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Existence of sectional-hyperbolic attractors for three-dimensional flows

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Abstract. We prove that every C^1 generic three-dimensional flow has either infinitely many sinks or finitely many sectional-hyperbolic attractors whose basins form a full Lebesgue measure set. In particular, all such flows exhibit sectional-hyperbolic attractors.

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§1. Introduction

Araujo announced in his 1988's thesis [2] that every C^1 generic surface diffeomorphism has either infinitely many attracting periodic orbits (often called *sinks*) or else finitely many hyperbolic attractors whose basins form a full Lebesgue measure of the ambient manifold. This would solve René Thom's question [31] about existence of hyperbolic attractors for C^1 generic surface diffeomorphisms. However, this thesis contains several gaps partially filled in the works of Pujals-Sambarino [32] among others (see [4] and references therein). A complete proof was never published although Sambarino claimed that a full proof can be obtained using Pujals-Sambarino methods. Very recently it was proved that every C^1 generic three-dimensional nonsingular flow has either infinitely many sinks or else finitely many hyperbolic attractors whose basins form a full Lebesgue measure in the ambient manifold [4]. In particular, Araujo's thesis [2] is true and not only all C^1 generic surface diffeomorphisms but also all C^1 generic three-dimensional nonsingular flows exhibit a hyperbolic attractor.

In light of these results it is natural to ask if [4] hold for three-dimensional flows with singularities. However, the answer is negative because there are open sets of C^1 flows in the sphere S^3 for which there are no hyperbolic attractors [28]. Nevertheless, such a counterexample exhibits a sectional-hyperbolic attractor (e.g. the geometric Lorenz one [17]) whose basin has full Lebesgue measure. It is then reasonable to ask if the result in [4] holds for three-dimensional flows but replacing the hyperbolic attractor alternative by the sectional-hyperbolic attractor one (we prefer the latter term instead of the common term *singular-hyperbolic attractor* which may suggest the existence of singularities).

In this paper we will give a positive answer for this last question. More precisely, we will prove that every C^1 generic three-dimensional flow has either infinitely many sinks or finitely many sectional-hyperbolic attractors whose basins form a full Lebesgue measure set (emphasizing here that some of the sectional-hyperbolic attractors in the second alternative may be hyperbolic). In particular, every C^1 generic three-dimensional flow carries a sectional-hyperbolic attractor (which, we insist, may be hyperbolic as in the Axiom A case). We stress that our results are false in dimension bigger than 3 (a counterexample can be obtained by suspending the examples in [10]). Let us present the precise statements.

Hereafter, the term *three-dimensional flow* will be referred to a C^1 vector field on a Riemannian compact connected boundaryless three-dimensional manifold M. The corresponding space equipped with the C^1 vector field topology will be denoted by $\mathfrak{X}^1(\mathrm{M})$. The flow of $X \in \mathfrak{X}^1(\mathrm{M})$ is denoted by X_t, $t \in \mathbb{R}$.

By a *critical point* of X we mean a point x satisfying $X_t(x) = x$ for some $t > 0$. If this is satisfied for every $t \geq 0$ we say that x is a *singularity*, otherwise it is a *periodic point*. For every periodic point we have a minimal $t > 0$ satisfying $X_t(x) = x$. The minimal of such t's is the period of x denoted by t_x (or $t_{x,X}$ to indicate X). We denote by $\mathrm{Crit}(X)$ the set of critical points, by $\mathrm{Sing}(X)$ the set of singularities and by $\mathrm{Per}(X)$ the set of periodic points thus $\mathrm{Crit}(X) = \mathrm{Sing}(X) \cup \mathrm{Per}(X)$. The *eigenvalues* of a critical point x are either those of the linear automorphism $DX_{t_x}(x) : T_xM \to T_xM$ not corresponding to the eigenvector $X(x)$ (periodic case) or those of $DX(x) : T_xM \to T_xM$ (singular case). We say that x is a *sink* if its eigenvalues either are less than 1 in modulus (periodic case) or else with negative real part (singular case). A *source* is a sink for the time-reversed flow $-X$. The set of sinks and sources of X will be denoted by $\mathrm{Sink}(X)$ and $\mathrm{Source}(X)$ respectively. A critical point is *hyperbolic* if it has no eigenvalues of modulus 1 (periodic case) or with zero real part (singular case).

For every point x we define its *omega-limit set*,

$$\omega(x) = \left\{ y \in M : y = \lim_{k\to\infty} X_{t_k}(x) \text{ for some sequence } t_k \to \infty \right\}.$$

The *basin* of any subset $\Lambda \subset M$ is defined by

$$W^s(\Lambda) = \{y \in M : \omega(y) \subset \Lambda\}.$$

(If necessary we shall write $\omega_X(x)$ or $W^s_X(\Lambda)$ to indicate the dependence on X.) We say that $\Lambda \subset M$ is *invariant* if $X_t(\Lambda) = \Lambda$ for all $t \in \mathbb{R}$; and *transitive* if there is $x \in \Lambda$ such that $\Lambda = \omega(x)$.

An *attractor* is a transitive set A exhibiting a neighborhood U such that

$$A = \bigcap_{t \geq 0} X_t(U).$$

Hereafter E^X will denote the map assigning to each $x \in M$ the subspace E^X_x of T_xM generated by $X(x)$.

A compact invariant set Λ is *hyperbolic* (for X) if there is a continuous tangent bundle splitting $T_\Lambda M = E^s \oplus E^X \oplus E^u$ and positive numbers K, λ such that

(i) The subbundles E^s and E^u are DX_t-*invariant*, i.e., $DX_t(x)E^l_x = E^l_{X_t(x)}$ for $x \in I$, $t \in \mathbb{R}$ and $l = s, u$.

(ii) E^s is *contracting*, i.e., $\|DX_t(x)v^s_x\| \leq Ke^{-\lambda t}\|v^s_x\|$ for every $x \in \Lambda$, $v^s_x \in E^s_x$ and $t \geq 0$.

(iii) E^u is *expanding*, i.e., $\|DX_t(x)v^u_x\| \geq K^{-1}e^{\lambda t}\|v^u_x\|$ for every $x \in \Lambda$, $v^u_x \in E^u_x$ and $t \geq 0$.

A *hyperbolic attractor (for X)* is an attractor which is also a hyperbolic set for X.

On the other hand, a *dominated splitting* for X over an invariant set I is a continuous DX_t-invariant tangent bundle splitting $T_IM = E \oplus F$ for which there are positive constants K, λ satisfying

$$\|DX_t(x)e_x\| \cdot \|f_x\| \leq Ke^{-\lambda t}\|DX_t(x)f_x\| \cdot \|e_x\|,$$

for all $x \in I$, $t \geq 0$, $e_x \in E_x$ and $f_x \in F_x$.

A compact invariant set Λ is *partially hyperbolic* if it has a *partially hyperbolic splitting*, i.e., a dominated splitting $T_\Lambda M = E \oplus F$ for X over Λ whose dominated subbundle E is contracting in the sense of (ii) above.

In the sequel we present a slight variation (suggested by S. Bautista) of the definition of sectional-hyperbolic set in [26]. The advantage of this variation is that it allows every hyperbolic set to be sectional-hyperbolic for both the flow and the reversed flow.

Let $\langle \cdot, \cdot \rangle$ denote the Riemannian structure of M. It induces a 2-*Riemannian metric* (in the sense of [29]) defined by

$$\langle u, v/w \rangle_p = \langle u, v \rangle_p \cdot \langle w, w \rangle_p - \langle u, w \rangle_p \cdot \langle v, w \rangle_p, \quad \forall p \in M, \forall u, v, w \in T_pM,$$

and a 2-*norm* [16] (often called *areal metric* [20]) defined by

$$\|u, v\| = \sqrt{\langle u, u/v \rangle_p}, \quad \forall p \in M, \forall u, v \in T_pM.$$

The latter is the area of the paralellogram generated by u and v in T_pM.

Let $T_\Lambda M = E \oplus F$ be a dominated splitting for X over Λ. We say that the central subbundle F of this splitting is *sectionally expanding* if

$$\|DX_t(x)u, DX_t(x)v\| \geq K^{-1}e^{\lambda t}\|u, v\|, \qquad \forall x \in \Lambda, u, v \in F_x, t \geq 0.$$

In other words, F is sectionally expanding precisely when the derivative of the flow *expands the area of the parallelograms along* F.

By a *sectional-hyperbolic splitting* for X over Λ we mean a partially hyperbolic splitting $T_\Lambda M = E \oplus F$ whose central subbundle F is sectionally expanding.

Definition 1.1. A compact invariant set Λ is *sectional-hyperbolic* for X if its singularities are hyperbolic and if there is a sectional-hyperbolic splitting for X over Λ. A sectional-hyperbolic attractor (for X) is an attractor which is also a sectional-hyperbolic set for X.

As already said, this definition is slight different from the original one [26] which requires for instance that the central subbundle be two-dimensional at least.

Remark 1.2. *Under this definition we have that every hyperbolic set is sectional-hyperbolic for both the flow and the reversed flow. In fact, denoting by* $T_\Lambda M = E^s \oplus E^X \oplus E^u$ *corresponding hyperbolic splitting we have that* $T_\Lambda M = E^s \oplus F$ *with* $F = E^X \oplus E^u$ *and* $T_\Lambda M = \hat{E}^s \oplus \hat{F}$ *with* $\hat{E}^s = E^u$ *and* $\hat{F} = E^s \oplus E^X$ *define sectional-hyperbolic splittings for* X *and* $-X$ *respectively over* Λ.

In particular, under this definition, all hyperbolic attractors (including sinks) are sectional-hyperbolic.

We say that a subset $\mathcal{R} \subset \mathfrak{X}^1(\mathrm{M})$ is *residual* if there is a countable collection of open and dense subsets $\{\mathcal{O}_n : n \in \mathbb{N}\}$ of $\mathfrak{X}^1(\mathrm{M})$ such that $\mathcal{R} = \bigcap_{n \in \mathbb{N}} O_n$. As in p.11 of [3], we will say that a C^1*-generic three-dimensional flow satisfies a property (P)* if there is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that (P) holds for every flow in $\mathcal{R}$.

With these definitions we can state our main result.

Theorem A. *A C^1 generic three-dimensional flow has either infinitely many sinks or finitely many sectional-hyperbolic attractors whose basins form a full Lebesgue measure set of M.*

We emphasize that some of the sectional-hyperbolic attractors in the second alternative of the above theorem may be hyperbolic as, for instance, sinks or more complicated hyperbolic attractors.

From this result we get immediately the existence of sectional-hyperbolic attractors for C^1 generic three-dimensional flows. More precisely, we have the following corollary.

Corollary B. *Every C^1 generic three-dimensional flow exhibits a sectional-hyperbolic attractor.*

§2. Proof of Theorem A

We will need a result about existence of spectral decomposition for certain invariant sets. To state it we will need some preliminars.

A critical point is a *saddle* if it has eigenvalues of modulus less and bigger than 1 simultaneously (periodic case) or with positive and negative real part simultaneously (singular case). The set of periodic saddles of X is denoted by $\mathrm{PSaddle}(X)$.

It is well known [19] that through any $x \in \mathrm{PSaddle}(X)$ it passes a pair of invariant manifolds, the so-called strong stable and unstable manifolds $W^{ss}(x)$ and $W^{uu}(x)$, tangent at x to the eigenspaces corresponding to the eigenvalue of modulus less and bigger than 1 respectively. Saturating these manifolds with the flow we obtain the stable and unstable manifolds $W^s(x)$ and $W^u(x)$ respectively. A *homoclinic point* associated to x is a point q where $W^s(x)$ and $W^u(x)$ meet. We say that q is a *transverse homoclinic point* if $T_qW^s(x) \cap T_qW^u(x)$ is one-dimensional, otherwise we call it *homoclinic tangency.* The *homoclinic class* associated to x, denoted by $H(x)$, is the closure of the set of transverse homoclinic points associated to x. We write $H_X(x)$ to indicate dependence on X. By a homoclinic class we mean the homoclinic class associated to some saddle of X. We denote by $\mathrm{Cl}(\cdot)$ the closure operator.

Definition 2.1. A non-empty subset $\mathcal{P} \subset \mathrm{PSaddle}(X)$ is *homoclinically closed* if $H(x) \subset \mathrm{Cl}(\mathcal{P})$ for every $x \in \mathcal{P}$.

Basic examples of homoclinically closed subsets are $\mathrm{PSaddle}(X)$ itself and also the set $\mathrm{PSaddle}_d(X)$ of *dissipative saddles*, i.e., those saddles for which the product of the eigenvalues is less than 1 in modulus. This follows from the Birkhoff-Smale Theorem [18].

Definition 2.2. We say that a compact invariant set of X *has a spectral decomposition* if it is a finite disjoint union of transitive sets $\{\Lambda_1, \cdots, \Lambda_k\}$ such that, for every $1 \leq i \leq k$, Λ_i is either a hyperbolic set or a sectional-hyperbolic attractor for X or a sectional-hyperbolic attractor for $-X$.

The following result will give a sufficient condition for existence of spectral decomposition for the closure of homoclinically closed subsets of saddles. Given $\Lambda \subset M$ we define $\Lambda^* = \Lambda \setminus \mathrm{Sing}(X)$. We define the vector bundle N^X over M^* whose fiber at $x \in M^*$ is the the orthogonal complement of E^X_x in T_xM. Denote by $\pi : T_{M^*}M \to N^X$ the orthogonal projection. We define the *linear Poincaré flow* (LPF), $P^X_t : N^X \to N^X$, by $P^X_t = \pi \circ DX_t$.

Definition 2.3. We say that an invariant set Λ of X *has a LPF-dominated splitting* if $\Lambda^* \neq \emptyset$ and there exist a continuous bundle decomposition $N^X = N^{s,X} \oplus N^{u,X}$ over Λ^* with $dimN^{s,X}_x = dimN^{u,X}_x = 1$ (for every $x \in \Lambda^*$) and $T > 0$ such that

$$\left\|P^X_T(x)/N^{s,X}_x\right\| \left\|P^X_{-T}(X_T(x))/N^{u,X}_{X_T(x)}\right\| \leq \frac{1}{2}, \quad \forall x \in \Lambda^*.$$

With these definitions we can state the following result.

Theorem 2.4. *There exists a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if $\mathcal{P} \subset \mathrm{PSaddle}(X)$ is homoclinically closed and if $\mathrm{Cl}(\mathcal{P})$ has a LPF-dominated splitting, then $\mathrm{Cl}(\mathcal{P})$ has a spectral decomposition.*

This result will be proved in Section 3.

Next we state some useful definitions.

Let X be a three-dimensional flow. Recall that a periodic point is a saddle if it has eigenvalues of modulus less and bigger than 1 simultaneously. Analogously for singularities by just replace 1 by 0 and the eigenvalues by their corresponding real parts. Denote by $\mathrm{Saddle}(X)$ the set of saddles of X.

A critical point x is *dissipative* if the product of its eigenvalues (in the periodic case) or the divergence $\mathrm{div}\, X(x)$ (in the singular case) is less than 1 (resp. 0). Denote by $\mathrm{Crit}_d(X)$ the set of dissipative critical points. Define the *dissipative region* by $\mathrm{Dis}(X) = \mathrm{Cl}(\mathrm{Crit}_d(X))$.

For every subset $\Lambda \subset M$ we define

$$W^s_w(\Lambda) = \{x \in M : \omega(x) \cap \Lambda \neq \emptyset\}.$$

(This is often called *weak basin of attraction* [7].)

With these definitions we can state the following result whose proof will be given in Appendix A.

Theorem 2.5. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that $W^s_w(\mathrm{Dis}(X))$ has full Lebesgue measure for all $X \in \mathcal{R}$.*

Given a homoclinic class $H = H_X(p)$ of a three-dimensional flow X we denote by $H_Y = H_Y(p_Y)$ the continuation of H, where p_Y is the analytic continuation of p for Y close to X (c.f. [30]).

The following lemma was proved in [4]. In its statement Leb denotes the normalized Lebesgue measure of M.

Lemma 2.6. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, then for every hyperbolic homoclinic class H there are an open neighborhood $\mathcal{O}_{X,H}$ of X and a residual subset $\mathcal{R}_{X,H}$ of $\mathcal{O}_{X,H}$ such that the following properties are equivalent:*

1. $\mathrm{Leb}(W^s_Y(H_Y)) = 0$ *for every* $Y \in \mathcal{R}_{X,H}$.
2. *H is not an attractor.*

We also need the following lemma essentially proved in [5]. We will use the notation

$$\max(Y, U) = \bigcap_{t\in\mathbb{R}} Y_t(U),$$

for all flow Y and all subset U of M.

Lemma 2.7. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, then every sectional-hyperbolic attractor with singularities of either X or $-X$ has zero Lebesgue measure.*

Proof. Given $U \subset M$ we define $\mathcal{U}(U)$ as the set of flows Y such that $\max(Y, U)$ is a sectional-hyperbolic set with singularities of Y. We shall assume that U is open. Since the existence of a sectional-hyperbolic splitting is an open property, we have that $\mathcal{U}(U)$ is open in $\mathfrak{X}^1(\mathrm{M})$.

Now define $\mathcal{U}(U)_n$ as the set of $Y \in \mathcal{U}(U)$ such that $\mathrm{Leb}(\max(Y, U)) < 1/n$. It was proved in [5] that $\mathcal{U}(U)_n$ is open and dense in $\mathcal{U}(U)$.

Define $\mathcal{R}(U)_n = \mathcal{U}(U)_n \cup (\mathfrak{X}^1(M) \setminus \mathrm{Cl}(\mathcal{U}(U))$ which is open and dense set in $\mathfrak{X}^1(M)$. Let $\{U_m\}$ be a countable basis of the topology, and $\{O_m\}$ be the set of finite unions of such U_m's. Define

$$\mathcal{L} = \bigcap_m \bigcap_n \mathcal{R}(O_m)_n.$$

This is clearly a residual subset of three-dimensional flows. We can assume without loss of generality that $\mathcal{L}$ is symmetric, i.e., $X \in \mathcal{L}$ if and only if $-X \in \mathcal{L}$. Take $X \in \mathcal{L}$. Let Λ be a sectional-hyperbolic attractor for X. Then, there exists m such that $\Lambda = \max(X, O_m)$. Thus $X \in \mathcal{U}(O_m)$ and so $X \in \mathcal{U}(O_m)_n$ for every n yielding $\mathrm{Leb}(\Lambda) = 0$. Analogously, since $\mathcal{L}$ is symmetric, we obtain that $\mathrm{Leb}(\Lambda) = 0$ for every sectional-hyperbolic attractor with singularities of $-X$. $\square$

Now we prove a result whose proof is similar to that of Theorem 3 in [4]. In its statement $\mathrm{PSaddle}_d(X)$ denotes the set of periodic dissipative saddles of X.

Theorem 2.8. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $Y \in \mathcal{R}$ and $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition, then every homoclinic class H associated to a dissipative periodic saddle satisfying $\mathrm{Leb}(W^s_Y(H)) > 0$ is a sectional-hyperbolic attractor for Y.*

Proof. Let 2^M_c be the set formed by the compact subsets of M. It is well known that 2^M_c is compact metric space if endowed with the Hausdorff distance,

$$d_h(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}.$$

Define the map $S : \mathfrak{X}^1(\mathrm{M}) \to 2^M_c$ by $S(X) = \mathrm{Cl}(\mathrm{PSaddle}_d(X))$. This map is continuous in a residual subset $\mathcal{N}$ of $\mathfrak{X}^1(\mathrm{M})$ (for the corresponding definitions and facts see [21], [22]).

Now, we observe that although the results in [1] were proved for diffeomorphisms they are valid for flows too. Then, there is a residual subset $\mathcal{R}^*$ of three-dimensional flows X such that for every sectional-hyperbolic attractor C for X (resp. $-X$) there are neighborhoods $U_{X,C}$ of C, $\mathcal{U}_{X,C}$ of X and a residual subset $\mathcal{R}^0_{X,C}$ of $\mathcal{U}_{X,C}$ such that for all $Y \in \mathcal{R}^0_{X,C}$ if $Z = Y$ (resp. $Z = -Y$) then

$$C_Y = \bigcap_{t\geq 0} Z_t(U_{X,C}) \text{ is a sectional-hyperbolic attractor for } Z. \tag{2.1}$$

Define $\mathcal{R} = \mathcal{N} \cap \mathcal{R}^*$. Clearly $\mathcal{R}$ is a residual subset of three-dimensional flows. Define

$$\mathcal{A} = \{X \in \mathcal{R} : \mathrm{Cl}(\mathrm{PSaddle}_d(X)) \text{ has no spectral decomposition}\}.$$

Fix $X \in \mathcal{R} \setminus \mathcal{A}$. Then, $X \in \mathcal{R}$ and $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition

$$\mathrm{Cl}(\mathrm{PSaddle}_d(X)) = \left(\bigcup_{i=1}^{r_X} H^i\right) \cup \left(\bigcup_{j=1}^{a_X} A^j\right) \cup \left(\bigcup_{k=1}^{b_X} R^k\right)$$

into hyperbolic homoclinic classes H_i ($1 \leq i \leq r_X$), sectional-hyperbolic attractors A^j for X ($1 \leq j \leq a_X$), and sectional-hyperbolic attractors R^k for $-X$ ($1 \leq k \leq b_X$).

As $X \in \mathcal{R}^*$, we can consider for each $1 \leq i \leq r_X$, $1 \leq j \leq a_X$ and $1 \leq k \leq b_X$ the neighborhoods $\mathcal{O}_{X,H^i}$, $\mathcal{U}_{X,A^j}$ and $\mathcal{U}_{X,R^k}$ of X as well as their residual subsets $\mathcal{R}_{X,H^i}$, $\mathcal{R}^0_{X,A^j}$ and $\mathcal{R}^0_{X,R^k}$ given by Lemma 2.6 and (2.1) respectively.

Define

$$\mathcal{O}_X = \left(\bigcap_{i=1}^{r_X} \mathcal{O}_{X,H^i}\right) \cap \left(\bigcap_{j=1}^{a_X} \mathcal{U}_{X,A^j}\right) \cap \left(\bigcap_{k=1}^{b_X} \mathcal{U}_{X,R^k}\right)$$

and

$$\mathcal{R}_X = \left(\bigcap_{i=1}^{r_X} \mathcal{R}_{X,H^i}\right) \cap \left(\bigcap_{j=1}^{a_X} \mathcal{R}^0_{X,A^j}\right) \cap \left(\bigcap_{k=1}^{b_X} \mathcal{R}^0_{X,R^k}\right).$$

Clearly $\mathcal{R}_X$ is residual in $\mathcal{O}_X$.

From the proof of Lemma 4 in [4] we obtain for each $1 \leq i \leq r_X$ a compact neighborhood $U_{X,i}$ of H^i such that

$$H^i_Y = \bigcap_{t \in \mathbb{R}} Y_t(U_{X,i}) \tag{2.2}$$

is hyperbolic and topologically equivalent to H^i, for all $Y \in \mathcal{O}_{X,H^i}$.

As $X \in \mathcal{N}$, S is continuous at X so we can further assume that

$$\mathrm{Cl}(\mathrm{PSaddle}_d(Y)) \subset \left(\bigcup_{i=1}^{r_X} U_{X,i}\right) \cup \left(\bigcup_{j=1}^{a_X} U_{X,A^j}\right) \cup \left(\bigcup_{k=1}^{b_X} U_{X,R^k}\right), \text{ for all } Y \in \mathcal{O}_X.$$

It follows that

$$\mathrm{Cl}(\mathrm{PSaddle}_d(Y)) = \left(\bigcup_{i=1}^{r_X} H^i_Y\right) \cup \left(\bigcup_{j=1}^{a_X} A^j_Y\right) \cup \left(\bigcup_{k=1}^{b_X} R^k_Y\right), \text{ for all } Y \in \mathcal{R}_X. \tag{2.3}$$

Next we take a sequence $X^i \in \mathcal{R} \setminus \mathcal{A}$ which is dense in $\mathcal{R} \setminus \mathcal{A}$.

If necessary we can replace $\mathcal{O}_{X^i}$ by $\mathcal{O}'_{X^i}$ where

$$\mathcal{O}'_{X^0} = \mathcal{O}_{X^0} \text{ and } \mathcal{O}'_{X^i} = \mathcal{O}_{X^i} \setminus \left(\bigcup_{j=0}^{i-1} \mathcal{O}_{X^j}\right), \text{ for } i \geq 1,$$

in order to assume that the collection $\{\mathcal{O}_{X^i} : i \in \mathbb{N}\}$ is pairwise disjoint.

Define

$$\mathcal{O}_{12} = \bigcup_{i \in \mathbb{N}} \mathcal{O}_{X^i} \quad \text{and} \quad \mathcal{R}'_{12} = \bigcup_{i \in \mathbb{N}} \mathcal{R}_{X^i}.$$

We claim that $\mathcal{R}'_{12}$ is residual in $\mathcal{O}_{12}$.

Indeed, for all $i \in \mathbb{N}$ write $\mathcal{R}_{X^i} = \bigcap_{n\in\mathbb{N}} \mathcal{O}_i^n$, where $\mathcal{O}_i^n$ is open-dense in $\mathcal{O}_{X^i}$ for every $n \in \mathbb{N}$. Since $\{\mathcal{O}_{X^i} : i \in \mathbb{N}\}$ is pairwise disjoint, we obtain

$$\bigcap_{n\in\mathbb{N}} \bigcup_{i\in\mathbb{N}} \mathcal{O}_i^n \subset \bigcup_{i\in\mathbb{N}} \bigcap_{n\in\mathbb{N}} \mathcal{O}_i^n = \bigcup_{i\in\mathbb{N}} \mathcal{R}_{X^i} = \mathcal{R}'_{12}.$$

As $\bigcup_{i\in\mathbb{N}} \mathcal{O}_i^n$ is open-dense in $\mathcal{O}_{12}$, $\forall n \in \mathbb{N}$, we obtain the claim.

Finally we define

$$\mathcal{R}_{11} = \mathcal{A} \cup \mathcal{R}'_{12}.$$

Since $\mathcal{R}$ is a residual subset of three-dimensional flows, we conclude as in Proposition 2.6 of [27] that $\mathcal{R}_{11}$ is also a residual subset of three-dimensional flows.

Take $Y \in \mathcal{R}_{11}$ such that $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition and let H be a homoclinic class associated to a dissipative saddle of Y. Then, $H \subset \mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ by Birkhoff-Smale's Theorem [18].

Since $\mathrm{Cl}(\mathrm{PSaddle}_d(Y))$ has a spectral decomposition, we have $Y \notin \mathcal{A}$ so $Y \in \mathcal{R}'_{12}$ thus $Y \in \mathcal{R}_X$ for some $X \in \mathcal{R} \setminus \mathcal{A}$. As $Y \in \mathcal{R}_X$, (2.3) implies $H = H_Y^i$ for some $1 \leq i \leq r_X$ or $H = A_Y^j$ for some $1 \leq j \leq a_X$ or $H = R_Y^k$ for some $1 \leq k \leq b_X$.

Now, suppose that $\mathrm{Leb}(W_Y^s(H)) > 0$. Since $Y \in \mathcal{R}_X$, we have $Y \in \mathcal{R}^0_{X,R^k}$ for all $1 \leq k \leq b_X$. As $W_Y^s(R_Y^k) \subset R_Y^k$ for every $1 \leq k \leq b_X$, we conclude by Lemma 2.7 that $H \neq R_Y^k$ for every $1 \leq k \leq b_X$.

If $H = A_Y^j$ for some $1 \leq j \leq a_X$ then H is a sectional-hyperbolic attractor and we are done. Otherwise, $H = H_Y^i$ for some $1 \leq i \leq r_X$. As $Y \in \mathcal{R}_X$, we have $Y \in \mathcal{R}_{X,H^i}$ and, since $Y \in \mathcal{R}'_{12}$, we conclude from Lemma 2.6 that H^i is an attractor. But by (2.2) we have that H_Y^i and H^i are topologically equivalent, so, H_Y^i is an attractor and hence H is a hyperbolic attractor. $\square$

Proof of Theorem A. Let $\mathcal{R}$ be the intersection of the residual subsets of $\mathfrak{X}^1(\mathrm{M})$ as in Theorems 2.4, 2.5 and Lemma 2.6. Take $X \in \mathcal{R}$ which has only a finite number of sinks. It follows as in [4] or [32] that $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a LPF-dominated splitting. Since $\mathrm{PSaddle}_d(X)$ is homoclinically closed, we conclude from Theorem 2.4 that $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$ has a spectral decomposition. On the other hand, Theorem 2.5 and Lemma 2.6 imply that the omega-limit set of almost every point meets $\mathrm{Cl}(\mathrm{PSaddle}_d(X))$. Then, [11] implies that such a limit set is contained in one of the elements of the spectral decomposition. Finally, by Theorem 2.8, the basin of an element in this decomposition has zero Lebesgue-measure except for the sectional-hyperbolic attractors of X. This completes the proof. $\square$

§3. Proof of Theorem 2.4

For this we need some preliminars.

A compact invariant set Λ is called *chain transitive* if for any $\epsilon > 0$, for any $x, y \in \Lambda$, there are finite sequences $(x_i)_{i=0}^n$ and $(t_i)_{i=0}^{n-1} \subset [1, \infty[$ such that $x_0 = x$, $x_n = y$ and $d(X_{t_i}(x_i), x_{i+1}) < \epsilon$ for $0 \leq i \leq n-1$. The following result is Lemma 3.1 in [8].

Lemma 3.1. *Every chain transitive set without singularities but with a LPF-dominated splitting of a C^1 generic three-dimensional flow is hyperbolic.*

The lemma below extends the conclusion above to any compact invariant set. More precisely, we obtain the following result.

Lemma 3.2. *Every compact invariant set without singularities but with a LPF-dominated splitting of a C^1 generic three-dimensional flow is hyperbolic.*

Proof. Clearly, every transitive set is chain transitive. Then, by Lemma 3.1, there is a residual subset $\mathcal{Q}_1$ of three-dimensional flows for which every transitive set without singularities but with a LPF-dominated splitting is hyperbolic. Fix $X \in \mathcal{Q}_1$ and a compact invariant set Λ without singularities but with a LPF-dominated splitting $N_\Lambda^X = N_\Lambda^{s,X} \oplus N_\Lambda^{u,X}$. Suppose by contradiction that Λ is not hyperbolic. Then, by Zorn's Lemma, there is a minimally nonhyperbolic set $\Lambda_0 \subset \Lambda$ (c.f. p.983 in [32]). Assume for a while that Λ_0 is not transitive. Then, $\omega(x)$ and $\alpha(x) = \omega_{-X}(x)$ are proper subsets of Λ_0, for every $x \in \Lambda_0$. Therefore, both sets are hyperbolic and then we have

$$\lim_{t\to\infty} \|P_t^X(x)/N_x^{s,X}\| = \lim_{t\to\infty} \|P_{-t}^X(x)/N_x^{u,X}\| = 0, \qquad \text{for all } x \in \Lambda_0,$$

which easily implies that Λ_0 is hyperbolic (see [13]). Since this is a contradiction, we conclude that Λ_0 is transitive. As $X \in \mathcal{Q}_1$ and Λ_0 has a LPF-dominated splitting (by restriction), we conclude that Λ_0 is hyperbolic, a contradiction once more proving the result. $\square$

Let Y be a three-dimensional flow. We say that $\sigma \in \mathrm{Sing}(Y)$ is *Lorenz-like for* Y if its eigenvalues $\lambda_1, \lambda_2, \lambda_3$ are real and satisfy $\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1$. This terminology is in honor of Professor Edward Norton Lorenz [23].

The invariant manifold theory [19] asserts the existence of *stable* and *unstable* manifolds denoted by $W^s(\sigma)$, $W^u(\sigma)$ (or $W^{s,Y}(\sigma)$, $W^{u,Y}(\sigma)$ to emphasize Y) tangent at σ to the eigenspaces associated to the eigenvalues $\{\lambda_2, \lambda_3\}$ and λ_1 respectively. There is an additional invariant manifold $W^{ss,Y}(\sigma)$, the *strong stable manifold*, contained in $W^{s,Y}(\sigma)$ and tangent at σ to the eigenspace associated to λ_2.

As in the remark after Lemma 2.13 in [8] we obtain the following.

Lemma 3.3. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if σ is a singularity accumulated by periodic orbits of X and if σ has three real eigenvalues λ_1, λ_2 and $\lambda - 3$ with $\lambda_2 < \lambda_3 < 0 < \lambda_1$ (resp. $\lambda_2 < 0 < \lambda_3 < \lambda_2$) then σ is Lorenz-like for X (resp. $-X$).*

We shall use the following standard definition.

Definition 3.4. *The index $Ind(\sigma)$, of a singularity σ, is the number of eigenvalues with negative real part counted with multiplicity.*

This definition will be considered in the following lemma.

Lemma 3.5. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if $\{O_n\}_{n\in\mathbb{N}}$ is a sequence of periodic orbits of X for which $\mathrm{Cl}(\bigcup_n O_n)$ has a LPF dominated splitting and if O_n converges to a compact invariant set H with respect to the Hausdorff metric, then for every $\sigma \in H \cap \mathrm{Sing}(X)$ with $Ind(\sigma) = 2$, σ is Lorenz-like for X and $H \cap W^{ss,X}(\sigma) = \{\sigma\}$ and, for every $\sigma \in H \cap \mathrm{Sing}(X)$ with $Ind(\sigma) = 1$, σ is Lorenz-like for $-X$ and $H \cap W^{ss,-X}(\sigma) = \{\sigma\}$.*

Proof. We only prove the result for $\sigma \in H \cap \mathrm{Sing}(X)$ with $Ind(\sigma) = 2$ because the other case is similar. Clearly $H \subset \mathrm{Cl}(\bigcup_n O_n)$ and so H has a LPF-dominated splitting. Then, Proposition 2.4 in [13] implies that σ has three different real eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Since $Ind(\sigma) = 2$ we have that these eigenvalues satisfy satisfying $\lambda_2 < \lambda_3 < 0 < \lambda_1$ (up to some order). But clearly σ is accumulated by the periodic orbits O_n, so, σ is Lorenz-like for X by Lemma 3.3.

To prove $H \cap W^{ss,X}(\sigma) = \{\sigma\}$ we assume by contradiction that there is

$$x \in H \cap W^{ss,X}(\sigma) \setminus \{\sigma\}.$$

Then, we can choose sequences $x_n \in O_n$ and $t_n \to \infty$ such that $x_n \to x$ and $X_{t_n}(x_n) \to y$ for some $y \in W^{u,X}(\sigma) \setminus \{\sigma\}$. Let $N^{s,X} \oplus N^{u,X}$ denote the LPF dominated splitting over $\mathrm{Cl}(\bigcup_n O_n)$. By Proposition 2.4 in [13] we have that $N^{s,X}_y$ is almost parallel to $E^{ss,X}_\sigma$. Since $N^{s,X}_{X_{t_n}(x_n)} \to N^{s,X}_y$, we also have that $N^{s,X}_{X_{t_n}(x_n)}$ is almost parallel to $E^{ss,X}_\sigma$ (for n large). But $\lambda_2 < \lambda_3$ and $N^{s,X}_{x_n} = P_{-t_n}(X_{t_n}(x_n))N^{s,X}_{X_{t_n}(x_n)}$. Since $N^{s,X}_{X_{t_n}(x_n)}$ is almost parallel to $E^{ss,X}_\sigma$ (for n large), we conclude that the angle between $N^s_{x_n}$ and $T_xW^s(\sigma) \cap N_x$ is bounded away from 0 (for n large). On the other hand, $N^{s,X}_x = N_x \cap W^{s,X}(\sigma)$ by Proposition 2.2 in [13]. Therefore, the angle between $N^s_{x_n}$ and $N^{s,X}_x$ is bounded away from 0 too. We conclude that $N^{s,X}_{x_n} \not\to N^{s,X}_x$ as $n \to \infty$. Since $x_n \to x$, we obtain a contradiction which proves the result. $\square$

Recall that a compact invariant set Λ of a flow X is *Lyapunov stable for* X if for every neighborhood U of Λ there is a neighborhood $V \subset U$ of Λ such that $X_t(V) \subset U$, for all $t \geq 0$.

Let Λ be a compact invariant set with singularities (all hyperbolic) of X. We say that Λ *has dense singular unstable* (resp. *stable) branches* if for every $\sigma \in \Lambda \cap \mathrm{Sing}(X)$ one has $\Lambda = \omega_X(q)$ (resp. $\Lambda = \omega_{-X}(q)$) for all $q \in W^u(\sigma) \setminus \sigma$ (resp. $q \in W^s(\sigma) \setminus \{\sigma\}$).

Applying Lemma 4.1 in [28] (or [11]) we obtain the following lemma.

Lemma 3.6. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if $\{O_n\}_{n\in\mathbb{N}}$ and if O_n converges to a compact invariant set H with respect to the Hausdorff metric, then the following properties holds $\forall \sigma \in H \cap \mathrm{Sing}(X)$:*

1. *If $Ind(\sigma) = 2$, then $\mathrm{Cl}(W^u(\sigma))$ is a Lyapunov stable set with dense singular unstable branches of X. Moreover, $\mathrm{Cl}(W^u(\sigma)) = H$.*
2. *If $Ind(\sigma) = 1$, then $\mathrm{Cl}(W^s(\sigma))$ is a Lyapunov stable set with dense singular stable branches of $-X$. Moreover, $\mathrm{Cl}(W^s(\sigma)) = H$.*

Combining lemmas 3.5 and 3.6 we obtain the following result.

Corollary 3.7. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if $\{O_n\}_{n\in\mathbb{N}}$, if $\mathrm{Cl}(\bigcup_n O_n)$ has a LPF dominated splitting and if O_n converges to a compact invariant set H with respect to the Hausdorff metric, then one of the following alternatives hold:*

1. *Every $\sigma \in H \cap \mathrm{Sing}(X)$ is Lorenz-like for X and $H \cap W^{ss,X}(\sigma) = \{\sigma\}$.*
2. *Every $\sigma \in H \cap \mathrm{Sing}(X)$ is Lorenz-like for $-X$ and $H \cap W^{ss,-X}(\sigma) = \{\sigma\}$.*

Next we formulate the key result by Crovisier and Yang. For simplicity, we say that a compact invariant set Γ of a flow Y is *dominated* if Γ has a dominated splitting for Y. By a *minimal repeller* we mean a minimal set which is also a repeller (i.e. an attractor for the time-reversed flow).

Theorem 3.8 (Theorem 1 in [12])**.** *Let Γ be a compact invariant set with a LPF-dominated splitting of a C^3 three-dimensional flow Y. If every periodic point in Γ is hyperbolic saddle, every $\sigma \in \Gamma \cap \mathrm{Sing}(Y)$ is Lorenz-like satisfying $W^{ss}(\sigma) \cap \Gamma = \{\sigma\}$ and Γ does not contain a minimal repeller whose dynamics is the suspension of an irrational rotation of the circle, then Γ is dominated.*

We shall use this result to prove the following lemma.

Lemma 3.9. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if Λ is a transitive set with a LPF-dominated splitting and if every singularity $\sigma \in \Lambda$ is Lorenz-like for X satisfying $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$, then Λ is dominated for X.*

Proof. The proof follows as in the proof of Lemma 3.1 in [8] but with Theorem 3.8 playing the role of Theorem B in [6]. We include details for the sake of completeness.

For any $K \subset M$ we define $CR(X, K)$ as the set of those points x for which there is a chain transitive set Λ satisfying $x \in \Lambda \subset K$. This is a compact invariant set of X contained in K.

Take a countable basis $\{U_n\}$ of M and let $\mathcal{O} = \{O_n\}$ be the sequence of all finite union of elements of $\{U_n\}$ (in this proof O does not mean periodic orbit). For each n we define

$$\mathcal{D}_n = \{X \in \mathfrak{X}^1(\mathrm{M}) : CR(X, \mathrm{Cl}(O_n)) \text{ is } \emptyset \text{ or dominated for } X\},$$

and

$$\mathcal{N}_n = \{X \in \mathfrak{X}^1(\mathrm{M}) : CR(Y, \mathrm{Cl}(O_n)) \text{ is neither } \emptyset \text{ nor dominated for } Y,$$
$$\forall Y \in \mathfrak{X}^1(\mathrm{M}) \text{ close to } X\}.$$

By Lemma 2.9 in [8] (which is true for dominated sets instead of hyperbolic sets) and Lemma 2.10 in [8] we have that $\mathcal{D}_n \cup \mathcal{N}_n$ is open and dense in $\mathfrak{X}^1(\mathrm{M})$. It follows that

$$\mathcal{G} = \bigcap_n (\mathcal{D}_n \cup \mathcal{N}_n)$$

is residual in $\mathfrak{X}^1(\mathrm{M})$. Let us prove that every $X \in \mathcal{G}$ satisfies the conclusion of the lemma.

Indeed, take Λ as in the hypothesis of the lemma and suppose by contradiction that Λ is not dominated for X. Since Λ is compact with a LPF-dominated splitting, and every singularity $\sigma \in \Lambda$ is Lorenz-like satisfying $W^{ss}(\sigma) \cap \Lambda = \{\sigma\}$, there is n such that $\Lambda \subset O_n$, $\max(X, O_n)$ has a LPF-dominated splitting and every singularity $\sigma \in \max(X, O_n)$ is Lorenz-like for X satisfying

$$W^{ss}(\sigma) \cap \max(X, O_n) = \{\sigma\}.$$

Since these last properties are open, there is a neighborhood $\mathcal{U}$ of X such that, for every $Y \in \mathcal{U}$, $\max(Y, \mathrm{Cl}(O_n))$ has a LPF-dominated splitting for Y and every $\sigma \in \max(Y, \mathrm{Cl}(O_n))$ is Lorenz-like satisfying

$$W^{ss,Y}(\sigma) \cap \max(Y, \mathrm{Cl}(O_n)) = \{\sigma\}.$$

Since Λ is not dominated for X (and $\emptyset \neq \Lambda \subset CR(X, \mathrm{Cl}(O_n))$), we see that $X \notin \mathcal{D}_n$. As $X \in \mathcal{G}$, we conclude that $X \in \mathcal{N}_n$. Now, it is not difficult to see that any minimal repeller whose dynamics is the suspension of a irrational rotation of the circle can be turned into a Morse-Smale dynamics by small C^3 perturbations. From this we have that the nonexistence of these minimal sets is generic in the C^3 topology. Therefore, we can take a C^3 Kupka-Smale flow $Y \in \mathcal{N}_n \cap \mathcal{U}$ having no minimal repellers whose dynamics is the suspension of a irrational rotation of the circle.

Since $Y \in \mathcal{N}_n$, one has that $CR(Y, \mathrm{Cl}(O_n))$ is not dominated. On the other hand, it is apparent that any sink (or source) cannot be accumulated by another sinks or sources. Therefore, $CR(Y, \mathrm{Cl}(O_n)) \cap (\mathrm{Sink}(Y) \cup \mathrm{Source}(Y))$ consists of isolated orbits. It follows that,

$$\Gamma = CR(Y, \mathrm{Cl}(O_n)) \setminus (\mathrm{Sink}(Y) \cup \mathrm{Source}(Y))$$

is a compact (and obviously invariant) for Y. If Γ were dominated, then we could defined a dominated splitting in the whole $CR(Y, \mathrm{Cl}(O_n))$. Since $CR(Y, \mathrm{Cl}(O_n))$ is not dominated, we conclude that Γ *is not dominated for* Y.

Nevertheless, $Y \in \mathcal{U}$ thus $\max(Y, \mathrm{Cl}(O_n))$ has a LPF-dominated splitting and, also, every $\sigma \in Sing(Y) \cap \max(Y, \mathrm{Cl}(O_n))$ is Lorenz-like satisfying $W^{ss,Y}(\sigma) \cap \max(Y, \mathrm{Cl}(O_n)) = \{\sigma\}$. As $\Gamma \subset CR(Y, \mathrm{Cl}(O_n)) \subset \max(Y, \mathrm{Cl}(O_n))$ we conclude the same for Γ instead of $\max(Y, \mathrm{Cl}(O_n))$. Since Γ has neither sinks nor sources, we conclude from Theorem 3.8 that Γ *is dominated for* Y. This is a contradiction so Λ is dominated for X. The proof follows. □

From this lemma we obtain the following proposition.

Proposition 3.10. *There is a residual subset $\mathcal{R}$ of $\mathfrak{X}^1(\mathrm{M})$ such that if $X \in \mathcal{R}$, if $\{O_n\}_{n\in\mathbb{N}}$ is a sequence of periodic orbits of X, if $\mathrm{Cl}(\bigcup_n O_n)$ has a LPF dominated splitting and if O_n converges to a compact invariant set H with respect to the Hausdorff metric, then H is a hyperbolic set (if $H \cap \mathrm{Sing}(X) = \emptyset$), a sectional-hyperbolic attractor for X (if H contains a singularity of index 2) or a sectional-hyperbolic attractor for $-X$ (if H contains a singularity of index 1).*

Proof. Let $\mathcal{R}$ be the intersection of the residual subsets of $\mathfrak{X}^1(\mathrm{M})$ as in Lemma 3.2, Corollary 3.7 and Lemma 3.9. Take any $X \in \mathcal{R}$.

If $H \cap \mathrm{Sing}(X) = \emptyset$, then H is hyperbolic by Lemma 3.2. Now, suppose that H contains a singularity σ of index 2. Clearly, H is nontrivial (i.e. not equal to a single orbit) and by Corollary 3.7 we also have that it is the chain-recurrent class of σ. Since H contains a singularity of index 2, we have from the first alternative of Corollary 3.7 that every $\sigma \in H \cap \mathrm{Sing}(X)$ is Lorenz-like and satisfies $H \cap W^{ss,X}(\sigma) = \{\sigma\}$. Then, we can apply Lemma 3.9 to conclude that H has a dominated splitting for X. Since H is the chain recurrent class of σ we conclude from Theorem C in [15] that H is a sectional-hyperbolic attractor for X. If H contains a singularity of index 1, then the same argument with $-X$ instead of X implies that H is a sectional-hyperbolic attractor for $-X$. This concludes the proof. □

Proof of Theorem 2.4. Let $\mathcal{R}$ be the residual subset of $\mathfrak{X}^1(M)$ as in Proposition 3.10. Take any $X \in \mathcal{R}$. Let $\mathcal{P} \subset \text{PSaddle}(X)$ be homoclinically closed. Suppose that $\text{Cl}(\mathcal{P})$ has a LPF-dominated splitting.

By taking the Hausdorff limit of sequences of periodic orbits in $\mathcal{P}$ accumulating on the singularities of X in $\text{Cl}(\mathcal{P})$ we obtain from Proposition 3.10 that every $\sigma \in \text{Cl}(\mathcal{P}) \cap \text{Sing}(X)$ belongs to a sectional-hyperbolic attractor for either X or $-X$.

On the other hand, $\mathcal{P}$ is homoclinically closed so

$$\text{Cl}(\mathcal{P}) = \text{Cl}\left(\bigcup\{H(p) : p \in \mathcal{P}\}\right). \tag{3.1}$$

(Notice that this union can be assumed to be disjoint by genericity, see [11].)

We claim that the family $\{H(p) : p \in \mathcal{P}\}$ is finite. Otherwise, there is an infinite sequence $p_k \in \mathcal{P}$ for which the corresponding homoclinic classes $H(p_k)$ are pairwise different (hence disjoint). Consider the closure $\text{Cl}(\bigcup_k H(p_k))$, which is a compact invariant set contained in $\text{Cl}(\mathcal{P})$. If this closure does not contain any singularity, then it would be a hyperbolic set by Lemma 3.1. Since the number of homoclinic classes contained in any hyperbolic set is finite, we obtain a contradiction proving that $\text{Cl}(\bigcup_k H(p_k))$ contains a singularity $\sigma \in \text{Cl}(\mathcal{P})$. But, as we have seen, any of these singularities belong to a sectional-hyperbolic attractor for either X or $-X$. Since there are finitely many singularities, it must exist distinct k, k' satisfying $H(p_k) = H(p_{k'})$. But this is an absurd, so the claim follows.

Combining the claim with (3.1) we obtain homoclinic classes $H_1, \cdots, H_k$ whose union is $\text{Cl}(\mathcal{P})$. Since every $\sigma \in \text{Cl}(\mathcal{P}) \cap \text{Sing}(X)$ belongs to a sectional-hyperbolic attractor for either X or $-X$, the ones with singularities are sectional-hyperbolic attractors for either X or $-X$. The remainder ones are hyperbolic by Lemma 3.1. This completes the proof. □

Appendix A

In this appendix we shall prove Theorem 2.5. The proof is similar to Theorem 2 in [4].

The proof needs some preliminars.

Let δ_p be the Dirac measure supported on a point p. Given a three-dimensional flow X and $t > 0$ we define the Borel probability measure

$$\mu_{p,t} = \frac{1}{t}\int_0^t \delta_{X_s(p)} ds.$$

(Notation $\mu_{p,t}^X$ indicates dependence on X.)

Denote by $\mathcal{M}(p, X)$ as the set of Borel probability measures $\mu = \lim_{k\to\infty} \mu_{p,t_k}$ for some sequence $t_k \to \infty$. Notice that each $\mu \in \mathcal{M}(p, X)$ is *invariant*, i.e., $\mu \circ X_{-t} = \mu$ for every $t \geq 0$. With these notations we have the following lemma.

Lemma A.1. *For every three-dimensional flow X there is a full Lebesgue measure set L_X of points x satisfying*

$$\int \operatorname{div} X d\mu \leq 0, \qquad \forall \mu \in \mathcal{M}(x, X).$$

Proof. For every $\delta > 0$ we define

$$\Lambda_\delta(X) = \{x : \exists N_x \in \mathbb{N} \text{ such that } |\det DX_t(x)| < (1+\delta)^t, \forall t \geq N_x\}.$$

We assert that $\operatorname{Leb}(\Lambda_\delta(X)) = 1$, for every $\delta > 0$. This assertion is similar to one for surface diffeomorphisms given by Araujo [2]. For completeness we include the proof.

Define

$$\Lambda_\rho(s) = \{x : \exists N_x \in \mathbb{N} \text{ such that } |\det DX_{ns}(x)| < (1+\rho)^{ns}, \forall n \geq N_x\}, \forall s, \rho > 0.$$

We claim that

$$\operatorname{Leb}(\Lambda_\rho(s)) = 1, \qquad \forall s, \rho > 0. \tag{A.1}$$

Indeed, take $\epsilon > 0$ and for each integer n we define

$$\Omega(n) = \{x : |\det DX_{ns}(x)| \geq (1+\rho)^{ns}\}.$$

On the one hand, we get easily that

$$\Lambda_\rho(s) = \bigcup_{N\in\mathbb{N}} \left(\bigcup_{n\geq N} \Omega(n) \right)^c,$$

where $(\cdot)^c$ above denotes the complement operation. On the other hand,

$$1 = \int |\det DX_{ns}(x)| dm \geq \int_{\Omega(n)} |\det DX_{ns}(x)| dm \geq (1+\rho)^{ns} \operatorname{Leb}(\Omega(n)),$$

yielding $\operatorname{Leb}(\Omega(n)) \leq \frac{1}{(1+\rho)^{ns}}$, for all n.

Take N large so that

$$\sum_{n=N}^{\infty} \frac{1}{(1+\rho)^{ns}} < \epsilon.$$

Therefore,

$$\mathrm{Leb}(\Lambda_\rho(s)) \geq 1 - m\left(\bigcup_{n\geq N} \Omega(n)\right) \geq 1 - \sum_{n=N}^{\infty} \frac{1}{(1+\rho)^{ns}} > 1-\epsilon.$$

As $\epsilon > 0$ is arbitrary we get (A.1). This proves the claim.

Now, we continue with the proof of the assertion.

Fix $0 < \rho < \delta$ and $\eta > 0$ such that

$$(1+\eta)(1+\rho)^t < (1+\delta)^t, \qquad \forall t \geq 1.$$

Choose $0 < s < 1$ satisfying

$$|\det DX_r(y) - 1| \leq \eta, \qquad \forall |r| \leq s, \forall y \in M.$$

Take $x \in \Lambda_\rho(s)$. Then, there is an integer $N_x > 1$ such that

$$|\det DX_{ns}(x)| < (1+\rho)^{ns}, \qquad \forall n \geq N_x.$$

Now, if $t \geq N_x$ there are $n \geq N_x$ and $0 \leq r < s$ such that

$$ns \leq t < ns + r.$$

Thus,

$$|\det DX_t(x)| = |\det DX_{t-ns}(X_{ns}(x))| \cdot |\det DX_{ns}(x)| < (1+\eta)(1+\rho)^{ns}.$$

Then, the choice of η, ρ above yields $|\det DX_t(x)| < (1+\delta)^t$ for all $t \geq N_x$. So,

$$\Lambda_\rho(s) \subset \Lambda_\delta(X).$$

But (A.1) implies $\mathrm{Leb}(\Lambda_\rho(s)) = 1$ so $\mathrm{Leb}(\Lambda_\delta(X)) = 1$ proving the assertion.

To continue with the proof of the lemma, we notice that $\Lambda_{\delta'}(X) \subset \Lambda_\delta(X)$ whenever $\delta' \leq \delta$. It then follows from the assertion that L_X has full Lebesgue measure, where

$$L_X = \bigcap_{k\in\mathbb{N}^+} \Lambda_{\frac{1}{k}}(X).$$

Now, take $x \in L_X$, $\mu \in \mathcal{M}(x, X)$ and $\epsilon > 0$. Fix $k > 0$ with $\log\left(1 + \frac{1}{k}\right) < \epsilon$. By definition we have $x \in \Lambda_{\frac{1}{k}}(X)$ and so there is $N_x \in \mathbb{N}^+$ such that

$$|\det DX_t(x)|^{\frac{1}{t}} < 1 + \frac{1}{k}, \qquad \forall t \geq N_x.$$

Take a sequence $\mu_{x,t_i} \to \mu$ with $t_i \to \infty$. Then, we can assume $t_i \geq N_x$ for all i.

From this and Liouville's Formula [24] we obtain,

$$\int \operatorname{div} X d\mu = \lim_{i\to\infty} \int \operatorname{div} X d\mu_{x,t_i} = \lim_{i\to\infty} \frac{1}{t_i} \int_0^{t_i} \operatorname{div} X(X_s(x)) ds =$$

$$\lim_{i\to\infty} \frac{1}{t_i} \log|\det DX_{t_i}(x)| \le \log\left(1+\frac{1}{k}\right) < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we are done. □

We shall use the following version of the classical Franks's Lemma [14] (c.f. Appendix A in [9]).

Lemma A.2 (Franks's Lemma for flows)**.** *For every three-dimensional flow X and every neighborhood $W(X)$ of X there is a neighborhood $W_0(X) \subset W(X)$ of X such that for any $T > 0$ there exists $\epsilon > 0$ such that for any $Z \in W_0(X)$ and $p \in \mathrm{Per}(Z)$, any tubular neighborhood U of $O_Z(p)$, any partition $0 = t_0 < t_1 < \dots < t_n = t_{p,Z}$, with $t_{i+1} - t_i < T$ and any family of linear maps $L_i : N_{Z_{t_i}(p)} \to N_{Z_{t_{i+1}}(p)}$ satisfying*

$$\left\| L_i - P^Z_{t_{i+1}-t_i}(Z_{t_i}(p)) \right\| < \epsilon, \quad \textit{for every } i \textit{ with } 0 \le i \le n-1,$$

there exists $Y \in W(X)$ with $Y = Z$ along $O_Z(p)$ and outside U such that

$$P^Y_{t_{i+1}-t_i}(Y_{t_i}(p)) = L_i, \quad \textit{for every } i \textit{ with } 0 \le i \le n-1.$$

Proof of Theorem 2.5. Let $S : \mathfrak{X}^1(\mathrm{M}) \to 2^M_c$ be the map defined by

$$S(X) = \mathrm{Cl}(\mathrm{Saddle}_d(X)) \cup \mathrm{Cl}(\mathrm{Sink}(X)).$$

It follows easily from the continuous dependence of the eigenvalues of a periodic point with respect to X that there is a residual subset $\mathcal{A} \subset \mathfrak{X}^1(\mathrm{M})$ where S is continuous [21],[22].

By the Kupka-Smale Theorem [18] there is a residual subset of Kupka-Smale three-dimensional flows $\mathcal{KS}$.

By the Ergodic Closing Lemma for flows (Theorem 3.9 in [33] or the Corollary in p.1270 of [25]) there is another residual subset $\mathcal{B}$ of three-dimensional flows X such that for every non-atomic ergodic measure μ of X there are sequences $Y^k \to X$ and $p_k \in \mathrm{Per}(Y^k)$ such that $\mu^{Y^k}_{p_k, t_{p_k, Y^k}} \to \mu$.

Define $\mathcal{D} = \mathcal{A} \cap \mathcal{KS} \cap \mathcal{B}$. Then, $\mathcal{D}$ is a residual subset of three-dimensional flows. Without loss of generality we can assume that $\operatorname{div} X(\sigma) \neq 0$ for every $(X, \sigma) \in \mathcal{D} \times \mathrm{Sing}(X)$ (this is a generic property).

To prove the result we only need to prove

$$L_X \subset W^s_w(\mathrm{Dis}(X)), \qquad \forall X \in \mathcal{D},$$

where L_X is the full Lebesgue measure set in Lemma A.1.

Fix $X \in \mathcal{D}$ and $x \in L_X$. Since $\mathcal{M}(x, X) \neq \emptyset$, the Ergodic Decomposition Theorem [24] and Lemma A.1 allow us to find an ergodic invariant measure μ supported on $\omega(x)$ satisfying $\int \operatorname{div} X d\mu \leq 0$.

If μ were atomic, we would have that $\mu = \delta_\sigma$ for some $\sigma \in \operatorname{Sing}(X)$. In particular, $\sigma \in \omega(x)$. But $\operatorname{div} X(\sigma) = \int \operatorname{div} X d\delta_\sigma \leq 0$. As $X \in \mathcal{D}$, we have $\operatorname{div} X(\sigma) \neq 0$ so $\operatorname{div} X(\sigma) < 0$ thus $\sigma \in \operatorname{Crit}_d(X)$ yielding $x \in W^s_w(\operatorname{Dis}(X))$.

Now, we will assume that μ is non-atomic and, by contradiction, that $x \notin W^s_w(\operatorname{Dis}(X))$. Then, $\omega(x) \cap \operatorname{Dis}(X) = \emptyset$. Since $X \in \mathcal{KS}$, we have $\operatorname{Dis}(X) = S(X)$ thus $\omega(x) \cap S(X) = \emptyset$. Since S is upper-semicontinuous at $X \in \mathcal{A}$, we can arrange neighborhoods U of $\omega(x)$ and $W(X)$ of X such that

$$U \cap (\operatorname{Saddle}_d(Z) \cup \operatorname{Sink}(Z)) = \emptyset, \qquad \forall Z \in W(X). \tag{A.2}$$

Put $W(X)$ and $T = 1$ in the Franks's Lemma for flows to obtain $\epsilon > 0$ and the neighborhood $W_0(X) \subset W(X)$ of X. Set

$$C = \sup\{\|P^Z_t(x)\| : (Z, x, t) \in W(X) \times M \times [0, 1]\}$$

and fix $\delta > 0$ such that

$$|1 - e^{-\frac{\delta}{2}}| < \frac{\epsilon}{C}.$$

Since $X \in \mathcal{B}$ and μ is non-atomic, there are sequences $Y^k \to X$ and $p_k \in \operatorname{Per}(Y^k)$ such that $\mu^{Y^k}_{p_k, t_{p_k, Y^k}} \to \mu$. Since $\int \operatorname{div} X d\mu \leq 0$ and μ is supported on $\omega(x) \subset U$, we can fix k such that

$$p_k \in U, \quad Y^k \in W_0(X) \quad \text{and} \quad |\det P^{Y^k}_{t_{p_k, Y^k}}(p_k)| < e^{t_{p_k, Y^k}\delta}.$$

Once we fix this k, write $t_{p_k, Y^k} = n + r$ for some $n \in \mathbb{N}^+$ and some $0 \leq r < 1$. This induces the partition $0 = t_0 < t_1 < \dots < t_{n+1} = t_{p_k, Y^k}$ given by $t_i = i$ for $1 \leq i \leq n$. It turns out that $t_{i+1} - t_i = 0$ (for $0 \leq i \leq n-1$) and $t_{n+1} - t_n = t_{p_k, Y^k} - n = r$ therefore, $t_{i+1} - t_i \leq 1$ for $0 \leq i \leq n$.

Define the linear maps $L_i : N^{Y^k}_{Y^k_{t_i}(p)} \to N^{Y^k}_{Y^k_{t_{i+1}}(p)}$ by

$$L_i = e^{-\frac{\delta}{2}} P^{Y^k}_{t_{i+1} - t_i}(Y^k_{t_i}(p_k)), \qquad \forall 0 \leq i \leq n.$$

A direct computation shows

$$\left\| L_i - P^{Y^k}_{t_{i+1} - t_i}(Y^k_{t_i}(p_k)) \right\| \leq |1 - e^{-\frac{\delta}{2}}| C < \epsilon, \qquad \forall 0 \leq i \leq n.$$

Then, by the Franks's Lemma for flows, there exists $Z \in W(X)$ with $Z = Y^k$ along $O_{Y^k}(p_k)$ such that

$$P^Z_{t_{i+1} - t_i}(Z_{t_i}(p_k)) = L_i, \qquad \text{for every } i \text{ with } 0 \leq i \leq n.$$

Consequently, $t_{p_k,Z} = t_{p_k,Y^k}$ and also $P^Z_{t_{p_k,Z}}(p_k) = e^{-t_{p_k,Y^k}\frac{\delta}{2}} P^{Y^k}_{t_{p_k,Y^k}}(p_k)$ thus

$$|\det P^Z_{t_{p_k,Z}}(p_k)| = e^{-t_{p_k,Y^k}\delta} |\det P^{Y^k}_{t_{p_k,Y^k}}(p_k)| < 1.$$

Up to a small perturbation if necessary we can assume that p_k has no eigenvalues of modulus 1. Then, $p_k \in \mathrm{PSaddle}_d(Z) \cup \mathrm{Sink}(Z)$ by the previous inequality which implies $p_k \in U \cap (\mathrm{PSaddle}_d(Z) \cup \mathrm{Sink}(Z))$. But $Z \in W(X)$ so we obtain a contradiction by (A.2) and the result follows. □

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