

On almost pseudo cyclic Ricci symmetric manifolds

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(Received October 23, 2009; Revised January 31, 2011)

Abstract. The object of the present paper is to introduce a type of non-flat Riemannian manifolds called *almost pseudo cyclic Ricci symmetric manifold* and study its geometric properties. Among others it is shown that an almost pseudo cyclic Ricci symmetric manifold is a special type of quasi-Einstein manifold. We also study conformally flat almost pseudo cyclic Ricci symmetric manifolds and prove that such a manifold is isometrically immersed in an Euclidean manifold as a hypersurface. The existence of such notion is ensured by a non-trivial example.

AMS 2010 Mathematics Subject Classification. 53B30, 53B50, 53C15, 53C25.

Key words and phrases. Almost pseudo Ricci symmetric manifold, almost pseudo cyclic Ricci symmetric manifold, scalar curvature, conformally flat, special conformally flat, concircular vector field, warped product.

§1. Introduction

By extending the definition of pseudo Ricci symmetric manifold, very recently M. C. Chaki and T. Kawaguchi [8] introduced the notion of almost pseudo Ricci symmetric manifold. A Riemannian manifold $(M^n, g)(n > 2)$ is called an almost pseudo Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the condition

$$(1.1) \quad (\nabla_X S)(Y, Z) = [A(X) + B(X)]S(Y, Z) \\ + A(Y)S(X, Z) + A(Z)S(Y, X) \quad \text{for all } X, Y, Z \in \chi(M),$$

where ∇ denotes the operator of covariant differentiation with respect to the metric tensor g and $\chi(M)$ is the Lie algebra of all smooth vector fields on M ; and A and B are nowhere vanishing 1-forms associated with the unique vector field U and V respectively such that $g(X, U) = A(X)$ and $g(X, V) = B(X)$ for all $X \in \chi(M)$. The vector fields U, V are called the generators of the manifold.

The 1-forms A and B are called associated 1-forms and an n -dimensional manifold of this kind is denoted by $A(PRS)_n$. If, in particular, $B = A$ then (1.1) reduces to the notion of pseudo Ricci symmetric manifold [6]. Extending the notion of almost pseudo Ricci symmetric manifold, in the present paper we introduce the notion of *almost pseudo cyclic Ricci symmetric manifold*. A Riemannian manifold $(M^n, g)(n > 2)$ (this condition is assumed throughout the paper) is said to be an almost pseudo cyclic Ricci symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the following:

$$(1.2) \quad (\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) \\ = [A(X) + B(X)]S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X),$$

where A and B are nowhere vanishing 1-forms associated to the unique vector field U and V respectively such that $A(X) = g(X, U)$ and $B(X) = g(X, V)$ for all X and ∇ denotes the operator of covariant differentiation with respect to the metric tensor g . Such an n -dimensional manifold is denoted by $A(PCRS)_n$.

The present paper is organized as follows. Section 2 is concerned with some basic properties of $A(PCRS)_n$. Every $A(PRS)_n$ is $A(PCRS)_n$ but not conversely. However, it is proved that an $A(PCRS)_n$ with Codazzi type Ricci tensor is an $A(PRS)_n$. Again it is shown that an $A(PCRS)_n$ is a special type of quasi-Einstein manifold ([7], [10], [11]) and also it is proved that the scalar curvature of an $A(PCRS)_n$ is always non-zero at every point of the manifold.

In Section 3 we investigate conformally flat $A(PCRS)_n$ and prove that such a manifold is of quasi-constant curvature. It is shown that in a conformally flat $A(PCRS)_n$, the unique unit vector field λ defined by $\frac{B(X)}{\sqrt{B(V)}} = T(X) = g(X, \lambda)$, for all $X \in \chi(M)$, is a unit proper concircular vector field and hence such a manifold is a subprojective manifold in the sense of Kagan ([1], [20]). Again it is proved that a conformally flat $A(PCRS)_n$ can be expressed as the warped product $I \times_{e^p} \overset{*}{M}$, where $(\overset{*}{M}, \overset{*}{g})$ is an $(n - 1)$ dimensional Einstein manifold [13].

As a generalization of subprojective manifold B. Y. Chen and K. Yano [9] introduced the notion of special conformally flat manifold. It is shown that a conformally flat $A(PCRS)_n$ with non-constant and negative scalar curvature is a special conformally flat manifold and also it is proved that such a simply connected manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.

A non-zero vector X on a semi-Riemannian manifold M is said to be time-like (resp., non-spacelike, null, spacelike) if it satisfies $g(X, X) < 0$ (resp. ≤ 0 , $= 0$, > 0) [15]. Since λ is a unit vector field on the Riemannian manifold $M = A(PCRS)_n$ with metric tensor g , it can be easily shown [15](p.148) that $\tilde{g} = g - 2T \otimes T$ is a Lorentz metric on M . Hence λ becomes timelike so that

the resulting Lorentz manifold (M, \tilde{g}) is time-orientable. The last section deals with a non-trivial example of $A(PCRS)_n$.

§2. Some basic properties of $A(PCRS)_n$

This section deals with various basic geometric properties of $A(PCRS)_n$. Let Q be the symmetric endomorphism of the tangent bundle of the manifold corresponding to the Ricci tensor S , i.e., $g(QX, Y) = S(X, Y)$ for all vector fields X, Y .

Let $\{e_i : i = 1, 2, \dots, n\}$ be an orthonormal basis of the tangent space at each point of the manifold. Then setting $Y = Z = e_i$ in (1.2) and then taking summation over i , $1 \leq i \leq n$, we obtain

$$(2.1) \quad 2dr(X) = r[A(X) + B(X)] + 2A(QX),$$

where r is the scalar curvature of the manifold. Again contracting (1.2) with respect to Z and X , and then replacing Y by X in the resulting equation, we get

$$(2.2) \quad 2dr(X) = rA(X) + 2A(QX) + B(QX).$$

From (2.1) and (2.2) it follows that

$$(2.3) \quad B(QX) = rB(X),$$

i.e.,

$$(2.4) \quad S(X, V) = rg(X, V).$$

This leads to the following:

Proposition 1. *The scalar curvature r of an $A(PCRS)_n$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector V .*

We now suppose that the Ricci tensor of an $A(PCRS)_n$ ($n > 2$) is of Codazzi type ([12],[16]). Then we have

$$(2.5) \quad (\nabla_X S)(Y, Z) = (\nabla_Y S)(Z, X) = (\nabla_Z S)(X, Y)$$

for all X, Y, Z and hence the defining condition (1.2) of an $A(PCRS)_n$ reduces to

$$(2.6) \quad (\nabla_X S)(Y, Z) = [\bar{A}(X) + \bar{B}(X)]S(Y, Z) + \bar{A}(Y)S(Z, X) + \bar{A}(Z)S(X, Y),$$

where $\bar{A}(X) = g(X, \frac{1}{3}U)$ and $\bar{B}(X) = g(X, \frac{1}{3}V)$ are nowhere vanishing 1-forms, and consequently the manifold is $A(PRS)_n$. This leads to the following:

Proposition 2. *If the Ricci tensor of an $A(PCRS)_n$ is of Codazzi type, then it is an $A(PRS)_n$.*

Interchanging X and Y in (1.2) we obtain

$$\begin{aligned} & (\nabla_Y S)(X, Z) + (\nabla_X S)(Z, Y) + (\nabla_Z S)(Y, X) \\ &= [A(Y) + B(Y)]S(X, Z) + A(X)S(Y, Z) + A(Z)S(Y, X). \end{aligned}$$

By virtue of (1.2) and above equation, it follows that

$$(2.7) \quad B(X)S(Y, Z) = B(Y)S(X, Z) \quad \forall X, Y, Z \in \chi(M).$$

Setting $Y = V$ in (2.7) we get

$$(2.8) \quad S(X, Z) = \frac{1}{B(V)}B(X)B(QZ).$$

In view of (2.3), (2.8) yields

$$(2.9) \quad S(X, Z) = rT(X)T(Z),$$

where $T(X) = g(X, \lambda) = \frac{1}{\sqrt{B(V)}}B(X)$, λ being a unit vector field associated with the nowhere vanishing 1-form T . From (2.9) it follows that if $r = 0$ at every point of the manifold, then $S(X, Z) = 0$, which is inadmissible by the definition of $A(PCRS)_n$. Hence we can state the following:

Proposition 3. *The scalar curvature of an $A(PCRS)_n$ is non-zero at every point of the manifold, and its Ricci tensor is of the form (2.9).*

A Riemannian manifold (M^n, g) ($n > 2$) is said to be quasi-Einstein manifold ([7], [10], [11]) if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies the following:

$$(2.10) \quad S(X, Y) = \alpha g(X, Y) + \beta D(X)D(Y),$$

where α, β are scalars of which $\beta \neq 0$ and D is a nowhere vanishing 1-form associated with the unique unit vector field σ defined by $g(X, \sigma) = D(X)$ for all $X \in \chi(M)$. Hence from (2.9) we can state the following:

Proposition 4. *An $A(PCRS)_n$ is a special type of quasi-Einstein manifold.*

If an $A(PCRS)_n$ ($n > 2$) is Einstein, then from (1.2) it follows that

$$(2.11) \quad \{A(X) + B(X)\}g(Y, Z) + A(Y)g(Z, X) + A(Z)g(X, Y) = 0,$$

as r is non-zero at every point of the manifold. From (2.11), we can easily get $B(X) = 0$, which is inadmissible by the definition of $A(PCRS)_n$. Hence we can state the following:

Proposition 5. *There does not exist any Einstein $A(PCRS)_n$ ($n > 2$).*

§3. Conformally flat $A(PCRS)_n$

A Riemannian manifold (M^n, g) ($n > 2$) is said to be the manifold of quasi-constant curvature ([2], [3], [4], [5], [17]) if it is conformally flat and its curvature tensor R of type $(0, 4)$ has the following form:

$$(3.1) \quad R(X, Y, Z, W) = a_1[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ + a_2[g(Y, Z)A(X)A(W) - g(X, Z)A(Y)A(W) \\ + g(X, W)A(Y)A(Z) - g(Y, W)A(X)A(Z)],$$

where A is a nowhere vanishing 1-form and a_1, a_2 are scalars of which $a_2 \neq 0$. Let a Riemannian manifold (M^n, g) ($n > 3$) be conformally flat. Then the curvature tensor R of type $(0, 4)$ is of the following form:

$$(3.2) \quad R(X, Y, Z, W) = \frac{1}{n-2}[\{S(Y, Z)g(X, W) - S(X, Z)g(Y, W) \\ + S(X, W)g(Y, Z) - S(Y, W)g(X, Z)\} \\ - \frac{r}{(n-1)}\{g(Y, Z)g(X, W) - g(X, Z)g(Y, W)\}].$$

Using (2.9) in (3.2) we obtain

$$(3.3) \quad R(X, Y, Z, W) = a[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)] \\ + b[g(X, W)T(Y)T(Z) - g(Y, W)T(X)T(Z) \\ + g(Y, Z)T(X)T(W) - g(X, Z)T(Y)T(W)],$$

where $a = -\frac{r}{(n-1)(n-2)}$ and $b = \frac{r}{n-2}$ are non-zero scalars. By virtue of (3.1), it follows from (3.3) that a conformally flat $A(PCRS)_n$ ($n > 3$) is a manifold of quasi-constant curvature. This leads to the following:

Theorem 1. *Every conformally flat $A(PCRS)_n$ ($n > 3$) is a manifold of quasi-constant curvature.*

Since in a 3-dimensional Riemannian manifold the conformal curvature tensor C vanishes, the relation (3.2) holds and hence (3.3) also holds. This leads to the following:

Corollary 1. *Every $A(PCRS)_3$ is a manifold of quasi-constant curvature.*

The conformal curvature tensor C of type $(1, 3)$ of a Riemannian manifold (M^n, g) ($n > 3$) is defined by

$$(3.4) \quad C(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X \\ - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\ + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y].$$

Differentiating (3.4) covariantly and then contracting we obtain

$$(3.5) \quad \begin{aligned} (divC)(X, Y)Z &= (divR)(X, Y)Z - \frac{1}{n-2}[(\nabla_X S)(Y, Z) \\ &\quad - (\nabla_Y S)(X, Z) + \frac{1}{2}dr(X)g(Y, Z) - \frac{1}{2}dr(Y)g(X, Z)] \\ &\quad + \frac{1}{(n-1)(n-2)}[dr(X)g(Y, Z) - dr(Y)g(X, Z)], \end{aligned}$$

where ‘div’ denotes the divergence. Again it is known that in a Riemannian manifold, we have

$$(divR)(X, Y)Z = (\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z).$$

Consequently by virtue of the above relation, (3.5) takes the form

$$(3.6) \quad \begin{aligned} (divC)(X, Y)Z &= \frac{n-3}{n-2}[(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \\ &\quad - \frac{1}{2(n-1)}\{dr(X)g(Y, Z) - dr(Y)g(X, Z)\}]. \end{aligned}$$

Let us consider a conformally flat $A(PCRS)_n (n > 3)$. Then we have

$$(divC)(X, Y)Z = 0$$

and hence (3.6) yields

$$(3.7) \quad \begin{aligned} &(\nabla_X S)(Y, Z) - (\nabla_Y S)(X, Z) \\ &= \frac{1}{2(n-1)}[dr(X)g(Y, Z) - dr(Y)g(X, Z)]. \end{aligned}$$

Again from (2.9) we have

$$(3.8) \quad \begin{aligned} &(\nabla_Y S)(X, Z) \\ &= dr(Y)T(X)T(Z) + r[(\nabla_Y T)(X)T(Z) + T(X)(\nabla_Y T)(Z)]. \end{aligned}$$

Using (3.8) in (3.7) we obtain

$$(3.9) \quad \begin{aligned} &dr(X)T(Y)T(Z) - dr(Y)T(X)T(Z) + r[(\nabla_X T)(Y)T(Z) \\ &\quad + T(Y)(\nabla_X T)(Z) - (\nabla_Y T)(X)T(Z) - T(X)(\nabla_Y T)(Z)] \\ &= \frac{1}{2(n-1)}[dr(X)g(Y, Z) - dr(Y)g(X, Z)]. \end{aligned}$$

Setting $Y = Z = e_i$ in (3.9) and taking summation over $i, 1 \leq i \leq n$, we get

$$(3.10) \quad dr(\lambda)T(X) + r\{(\nabla_\lambda T)(X) + T(X) \sum_{i=1}^n (\nabla_{e_i} T)(e_i)\} = \frac{1}{2}dr(X).$$

Again putting $Y = Z = \lambda$ in (3.9) we obtain

$$(3.11) \quad r(\nabla_\lambda T)(X) = \frac{2n-3}{2(n-1)}[dr(X) - dr(\lambda)T(X)].$$

Using (3.11) in (3.10) we get

$$(3.12) \quad rT(X) \sum_{i=1}^n (\nabla_{e_i} T)(e_i) + \frac{n-2}{2(n-1)}dr(X) + \frac{1}{2(n-1)}dr(\lambda)T(X) = 0.$$

Setting $X = \lambda$ in (3.12) we get

$$(3.13) \quad r \sum_{i=1}^n (\nabla_{e_i} T)(e_i) = -\frac{1}{2}dr(\lambda).$$

From (3.12) and (3.13), it follows that

$$(3.14) \quad dr(X) = dr(\lambda)T(X).$$

Again plugging $Z = \lambda$ in (3.9) and then using (3.14), we obtain

$$r\{(\nabla_X T)(Y) - (\nabla_Y T)(X)\} = 0,$$

which implies that

$$(3.15) \quad (\nabla_X T)(Y) - (\nabla_Y T)(X) = 0,$$

since $r \neq 0$. The relation (3.15) implies that the 1-form T is closed. In view of (3.14) it follows from (3.11) that

$$(3.16) \quad (\nabla_\lambda T)(Z) = 0,$$

which implies that $\nabla_\lambda \lambda = 0$. The existence of the integral curve of λ is ensured by Proposition 49 of ([15], p. 28). Hence we can state the following:

Theorem 2. *In a conformally flat $A(PCRS)_n$ ($n > 3$), the integral curves of the vector field λ are geodesics.*

Also setting $Y = \lambda$ in (3.9) we obtain by virtue of (3.14) and (3.16) that

$$(3.17) \quad (\nabla_X T)(Z) = \frac{1}{2(n-1)r}dr(\lambda)[T(X)T(Z) - g(X, Z)].$$

Let us now consider a non-zero scalar function $f = \frac{1}{2(n-1)r}dr(\lambda)$, where the scalar curvature r is non-constant. Then we have

$$(3.18) \quad \nabla_X f = -\frac{1}{2(n-1)r^2}dr(\lambda)dr(X) + \frac{1}{2(n-1)r}d^2r(\lambda, X).$$

From (3.14) it follows that

$$(3.19) \quad d^2r(X, Y) = d^2r(\lambda, Y)T(X) + dr(\lambda)(\nabla_Y T)(X).$$

Since a second covariant differential of any function is a symmetric 2-form, (3.15) and (3.19) give

$$(3.20) \quad d^2r(\lambda, Y)T(X) = d^2r(\lambda, X)T(Y).$$

Replacing Y by λ in (3.20) we have

$$(3.21) \quad d^2r(\lambda, X) = d^2r(\lambda, \lambda)T(X) = -\psi T(X),$$

where $\psi = -d^2r(\lambda, \lambda)$ is a scalar function. Using (3.14) and (3.21) in (3.18) we obtain

$$(3.22) \quad \nabla_X f = \mu T(X),$$

where

$$\mu = -\frac{1}{2(n-1)r^2}[r\psi + \{dr(\lambda)\}^2]$$

is a scalar. Since r is non-zero and non-constant function, $dr(X) \neq 0$ for all X and hence $dr(\lambda) \neq 0$, and therefore $\psi \neq 0$. Consequently μ is a non-zero scalar. We now consider an 1-form ω defined by

$$\omega(X) = \frac{1}{2(n-1)r}dr(\lambda)T(X) = fT(X).$$

Then by virtue of (3.15) and (3.22) we have

$$d\omega(X, Y) = 0.$$

Hence the 1-form ω is closed. Therefore (3.17) can be rewritten as

$$(3.23) \quad (\nabla_X T)(Z) = -fg(X, Z) + \omega(X)T(Z),$$

which implies that the vector field λ is a proper unit concircular vector field ([18], [19]). Again in a conformally flat 3-dimensional Riemannian manifold, the relation (3.7) holds for $n = 3$. Hence proceeding similarly as above it is easy to check that in an $A(PCRS)_3$, λ is a unit proper concircular vector field. Thus we can state the following:

Theorem 3. *In a conformally flat $A(PCRS)_n$ with non-constant scalar curvature the vector field λ defined by $g(X, \lambda) = T(X)$ is a unit proper concircular vector field.*

Again if a conformally flat Riemannian manifold $(M^n, g), n > 3$, admits a proper concircular vector field, then the manifold is a subprojective manifold of Kagan ([1], [20]) and hence by virtue of Theorem 3, we can state the following:

Theorem 4. *Every conformally flat $A(PCRS)_n, n > 3$, with non-constant scalar curvature is a subprojective manifold in the sense of Kagan.*

In [19] K. Yano proved that a Riemannian manifold admits a concircular vector field, if and only if there exists a coordinate system with respect to which the fundamental quadratic differential form can be written as

$$ds^2 = dx'^2 + e^p g_{ij}^* dx^i dx^j,$$

where $g_{ij}^* = g_{ij}[x^k]$ are the functions of x^k only ($i, j, k = 2, 3, \dots, n$) and $p = p(x')$ is a non-constant function of x' only. Hence if an $A(PCRS)_n, n > 3$, is conformally flat, then it is a warped product $I \times_{e^p} M^*$, where (M^*, g^*) is an $(n-1)$ dimensional Riemannian manifold, and $I \subset \mathbf{R}$ is an open interval. Again, A. Gebarowski [13] proved that the warped product $I \times_{e^p} M^*$ satisfies the condition (3.7) if and only if M^* is an Einstein manifold. Since a conformally flat $A(PCRS)_n$ satisfies (3.7), it must be a warped product $I \times_{e^p} M^*$, where M^* is an Einstein manifold. Hence we can state the following:

Theorem 5. *Every conformally flat $A(PCRS)_n, n > 3$, with non-constant scalar curvature can be expressed as a warped product $I \times_{e^p} M^*$, where (M^*, g^*) is an $(n-1)$ dimensional Einstein manifold.*

As a generalization of subprojective manifold, B. Y. Chen and K. Yano [9] introduced the notion of special conformally flat manifold defined as follows: a conformally flat Riemannian manifold is said to be a special conformally flat manifold if the tensor field H of type $(0, 2)$ defined by

$$(3.24) \quad H(X, Y) = -\frac{1}{n-2}S(X, Y) + \frac{r}{2(n-1)(n-2)}g(X, Y)$$

is expressible in the form

$$(3.25) \quad H(X, Y) = -\frac{\alpha^2}{2}g(X, Y) + \beta(X\alpha)(Y\alpha),$$

where α, β are two scalars such that α is positive. In view of (2.9), (3.24) can be written as

$$(3.26) \quad H(X, Y) = -\frac{r}{n-2}T(X)T(Y) + \frac{r}{2(n-1)(n-2)}g(X, Y).$$

Let us now take

$$(3.27) \quad \alpha^2 = -\frac{r}{(n-1)(n-2)} > 0, \quad \text{provided } r < 0.$$

Then

$$2\alpha(X\alpha) = -\frac{dr(X)}{(n-1)(n-2)},$$

which implies by virtue of (3.14) that

$$2\alpha(X\alpha) = -\frac{dr(\lambda)T(X)}{(n-1)(n-2)}.$$

Hence

$$T(X)T(Y) = -\frac{4(n-1)(n-2)r}{\delta^2}(X\alpha)(Y\alpha),$$

where $\delta = dr(\lambda) \neq 0$. Therefore, by virtue of (3.27), (3.26) can be expressed as

$$H(X, Y) = -\frac{\alpha^2}{2}g(X, Y) + \beta(X\alpha)(Y\alpha),$$

where $\beta = \frac{4(n-1)r^2}{\delta^2} \neq 0$ at each point of the manifold, provided r is non-constant. Hence the manifold is a special conformally flat manifold. This leads to the following:

Theorem 6. *A conformally flat $A(PCRS)_n, n > 3$, with non-constant and negative scalar curvature is a special conformally flat manifold.*

Again in [9] it is proved that every simply-connected special conformally flat manifold can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface. Therefore by virtue of Theorem 6, we can state the following:

Theorem 7. *Every simply-connected conformally flat $A(PCRS)_n, n > 3$, with non-constant and negative scalar curvature can be isometrically immersed in an Euclidean manifold E^{n+1} as a hypersurface.*

§4. An example of $A(PCRS)_4$

Example . Let $M = \mathbf{R}^3 \times (0, \infty) \subseteq \mathbf{R}^4$. We take the identity map I_d on M such that $N = I_d(M) = \mathbf{R}^3 \times (0, \infty)$. Then $\{E_1 = (1, 0, 0, 0), E_2 = (0, 1, 0, 0), E_3 = (0, 0, 1, 0), E_4 = (0, 0, 0, 1)\}$ form a basis at each point $p \in N$. We define the Lorentz metric g on N by giving its components relative to the basis $\{E_i(p)\}$ at each point $p = (x^1, x^2, x^3, x^4)$ as follows:

$$(4.1) \quad g_{ij}(p) = g_{ij}(x^1, x^2, x^3, x^4) = \begin{cases} 0 & \text{if } i \neq j; \\ (x^4)^{\frac{4}{3}} & \text{if } i = j = 1, 2, 3; \\ -1 & \text{if } i = j = 4, \end{cases}$$

where $x^i, i = 1, 2, 3, 4$ are the standard coordinates of \mathbf{R}^4 . Then the manifold N with metric (4.1) represents the Einstein-deSitter spacetime which represents the simplest cosmological model in general relativity ([14], p. 89). Hence for $X, Y \in T_p N$,

$$\begin{aligned} g_p(X, Y) &= g_p(X^i E_i, Y^j E_j) \\ &= (x^4)^{\frac{4}{3}} (X^1 Y^1 + X^2 Y^2 + X^3 Y^3) - X^4 Y^4, \end{aligned}$$

where x^4 is the ‘height’ of p . We note that this is a Lorentz metric since smoothness is obvious on $x^4 > 0$ and, for each $p \in N$, one can define a new basis $\{e_i : i = 1, 2, 3, 4\}$ for $T_p N$ by $e_i = (x^4)^{-\frac{2}{3}} E_i$ for $i = 1, 2, 3$ and $e_4 = E_4$. Then $g(e_i, e_j) = \eta_{ij}$, where

$$\eta_{ij} = \begin{cases} 1 & \text{for } i = j = 1, 2, 3; \\ -1 & \text{for } i = j = 4; \\ 0 & \text{otherwise.} \end{cases}$$

Also we observe that for $p \in N$ the null cone at p is

$$\left\{ X = X^i E_i \in T_p N : (x^4)^{\frac{4}{3}} ((X^1)^2 + (X^2)^2 + (X^3)^2) = (X^4)^2 \right\}$$

which one might interpret geometrically as saying that the null cones in N ‘get steeper’ as p ‘gets higher’. Then the only non-zero Christoffel symbols for the standard coordinate patch on N are given by ([14], p. 95, Exercise 3.3.4)

$$(4.2) \quad \Gamma_{14}^1 = \Gamma_{41}^1 = \Gamma_{24}^2 = \Gamma_{42}^2 = \Gamma_{34}^3 = \Gamma_{43}^3 = \frac{2}{3(x^4)}$$

and the components of the Ricci tensor relative to the standard coordinate patch are given by

$$S_{ij} = \begin{cases} \frac{2}{3(x^4)^2} & \text{for } i = j = 4, \\ 0 & \text{otherwise.} \end{cases}$$

Also the scalar curvature of N is given by

$$r = -\frac{2}{3(x^4)^2} \neq 0.$$

In terms of local coordinates the defining condition (1.2) of an $A(PCRS)_4$ can be written as

$$(4.3) \quad \begin{aligned} &S_{ij,k} + S_{jk,i} + S_{ki,j} \\ &= (A_k + B_k)S_{ij} + A_i S_{jk} + A_j S_{ik}, \quad i, j, k = 1, 2, 3, 4. \end{aligned}$$

In view of (4.2) and (4.3) it follows that

$$S_{ij, k} = \begin{cases} -\frac{4}{3(x^4)^3} & \text{for } i = j = k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

In terms of local coordinates, if we consider the components of the 1-forms A and B as follows:

$$(4.4) \quad A_i = \begin{cases} -\frac{4}{3x^4} & \text{for } i = 4, \\ 0 & \text{otherwise ;} \end{cases}$$

$$(4.5) \quad B_i = \begin{cases} -\frac{2}{x^4} & \text{for } i = 4, \\ 0 & \text{otherwise ;} \end{cases}$$

then in view of (4.4) - (4.5), it is easy to check that (4.3) holds for all $i, j, k = 1, 2, 3, 4$. Therefore N equipped with metric (4.1) is an $A(PCRS)_4$ which is not $A(PRS)_4$. This leads to the following:

Theorem 8. *An Einstein-deSitter spacetime N equipped with the metric (4.1) is an $A(PCRS)_4$ which is not an $A(PRS)_4$.*

Remark 1. *The metric defined in (4.1) is not positive definite.*

Acknowledgement: The authors wish to express their sincere thanks to the referees for the valuable comments towards the improvement of the paper. The first author (A. A. Shaikh) gratefully acknowledges the financial support of CSIR, New Delhi, India [Project F. No. 25(0171)/09/EMR-II].

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