimes_\Gamma c_{s',t'} i^w j^{w'} \text{ for } s,t,s',t' \text{ odd},\\ -\zeta \displaystyle\sum_{u+v+w=p-2} \zeta^u i^u c_{s,t} i^v \otimes_\Gamma c_{s',t'} i^w \text{ for } s \text{ odd}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\ \displaystyle\sum_{u'+v'+w'=p-2} \zeta^{-u'} j^{u'} c_{s,t} j^{v'} \otimes_\Gamma c_{s',t'} j^{w'} \text{ for } s \text{ odd}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\ \displaystyle\sum_{u+v+w=p-2} i^u c_{s,t} i^v \otimes_\Gamma \zeta^{-w} c_{s',t'} i^w \text{ for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ odd},\\ -\zeta \displaystyle\sum_{u'+v'+w'=p-2} j^{u'} c_{s,t} j^{v'} \otimes_\Gamma \zeta^{w'} c_{s',t'} j^{w'} \text{ for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ odd},\\ \displaystyle\sum_{u+v+w=p-2} i^u c_{s,t} i^v \otimes_\Gamma c_{s',t'} i^w \text{ for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ even},\\ \displaystyle\sum_{u'+v'+w'=p-2} j^{u'} c_{s,t} j^{v'} \otimes_\Gamma c_{s',t'} j^{w'} \text{ for } s \text{ even}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\ -\zeta^{-1} c_{s,t} \otimes_\Gamma c_{s',t'} \text{ for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\ c_{s,t} \otimes_\Gamma c_{s',t'} \text{ otherwise.} \end{cases}$$

Then, $\Phi = \{\Phi_{s,t;s',t'}\}$ satisfies the following relations:

$$\begin{aligned} &\Phi_{s,t;s',t'} \circ \partial_{s+s'+1,t+t'} \\ &\qquad = \partial_{s+1,t} \otimes \iota \circ \Phi_{s+1,t;s',t'} + (-1)^{s+t} \iota \otimes \partial_{s'+1,t'} \circ \Phi_{s,t;s'+1,t'}, \\ &\Phi_{s,t;s',t'} \circ \delta_{s+s',t+t'+1} \\ &\qquad = \delta_{s,t+1} \otimes \iota \circ \Phi_{s,t+1;s',t'} + (-1)^{s+t} \iota \otimes \delta_{s',t'+1} \circ \Phi_{s,t;s',t'+1}, \\ &\quad \varepsilon \otimes \varepsilon \circ \Phi_{0,0;0,0} = \varepsilon. \end{aligned}$$

Therefore, $\Phi_{m,n} := \sum_{s+t=m,s'+t'=n} \Phi_{s,t;s't'}$ is a 'diagonal approximation', that is, this satisfies

$$\begin{aligned} &\Phi_{m,n} \circ \Delta_{m+n+1} = (\Delta_{m+1} \otimes \iota) \circ \Phi_{m+1,n} + (-1)^m (\iota \otimes \Delta_{n+1}) \circ \Phi_{m,n+1}, \\ &\quad (\varepsilon \otimes \varepsilon) \circ \Phi_{0,0} = \varepsilon. \end{aligned}$$

Using Φ, we define the cup product

$$HH^m(\Gamma) \otimes HH^n(\Gamma) \overset{\smile}{\longrightarrow} HH^{m+n}(\Gamma); \quad \alpha \otimes \beta \mapsto a \smile \beta$$

by

$$\alpha \smile \beta = (\alpha \otimes_\Gamma \beta) \circ \Phi_{s,t;s',t'} : \Gamma_{s+t,s'+t'} \to \Gamma_{s,t} \otimes_\Gamma \Gamma_{s',t'} \to \Gamma \otimes_\Gamma \Gamma = \Gamma.$$

for $\alpha \in \Gamma^{s,t}$ with $s+t = m$ and $\beta \in \Gamma^{s',t'}$ with $s'+t' = n$. Hence $\Gamma^{s,t} \otimes \Gamma^{s',t'} \stackrel{\smile}{\longrightarrow} \Gamma^{s+t,s'+t'}$ is explicitly presented by

$$\alpha \smile \beta =$$
$$\begin{cases} \sum_{\substack{u+v+w=p-2,\\ u'+v'+w'=p-2}} \zeta^{(v+1)(v'+1)+2-uw'} i^u j^{u'} \alpha i^v j^{v'} \beta i^w j^{w'} & \text{for } s,t,s',t' \text{ odd},\\ -\zeta \sum_{u+v+w=p-2} \zeta^u i^u \alpha i^v \beta i^w & \text{for } s \text{ odd}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\ \sum_{u'+v'+w'=p-2} \zeta^{-u'} j^{u'} \alpha j^{v'} \beta j^{w'} & \text{for } s \text{ odd}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\ \sum_{u+v+w=p-2} i^u \alpha i^v \zeta^{-w} \beta i^w & \text{for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ odd},\\ -\zeta \sum_{u'+v'+w'=p-2} j^{u'} \alpha j^{v'} \zeta^{w'} \beta j^{w'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ odd},\\ \sum_{u+v+w=p-2} i^u \alpha i^v \beta i^w & \text{for } s \text{ odd}, t \text{ even}, s' \text{ odd}, t' \text{ even},\\ \sum_{u'+v'+w'=p-2} j^{u'} \alpha j^{v'} \beta j^{w'} & \text{for } s \text{ even}, t \text{ odd}, s' \text{ even}, t' \text{ odd},\\ -\zeta^{-1} \alpha\beta & \text{for } s \text{ even}, t \text{ odd}, s' \text{ odd}, t' \text{ even},\\ \alpha\beta & \text{otherwise}. \end{cases}$$

for $\alpha \in \Gamma^{s,t}$ and $\beta \in \Gamma^{s',t'}$. In the above, we identify $\Gamma^{s,t}$ with Γ and so on. As long as there is no confusion, we often denote $\alpha \smile \beta$ by $\alpha\beta$ for simplicity. It is well known that the anti-commutativity $\alpha\beta = (-1)^{mn}\beta\alpha$ holds for $\alpha \in HH^m(\Gamma)$ and $\beta \in HH^n(\Gamma)$. That is, the Hochschild cohomology ring $HH^*(\Gamma)$ is a graded commutative ring.

4.2. Generators of $HH^*(\Gamma)$ as an R-algebra and the relations

In this subsection, we determine the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$ using cup product on generators of $HH^m(\Gamma)$. By the way, the ring structure of the Hochschild cohomology ring $HH^*(\Gamma)$ in the case $p = 2$ was already known in [2]. So, we mainly treat the case $p \geq 3$.

We denote the representatives of each element of $HH^m(\Gamma)$ by $(*, *, \ldots, *) \in Q^m = \Gamma^{m,0} \oplus \Gamma^{m-1,1} \oplus \cdots \oplus \Gamma^{0,m}$. Then, referring to Theorem 2, generators of $HH^m(\Gamma)$ for $m = 1, 2, 3$ as an R-module are as follows including the case $p = 2$:

Generators of $HH^1(\Gamma)$:

$$\sigma_l := (ij^l, 0) \quad \text{for } 1 \le l \le p-1,$$
$$\tau_k := (0, i^k j) \quad \text{for } 1 \le k \le p-1,$$
$$\theta_{k,l} := (i^{k+1}j^l, -\frac{\omega_k}{\omega_l} i^k j^{l+1}) \quad \text{for } 1 \le k, l \le p-1 \text{ with } (k,l) \neq (p-1, p-1),$$
$$\pi := (a' j^{p-1}, -b' i^{p-1}).$$

Generators of $HH^2(\Gamma)$:

$$\varphi := (1, 0, 0),$$
$$\psi := (0, 1, 0),$$
$$\chi := (0, 0, 1),$$
$$\rho_k := (\frac{p}{\omega_{p-1+k}} i^{p-1+k}, i^k j, 0) \quad \text{for } 0 \le k (\neq 1) \le p-1,$$
$$\eta_l := (0, ij^l, \frac{p}{\omega_{p-1+l}} j^{p-1+l}) \quad \text{for } 0 \le l (\neq 1) \le p-1,$$
$$\mu_{k,l} := (0, i^k j^l, 0) \quad \text{for } 0 \le k, l (\neq 1) \le p-1 \text{ with } (k,l) \neq (0,0).$$

Generators of $HH^3(\Gamma)$:

$$(ij^l, 0, 0, 0) \quad \text{for } 1 \le l \le p-1,$$
$$(0, i^k j, 0, 0) \quad \text{for } 1 \le k \le p-1,$$
$$(0, 0, ij^l, 0) \quad \text{for } 1 \le l \le p-1,$$
$$(0, 0, 0, i^k j) \quad \text{for } 1 \le k \le p-1,$$
$$(i^{k+1}j^l, -\frac{\omega_k}{\omega_l} i^k j^{l+1}, 0, 0) \quad \text{for } 1 \le k, l \le p-1 \text{ with } (k,l) \neq (p-1, p-1),$$
$$(0, 0, i^{k+1}j^l, -\frac{\omega_k}{\omega_l} i^k j^{l+1}) \quad \text{for } 1 \le k, l \le p-1 \text{ with } (k,l) \neq (p-1, p-1),$$
$$(a' j^{p-1}, -b' i^{p-1}, 0, 0),$$
$$(0, 0, a' j^{p-1}, -b' i^{p-1}),$$
$$\kappa := (0, a' j, -b' i, 0).$$

Let $x = (x^{m,0}, \ldots, x^{0,m}) \in HH^m(\Gamma)$. Then, it is easy to check that the elements $(x^{m,0}, \ldots, x^{0,m}, 0, 0)$ and $(0, 0, x^{m,0}, \ldots, x^{0,m}) \in HH^{m+2}(\Gamma)$ are given by $x\varphi$ and $x\chi$ respectively. In particular, if x is a generator, then $x\varphi$ and $x\chi$ are also generators. Therefore, we see that the generators of $HH^m(\Gamma)$ for any $m \ge 3$ except κ are given by the cup products of the generators above of $HH^1(\Gamma)$ and $HH^2(\Gamma)$ and $\kappa \in HH^3(\Gamma)$. On the other hand, the relation $\sigma_l\tau_k = \mu_{k+1,l+1}$ holds for $1 \leq k, l < p-1$.

Therefore we have the following main theorem.

Theorem 3. *Let p be an odd prime and a, b nonzero integers, and set $d = \gcd(a, b)$, $a' = a/d$, $b' = b/d$. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is the graded commutative ring generated by at most the following $p^2 + 4p - 3$ elements:*

$\sigma_l, \tau_k, \theta_{k',l'}, \pi \in HH^1(\Gamma)$ *for* $1 \leq k, k', l, l' \leq p-1$ *with* $(k', l') \neq (p-1, p-1)$,
$\varphi, \psi, \chi, \mu_{k,0}, \mu_{0,l}, \rho_{k'}, \eta_{l'} \in HH^2(\Gamma)$ *for* $2 \leq k, l \leq p-1, 0 \leq k', l'(\neq 1) \leq p-1$,
$\kappa \in HH^3(\Gamma)$.

The list of the relations of the generators above is as follows:
The relations in $HH^1(\Gamma)$:

$$\omega\tau_k = \omega\sigma_l = d\omega\pi = \omega\theta_{k',l'} = 0.$$

The relations in $HH^2(\Gamma)$:

$$ap\varphi = d\omega\psi = bp\chi = \omega\rho_{k'} = \omega\eta_{l'} = \omega\mu_{k,0} = \omega\mu_{0,l} = \pi\pi = 0.$$

$$\tau_{k'}\tau_k = \begin{cases} \frac{p}{\omega_k}\zeta^k ab\chi & \text{if } k + k' = p, \\ 0 & \text{if } k + k' \neq p. \end{cases}$$

$$\sigma_l\sigma_{l'} = \begin{cases} \frac{p}{\omega_l}\zeta^l ab\varphi & \text{if } l + l' = p, \\ 0 & \text{if } l + l' \neq p. \end{cases}$$

$$\tau_k\pi = \begin{cases} -\zeta^{-1}a'b\eta_0 & \text{if } k = 1, \\ -\zeta^{-1}a'b\mu_{k,0} & \text{if } 1 < k. \end{cases}$$

$$\sigma_l\pi = \begin{cases} -\zeta^{-1}b'a\rho_0 & \text{if } l = 1, \\ -\zeta^{-l}b'a\mu_{0,l} & \text{if } 1 < l. \end{cases}$$

$$\sigma_l\tau_k = \begin{cases} b\mu_{k+1,0} & \text{if } k < p-1 \text{ and } l = p-1, \\ a\mu_{0,l+1} & \text{if } k = p-1 \text{ and } l < p-1, \\ ab\psi & \text{if } k = p-1 \text{ and } l = p-1. \end{cases}$$

$\theta_{k,l}\theta_{k',l'} =$

$$
\begin{cases}
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}\sigma_{l+l'}\tau_{k+k'} & \textit{if } 0 < k+k' < p-1 \\
 & \quad \textit{and } 0 < l+l' < p-1, \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}b\mu_{k+k'+1,0} & \textit{if } 0 < k+k' < p-1 \\
 & \quad \textit{and } l+l' = p-1, \\
\frac{\omega_{k+k'}}{\omega_l}\zeta^{l(k'+1)}b\rho_{k+k'+1} & \textit{if } 0 < k+k' < p-1 \textit{ and } l+l' = p, \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}b\sigma_{l+l'-p}\tau_{k+k'} & \textit{if } 0 < k+k' < p-1 \textit{ and } p < l+l', \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}a\mu_{0,l+l'+1} & \textit{if } k+k' = p-1 \\
 & \quad \textit{and } 0 < l+l' < p-1, \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\psi & \textit{if } k+k' = p-1 \textit{ and } l+l' = p-1, \\
\frac{\omega_{k+k'}}{\omega_l}\zeta^{l(k'+1)}ab\rho_0 & \textit{if } k+k' = p-1 \textit{ and } l+l' = p, \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\mu_{0,l+l'+1-p} & \textit{if } k+k' = p-1 \textit{ and } p < l+l', \\
\frac{\omega_k\omega_{l+l'}}{\omega_l\omega_{l'}}\zeta^{-k(l+1)}a\eta_{l+l'+1} & \textit{if } k+k' = p \textit{ and } 0 < l+l' < p, \\
\frac{\omega_k\omega_{l+l'}}{\omega_l\omega_{l'}}\zeta^{-k(l+1)}ab\eta_0 & \textit{if } k+k' = p \textit{ and } l+l' = p-1, \\
\frac{p}{\omega_l\omega_{l'}}\zeta^{kl'}ab(\omega_{l'}a\varphi + \omega_{k'}b\chi) & \textit{if } k+k' = p \textit{ and } l+l' = p, \\
\frac{\omega_k\omega_{l+l'}}{\omega_l\omega_{l'}}\zeta^{-k(l+1)}ab\eta_{l+l'+1-p} & \textit{if } k+k' = p \textit{ and } p < l+l', \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}a\sigma_{l+l'}\tau_{k+k'-p} & \textit{if } p < k+k' \textit{ and } 0 < l+l' < p, \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\mu_{k+k'+1-p,0} & \textit{if } p < k+k' \textit{ and } l+l' = p-1, \\
\frac{\omega_{k+k'}}{\omega_l}\zeta^{l(k'+1)}ab\rho_{k+k'+1-p} & \textit{if } p < k+k' \textit{ and } l+l' = p, \\
(\frac{\omega_k}{\omega_l}\zeta^{k'+l} - \frac{\omega_{k'}}{\omega_{l'}})\zeta^{k'l}ab\sigma_{l+l'-p}\tau_{k+k'-p} & \textit{if } p < k+k' \textit{ and } p < l+l'.
\end{cases}
$$

$$
\pi\theta_{k,l} = \begin{cases}
\frac{p}{\omega_1}ab(-\zeta^{-1}a'\varphi + b'\chi) & \textit{if } k = 1 \textit{ and } l = 1, \\
\frac{\omega_{l-1}}{\omega_l}a'b\eta_l & \textit{if } k = 1 \textit{ and } 1 < l, \\
-\frac{\omega_{k-1}}{\omega_1}\zeta^{-k}b'a\rho_k & \textit{if } 1 < k \textit{ and } l = 1, \\
(\zeta^{-1} - \frac{\omega_k}{\omega_l}\zeta^{-k})a'b'd\sigma_{l-1}\tau_{k-1} & \textit{if } 1 < k \textit{ and } 1 < l.
\end{cases}
$$

$$
\tau_{k'}\theta_{k,l} = \begin{cases}
-\zeta^k\sigma_l\tau_{k+k'} & \textit{if } 0 < k+k' < p \textit{ and } l < p-1, \\
-\zeta^k b\mu_{k+k'+1,0} & \textit{if } 0 < k+k' < p \textit{ and } l = p-1, \\
-\zeta^k a\eta_{l+1} & \textit{if } k+k' = p \textit{ and } l < p-1, \\
-\zeta^k ab\eta_0 & \textit{if } k+k' = p \textit{ and } l = p-1, \\
-\zeta^k a\sigma_l\tau_{k+k'-p} & \textit{if } p < k+k' \textit{ and } l < p-1, \\
-\zeta^k ab\mu_{k+k'+1-p,0} & \textit{if } p < k+k' \textit{ and } l = p-1.
\end{cases}
$$

$$\sigma_{l'}\theta_{k,l} = \begin{cases} -\frac{\omega_k}{\omega_l}\zeta^{kl'}\sigma_{l+l'}\tau_k & \text{if } 0 < l+l' < p \text{ and } k < p-1, \\ -\frac{\omega_{p-1}}{\omega_l}\zeta^{-l'}a\mu_{0,l+l'+1} & \text{if } 0 < l+l' < p \text{ and } k = p-1, \\ \frac{\omega_k}{\omega_{l'}}\zeta^{l'(k+1)}b\rho_{k+1} & \text{if } l+l' = p \text{ and } k < p-1, \\ \frac{\omega_{p-1}}{\omega_{l'}}ab\rho_0 & \text{if } l+l' = p \text{ and } k = p-1, \\ -\frac{\omega_k}{\omega_l}\zeta^{kl'}b\sigma_{l+l'-p}\tau_k & \text{if } p < l+l' \text{ and } k < p-1, \\ -\frac{\omega_{p-1}}{\omega_l}\zeta^{-l'}ab\mu_{0,l+l'+1-p} & \text{if } p < l+l' \text{ and } k = p-1. \end{cases}$$

The relations in $HH^3(\Gamma)$:

$$dp\kappa = \pi\psi = 0.$$

$$\tau_k\psi = \begin{cases} \frac{p}{\omega_1}\sigma_{p-1}\chi & \text{if } k = 1, \\ 0 & \text{if } 1 < k. \end{cases}$$

$$\sigma_l\psi = \begin{cases} \frac{p}{\omega_1}\tau_{p-1}\varphi & \text{if } l = 1, \\ 0 & \text{if } 1 < l. \end{cases}$$

$$\tau_k\rho_{k'} = \begin{cases} -\frac{p}{\omega_1}d\kappa & \text{if } k+k'-1 = 0 \,(i.e.\ k=1, k'=0), \\ \frac{p}{\omega_{p-1+k'}}\zeta^{k'-1}a\tau_{k+k'-1}\varphi & \text{if } 0 < k+k'-1 < p, \\ -\frac{p}{\omega_k}ad\kappa & \text{if } k+k'-1 = p, \\ \frac{p}{\omega_{k'-1}}\zeta^{k'-1}a^2\tau_{k+k'-1-p}\varphi & \text{if } p < k+k'-1. \end{cases}$$

$$\sigma_l\eta_{l'} = \begin{cases} \frac{p}{\omega_1}\zeta d\kappa & \text{if } l+l'-1 = 0 \,(i.e.\ l=1, l'=0), \\ \frac{p}{\omega_{p-1+l'}}b\sigma_{l+l'-1}\chi & \text{if } 0 < l+l'-1 < p, \\ \frac{p}{\omega_l}\zeta^l bd\kappa & \text{if } l+l'-1 = p, \\ \frac{p}{\omega_{p-1+l'}}b^2\sigma_{l+l'-1-p}\chi & \text{if } p < l+l'-1. \end{cases}$$

$$\sigma_l\rho_k = \begin{cases} \frac{p}{\omega_{p-1}}\zeta^{-l}\theta_{p-1,l}\varphi & \text{if } k = 0, \\ \frac{p}{\omega_{k-1}}\zeta^{l(k-1)}a\theta_{k-1,l}\varphi & \text{if } 0 < k. \end{cases}$$

$$\tau_k\eta_l = \begin{cases} -\frac{p}{\omega_k}\theta_{k,p-1}\chi & \text{if } l = 0, \\ -\frac{p}{\omega_k}b\theta_{k,l-1}\chi & \text{if } 0 < l. \end{cases}$$

$$\tau_k\mu_{k',0} = \begin{cases} \frac{p}{\omega_k}a\sigma_{p-1}\chi & \text{if } k+k'-1 = p \ (i.e.\ k=1, k'=0), \\ 0 & \text{if } k+k'-1 \neq p. \end{cases}$$

$$\sigma_l\mu_{0,l'} = \begin{cases} \frac{p}{\omega_l}b\tau_{p-1}\varphi & \text{if } l+l'-1 = p \,(i.e.\ l=1, l'=0), \\ 0 & \text{if } l+l'-1 \neq p. \end{cases}$$

$$\tau_k\mu_{0,l} = \begin{cases} \frac{p}{\omega_1}b\sigma_{l-1}\chi & \text{if } k = 1, \\ 0 & \text{if } 1 < k. \end{cases}$$

$$\sigma_l\mu_{k,0} = \begin{cases} \frac{p}{\omega_1}\zeta^k a\tau_{k-1}\varphi & \textit{if } l = 1, \\ 0 & \textit{if } 1 < l. \end{cases}$$

$$\pi\mu_{k,0} = \begin{cases} \frac{p}{\omega_1}\zeta^{-1} a'b\sigma_{p-2}\chi & \textit{if } k = 2, \\ 0 & \textit{if } 2 < k. \end{cases}$$

$$\pi\mu_{0,l} = \begin{cases} -\frac{p}{\omega_1}\zeta ab'\tau_{p-2}\varphi & \textit{if } k = 2, \\ 0 & \textit{if } 2 < l. \end{cases}$$

$$\pi\rho_k = \begin{cases} \frac{p}{\omega_{p-1}}\zeta a'\theta_{p-2,p-1}\varphi & \textit{if } k = 0, \\ \frac{p}{\omega_1}\zeta^{-1} a'(a\sigma_{p-1}\varphi + b\sigma_{p-1}\chi) & \textit{if } k = 2, \\ \frac{p}{\omega_{k-1}}\zeta^{1-k} aa'\theta_{k-2,p-1}\varphi & \textit{if } 2 < k. \end{cases}$$

$$\pi\eta_l = \begin{cases} \frac{p}{\omega_{p-1}}\frac{\omega_{p-2}}{\omega_{p-1}} b'\theta_{p-1,p-2}\chi & \textit{if } l = 0, \\ -\frac{p}{\omega_1} b'(a\zeta\tau_{p-1}\varphi + b\tau_{p-1}\chi) & \textit{if } l = 2, \\ \frac{p}{\omega_{p-1}}\frac{\omega_{l-2}}{\omega_{p-1}} bb'\theta_{p-1,l-2}\chi & \textit{if } 2 < l. \end{cases}$$

$$\theta_{k,l}\mu_{k',0} = \begin{cases} \frac{p}{\omega_1}\zeta^{k'} a\tau_{k+k'-1}\varphi & \textit{if } 1 < k+k'-1 < p \textit{ and } l = 1, \\ 0 & \textit{if } 1 < k+k'-1 < p \textit{ and } 1 < l, \\ \frac{p}{\omega_1}\zeta^{-(k-1)} ad\kappa & \textit{if } k+k'-1 = p \textit{ and } l = 1, \\ -\frac{p}{\omega_l}\zeta^{-l(k-1)} ab\sigma_{l-1}\chi & \textit{if } k+k'-1 = p \textit{ and } 1 < l, \\ \frac{p}{\omega_1}\zeta^{k'} a^2\tau_{k+k'-1-p}\varphi & \textit{if } p < k+k'-1 \textit{ and } l = 1, \\ 0 & \textit{if } p < k+k'-1 \textit{ and } 1 < l. \end{cases}$$

$$\theta_{k,l}\mu_{0,l'} = \begin{cases} -\frac{p}{\omega_l} b\sigma_{l+l'-1}\chi & \textit{if } 1 < l+l'-1 < p \textit{ and } k = 1, \\ 0 & \textit{if } 1 < l+l'-1 < p \textit{ and } 1 < k, \\ \frac{p}{\omega_l} bd\kappa & \textit{if } l+l'-1 = p \textit{ and } k = 1, \\ \frac{p}{\omega_l} ab\tau_{k-1}\varphi & \textit{if } l+l'-1 = p \textit{ and } 1 < k, \\ -\frac{p}{\omega_l} b^2\sigma_{l+l'-1-p}\chi & \textit{if } p < l+l'-1 \textit{ and } k = 1, \\ 0 & \textit{if } p < l+l'-1 \textit{ and } 1 < k. \end{cases}$$

$$\theta_{k,l}\rho_{k'} = \begin{cases} \frac{p}{\omega_{p-1}}\zeta^{-l} a\sigma_l\varphi - \frac{p}{\omega_l} b\sigma_l\chi & \textit{if } k+k'-1 = 0 \\ & \quad (\textit{i.e. } k = 1, k' = 0), \\ \frac{p}{\omega_{k'-1}}\zeta^{l(k'-1)} a\theta_{k+k'-1,l}\varphi & \textit{if } 0 < k+k'-1 < p, \\ \frac{p}{\omega_{k'-1}}\zeta^{l(k'-1)} a^2\sigma_l\varphi - \frac{p}{\omega_l}\zeta^{k'l} ab\sigma_l\chi & \textit{if } k+k'-1 = p, \\ \frac{p}{\omega_{k'-1}}\zeta^{l(k'-1)} a^2\theta_{k+k'-1-p,l}\varphi & \textit{if } p < k+k'-1. \end{cases}$$

$$\theta_{k,l}\eta_{l'} = \begin{cases} \frac{p}{\omega_1}\zeta d(a'\tau_k\varphi + \frac{\omega_k}{\omega_1}b'\tau_k\chi) & \text{if } l+l'-1=0 \ (i.e.\ l=1, l'=0), \\ \frac{p}{\omega_{p-1+l'}}\frac{\omega_{l+l'-1}}{\omega_l}b\theta_{k,l+l'-1}\chi & \text{if } 0<l+l'-1<p, \\ \frac{p}{\omega_l}\zeta^l bd(a'\tau_k\varphi + \frac{\omega_k}{\omega_l}b'\tau_k\chi) & \text{if } l+l'-1=p, \\ \frac{p}{\omega_{p-1+l'}}\frac{\omega_{l+l'-1}}{\omega_l}b^2\theta_{k,l+l'-1-p}\chi & \text{if } p<l+l'-1. \end{cases}$$

$$\theta_{k,l}\psi = \begin{cases} \frac{p}{\omega_1}d\kappa & \text{if } k=1 \text{ and } l=1, \\ -\frac{p}{\omega_l}b\sigma_{l-1}\chi & \text{if } k=1 \text{ and } 1<l, \\ \frac{p}{\omega_1}a\tau_{k-1}\varphi & \text{if } 1<k \text{ and } l=1, \\ 0 & \text{if } 1<k \text{ and } 1<l. \end{cases}$$

The relations in $HH^4(\Gamma)$:

$$\psi\psi = \psi\mu_{k,0} = \psi\mu_{0,l} = \mu_{k,0}\mu_{k',0} = \mu_{0,l}\mu_{0,l'} = 0.$$

$$\pi\kappa = a'b'(a\psi\varphi - \zeta^{-1}b\psi\chi).$$

$$\tau_k\kappa = \begin{cases} b'\rho_{k+1}\chi & \text{if } k<p-1, \\ b'a\rho_0\chi & \text{if } k=p-1. \end{cases}$$

$$\sigma_l\kappa = \begin{cases} a'\eta_{l+1}\varphi & \text{if } l<p-1, \\ a'b\eta_0\varphi & \text{if } l=p-1. \end{cases}$$

$$\theta_{k,l}\kappa = \begin{cases} a'\sigma_l\tau_k\varphi - \frac{\omega_k}{\omega_l}\zeta^l b'\sigma_l\tau_k\chi & \text{if } k<p-1 \text{ and } l<p-1, \\ a'b\mu_{k+1,0}\varphi - \frac{\omega_k}{\omega_{p-1}}\zeta^{p-1}b'b\mu_{k+1,0}\chi & \text{if } k<p-1 \text{ and } l=p-1, \\ aa'\mu_{0,l+1}\varphi - \frac{\omega_{p-1}}{\omega_l}\zeta^l ab'\mu_{0,l+1}\chi & \text{if } k=p-1 \text{ and } l<p-1. \end{cases}$$

$$\psi\rho_k = \begin{cases} \frac{p}{\omega_{p-1}}\mu_{p-1,0}\varphi & \text{if } k=0, \\ \frac{p}{\omega_1}a\eta_0\varphi & \text{if } k=2, \\ \frac{p}{\omega_{k-1}}a\mu_{k-1,0}\varphi & \text{if } 2<k. \end{cases}$$

$$\psi\eta_l = \begin{cases} \frac{p}{\omega_{p-1}}\mu_{0,p-1}\chi & \text{if } l=0, \\ \frac{p}{\omega_1}b\rho_0\chi & \text{if } l=2, \\ \frac{p}{\omega_{l-1}}b\mu_{0,l-1}\chi & \text{if } 2<l. \end{cases}$$

$$\rho_k\mu_{k',0} = \begin{cases} \frac{p}{\omega_{p-1}}a\eta_0\varphi & \text{if } k+k'-2=0 \ (i.e.\ k=0, k'=2), \\ \frac{p}{\omega_{p-1}}a\mu_{k+k'-1,0}\varphi & \text{if } 0<k+k'-2<p, \\ \frac{p}{\omega_{k-1}}a^2\eta_0\varphi & \text{if } k+k'-2=p, \\ \frac{p}{\omega_{p-1}}a^2\mu_{k+k'-1-p,0}\varphi & \text{if } p<k+k'-2. \end{cases}$$

$$\eta_l\mu_{0,l'} = \begin{cases} -\frac{p}{\omega_{p-1}}b\rho_0\chi & \textit{if } l+l'-2=0 \ (\textit{i.e. } l=0, l'=2), \\ -\frac{p}{\omega_{p-1}}b\mu_{0,l+l'-1}\varphi & \textit{if } 0<l+l'-2<p, \\ -\frac{p}{\omega_{l-1}}b^2\rho_0\chi & \textit{if } l+l'-2=p, \\ -\frac{p}{\omega_{p-1}}b^2\mu_{0,l+l'-1-p}\varphi & \textit{if } p<l+l'-2. \end{cases}$$

$$\rho_k\mu_{0,l} = \begin{cases} \frac{p}{\omega_{p-1}}\sigma_{l-1}\tau_{p-2}\varphi & \textit{if } k=0, \\ \frac{p}{\omega_1}a\eta_l\varphi & \textit{if } k=2, \\ \frac{p}{\omega_{k-1}}a\sigma_{l-1}\tau_{k-2}\varphi & \textit{if } 2<k. \end{cases}$$

$$\eta_l\mu_{k,0} = \begin{cases} \frac{p}{\omega_{p-1}}\zeta^{-k}\sigma_{p-2}\tau_{k-1}\chi & \textit{if } l=0, \\ \frac{p}{\omega_1}\zeta^k b\rho_k\chi & \textit{if } l=2, \\ \frac{p}{\omega_{l-1}}\zeta^{k(l-1)}b\sigma_{l-2}\tau_{k-1}\chi & \textit{if } 2<l. \end{cases}$$

$$\rho_k\rho_{k'} = \begin{cases} \frac{p}{\omega_{p-1}}\frac{\omega_{p-2}}{\omega_{p-1}}\rho_{p-1}\varphi & \textit{if } k=k'=0, \\ -(\frac{p}{\omega_{k-1}}a)^2\zeta^{k-1}\varphi\varphi + \frac{p(p-1)}{2}\frac{p}{\omega_{k-1}}ab\varphi\chi & \textit{if } k+k'-2=0, \\ \frac{p}{\omega_{p-1+k}}\frac{\omega_{p-2+k+k'}}{\omega_{p-1+k'}}a\rho_{k+k'-1}\varphi & \textit{if } 0<k+k'-2<p-1, \\ \frac{p}{\omega_{p-1+k}}\frac{\omega_{p-1}}{\omega_{p-k}}a^2\rho_0\varphi & \textit{if } k+k'-2=p-1, \\ -(\frac{p}{\omega_{k-1}}a)^2\zeta^{k-1}a\varphi\varphi + \frac{p(p-1)}{2}\frac{p}{\omega_{k-1}}a^2b\varphi\chi & \textit{if } k+k'-2=p, \\ \frac{p}{\omega_{p-1+k}}\frac{\omega_{p-2+k+k'}}{\omega_{p-1+k'}}a^2\rho_{k+k'-1-p}\varphi & \textit{if } p<k+k'-2. \end{cases}$$

$$\rho_k\eta_l = \begin{cases} \frac{p}{\omega_{p-1}}a\psi\varphi + \frac{p}{\omega_{p-1}}b\psi\chi & \textit{if } k=0 \textit{ and } l=0, \\ \frac{p}{\omega_{p-1}}a\mu_{0,l}\varphi + \frac{p}{\omega_{l-1}}b\mu_{0,l}\chi & \textit{if } k=0 \textit{ and } 0<l, \\ \frac{p}{\omega_{k-1}}a\mu_{k,0}\varphi + \frac{p}{\omega_{p-1}}b\mu_{k,0}\chi & \textit{if } 0<k \textit{ and } l=0, \\ \frac{p}{\omega_{k-1}}a\sigma_{l-1}\tau_{k-1}\varphi + \frac{p}{\omega_{l-1}}b\sigma_{l-1}\tau_{k-1}\chi & \textit{if } 0<k \textit{ and } 0<l. \end{cases}$$

$$\eta_l\eta_{l'} = \begin{cases} \frac{p}{\omega_{p-1}}\frac{\omega_{p-2}}{\omega_{p-1}}\eta_{p-1}\chi & \textit{if } l=l'=0, \\ -\frac{p(p-1)}{2}\frac{p}{\omega_{l-1}}\zeta^{l-1}ab\varphi\chi + (\frac{p}{\omega_{l-1}}b)^2\zeta^{l-l}\chi\chi & \textit{if } l+l'-2=0, \\ \frac{p}{\omega_{p-1+l}}\frac{\omega_{p-2+l+l'}}{\omega_{p-1+l'}}b\eta_{l+l'-1}\chi & \textit{if } 0<l+l'-2<p-1, \\ \frac{p}{\omega_{p-1+l}}\frac{\omega_{p-1}}{\omega_{p-l}}b^2\eta_0\chi & \textit{if } l+l'-2=p-1, \\ -\frac{p(p-1)}{2}\frac{p}{\omega_{l-1}}\zeta^{l-1}ab^2\varphi\chi + (\frac{p}{\omega_{l-1}}b)^2\zeta^{l-1}b\chi\chi & \textit{if } l+l'-2=p, \\ \frac{p}{\omega_{p-1+l}}\frac{\omega_{p-2+l+l'}}{\omega_{p-1+l'}}b^2\eta_{l+l'-1-p}\chi & \textit{if } p<l+l'-2. \end{cases}$$

$$\mu_{k,0}\mu_{0,l} = \begin{cases} \frac{p^2}{\omega_1^2}\zeta ab\varphi\chi & \textit{if } k=2 \textit{ and } l=2, \\ 0 & \textit{if } 2<k \textit{ or } 2<l. \end{cases}$$

The relations in $HH^5(\Gamma)$:

$$\psi\kappa = -\frac{p}{\omega_1}\zeta\varphi\chi\pi.$$

$$\rho_k\kappa = \begin{cases} \frac{p}{\omega_{p-1}} a'\tau_{p-1}\varphi\varphi + \frac{p(p-1)}{2} b'\tau_{p-1}\varphi\chi & \textit{if } k = 0, \\ \frac{p}{\omega_{k-1}} aa'\tau_{k-1}\varphi\varphi + \frac{p(p-1)}{2} ab'\tau_{k-1}\varphi\chi & \textit{if } 0 < k. \end{cases}$$

$$\eta_l\kappa = \begin{cases} \frac{p(p-1)}{2} a'\sigma_{p-1}\varphi\chi + \frac{p}{\omega_1} b'\sigma_{p-1}\chi\chi & \textit{if } l = 0, \\ \frac{p(p-1)}{2} ba'\sigma_{l-1}\varphi\chi + \frac{p}{\omega_{1-l}} bb'\sigma_{l-1}\chi\chi & \textit{if } 0 < l. \end{cases}$$

$$\mu_{k,0}\kappa = \frac{p}{\omega_{k-1}} a'\theta_{k-1,p-1}\varphi\chi.$$

$$\mu_{0,l}\kappa = \frac{p}{\omega_{p-1}} b'\theta_{p-1,l-1}\varphi\chi.$$

The relation in $HH^6(\Gamma)$:

$$\kappa\kappa = \frac{p(p-1)}{2} a'b'\varphi\chi(a\varphi + b\chi).$$

Last, we consider the Hochschild cohomology ring $HH^*(\Gamma)$ in the special case $|a| = |b| = 1$.

If $p \geq 3$, then we have the following relations from Theorem 3:

$$\begin{aligned}
\sigma_{p-1}\tau_k &= \mu_{k+1,0} \quad \text{for } 1 \leq k < p-1, \\
\sigma_l\tau_{p-1} &= \mu_{0,l+1} \quad \text{for } 1 \leq l < p-1, \\
\sigma_{p-1}\tau_{p-1} &= \psi, \\
\sigma_k\theta_{k,p-k} &= \zeta^{k(k+1)}\rho_{k+1} \quad \text{for } 1 \leq k < p-1, \\
\sigma_{p-1}\theta_{p-1,1} &= \rho_0, \\
\tau_{p-1}\theta_{1,l} &= -\zeta\eta_{l+1} \quad \text{for } 1 \leq l < p-1, \\
\tau_{p-1}\theta_{1,p-1} &= -\zeta\eta_0.
\end{aligned}$$

Hence, we have the following corollary:

Corollary 4. *Let* $p \geq 3$ *be a prime number and* $|a| = |b| = 1$. *Then the Hochschild cohomology ring* $HH^*(\Gamma)$ *is the graded commutative ring generated by the following* $p^2 + 2$ *elements:*

$\sigma_l, \tau_k, \theta_{k',l'}, \pi \in HH^1(\Gamma)$ *for* $1 \leq k, k', l, l' \leq p-1$ *with* $(k', l') \neq (p-1, p-1)$,
$\varphi, \chi \in HH^2(\Gamma)$, $\quad \kappa \in HH^3(\Gamma)$.

§5. The ring structure of $HH^*(\Gamma)$ in the case $p = 2$

In the last section, we deal with the case $p = 2$. Then Γ is a generalized quaternion algebra over $\mathbb{Z}$:

$$\Gamma = \mathbb{Z}1 \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}ij, \quad i^2 = a, j^2 = b, ji = -ij \ (a, b \in \mathbb{Z}, \neq 0).$$

In that case, $\zeta = -1$ and $R = \mathbb{Z}$ and the diagonal approximation map Φ is

$$\Phi_{s,t;s',t'}(c_{s+t,s'+t'}) = c_{s,t} \otimes_\Gamma c_{s',t'},$$

hence, the cup product $\smile$ is

$$\alpha \smile \beta = \alpha\beta$$

for $\alpha \in \Gamma^{s,t}$ and $\beta \in \Gamma^{s',t'}$. Furthermore, we note that the following relations hold:

$$\begin{aligned}
\pi\pi &= (a'b'a, 0, a'b'b) = a'b'a\varphi + a'b'b\chi, \\
\pi\psi &= (0, a'j, -b'i, 0) = \kappa, \\
\psi\psi &= (0, 0, 1, 0, 0),
\end{aligned}$$

where d is the greatest common divisor of a and b, and set $a' = a/d$, $b' = b/d$.

Hence we have the following theorem. This result was already known in [2], and also [8] for a special case.

Theorem 5. *Let $p = 2$ and a, b any nonzero integers. Then the Hochschild cohomology ring $HH^*(\Gamma)$ is the graded commutative ring generated by at most the eight elements*

$$\sigma_1, \tau_1, \pi \in HH^1(\Gamma), \quad \varphi, \psi, \chi, \eta_0, \rho_0 \in HH^2(\Gamma)$$

with the following relations.
The relations in $HH^1(\Gamma)$:

$$2\sigma_1 = 2\tau_1 = 2d\pi = 0.$$

The relations in $HH^2(\Gamma)$:

$$\begin{aligned}
&2a\varphi = 2d\psi = 2b\chi = 2\rho_0 = 2\eta_0 = 0, \\
&\sigma_1\sigma_1 = ab\varphi, \quad \sigma_1\tau_1 = ab\psi, \quad \sigma_1\pi = b'a\rho_0, \\
&\tau_1\tau_1 = ab\chi, \quad \tau_1\pi = a'b\eta_0, \quad \pi\pi = a'b'(a\varphi + b\chi).
\end{aligned}$$

The relations in $HH^3(\Gamma)$:

$$\begin{aligned}
&\tau_1\varphi = \sigma_1\psi, \quad \tau_1\psi = \sigma_1\chi, \quad \tau_1\eta_0 = d\pi\chi, \\
&\tau_1\rho_0 = \sigma_1\eta_0 = d\pi\psi, \quad \sigma_1\rho_0 = d\pi\varphi, \quad \pi\rho_0 = a'\sigma_1\varphi + b'\sigma_1\chi, \\
&\pi\eta_0 = a'\tau_1\varphi + b'\tau_1\chi.
\end{aligned}$$

The relations in $HH^4(\Gamma)$:

$$\begin{aligned}
&\varphi\chi = \psi\psi, \quad \varphi\eta_0 = \psi\rho_0, \quad \psi\eta_0 = \chi\rho_0, \\
&\rho_0\rho_0 = a\varphi\varphi + b\psi\psi, \quad \rho_0\eta_0 = a\varphi\psi + b\psi\chi, \quad \eta_0\eta_0 = a\psi\psi + b\chi\chi.
\end{aligned}$$

In particular, if $|a| = |b| = 1$, then we have the following result of [6] from Theorem 5:

Corollary 6. *If $p = 2$ and $|a| = |b| = 1$, then we have the ring isomorphism*

$$HH^*(\Gamma) \cong \mathbb{Z}[x, y, z]/(2x, 2y, 2z, x^2 + y^2 + z^2).$$

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