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# Tests for mean vector with two-step monotone missing data

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**Abstract.** We consider the problem of testing for multivariate mean vector when the data have two-step monotone pattern missing observations. We obtain two test statistics for this problem: a test statistic similar to Hotelling's $T^2$ test statistic and the likelihood ratio test statistic. We propose the approximate upper percentiles of these statistics. The accuracy of the approximation is investigated by Monte Carlo simulation. A test statistic for the components of mean vector is outlined. Approximate simultaneous confidence intervals are obtained and the proposed method is illustrated using an example.

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*Key words and phrases.* Hotelling's $T^2$ type statistic, likelihood ratio test statistic, maximum likelihood estimator, simultaneous confidence intervals, two-step monotone missing data.

## §1. Introduction

In statistical data analyses, missing data is an inevitable problem in many practical situations. For example, in clinical trials that are conducted over several years, missing data often occurs when patients drop out mid-study. Many statistical methods have been developed to analyze data with missing values (see, e.g., Anderson (1957), Bhargava (1962), Little and Rubin (2002), McLachlan and Krishnan (1997)). For a general missing pattern, Srivastava (1985) discussed the likelihood ratio test (LRT) for mean vector in one-sample problem and the LRT for mean vectors in two-sample problem. Srivastava and Carter (1986) and Shutoh et al. (2010) obtained the maximum likelihood estimators (MLEs) of the mean vector and the covariance matrix by the Newton-Raphson method and provided the LRT for the same. Seo and Srivastava (2000) derived a test of equality of means and simultaneous confidence intervals for monotone missing data in one-sample problem under a covariance matrix with intraclass correlation. As an extension of Seo and Srivastava (2000), Koizumi and Seo (2009a, 2009b) considered testing the equality of means and simultaneous confidence intervals in $l$-sample problem for $k$-step monotone missing data. They gave the exact distribution of test statistics under the null hypothesis.

On the other hand, Anderson (1957) developed an approach to derive the MLEs of the mean and the covariance vector by solving the likelihood equations for monotone missing data with several missing patterns. Anderson and Olkin (1985) derived the MLEs for two-step monotone missing data in one-sample problem. Kanda and Fujikoshi (1998) discussed the distribution of the MLEs in the cases of two-step, three-step, and general $k$-step monotone missing data.

In this paper, we consider two-step monotone missing data drawn from a multivariate normal population that is of the form

$$\begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1p_1} & x_{1p_1+1} & \cdots & x_{1p} \\ x_{21} & x_{22} & \cdots & x_{2p_1} & x_{2p_1+1} & \cdots & x_{2p} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{N_11} & x_{N_12} & \cdots & x_{N_1p_1} & x_{N_1p_1+1} & \cdots & x_{N_1p} \\ x_{N_1+11} & x_{N_1+12} & \cdots & x_{N_1+1p_1} & * & \cdots & * \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ x_{N1} & x_{N2} & \cdots & x_{Np_1} & * & \cdots & * \end{pmatrix},$$

where $N = N_1 + N_2$ and $p = p_1 + p_2$. "$*$" indicates a missing observation. That is, we have complete data for $N_1$ observations with $p$ dimensions and incomplete data for $N_2$ observations with $p_1$ dimensions.

Let $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{N_1}$ be distributed as the multivariate normal $N_p(\boldsymbol{\mu},\boldsymbol{\Sigma})$ and $\boldsymbol{x}_{1N_1+1},\ldots,\boldsymbol{x}_{1N}$ be distributed as the multivariate normal $N_{p_1}(\boldsymbol{\mu}_1,\boldsymbol{\Sigma}_{11})$, where each $\boldsymbol{x}_j, j = 1,\ldots,N_1$ is $p \times 1$ and each $\boldsymbol{x}_{1j}, j = N_1+1,\ldots,N$ is $p_1 \times 1$, and

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \qquad \boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21} & \boldsymbol{\Sigma}_{22} \end{pmatrix}.$$

We partition $\boldsymbol{x}_j$ into a $p_1 \times 1$ random vector and a $p_2 \times 1$ random vector as $\boldsymbol{x}_j = (\boldsymbol{x}'_{1j}, \boldsymbol{x}'_{2j})'$, where $\boldsymbol{x}_{ij} : p_i \times 1,\ i = 1,2,\ j = 1,\ldots,N_1$. The two-step monotone missing data can be written in a vector expression as below:

$$\begin{pmatrix} \boldsymbol{x}'_{11} & \boldsymbol{x}'_{21} \\ \boldsymbol{x}'_{12} & \boldsymbol{x}'_{22} \\ \vdots & \vdots \\ \boldsymbol{x}'_{1N_1} & \boldsymbol{x}'_{2N_1} \\ \boldsymbol{x}'_{1N_1+1} & * \\ \vdots & \vdots \\ \boldsymbol{x}'_{1N} & * \end{pmatrix}.$$

Therefore, the joint density function of the observed data set $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{N_1}$, $\boldsymbol{x}_{1N_1+1}, \ldots, \boldsymbol{x}_{1N}$ can be written as

$$\prod_{j=1}^{N_1} f(\boldsymbol{x}_j; \boldsymbol{\mu}, \boldsymbol{\Sigma}) \times \prod_{j=N_1+1}^{N} f(\boldsymbol{x}_{1j}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11}),$$

where $f(\boldsymbol{x}_j; \boldsymbol{\mu}, \boldsymbol{\Sigma})$ are the density functions of $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $f(\boldsymbol{x}_{1j}; \boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$ are the density functions of $N_{p_1}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_{11})$.

We define the sample means:

$$\overline{\boldsymbol{x}}_T = \frac{1}{N} \sum_{j=1}^{N} \boldsymbol{x}_{1j}, \quad \overline{\boldsymbol{x}}_1^{(1)} = \frac{1}{N_1} \sum_{j=1}^{N_1} \boldsymbol{x}_{1j},$$

$$\overline{\boldsymbol{x}}_2^{(1)} = \frac{1}{N_1} \sum_{j=1}^{N_1} \boldsymbol{x}_{2j}, \quad \overline{\boldsymbol{x}}^{(2)} = \frac{1}{N_2} \sum_{j=N_1+1}^{N} \boldsymbol{x}_{1j},$$

and the sample covariance matrices:

$$\boldsymbol{S}^{(1)} = \frac{1}{N_1 - 1} \sum_{j=1}^{N_1} \left(\boldsymbol{x}_j - \overline{\boldsymbol{x}}^{(1)}\right)\left(\boldsymbol{x}_j - \overline{\boldsymbol{x}}^{(1)}\right)' = \begin{pmatrix} \boldsymbol{S}_{11}^{(1)} & \boldsymbol{S}_{12}^{(1)} \\ \boldsymbol{S}_{21}^{(1)} & \boldsymbol{S}_{22}^{(1)} \end{pmatrix},$$

$$\boldsymbol{S}^{(2)} = \frac{1}{N_2 - 1} \sum_{j=N_1+1}^{N} \left(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}^{(2)}\right)\left(\boldsymbol{x}_{1j} - \overline{\boldsymbol{x}}^{(2)}\right)'.$$

We consider the problem of testing $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ against $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$ when the data have two-step monotone pattern missing observations. Krishnamoorthy and Pannala (1999) gave a statistic similar to Hotelling's $T^2$ test statistic. They derived F-approximations of the $T^2$ type statistic by the method of moments and using simulations illustrated that the $T^2$ type statistic is as powerful as the LRT. Chang and Richards (2009) also studied the asymptotic distribution of the $T^2$ type statistic. Romer and Richards (2010) obtained a new derivation of a stochastic representation for the MLE of mean vector established by Chang and Richards (2009). Krishnamoorthy and Pannala (1999) and Chang and Richards (2009) assumed that the data are missing completely at random (MCAR). They derived the covariance matrix of the MLE of mean vector that is valid only under the assumption of MCAR. Kanda and Fujikoshi (1998) derived the covariance matrix of the MLE of mean vector without the assumption of MCAR. In this paper, we give the $T^2$ type statistic using Kanda and Fujikoshi (1998). We propose the approximate upper percentile of the $T^2$ type statistic using the upper percentile of Hotelling's $T^2$ statistic for non-missing data. The $T^2$ type statistic is asymptotically distributed as $\chi^2$ when the sample size is large. The proposed method gives a good approximation even when the sample size is not large. We also obtain the LRT statistic and its approximate upper percentile. In the following section, we introduce the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ in general. We derive the MLE of $\boldsymbol{\Sigma}$ under $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 (= \mathbf{0})$ following Kanda and Fujikoshi (1998). In Section 3, we obtain the $T^2$ type statistic and the LRT statistic for the null hypothesis and their approximate upper percentiles. In Section 4, the test statistic for the components of mean vector is outlined. Section 5 gives simultaneous confidence intervals for $\boldsymbol{\mu}$. The accuracy of the approximate upper percentiles of the test statistics is investigated by Monte Carlo simulation in Section 6. A numerical example is provided to show the approximate simultaneous confidence intervals in Section 7.

## §2. Maximum likelihood estimators

### 2.1. MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$

Let the MLEs of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ denote by $\widehat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$, which are partitioned in the same way as $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. We assume that the observation vectors are distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $N_1 > p$, which is a necessary and sufficient condition for the existence and uniqueness of the MLEs of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Anderson and Olkin (1985) derived the MLEs of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (see Kanda and Fujikoshi (1998), Chang and Richards (2009)) as follows:

$$\widehat{\boldsymbol{\mu}} = \begin{pmatrix} \widehat{\boldsymbol{\mu}}_1 \\ \widehat{\boldsymbol{\mu}}_2 \end{pmatrix} = \begin{pmatrix} \overline{\boldsymbol{x}}_T \\ \overline{\boldsymbol{x}}_2^{(1)} - \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\left(\overline{\boldsymbol{x}}_1^{(1)} - \widehat{\boldsymbol{\mu}}_1\right) \end{pmatrix},$$

$$\widehat{\boldsymbol{\Sigma}} = \begin{pmatrix} \widehat{\boldsymbol{\Sigma}}_{11} & \widehat{\boldsymbol{\Sigma}}_{12} \\ \widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\boldsymbol{\Sigma}}_{22} \end{pmatrix} = \begin{pmatrix} \frac{1}{N}\left(\boldsymbol{W}_{11}^{(1)} + \boldsymbol{W}^{(2)}\right) & \widehat{\boldsymbol{\Sigma}}_{11}\left(\boldsymbol{W}_{11}^{(1)}\right)^{-1}\boldsymbol{W}_{12}^{(1)} \\ \boldsymbol{W}_{21}^{(1)}\left(\boldsymbol{W}_{11}^{(1)}\right)^{-1}\widehat{\boldsymbol{\Sigma}}_{11} & \frac{1}{N_1}\boldsymbol{W}_{22\cdot 1}^{(1)} + \widehat{\boldsymbol{\Sigma}}_{21}\widehat{\boldsymbol{\Sigma}}_{11}^{-1}\widehat{\boldsymbol{\Sigma}}_{12} \end{pmatrix},$$

where

$$\boldsymbol{W}^{(1)} = (N_1 - 1)\boldsymbol{S}^{(1)} = \begin{pmatrix} \boldsymbol{W}_{11}^{(1)} & \boldsymbol{W}_{12}^{(1)} \\ \boldsymbol{W}_{21}^{(1)} & \boldsymbol{W}_{22}^{(1)} \end{pmatrix},$$

$$\boldsymbol{W}^{(2)} = (N_2 - 1)\boldsymbol{S}^{(2)} + \frac{N_1 N_2}{N}\left(\overline{\boldsymbol{x}}_1^{(1)} - \overline{\boldsymbol{x}}^{(2)}\right)\left(\overline{\boldsymbol{x}}_1^{(1)} - \overline{\boldsymbol{x}}^{(2)}\right)',$$

$$\boldsymbol{W}_{22\cdot 1}^{(1)} = \boldsymbol{W}_{22}^{(1)} - \boldsymbol{W}_{21}^{(1)}\left(\boldsymbol{W}_{11}^{(1)}\right)^{-1}\boldsymbol{W}_{12}^{(1)}.$$

These MLEs are derived using the usual transformed parameters

$$\boldsymbol{\eta} = \begin{pmatrix} \boldsymbol{\eta}_1 \\ \boldsymbol{\eta}_2 \end{pmatrix} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\mu}_1 \end{pmatrix},$$

$$\boldsymbol{\Psi} = \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{\Psi}_{12} \\ \boldsymbol{\Psi}_{21} & \boldsymbol{\Psi}_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_{11} & \boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12} \\ \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1} & \boldsymbol{\Sigma}_{22\cdot 1} \end{pmatrix},$$

which have one-to-one correspondence with $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, where $\boldsymbol{\Sigma}_{22\cdot 1} = \boldsymbol{\Sigma}_{22} - \boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}$. Multiplying the observation vectors $\boldsymbol{x}_j$ by the transformation matrix

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{I}_{p_1} & \boldsymbol{O} \\ -\boldsymbol{\Psi}_{21} & \boldsymbol{I}_{p_2} \end{pmatrix}$$

on the left side, the mean vector and the covariance matrix of the transformed observation vectors are

$$\boldsymbol{A}\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 - \boldsymbol{\Psi}_{21}\boldsymbol{\mu}_1 \end{pmatrix} = \boldsymbol{\eta}, \qquad \boldsymbol{A}\boldsymbol{\Sigma}\boldsymbol{A}' = \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Psi}_{22} \end{pmatrix},$$

respectively. The MLEs of $(\boldsymbol{\eta}, \boldsymbol{\Psi})$ are expressed as

$$\widehat{\boldsymbol{\eta}}_1 = \widehat{\boldsymbol{\mu}}_1, \quad \widehat{\boldsymbol{\eta}}_2 = \overline{\boldsymbol{x}}_2^{(1)} - \widehat{\boldsymbol{\Psi}}_{21}\overline{\boldsymbol{x}}_1^{(1)},$$
$$\widehat{\boldsymbol{\Psi}}_{11} = \widehat{\boldsymbol{\Sigma}}_{11}, \quad \widehat{\boldsymbol{\Psi}}_{12} = \left(\boldsymbol{W}_{11}^{(1)}\right)^{-1}\boldsymbol{W}_{12}^{(1)}, \quad \widehat{\boldsymbol{\Psi}}_{22} = \tfrac{1}{N_1}\boldsymbol{W}_{22\cdot 1}^{(1)}.$$

Kanda and Fujikoshi (1998) derived the next result.

**Theorem 1.** *(Kanda and Fujikoshi (1998))*
*The mean vector and the covariance matrix of $\widehat{\boldsymbol{\mu}}$ are given by*

$$\mathrm{E}[\widehat{\boldsymbol{\mu}}] = \boldsymbol{\mu},$$

$$\mathrm{Cov}[\widehat{\boldsymbol{\mu}}] = \begin{pmatrix} \frac{1}{N}\boldsymbol{\Sigma}_{11} & \frac{1}{N}\boldsymbol{\Sigma}_{12} \\ \frac{1}{N}\boldsymbol{\Sigma}_{21} & \mathrm{Cov}[\widehat{\boldsymbol{\mu}}_2] \end{pmatrix},$$

*respectively, where*

$$\mathrm{Cov}[\widehat{\boldsymbol{\mu}}_2] = \frac{1}{N_1}\left(\boldsymbol{\Sigma}_{22} - \frac{N_2}{N}\boldsymbol{\Sigma}_{21}\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{\Sigma}_{12}\right) + \frac{N_2 p_1}{NN_1(N_1-p_1-2)}\boldsymbol{\Sigma}_{22\cdot 1} \qquad (N_1 > p_1+2).$$

### 2.2. MLE of $\boldsymbol{\Sigma}$ under $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0(= \mathbf{0})$

In this section, we derive the MLE of $\boldsymbol{\Sigma}$ under $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0(= \mathbf{0})$ to obtain the LRT statistic, following Kanda and Fujikoshi (1998). Let $\boldsymbol{x}_j = (\boldsymbol{x}_{1j}', \boldsymbol{x}_{2j}')'$ be distributed as $N_p(\mathbf{0}, \boldsymbol{\Sigma}), j{=}1, \ldots, N_1$ and $\boldsymbol{x}_{1j}$ be distributed as $N_{p_1}(\mathbf{0}, \boldsymbol{\Sigma}_{11}), j{=}N_1{+}1, \ldots, N$, then, the likelihood function is

$$\begin{aligned} L(\mathbf{0}, \boldsymbol{\Sigma}) &= \prod_{j=1}^{N_1} \frac{1}{(2\pi)^{p/2}|\boldsymbol{\Sigma}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{x}_j'\boldsymbol{\Sigma}^{-1}\boldsymbol{x}_j\right) \\ &\quad \times \prod_{j=N_1+1}^{N} \frac{1}{(2\pi)^{p_1/2}|\boldsymbol{\Sigma}_{11}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{x}_{1j}'\boldsymbol{\Sigma}_{11}^{-1}\boldsymbol{x}_{1j}\right). \end{aligned}$$

Multiplying the observation vectors by $\boldsymbol{A}$ on the left side, we have

$$\boldsymbol{A}\boldsymbol{x}_j = \begin{pmatrix} \boldsymbol{x}_{1j} \\ \boldsymbol{x}_{2j} - \boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j} \end{pmatrix} \sim N_p\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} \boldsymbol{\Psi}_{11} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Psi}_{22} \end{pmatrix}\right), \ j{=}1, \ldots, N_1.$$

We note that $\boldsymbol{\Sigma}$ is one to one correspondence to $\boldsymbol{\Psi}$. For the parameter $\boldsymbol{\Psi}$, the likelihood function can be written as

$$\begin{aligned} L(\mathbf{0}, \boldsymbol{\Psi}) &= \prod_{j=1}^{N} \frac{1}{(2\pi)^{p_1/2}|\boldsymbol{\Psi}_{11}|^{1/2}} \exp\left(-\frac{1}{2}\boldsymbol{x}_{1j}'\boldsymbol{\Psi}_{11}^{-1}\boldsymbol{x}_{1j}\right) \\ &\times \prod_{j=1}^{N_1} \frac{1}{(2\pi)^{p_2/2}|\boldsymbol{\Psi}_{22}|^{1/2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{x}_{2j} - \boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\right)'\boldsymbol{\Psi}_{22}^{-1}\left(\boldsymbol{x}_{2j} - \boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\right)\right\}. \end{aligned}$$

Thus, the log likelihood function is

$$\log L(\mathbf{0}, \boldsymbol{\Psi}) = -\left(\frac{p_1 N}{2} + \frac{p_2 N_1}{2}\right)\log(2\pi) - \frac{N}{2}\log|\boldsymbol{\Psi}_{11}| - \frac{N_1}{2}\log|\boldsymbol{\Psi}_{22}|$$

$$+\sum_{j=1}^{N}\left(-\frac{1}{2}\boldsymbol{x}_{1j}'\boldsymbol{\Psi}_{11}^{-1}\boldsymbol{x}_{1j}\right)+\sum_{j=1}^{N_1}\left\{-\frac{1}{2}\left(\boldsymbol{x}_{2j}-\boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\right)'\boldsymbol{\Psi}_{22}^{-1}\left(\boldsymbol{x}_{2j}-\boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\right)\right\}.$$

The partial derivative of $\log L(\mathbf{0},\boldsymbol{\Psi})$ with respect to $\boldsymbol{\Psi}_{11}$ is

$$\frac{\partial \log L(\boldsymbol{\eta},\boldsymbol{\Psi})}{\partial \boldsymbol{\Psi}_{11}} = -\frac{N}{2}\boldsymbol{\Psi}_{11}^{-1}+\sum_{j=1}^{N}\frac{1}{2}\boldsymbol{\Psi}_{11}^{-1}\boldsymbol{x}_{1j}\boldsymbol{x}_{1j}'\boldsymbol{\Psi}_{11}^{-1}.$$

Solving the partial derivative of $\log L(\mathbf{0},\boldsymbol{\Psi})=0$, we obtain the MLE of $\boldsymbol{\Psi}_{11}$ as

$$\widetilde{\boldsymbol{\Psi}}_{11}=\frac{1}{N}\sum_{j=1}^{N}\boldsymbol{x}_{1j}\boldsymbol{x}_{1j}'.$$

Similarly, the partial derivative of $\log L(\mathbf{0},\boldsymbol{\Psi})$ with respect to $\boldsymbol{\Psi}_{21}$ is

$$\frac{\partial \log L\left(\boldsymbol{\eta},\boldsymbol{\Psi}\right)}{\partial \boldsymbol{\Psi}_{21}} = \sum_{j=1}^{N_1}\left(\boldsymbol{\Psi}_{22}^{-1}\boldsymbol{x}_{2j}\boldsymbol{x}_{1j}'-\boldsymbol{\Psi}_{22}^{-1}\boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\boldsymbol{x}_{1j}'\right),$$

and the partial derivative of $\log L(\mathbf{0},\boldsymbol{\Psi})$ with respect to $\boldsymbol{\Psi}_{22}$ is

$$\frac{\partial \log L(\boldsymbol{\Psi})}{\partial \boldsymbol{\Psi}_{22}} = -\frac{N_1}{2}\boldsymbol{\Psi}_{22}^{-1}+\sum_{j=1}^{N_1}\frac{1}{2}\boldsymbol{\Psi}_{22}^{-1}\left(\boldsymbol{x}_{2j}-\boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\right)\left(\boldsymbol{x}_{2j}-\boldsymbol{\Psi}_{21}\boldsymbol{x}_{1j}\right)'\boldsymbol{\Psi}_{22}^{-1}.$$

We obtain the MLEs of $\boldsymbol{\Psi}_{21}$ and $\boldsymbol{\Psi}_{22}$:

$$\widetilde{\boldsymbol{\Psi}}_{21}=\sum_{j=1}^{N_1}\boldsymbol{x}_{2j}\boldsymbol{x}_{1j}'\left(\sum_{j=1}^{N_1}\boldsymbol{x}_{1j}\boldsymbol{x}_{1j}'\right)^{-1},$$

and

$$\begin{aligned}\widetilde{\boldsymbol{\Psi}}_{22} &= \frac{1}{N_1}\sum_{j=1}^{N_1}\left(\boldsymbol{x}_{2j}-\widetilde{\boldsymbol{\Psi}}_{21}\boldsymbol{x}_{1j}\right)\left(\boldsymbol{x}_{2j}-\widetilde{\boldsymbol{\Psi}}_{21}\boldsymbol{x}_{1j}\right)' \\ &= \frac{1}{N_1}\left\{\sum_{j=1}^{N_1}\boldsymbol{x}_{2j}\boldsymbol{x}_{2j}'-\left(\sum_{j=1}^{N_1}\boldsymbol{x}_{2j}\boldsymbol{x}_{1j}'\right)\left(\sum_{j=1}^{N_1}\boldsymbol{x}_{1j}\boldsymbol{x}_{1j}'\right)^{-1}\left(\sum_{j=1}^{N_1}\boldsymbol{x}_{1j}\boldsymbol{x}_{2j}'\right)\right\}.\end{aligned}$$

The MLE of $\boldsymbol{\Psi}$ is expressed as

$$\widetilde{\boldsymbol{\Psi}}=\begin{pmatrix}\widetilde{\boldsymbol{\Psi}}_{11} & \widetilde{\boldsymbol{\Psi}}_{12}\\ \widetilde{\boldsymbol{\Psi}}_{21} & \widetilde{\boldsymbol{\Psi}}_{22}\end{pmatrix}=\begin{pmatrix}\frac{1}{N}(\boldsymbol{W}_{11}^{(1)}+\boldsymbol{V}^{(2)}) & (\boldsymbol{V}_{11}^{(1)})^{-1}\boldsymbol{V}_{12}^{(1)}\\ \boldsymbol{V}_{21}^{(1)}(\boldsymbol{V}_{11}^{(1)})^{-1} & \frac{1}{N}\boldsymbol{V}_{22\cdot 1}^{(1)}\end{pmatrix},$$

where

$$\begin{aligned} \boldsymbol{V}^{(2)} &= \boldsymbol{W}^{(2)} + N\overline{\boldsymbol{x}}_T\overline{\boldsymbol{x}}_T', \quad \boldsymbol{V}_{11}^{(1)} = \boldsymbol{W}_{11}^{(1)} + N_1\overline{\boldsymbol{x}}_1^{(1)}\overline{\boldsymbol{x}}_1^{(1)\prime}, \\ \boldsymbol{V}_{12}^{(1)} &= \boldsymbol{W}_{12}^{(1)} + N_1\overline{\boldsymbol{x}}_1^{(1)}\overline{\boldsymbol{x}}_2^{(1)\prime}, \quad \boldsymbol{V}_{22}^{(1)} = \boldsymbol{W}_{22}^{(1)} + N_1\overline{\boldsymbol{x}}_2^{(1)}\overline{\boldsymbol{x}}_2^{(1)\prime}. \end{aligned}$$

## §3. Test statistics for mean vector

In this section, we provide a test statistic for testing the following hypothesis:

$$H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0,$$

where $\boldsymbol{\mu}_0$ is known.

### 3.1. $T^2$ type statistic

When data are non-missing, Hotelling's $T^2$ statistic is widely used to test the hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ against $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$. For two-step monotone missing data, it is easy to construct a test statistic based on Hotelling's $T^2$ statistic structure:

$$T^2 = (\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0)'\widehat{\boldsymbol{\Gamma}}^{-1}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu}_0),$$

where $\widehat{\boldsymbol{\Gamma}}$ is the estimator of $\boldsymbol{\Gamma} = \mathrm{Cov}[\widehat{\boldsymbol{\mu}}]$, that is,

$$\widehat{\boldsymbol{\Gamma}} = \widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}] = \begin{pmatrix} \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{11} & \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{12} \\ \frac{1}{N}\widehat{\boldsymbol{\Sigma}}_{21} & \widehat{\mathrm{Cov}}[\widehat{\boldsymbol{\mu}}_2] \end{pmatrix}.$$

We call this statistic the $T^2$ type statistic. Under $H_0$, the $T^2$ type statistic is asymptotically distributed as $\chi^2$ with degree of freedom $p$ when $N_1, N \to \infty$ with $N_1/N \to \delta \in (0, 1]$ (see Chang and Richards (2009)). However, the $\chi^2$ distribution is not a good approximation to the upper percentile of the $T^2$ type statistic when the sample size is not large.

The $T^2$ type statistic is a generalization of Hotelling's test statistic for two-step monotone missing data. If the data are non-missing, $N_2$=0, the $T^2$ type statistic is equal to Hotelling's test statistic. If we assume that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, Hotelling's $T^2$ statistic is related to the $F$ distribution by

$$T_N^2 \quad \sim \frac{(N-1)p}{N-p}F_{p,N-p}.$$

If we have $N_1$ non-missing observations with $p$ dimensions, Hotelling's $T^2$ statistic is related to the $F$ distribution by

$$T^2_{N_1} \sim \frac{(N_1-1)p}{N_1-p} F_{p,N_1-p}.$$

Considering the data structure, the two-step monotone missing data are larger than the non-missing data with $N_1$ observations, but smaller than the non-missing data with $N$ observations. The test statistic for the two-step monotone missing data should also lie between the two test statistics of non-missing data. We obtain the approximate upper percentile of the $T^2$ type statistic.

**Theorem 2.** *Suppose that the data have two-step monotone pattern missing observations. Then the approximate upper* $100\alpha$ *percentile of the* $T^2$ *type statistic is given by*

$$\begin{aligned} F^*_\alpha &= T^2_{N_1,\alpha} - \frac{Np - N_2 p_2}{Np}\left(T^2_{N_1,\alpha} - T^2_{N,\alpha}\right) \\ &= cT^2_{N_1,\alpha} + (1-c)T^2_{N,\alpha}\ , \end{aligned}$$

*where*

$$c = \frac{N_2 p_2}{Np}, \quad T^2_{N_1,\alpha} = \frac{(N_1-1)p}{N_1-p} F_{p,N_1-p,\alpha}, \quad T^2_{N,\alpha} = \frac{(N-1)p}{N-p} F_{p,N-p,\alpha}$$

*and* $F_{p,q,\alpha}$ *is the upper* $100\alpha$ *percentile of the* $F$ *distribution with* $p$ *and* $q$ *degrees of freedom.*

### 3.2. Likelihood ratio test statistic

Using the MLEs derived in Section 2.2, we obtain the LRT statistic for testing the hypothesis $H_0 : \boldsymbol{\mu} = \boldsymbol{\mu}_0$ against $H_1 : \boldsymbol{\mu} \neq \boldsymbol{\mu}_0$. Without loss of generality, we can assume that $\boldsymbol{\mu}_0 = \mathbf{0}$ . The LRT statistic, $-2\log\lambda$, is asymptotically distributed as $\chi^2$ with $p$ degrees of freedom, where

$$\begin{aligned} \lambda &= \frac{L(\boldsymbol{\mu}_0, \widetilde{\boldsymbol{\Sigma}})}{L(\widehat{\boldsymbol{\mu}}, \widehat{\boldsymbol{\Sigma}})} = \frac{L(\mathbf{0}, \widetilde{\boldsymbol{\Psi}})}{L(\widehat{\boldsymbol{\eta}}, \widehat{\boldsymbol{\Psi}})} \\ &= \frac{|\widehat{\boldsymbol{\Psi}}_{11}|^{N/2}}{|\widetilde{\boldsymbol{\Psi}}_{11}|^{N/2}} \times \frac{|\widehat{\boldsymbol{\Psi}}_{22}|^{N_1/2}}{|\widetilde{\boldsymbol{\Psi}}_{22}|^{N_1/2}}. \end{aligned}$$

When the data are non-missing, if we assume that $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are distributed as $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the likelihood ratio can be written using Hotelling's $T^2$ statistic as

$$\lambda^{\frac{2}{N}} = \left(1 + \frac{T^2}{N-1}\right)^{-1}.$$

The LRT statistic is

$$-2\log\lambda = N\log\left(1+\frac{T^2}{N-1}\right).$$

The LRT statistic for the non-missing data with $N$ observations can be written using Hotelling's $T^2$ statistic, $T_N^2$, as

$$Q_N = -2\log\lambda_N = N\log\left(1+\frac{T_N^2}{N-1}\right),$$

and the LRT statistic for the non-missing data with $N_1$ observations can be written using Hotelling's $T^2$ statistic, $T_{N_1}^2$, as

$$Q_{N_1} = -2\log\lambda_{N_1} = N_1\log\left(1+\frac{T_{N_1}^2}{N_1-1}\right).$$

Using the same idea for the $T^2$ type statistic, we obtain the approximate upper percentile of the LRT statistic.

**Theorem 3.** *Suppose that the data have two-step monotone pattern missing observations. Then the approximate upper* $100\alpha$ *percentile of the LRT statistic is given by*

$$\begin{aligned} Q_\alpha^* &= Q_{N_1,\alpha} - \frac{Np-N_2p_2}{Np}\left(Q_{N_1,\alpha}-Q_{N,\alpha}\right) \\ &= cQ_{N_1,\alpha} + (1-c)Q_{N,\alpha}\ , \end{aligned}$$

*where*

$$c = \frac{N_2p_2}{Np}, \quad Q_{N_1,\alpha} = N_1\log\left(1+\frac{T_{N_1,\alpha}^2}{N_1-1}\right), \quad Q_{N,\alpha} = N\log\left(1+\frac{T_{N,\alpha}^2}{N-1}\right),$$

$$T_{N_1,\alpha}^2 = \frac{(N_1-1)p}{N_1-p}F_{p,N_1-p,\alpha}, \quad T_{N,\alpha}^2 = \frac{(N-1)p}{N-p}F_{p,N-p,\alpha}.$$

## §4. Test statistic for components of mean vector

In this section, we provide a test statistic for the following hypothesis:

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_p \quad \text{vs.} \quad H_1 \neq H_0.$$

This hypothesis can be written as

$$H_0 : \boldsymbol{C\mu} = \boldsymbol{0} \text{ vs. } H_1 \neq H_0,$$

where

$$\underset{(p-1)\times p}{\boldsymbol{C}} = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 0 & 1 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{pmatrix}.$$

When the data have no missing observations, Hotelling's $T^2$ statistic is

$$T^2 = N(\boldsymbol{C}\overline{\boldsymbol{x}})'(\boldsymbol{CSC}')^{-1}(\boldsymbol{C}\overline{\boldsymbol{x}}),$$

where $\boldsymbol{S}$ is a sample covariance matrix. Under the null hypothesis, Hotelling's $T^2$ statistic is related to the $F$ distribution by

$$T^2 \sim \frac{(N-1)(p-1)}{(N-p+1)} F_{p-1,N-p+1}.$$

Given two-step monotone missing data, we can construct the $T^2$ type statistic, expanding the case in which the data are not missing. Further, without lost of generality, we assume that $\boldsymbol{\Sigma} = \boldsymbol{I}$ when we consider the $T^2$ type statistic. We set $\boldsymbol{C}_i, i = 1, 2$ to be a $(p_i-1)\times p_i$ matrix such that $\boldsymbol{C}_i\boldsymbol{1} = \boldsymbol{0}$ and $\boldsymbol{C}_i\boldsymbol{C}_i' = \boldsymbol{I}_{p_i-1}$ as

$$\boldsymbol{C}_i = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & \dots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{\sqrt{p_i(p_i-1)}} & \frac{1}{\sqrt{p_i(p_i-1)}} & \frac{1}{\sqrt{p_i(p_i-1)}} & \dots & -\frac{p_i-1}{\sqrt{p_i(p_i-1)}} \end{pmatrix},$$

where $\boldsymbol{1} = (1, 1, \dots, 1)'$. Considering that $\boldsymbol{y}_j^{(1)} = \boldsymbol{C}_1\boldsymbol{x}_j^{(1)}, j = 1, 2, \dots, N_1$ and $\boldsymbol{y}_j^{(2)} = \boldsymbol{C}_2\boldsymbol{x}_j^{(2)}, j = N_1+1, \dots, N$, $\boldsymbol{y}_j^{(1)}$ are distributed as $N_{p-1}(\boldsymbol{\mu}^*, \boldsymbol{I})$ and $\boldsymbol{y}_j^{(2)}$ are distributed as $N_{p_1-1}(\boldsymbol{\mu}_1^*, \boldsymbol{I})$ , where $\boldsymbol{\mu}^* = \boldsymbol{C}_1\boldsymbol{\mu}$, $\boldsymbol{\mu}_1^* = \boldsymbol{C}_2\boldsymbol{\mu}_1$ The $T^2$ type statistic for $H_0:\ \boldsymbol{\mu}^*(= \boldsymbol{C}_1\boldsymbol{\mu}) = \boldsymbol{0}$ can be constructed as

$$T^{*2} = (\widehat{\boldsymbol{\mu}}^*)'(\widehat{\boldsymbol{\Gamma}}^*)^{-1}(\widehat{\boldsymbol{\mu}}^*),$$

where $\widehat{\boldsymbol{\mu}}^*$ is the MLE of $\boldsymbol{\mu}^*$ and $\widehat{\boldsymbol{\Gamma}}^*$ is the estimator of $\boldsymbol{\Gamma}^* = \mathrm{Cov}(\widehat{\boldsymbol{\mu}}^*)$. $\mathrm{Cov}(\widehat{\boldsymbol{\mu}}^*)$ can be given by Theorem 1 in Section 2.1.

It can be easily shown that the test for the components of mean vector with $p$ dimensions is equivalent to the test for mean vector with $p-1$ dimensions. Therefore, we can use the same $F_\alpha^*$ values derived in Section 3.1 for the approximate upper percentile of the test statistic.

As a remark, we can use the proposed approximation method for $H_0 : \mu_1 = \mu_2$, which is the hypothesis testing for the components of mean vector when $p = 2$ $(p_1 = p_2 = 1)$.

## §5. Simultaneous confidence intervals

Using the $T^2$ type statistic in Section 3.1, we obtain the simultaneous confidence intervals for any and all linear compounds of the mean. Suppose that we have a sample of $N$ observations with two-step monotone missing observations with mean vector $\boldsymbol{\mu}$. Then, for any vector $\boldsymbol{a}' = (a_1, \ldots, a_p)$,

$$T^2(\boldsymbol{a}) = \frac{[\boldsymbol{a}'(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})]^2}{\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}\boldsymbol{a}} \leq (\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})'\widehat{\boldsymbol{\Gamma}}^{-1}(\widehat{\boldsymbol{\mu}} - \boldsymbol{\mu})$$

and from the distribution of the $T^2$ type statistic it follows that the probability statement

$$P[T^2(\boldsymbol{a}) \leq t^2_{p,\alpha} \text{ for all } \boldsymbol{a}] = 1 - \alpha$$

holds for all $\boldsymbol{a}$, where $t^2_{p,\alpha}$ denotes the upper $100\alpha$ percentile of the $T^2$ type statistic. Then we obtain the simultaneous confidence intervals for $\boldsymbol{a}'\boldsymbol{\mu}$

$$\boldsymbol{a}'\widehat{\boldsymbol{\mu}} - \sqrt{\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}\boldsymbol{a}t^2_{p,\alpha}} \leq \boldsymbol{a}'\boldsymbol{\mu} \leq \boldsymbol{a}'\widehat{\boldsymbol{\mu}} + \sqrt{\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}\boldsymbol{a}t^2_{p,\alpha}},\ \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}.$$

Since the asymptotic distribution of the $T^2$ type statistic is $\chi^2$, asymptotic simultaneous confidence intervals can be given using the upper $100\alpha$ percentile of the $\chi^2$ distribution, $\chi^2_{p,\alpha}$, instead of $t^2_{p,\alpha}$. However, as stated in Section 3.1, $F^*_\alpha$ is a better approximation to the upper $100\alpha$ percentiles of the $T^2$ type statistic. The approximate simultaneous confidence intervals for $\boldsymbol{a}'\boldsymbol{\mu}$ can be improved using $F^*_\alpha$ :

$$\boldsymbol{a}'\widehat{\boldsymbol{\mu}} - \sqrt{\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}\boldsymbol{a}F^*_\alpha} \leq \boldsymbol{a}'\boldsymbol{\mu} \leq \boldsymbol{a}'\widehat{\boldsymbol{\mu}} + \sqrt{\boldsymbol{a}'\widehat{\boldsymbol{\Gamma}}\boldsymbol{a}F^*_\alpha},\ \forall \boldsymbol{a} \in \boldsymbol{R}^p - \{\boldsymbol{0}\}.$$

## §6. Simulation studies

We compute the upper $100\alpha$ percentiles of the $T^2$ type statistic and the LRT statistic by Monte Carlo simulation ($10^6$ runs) for $\alpha = 0.05, 0.01$ and various conditions of $p, N_1, N_2$. We generate artificial two-step monotone missing data from $N_p(\boldsymbol{0}, \boldsymbol{I}_p)$. We examine the asymptotic distributions of these statistics when $\rho_i = n_i/n \to$ positive constants as $N_i$s tend to infinity$(i = 1, 2)$, where $n_i = N_i - 1$ and $n = n_1 + n_2$. We also examine the cases in which $\rho_1 = 1$ as $N_1$ is large and $N_2$ is fixed. Then we evaluate the accuracy of the proposed approximate upper percentiles of the test statistics.

The simulated upper percentiles of the $T^2$ type statistic and $F^*_\alpha$ values are given in Table 1 for three conditions $\rho_1 = \rho_2 = 1/2, \rho_1 = 2/3$ and $\rho_2 = 1/3, \rho_1 = 1/3$ and $\rho_2 = 2/3$. It can be seen from Table 1 that the simulated upper percentiles of the $T^2$ type statistic are closer to the upper percentiles of $\chi_p^2$ distribution as $N_1$ and $N_2$ get larger. Meanwhile, $F_\alpha^*$ values are much closer to the simulated upper percentiles of the $T^2$ type statistic than the upper percentiles of $\chi_p^2$ distribution even when the sample sizes are not large. Table 2 shows the results for $\rho_1 = 1$. We can see that the simulated upper percentiles of the $T^2$ type statistic are close to the upper percentiles of $\chi^2$ distribution when the sample sizes get larger. $F_\alpha^*$ is a good approximation to the upper percentile of the $T^2$ type statistic. Here, we note that the obtained upper percentiles of the $T^2$ type statistic are slightly overestimated in simulation when $N_2$ is very small relative to $N_1$.

Tables 3 and 4 present the type I error rate when the null hypothesis is rejected using $F_\alpha^*$ and $\chi_p^2$ under the simulated $T^2$ type statistic. The rejection regions of $F_\alpha^*$ and $\chi_p^2$ are bigger than the true rejection regions when the sample sizes are small. However, $F_\alpha^*$ always gives smaller rejection regions compared to $\chi_p^2$. It is clear from these tables that $F_\alpha^*$ is a very good approximation to the upper percentile of the $T^2$ type statistic.

As stated in Section 4, the simulation results for the $T^2$ type statistic can be applied to the test for the components of mean vector since the test for the components of mean vector with $p$ dimensions is equivalent to the test for mean vector with $p - 1$ dimensions.

Tables 5 and 6 present the simulated upper percentiles of the LRT statistic and $Q_\alpha^*$ values. We can see that the simulated upper percentiles of the LRT statistic are close to the upper percentiles of $\chi_p^2$ distribution when the sample sizes get larger and that $Q_\alpha^*$ is a good approximation to the upper percentile of the LRT statistic. Tables 7 and 8 present the type I error rate when the null hypothesis is rejected using $Q_\alpha^*$ and $\chi_p^2$ under the simulated LRT statistic. The type I error rates show that $Q_\alpha^*$ is a very good approximation to the upper percentiles of the LRT statistic.

Table 1: Upper percentiles of $T^2$ type statistic and $F_\alpha^*$ value

| | | | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $\rho_1$ | $\rho_2$ | $N$ | $N_1$ | $N_2$ | $T^2$ | $F_\alpha^*$ | $T^2$ | $F_\alpha^*$ |
| 4 | 2 | 2 | 1/2 | 1/2 | 20 | 10 | 10 | 23.81 | 17.51 | 47.95 | 30.72 |
| $\chi^2_{4,0.05} = 9.49$ | | | | | 40 | 20 | 20 | 13.47 | 12.13 | 20.87 | 18.31 |
| $\chi^2_{4,0.01} = 13.28$ | | | | | 100 | 50 | 50 | 10.73 | 10.37 | 15.44 | 14.90 |
| | | | | | 200 | 100 | 100 | 10.06 | 9.91 | 14.30 | 14.04 |
| | | | | | 300 | 150 | 150 | 9.86 | 9.76 | 13.90 | 13.77 |
| | | | | | 400 | 200 | 200 | 9.78 | 9.69 | 13.75 | 13.65 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 13.94 | 12.58 | 44.87 | 30.61 |
| | | | | | 60 | 40 | 20 | 11.27 | 10.81 | 16.47 | 15.71 |
| | | | | | 120 | 80 | 40 | 10.30 | 10.10 | 14.71 | 14.40 |
| | | | | | 240 | 160 | 80 | 9.90 | 9.79 | 13.96 | 13.81 |
| | | | | | 480 | 320 | 160 | 9.67 | 9.63 | 13.59 | 13.54 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 22.16 | 17.22 | 21.75 | 19.17 |
| | | | | | 60 | 20 | 40 | 12.99 | 11.89 | 20.07 | 17.88 |
| | | | | | 120 | 40 | 80 | 10.90 | 10.51 | 15.83 | 15.15 |
| | | | | | 240 | 80 | 160 | 10.13 | 9.96 | 14.41 | 14.14 |
| | | | | | 480 | 160 | 320 | 9.80 | 9.72 | 13.79 | 13.69 |
| 8 | 4 | 4 | 1/2 | 1/2 | 20 | 10 | 10 | 510.79 | 201.40 | 2633.73 | 937.11 |
| $\chi^2_{8,0.05} = 15.51$ | | | | | 40 | 20 | 20 | 31.42 | 25.43 | 49.03 | 37.11 |
| $\chi^2_{8,0.01} = 20.09$ | | | | | 100 | 50 | 50 | 19.19 | 18.23 | 26.00 | 24.43 |
| | | | | | 200 | 100 | 100 | 17.15 | 16.75 | 22.60 | 22.03 |
| | | | | | 300 | 150 | 150 | 16.53 | 16.31 | 21.64 | 21.34 |
| | | | | | 400 | 200 | 200 | 16.26 | 16.10 | 21.26 | 21.01 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 33.29 | 27.07 | 52.30 | 39.84 |
| | | | | | 60 | 40 | 20 | 21.07 | 19.70 | 29.13 | 26.82 |
| | | | | | 120 | 80 | 40 | 17.86 | 17.35 | 23.76 | 22.99 |
| | | | | | 240 | 160 | 80 | 16.60 | 16.38 | 21.72 | 21.45 |
| | | | | | 480 | 320 | 160 | 16.03 | 15.93 | 20.88 | 20.75 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 460.49 | 249.30 | 52.13 | 39.84 |
| | | | | | 60 | 20 | 40 | 29.68 | 24.87 | 46.58 | 36.39 |
| | | | | | 120 | 40 | 80 | 19.93 | 18.75 | 27.18 | 25.32 |
| | | | | | 240 | 80 | 160 | 17.32 | 16.93 | 22.88 | 22.32 |
| | | | | | 480 | 160 | 320 | 16.33 | 16.18 | 21.34 | 21.13 |
| 20 | 10 | 10 | 1/2 | 1/2 | 100 | 50 | 50 | 54.91 | 47.39 | 71.55 | 60.08 |
| $\chi^2_{20,0.05} = 31.41$ | | | | | 200 | 100 | 100 | 39.48 | 37.56 | 48.57 | 45.95 |
| $\chi^2_{20,0.05} = 37.57$ | | | | | 300 | 150 | 150 | 36.25 | 35.23 | 44.07 | 42.74 |
| | | | | | 400 | 200 | 200 | 34.88 | 34.18 | 42.23 | 41.30 |
| | | | | | 500 | 250 | 250 | 34.11 | 33.58 | 41.23 | 40.49 |
| | | | | | 600 | 300 | 300 | 33.66 | 33.20 | 40.56 | 39.97 |
| | | | 2/3 | 1/3 | 240 | 160 | 80 | 36.48 | 35.54 | 44.35 | 43.15 |
| | | | | | 480 | 320 | 160 | 33.74 | 33.35 | 40.66 | 40.17 |
| | | | | | 960 | 640 | 320 | 32.52 | 32.35 | 39.02 | 38.83 |
| | | | | | 1920 | 1280 | 640 | 31.99 | 31.87 | 38.27 | 38.19 |
| | | | 1/3 | 2/3 | 240 | 80 | 160 | 41.07 | 38.79 | 50.94 | 47.72 |
| | | | | | 480 | 160 | 320 | 35.35 | 34.59 | 42.87 | 41.87 |
| | | | | | 960 | 320 | 640 | 33.24 | 32.90 | 39.97 | 39.57 |
| | | | | | 1920 | 640 | 1280 | 32.32 | 32.13 | 38.77 | 38.54 |

Table 2: Upper percentiles of $T^2$ type statistic and $F_\alpha^*$ value when $N_2$ is fixed

| | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $N$ | $N_1$ | $N_2$ | $T^2$ | $F_\alpha^*$ | $T^2$ | $F_\alpha^*$ |
| 4 | 2 | 2 | 20 | 10 | 10 | 23.81 | 17.51 | 47.95 | 30.72 |
| $\chi^2_{4,0.05} = 9.49$ | | | 30 | 20 | 10 | 13.94 | 12.58 | 21.75 | 19.17 |
| $\chi^2_{4,0.01} = 13.28$ | | | 60 | 50 | 10 | 11.04 | 10.71 | 16.09 | 15.53 |
| | | | 110 | 100 | 10 | 10.26 | 10.11 | 14.62 | 14.41 |
| | | | 60 | 10 | 50 | 20.95 | 17.57 | 42.69 | 31.90 |
| | | | 70 | 20 | 50 | 12.90 | 11.85 | 19.87 | 17.82 |
| | | | 100 | 50 | 50 | 10.73 | 10.37 | 15.44 | 14.90 |
| | | | 150 | 100 | 50 | 10.14 | 9.97 | 14.44 | 14.16 |
| | | | 110 | 10 | 100 | 20.48 | 17.88 | 41.43 | 32.81 |
| | | | 120 | 20 | 100 | 12.54 | 11.82 | 19.30 | 17.79 |
| | | | 150 | 50 | 100 | 10.57 | 10.28 | 15.21 | 14.73 |
| | | | 200 | 100 | 100 | 10.06 | 9.91 | 14.30 | 14.04 |
| 8 | 4 | 4 | 20 | 10 | 10 | 510.79 | 201.40 | 2648.20 | 937.11 |
| $\chi^2_{8,0.05} = 15.51$ | | | 30 | 20 | 10 | 33.29 | 27.07 | 52.13 | 39.84 |
| $\chi^2_{8,0.01} = 20.09$ | | | 60 | 50 | 10 | 20.14 | 19.34 | 27.42 | 26.23 |
| | | | 110 | 100 | 10 | 17.61 | 17.37 | 23.38 | 23.02 |
| | | | 60 | 10 | 50 | 419.47 | 301.80 | 2174.61 | 1505.83 |
| | | | 70 | 20 | 50 | 29.29 | 24.86 | 45.76 | 36.45 |
| | | | 100 | 50 | 50 | 19.19 | 18.23 | 25.89 | 24.43 |
| | | | 150 | 100 | 50 | 17.34 | 16.95 | 22.89 | 22.35 |
| | | | 110 | 10 | 100 | 401.03 | 326.45 | 2094.58 | 1638.67 |
| | | | 120 | 20 | 100 | 28.25 | 25.06 | 43.94 | 37.01 |
| | | | 150 | 50 | 100 | 18.76 | 17.96 | 25.26 | 24.00 |
| | | | 200 | 100 | 100 | 17.15 | 16.75 | 22.62 | 22.03 |
| 20 | 10 | 10 | 100 | 50 | 50 | 54.91 | 47.39 | 71.55 | 60.08 |
| $\chi^2_{20,0.05} = 31.41$ | | | 150 | 100 | 50 | 40.43 | 38.58 | 49.88 | 47.36 |
| $\chi^2_{20,0.01} = 37.57$ | | | 200 | 150 | 50 | 37.19 | 36.25 | 45.31 | 44.14 |
| | | | 150 | 50 | 100 | 52.38 | 46.28 | 68.26 | 58.66 |
| | | | 200 | 100 | 100 | 39.48 | 37.56 | 48.57 | 45.95 |
| | | | 250 | 150 | 100 | 36.62 | 35.57 | 44.53 | 43.21 |

Table 3: Type I error rate using $F^*_\alpha$ and $\chi^2_p$ values under $T^2$ type statistic

| | | | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $\rho_1$ | $\rho_2$ | $N$ | $N_1$ | $N_2$ | $F^*_\alpha$ | $\chi^2_p$ | $F^*_\alpha$ | $\chi^2_p$ |
| 4 | 2 | 2 | 1/2 | 1/2 | 20 | 10 | 10 | 0.094 | 0.264 | 0.029 | 0.156 |
| | | | | | 40 | 20 | 20 | 0.068 | 0.131 | 0.017 | 0.052 |
| | | | | | 100 | 50 | 50 | 0.057 | 0.076 | 0.012 | 0.021 |
| | | | | | 200 | 100 | 100 | 0.053 | 0.062 | 0.011 | 0.015 |
| | | | | | 300 | 150 | 150 | 0.052 | 0.058 | 0.011 | 0.013 |
| | | | | | 400 | 200 | 200 | 0.052 | 0.056 | 0.011 | 0.012 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 0.068 | 0.140 | 0.017 | 0.058 |
| | | | | | 60 | 40 | 20 | 0.058 | 0.088 | 0.013 | 0.027 |
| | | | | | 120 | 80 | 40 | 0.054 | 0.067 | 0.011 | 0.017 |
| | | | | | 240 | 160 | 80 | 0.052 | 0.058 | 0.011 | 0.013 |
| | | | | | 480 | 320 | 160 | 0.051 | 0.054 | 0.010 | 0.011 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 0.085 | 0.243 | 0.025 | 0.139 |
| | | | | | 60 | 20 | 40 | 0.066 | 0.121 | 0.016 | 0.047 |
| | | | | | 120 | 40 | 80 | 0.057 | 0.080 | 0.012 | 0.023 |
| | | | | | 240 | 80 | 160 | 0.053 | 0.064 | 0.011 | 0.015 |
| | | | | | 480 | 160 | 320 | 0.052 | 0.057 | 0.010 | 0.012 |
| 8 | 4 | 4 | 1/2 | 1/2 | 20 | 10 | 10 | 0.120 | 0.773 | 0.028 | 0.690 |
| | | | | | 40 | 20 | 20 | 0.094 | 0.334 | 0.029 | 0.176 |
| | | | | | 100 | 50 | 50 | 0.063 | 0.118 | 0.014 | 0.040 |
| | | | | | 200 | 100 | 100 | 0.056 | 0.079 | 0.012 | 0.021 |
| | | | | | 300 | 150 | 150 | 0.053 | 0.068 | 0.011 | 0.017 |
| | | | | | 400 | 200 | 200 | 0.053 | 0.063 | 0.011 | 0.015 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 0.094 | 0.334 | 0.027 | 0.199 |
| | | | | | 60 | 40 | 20 | 0.066 | 0.154 | 0.016 | 0.061 |
| | | | | | 120 | 80 | 40 | 0.057 | 0.093 | 0.012 | 0.027 |
| | | | | | 240 | 160 | 80 | 0.053 | 0.069 | 0.011 | 0.017 |
| | | | | | 480 | 320 | 160 | 0.052 | 0.059 | 0.010 | 0.013 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 0.089 | 0.742 | 0.019 | 0.653 |
| | | | | | 60 | 20 | 40 | 0.086 | 0.280 | 0.025 | 0.156 |
| | | | | | 120 | 40 | 80 | 0.065 | 0.015 | 0.015 | 0.048 |
| | | | | | 240 | 80 | 160 | 0.056 | 0.083 | 0.012 | 0.023 |
| | | | | | 480 | 160 | 320 | 0.052 | 0.064 | 0.011 | 0.015 |
| 20 | 10 | 10 | 1/2 | 1/2 | 100 | 50 | 50 | 0.104 | 0.424 | 0.030 | 0.257 |
| | | | | | 200 | 100 | 100 | 0.069 | 0.178 | 0.016 | 0.068 |
| | | | | | 300 | 150 | 150 | 0.061 | 0.122 | 0.013 | 0.039 |
| | | | | | 400 | 200 | 200 | 0.058 | 0.099 | 0.012 | 0.028 |
| | | | | | 500 | 250 | 250 | 0.056 | 0.088 | 0.012 | 0.023 |
| | | | | | 600 | 300 | 300 | 0.055 | 0.081 | 0.012 | 0.020 |
| | | | 2/3 | 1/3 | 240 | 160 | 80 | 0.060 | 0.126 | 0.013 | 0.041 |
| | | | | | 480 | 320 | 160 | 0.054 | 0.082 | 0.011 | 0.021 |
| | | | | | 960 | 640 | 320 | 0.052 | 0.064 | 0.011 | 0.015 |
| | | | | | 1920 | 1280 | 640 | 0.051 | 0.057 | 0.010 | 0.012 |
| | | | 1/3 | 2/3 | 240 | 80 | 160 | 0.071 | 0.206 | 0.017 | 0.086 |
| | | | | | 480 | 160 | 320 | 0.058 | 0.107 | 0.013 | 0.032 |
| | | | | | 960 | 320 | 640 | 0.054 | 0.074 | 0.011 | 0.018 |
| | | | | | 1920 | 640 | 1280 | 0.052 | 0.062 | 0.011 | 0.014 |

Table 4: Type I error rate using $F_\alpha^*$ and $\chi_p^2$ values under $T^2$ type statistic when $N_2$ is fixed

| | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $N$ | $N_1$ | $N_2$ | $F_\alpha^*$ | $\chi_p^2$ | $F_\alpha^*$ | $\chi_p^2$ |
| 4 | 2 | 2 | 20 | 10 | 10 | 0.094 | 0.264 | 0.029 | 0.156 |
| | | | 30 | 20 | 10 | 0.068 | 0.140 | 0.017 | 0.058 |
| | | | 60 | 50 | 10 | 0.055 | 0.082 | 0.012 | 0.024 |
| | | | 110 | 100 | 10 | 0.052 | 0.066 | 0.011 | 0.016 |
| | | | 60 | 10 | 50 | 0.072 | 0.223 | 0.020 | 0.125 |
| | | | 70 | 20 | 50 | 0.064 | 0.119 | 0.015 | 0.045 |
| | | | 100 | 50 | 50 | 0.057 | 0.076 | 0.012 | 0.021 |
| | | | 150 | 100 | 50 | 0.053 | 0.064 | 0.011 | 0.016 |
| | | | 110 | 10 | 100 | 0.066 | 0.214 | 0.018 | 0.118 |
| | | | 120 | 20 | 100 | 0.060 | 0.112 | 0.014 | 0.041 |
| | | | 150 | 50 | 100 | 0.055 | 0.073 | 0.012 | 0.020 |
| | | | 200 | 100 | 100 | 0.053 | 0.062 | 0.011 | 0.015 |
| 8 | 4 | 4 | 20 | 10 | 10 | 0.120 | 0.773 | 0.028 | 0.690 |
| | | | 30 | 20 | 10 | 0.094 | 0.334 | 0.027 | 0.199 |
| | | | 60 | 50 | 10 | 0.059 | 0.136 | 0.013 | 0.050 |
| | | | 110 | 100 | 10 | 0.054 | 0.088 | 0.011 | 0.025 |
| | | | 60 | 10 | 50 | 0.068 | 0.710 | 0.014 | 0.619 |
| | | | 70 | 20 | 50 | 0.083 | 0.274 | 0.023 | 0.151 |
| | | | 100 | 50 | 50 | 0.063 | 0.118 | 0.014 | 0.040 |
| | | | 150 | 100 | 50 | 0.056 | 0.083 | 0.012 | 0.023 |
| | | | 110 | 10 | 100 | 0.061 | 0.697 | 0.013 | 0.605 |
| | | | 120 | 20 | 100 | 0.073 | 0.254 | 0.019 | 0.137 |
| | | | 150 | 50 | 100 | 0.061 | 0.110 | 0.014 | 0.036 |
| | | | 200 | 100 | 100 | 0.056 | 0.079 | 0.012 | 0.021 |
| 20 | 10 | 10 | 100 | 50 | 50 | 0.104 | 0.424 | 0.030 | 0.257 |
| | | | 150 | 100 | 50 | 0.068 | 0.196 | 0.016 | 0.079 |
| | | | 200 | 150 | 50 | 0.059 | 0.138 | 0.013 | 0.047 |
| | | | 150 | 50 | 100 | 0.094 | 0.383 | 0.026 | 0.221 |
| | | | 200 | 100 | 100 | 0.069 | 0.178 | 0.016 | 0.068 |
| | | | 250 | 150 | 100 | 0.061 | 0.127 | 0.013 | 0.042 |

Table 5: Upper percentiles of LRT statistic and $Q_\alpha^*$ value

| | | | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $\rho_1$ | $\rho_2$ | $N$ | $N_1$ | $N_2$ | LRT | $Q_\alpha^*$ | LRT | $Q_\alpha^*$ |
| 4 | 2 | 2 | 1/2 | 1/2 | 20 | 10 | 10 | 13.32 | 11.89 | 18.84 | 16.68 |
| $\chi^2_{4,0.05} = 9.49$ | | | | | 40 | 20 | 20 | 10.95 | 10.50 | 15.39 | 14.70 |
| $\chi^2_{4,0.01} = 13.28$ | | | | | 100 | 50 | 50 | 10.01 | 9.86 | 14.03 | 13.80 |
| | | | | | 200 | 100 | 100 | 9.75 | 9.67 | 13.63 | 13.53 |
| | | | | | 300 | 150 | 150 | 9.65 | 9.61 | 13.51 | 13.45 |
| | | | | | 400 | 200 | 200 | 9.60 | 9.58 | 13.42 | 13.40 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 11.06 | 10.67 | 15.49 | 14.94 |
| | | | | | 60 | 40 | 20 | 10.19 | 10.04 | 14.25 | 14.05 |
| | | | | | 120 | 80 | 40 | 9.82 | 9.75 | 13.73 | 13.65 |
| | | | | | 240 | 160 | 80 | 9.65 | 9.62 | 13.53 | 13.46 |
| | | | | | 480 | 320 | 160 | 9.58 | 9.55 | 13.36 | 13.37 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 13.17 | 11.68 | 18.67 | 16.39 |
| | | | | | 60 | 20 | 40 | 10.86 | 10.40 | 15.24 | 14.56 |
| | | | | | 120 | 40 | 80 | 10.12 | 9.91 | 14.16 | 13.87 |
| | | | | | 240 | 80 | 160 | 9.78 | 9.69 | 13.67 | 13.56 |
| | | | | | 480 | 160 | 320 | 9.63 | 9.59 | 13.48 | 13.42 |
| 8 | 4 | 4 | 1/2 | 1/2 | 20 | 10 | 10 | 42.15 | 26.87 | 58.36 | 35.76 |
| $\chi^2_{8,0.05} = 15.51$ | | | | | 40 | 20 | 20 | 20.60 | 18.68 | 26.87 | 24.26 |
| $\chi^2_{8,0.01} = 20.09$ | | | | | 100 | 50 | 50 | 17.02 | 16.57 | 22.09 | 21.48 |
| | | | | | 200 | 100 | 100 | 16.24 | 16.01 | 21.00 | 20.75 |
| | | | | | 300 | 150 | 150 | 15.96 | 15.84 | 20.67 | 20.52 |
| | | | | | 400 | 200 | 200 | 15.84 | 15.76 | 20.54 | 20.41 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 20.80 | 19.22 | 27.14 | 24.96 |
| | | | | | 60 | 40 | 20 | 17.58 | 17.10 | 22.85 | 22.16 |
| | | | | | 120 | 80 | 40 | 16.47 | 16.25 | 21.31 | 21.06 |
| | | | | | 240 | 160 | 80 | 15.94 | 15.87 | 20.65 | 20.56 |
| | | | | | 480 | 320 | 160 | 15.72 | 15.68 | 20.36 | 20.32 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 41.78 | 27.08 | 58.04 | 36.23 |
| | | | | | 60 | 20 | 40 | 20.39 | 18.40 | 26.62 | 23.89 |
| | | | | | 120 | 40 | 80 | 17.39 | 16.73 | 22.56 | 21.68 |
| | | | | | 240 | 80 | 160 | 16.35 | 16.08 | 21.19 | 20.83 |
| | | | | | 480 | 160 | 320 | 15.91 | 15.78 | 20.61 | 20.45 |
| 20 | 10 | 10 | 1/2 | 1/2 | 100 | 50 | 50 | 40.25 | 36.95 | 48.29 | 44.25 |
| $\chi^2_{20,0.05} = 31.41$ | | | | | 200 | 100 | 100 | 34.95 | 33.83 | 41.81 | 40.46 |
| $\chi^2_{20,0.01} = 37.57$ | | | | | 300 | 150 | 150 | 33.65 | 32.96 | 40.26 | 39.42 |
| | | | | | 400 | 200 | 200 | 33.00 | 32.55 | 32.55 | 38.93 |
| | | | | | 500 | 250 | 250 | 32.66 | 32.31 | 39.04 | 38.65 |
| | | | | | 600 | 300 | 300 | 32.46 | 32.16 | 38.77 | 38.46 |
| | | | 2/3 | 1/3 | 240 | 160 | 80 | 33.56 | 33.09 | 40.15 | 39.57 |
| | | | | | 480 | 320 | 160 | 32.44 | 32.22 | 38.84 | 38.53 |
| | | | | | 960 | 640 | 320 | 31.93 | 31.81 | 38.19 | 38.04 |
| | | | | | 1920 | 1280 | 640 | 31.65 | 31.61 | 37.89 | 37.80 |
| | | | 1/3 | 2/3 | 240 | 80 | 160 | 35.89 | 34.21 | 43.01 | 40.93 |
| | | | | | 480 | 160 | 320 | 33.37 | 32.70 | 40.00 | 39.11 |
| | | | | | 960 | 320 | 640 | 32.30 | 32.03 | 38.65 | 38.31 |
| | | | | | 1920 | 640 | 1280 | 31.88 | 31.72 | 38.08 | 37.93 |

Table 6: Upper percentiles of LRT statistic and $Q^*_\alpha$ value when $N_2$ is fixed

| | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $N$ | $N_1$ | $N_2$ | LRT | $Q^*_\alpha$ | LRT | $Q^*_\alpha$ |
| 4 | 2 | 2 | 20 | 10 | 10 | 13.32 | 11.89 | 18.84 | 16.68 |
| $\chi^2_{4,0.05} = 9.49$ | | | 30 | 20 | 10 | 11.06 | 10.67 | 15.49 | 14.94 |
| $\chi^2_{4,0.01} = 13.28$ | | | 60 | 50 | 10 | 10.06 | 10.00 | 14.06 | 14.00 |
| | | | 110 | 100 | 10 | 9.78 | 9.76 | 13.66 | 13.65 |
| | | | 60 | 10 | 50 | 13.00 | 11.63 | 18.49 | 16.32 |
| | | | 70 | 20 | 50 | 10.81 | 10.38 | 15.17 | 14.53 |
| | | | 100 | 50 | 50 | 10.01 | 9.86 | 14.03 | 13.80 |
| | | | 150 | 100 | 50 | 9.76 | 9.70 | 13.63 | 13.57 |
| | | | 110 | 10 | 100 | 12.94 | 11.65 | 18.42 | 16.35 |
| | | | 120 | 20 | 100 | 10.77 | 10.35 | 15.13 | 14.49 |
| | | | 150 | 50 | 100 | 9.95 | 9.82 | 13.91 | 13.74 |
| | | | 200 | 100 | 100 | 9.75 | 9.67 | 13.63 | 13.53 |
| 8 | 4 | 4 | 20 | 10 | 10 | 42.15 | 26.87 | 58.36 | 35.76 |
| $\chi^2_{8,0.05} = 15.51$ | | | 30 | 20 | 10 | 20.80 | 19.22 | 27.14 | 24.96 |
| $\chi^2_{8,0.01} = 20.09$ | | | 60 | 50 | 10 | 17.20 | 16.98 | 22.30 | 22.01 |
| | | | 110 | 100 | 10 | 16.32 | 16.26 | 21.19 | 21.07 |
| | | | 60 | 10 | 50 | 41.49 | 28.07 | 57.70 | 37.77 |
| | | | 70 | 20 | 50 | 20.32 | 18.35 | 26.61 | 23.83 |
| | | | 100 | 50 | 50 | 17.02 | 16.57 | 22.09 | 21.48 |
| | | | 150 | 100 | 50 | 16.25 | 16.09 | 21.06 | 20.85 |
| | | | 110 | 10 | 100 | 41.30 | 28.70 | 57.52 | 38.70 |
| | | | 120 | 20 | 100 | 20.20 | 18.31 | 26.42 | 23.79 |
| | | | 150 | 50 | 100 | 16.95 | 16.46 | 21.97 | 21.33 |
| | | | 200 | 100 | 100 | 16.24 | 16.01 | 21.00 | 20.75 |
| 20 | 10 | 10 | 100 | 50 | 50 | 40.25 | 36.95 | 48.29 | 44.25 |
| $\chi^2_{20,0.05} = 31.41$ | | | 150 | 100 | 50 | 35.13 | 34.22 | 42.04 | 40.94 |
| $\chi^2_{20,0.01} = 37.57$ | | | 200 | 150 | 50 | 33.81 | 33.36 | 40.43 | 39.91 |
| | | | 150 | 50 | 100 | 39.92 | 36.44 | 47.90 | 43.64 |
| | | | 200 | 100 | 100 | 34.95 | 33.83 | 41.81 | 40.46 |
| | | | 250 | 150 | 100 | 33.70 | 33.10 | 40.31 | 39.59 |

Table 7: Type I error rate using $Q^*_\alpha$ and $\chi^2_p$ values under LRT statistic

| | | | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $\rho_1$ | $\rho_2$ | $N$ | $N_1$ | $N_2$ | $Q^*_\alpha$ | $\chi^2_p$ | $Q^*_\alpha$ | $\chi^2_p$ |
| 4 | 2 | 2 | 1/2 | 1/2 | 20 | 10 | 10 | 0.075 | 0.146 | 0.019 | 0.051 |
| | | | | | 40 | 20 | 20 | 0.059 | 0.084 | 0.013 | 0.022 |
| | | | | | 100 | 50 | 50 | 0.053 | 0.061 | 0.011 | 0.014 |
| | | | | | 200 | 100 | 100 | 0.052 | 0.055 | 0.010 | 0.012 |
| | | | | | 300 | 150 | 150 | 0.051 | 0.053 | 0.010 | 0.011 |
| | | | | | 400 | 200 | 200 | 0.050 | 0.052 | 0.010 | 0.011 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 0.057 | 0.086 | 0.012 | 0.022 |
| | | | | | 60 | 40 | 20 | 0.053 | 0.065 | 0.011 | 0.015 |
| | | | | | 120 | 80 | 40 | 0.051 | 0.057 | 0.010 | 0.012 |
| | | | | | 240 | 160 | 80 | 0.051 | 0.053 | 0.010 | 0.011 |
| | | | | | 480 | 320 | 160 | 0.051 | 0.052 | 0.010 | 0.010 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 0.077 | 0.141 | 0.020 | 0.049 |
| | | | | | 60 | 20 | 40 | 0.059 | 0.081 | 0.013 | 0.021 |
| | | | | | 120 | 40 | 80 | 0.054 | 0.064 | 0.011 | 0.014 |
| | | | | | 240 | 80 | 160 | 0.052 | 0.056 | 0.010 | 0.012 |
| | | | | | 480 | 160 | 320 | 0.051 | 0.053 | 0.010 | 0.011 |
| 8 | 4 | 4 | 1/2 | 1/2 | 20 | 10 | 10 | 0.217 | 0.570 | 0.093 | 0.396 |
| | | | | | 40 | 20 | 20 | 0.079 | 0.162 | 0.020 | 0.057 |
| | | | | | 100 | 50 | 50 | 0.057 | 0.079 | 0.012 | 0.019 |
| | | | | | 200 | 100 | 100 | 0.054 | 0.063 | 0.011 | 0.014 |
| | | | | | 300 | 150 | 150 | 0.052 | 0.058 | 0.011 | 0.012 |
| | | | | | 400 | 200 | 200 | 0.051 | 0.056 | 0.010 | 0.012 |
| | | | 2/3 | 1/3 | 30 | 20 | 10 | 0.073 | 0.168 | 0.018 | 0.059 |
| | | | | | 60 | 40 | 20 | 0.058 | 0.091 | 0.012 | 0.024 |
| | | | | | 120 | 80 | 40 | 0.054 | 0.067 | 0.011 | 0.015 |
| | | | | | 240 | 160 | 80 | 0.051 | 0.057 | 0.010 | 0.012 |
| | | | | | 480 | 320 | 160 | 0.051 | 0.054 | 0.010 | 0.011 |
| | | | 1/3 | 2/3 | 30 | 10 | 20 | 0.206 | 0.557 | 0.086 | 0.386 |
| | | | | | 60 | 20 | 40 | 0.081 | 0.157 | 0.021 | 0.054 |
| | | | | | 120 | 40 | 80 | 0.061 | 0.086 | 0.013 | 0.022 |
| | | | | | 240 | 80 | 160 | 0.054 | 0.065 | 0.011 | 0.015 |
| | | | | | 480 | 160 | 320 | 0.052 | 0.057 | 0.011 | 0.012 |
| 20 | 10 | 10 | 1/2 | 1/2 | 100 | 50 | 50 | 0.090 | 0.216 | 0.023 | 0.081 |
| | | | | | 200 | 100 | 100 | 0.064 | 0.104 | 0.014 | 0.028 |
| | | | | | 300 | 150 | 150 | 0.058 | 0.081 | 0.012 | 0.020 |
| | | | | | 400 | 200 | 200 | 0.056 | 0.071 | 0.011 | 0.016 |
| | | | | | 500 | 250 | 250 | 0.054 | 0.066 | 0.011 | 0.015 |
| | | | | | 600 | 300 | 300 | 0.054 | 0.064 | 0.011 | 0.014 |
| | | | 2/3 | 1/3 | 240 | 160 | 80 | 0.056 | 0.081 | 0.012 | 0.019 |
| | | | | | 480 | 320 | 160 | 0.053 | 0.064 | 0.011 | 0.014 |
| | | | | | 960 | 640 | 320 | 0.051 | 0.056 | 0.010 | 0.012 |
| | | | | | 1920 | 1280 | 640 | 0.051 | 0.053 | 0.010 | 0.011 |
| | | | 1/3 | 2/3 | 240 | 80 | 160 | 0.070 | 0.121 | 0.016 | 0.035 |
| | | | | | 480 | 160 | 320 | 0.058 | 0.077 | 0.013 | 0.018 |
| | | | | | 960 | 320 | 640 | 0.053 | 0.062 | 0.011 | 0.014 |
| | | | | | 1920 | 640 | 1280 | 0.052 | 0.056 | 0.010 | 0.012 |

Table 8: Type I error rate using $Q_\alpha^*$ and $\chi_p^2$ values under LRT statistic when $N_2$ is fixed

| | | | | | | $\alpha = 0.05$ | | $\alpha = 0.01$ | |
|---|---|---|---|---|---|---|---|---|---|
| $p$ | $p_1$ | $p_2$ | $N$ | $N_1$ | $N_2$ | $Q_\alpha^*$ | $\chi_p^2$ | $Q_\alpha^*$ | $\chi_p^2$ |
| 4 | 2 | 2 | 20 | 10 | 10 | 0.075 | 0.146 | 0.019 | 0.051 |
| | | | 30 | 20 | 10 | 0.057 | 0.086 | 0.012 | 0.022 |
| | | | 60 | 50 | 10 | 0.051 | 0.062 | 0.010 | 0.014 |
| | | | 110 | 100 | 10 | 0.050 | 0.056 | 0.010 | 0.012 |
| | | | 60 | 10 | 50 | 0.074 | 0.136 | 0.019 | 0.046 |
| | | | 70 | 20 | 50 | 0.058 | 0.080 | 0.013 | 0.020 |
| | | | 100 | 50 | 50 | 0.053 | 0.061 | 0.011 | 0.014 |
| | | | 150 | 100 | 50 | 0.051 | 0.056 | 0.010 | 0.012 |
| | | | 110 | 10 | 100 | 0.073 | 0.134 | 0.018 | 0.045 |
| | | | 120 | 20 | 100 | 0.058 | 0.079 | 0.013 | 0.020 |
| | | | 150 | 50 | 100 | 0.053 | 0.060 | 0.011 | 0.013 |
| | | | 200 | 100 | 100 | 0.052 | 0.055 | 0.010 | 0.012 |
| 8 | 4 | 4 | 20 | 10 | 10 | 0.217 | 0.570 | 0.093 | 0.396 |
| | | | 30 | 20 | 10 | 0.073 | 0.168 | 0.018 | 0.059 |
| | | | 60 | 50 | 10 | 0.053 | 0.082 | 0.011 | 0.020 |
| | | | 110 | 100 | 10 | 0.051 | 0.065 | 0.010 | 0.015 |
| | | | 60 | 10 | 50 | 0.183 | 0.548 | 0.072 | 0.377 |
| | | | 70 | 20 | 50 | 0.081 | 0.155 | 0.021 | 0.053 |
| | | | 100 | 50 | 50 | 0.057 | 0.079 | 0.012 | 0.019 |
| | | | 150 | 100 | 50 | 0.052 | 0.063 | 0.011 | 0.014 |
| | | | 110 | 10 | 100 | 0.374 | 0.543 | 0.065 | 0.170 |
| | | | 120 | 20 | 100 | 0.079 | 0.151 | 0.020 | 0.051 |
| | | | 150 | 50 | 100 | 0.058 | 0.077 | 0.012 | 0.019 |
| | | | 200 | 100 | 100 | 0.054 | 0.063 | 0.011 | 0.014 |
| 20 | 10 | 10 | 100 | 50 | 50 | 0.090 | 0.216 | 0.023 | 0.081 |
| | | | 150 | 100 | 50 | 0.061 | 0.107 | 0.013 | 0.029 |
| | | | 200 | 150 | 50 | 0.055 | 0.084 | 0.011 | 0.021 |
| | | | 150 | 50 | 100 | 0.093 | 0.208 | 0.024 | 0.077 |
| | | | 200 | 100 | 100 | 0.064 | 0.104 | 0.014 | 0.028 |
| | | | 250 | 150 | 100 | 0.057 | 0.082 | 0.012 | 0.020 |

## §7. Numerical example

We illustrate how $F^*_\alpha$ improves the approximation of simultaneous confidence intervals using an example. The sample data consist of serum cholesterol values that were measured under treatment at five different time points, baseline and months 6, 12, 20, and 24 (Wei and Lachin (1984)). The original data has 36 complete observations. We randomly selected 30 observations and deleted values for ten observations for months 20 and 24 to create two-step monotone missing data. We are interested in the change from the baseline at each post-baseline time point. Thus, we have the two-step monotone missing data of $N_1 = 20, N_2 = 10$, and $p_1 = p_2 = 2$. The hypothesis $H_0 : \boldsymbol{\mu} = 0$ is considered for this data. We obtained $T^2 = 19.62$. Since $t^2_{4,0.05} = 13.94$ from the simulation study, the null hypothesis is rejected at the significance level of 0.05. When we use $F^*_{0.05} = 12.58$ or $\chi^2_{4,0.05} = 9.46$, the null hypothesis is also rejected. 95 % simultaneous confidence intervals for the change from the baseline at each time point are shown in Figure 1. Considering the confidence intervals using the upper $100\alpha$ percentile of the $T^2$ type statistic to be true results, Figure 1 shows that $F^*_\alpha$ gives the same results as the $T^2$ type statistic, while the $\chi^2$ distribution leads to incorrect conclusions at months 6 and 20.

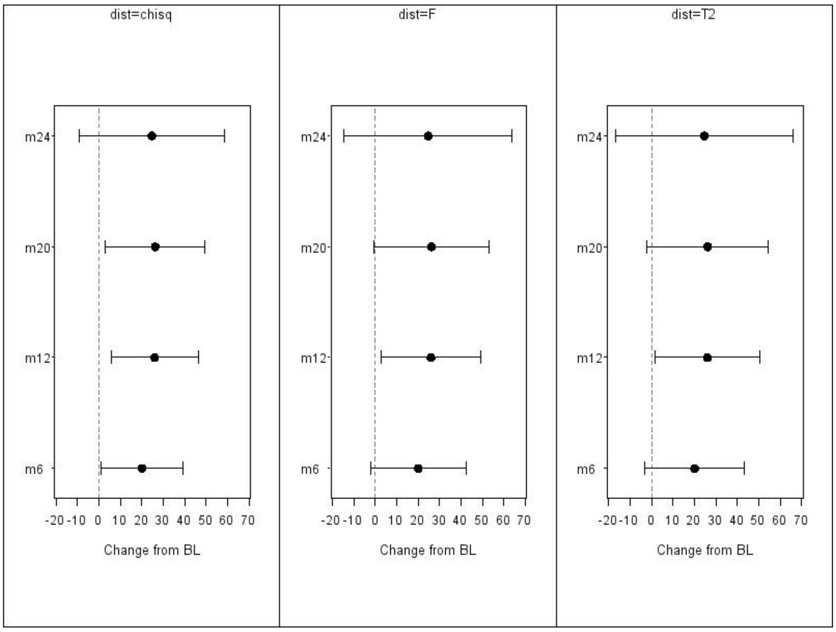


Figure 1: Mean and 95 % simultaneous confidence interval for change from baseline

## §8. Conclusion remarks

In this paper, we have developed the approximate upper percentiles of Hotelling's $T^2$ type statistic and the likelihood ratio test for mean vector based on two-step monotone missing data. The approximate values can be calculated easily and the approximation is much better than the chi-squared approximation even when the sample size is small. The approximate values can also be used for the test of the components of mean vector and for the approximate simultaneous confidence intervals.

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