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On factorizable classes of second order linear ordinary differential equations with rational functions coefficients

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Abstract. This paper addresses necessary and sufficient factorizability conditions for classes of second order linear ordinary differential equations (ODEs) characterized by the degrees of their corresponding polynomial functions coefficients. A pure algebraic method is used to solve a system of linear algebraic equations whose solutions satisfy a compatibility criterion and generate two first order differential operators factorizing the considered second order differential operator. Concrete examples are probed, including special cases of Böcher ODEs like Heun, extensions of Wangerin and Heine's differential equations.

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§1. Introduction

The mathematical description of natural and physical phenomena very often leads to differential equations (DEs). DEs can be also derived from the transformations performed on equations modeling given systems. For example, many important equations, pertaining to physical and technical applications, are reducible to Helmhotz equation [1] if time dependence is separated. This property extends to equations generally describing quite the propagation of waves like the diffusion equation, the wave equation, the damped wave equation, the transmission line equation and the vector wave equation. The Helmhotz and Laplace equations are expressible, using a separation of variables in appropriate coordinate systems, into linear ordinary differential equations

(ODEs) which belong to the class of Böcher equations [2]. For instance, Heine and Wangerin equations [2, 3], which are special cases of Böcher equations, appear when the Laplace's equation is solved in the bi-cyclide and flat-ring cyclide coordinate systems, respectively.

A great number of methods, including algorithmic and symbolic computational (e.g. Maple and Mathematica codes) approaches, have been elaborated in order to study and solve the DEs. However, their efficiency remains limited to particular forms of DEs with specific properties. Thus, tanh method [4] and Hirota's bilinear method [5, 6, 7] aim at constructing particular solutions of soliton, traveling waves and Wronskian types. The supersymmetry factorization [8, 9, 10] is adapted to solve and to determine the spectrum of certain classes of differential operators. The Lie method for symmetry reduction [11, 12] is used for reducing the order when the considered equation has some infinitesimals. Even the Beke's method [14] and van Hoiej's methods [15] for factorization of ordinary linear differential operators are also restrictive in their application. Unfortunately, none of these methods of factorization does give indication on what type of ODE's is factorizable or not. In other words, the existing factorization methods and symbolic computation codes do not tell us what kinds of linear equations are factorizable. More specifically, they do not answer to the question: given a second order linear differential equation, does it admit a factorizable form? The principal goal of this paper is to partially fulfill such a lack by proceeding to a systematic classification of factorizable second order linear ODEs with polynomial coefficients whose degrees satisfy some particular relations, using an algebraic method [13-17] of differential operator decomposition into a product of lower order differential operators.

Consider the n-order linear ODEs of the form:

$$\mathcal{P}(n,\mathcal{D})u = 0, \quad \mathcal{D} := \frac{d}{dx}, \tag{1.1}$$

where u is an unknown function, differentiable in an open subset Γ of $\mathbb{R}$, and $\mathcal{P}(n,\mathcal{D})$ is an n-order differential operator defined by:

$$\mathcal{P}(n,\mathcal{D}) = \sum_{k=0}^{n} g_k(x)\mathcal{D}^k, \tag{1.2}$$

the g_k being differentiable functions in an open subset $\Omega \supset \Gamma$ of $\mathbb{R}$. The method of factorization consists in seeking a decomposition of the differential operator (1.2) in the following form:

$$\mathcal{P}(n,\mathcal{D}) = \prod_{i=1}^{l} \mathcal{Q}_i(n_i,\mathcal{D}), \quad \text{with} \quad \sum_{i=1}^{l} n_i = n \quad \text{and} \tag{1.3}$$

$$\mathcal{Q}_i(n_i, \mathcal{D}) = \sum_{j=0}^{n_i} \mathcal{L}_{ij}(x)\mathcal{D}^j, \tag{1.4}$$

where the $\mathcal{L}_{ij}$ are differentiable functions in the open subset Ω of $\mathbb{R}$.

Proposition 1. *Let $\mathcal{P}(n, \mathcal{D})$ be an operator which can be decomposed into the form (1.3). If the function u_0 is a solution of*

$$\mathcal{Q}_l(n_l, \mathcal{D})u_0 = 0, \tag{1.5}$$

and $u_1, \ldots, u_{l-1}$ are solutions of the system

$$\prod_{k=l-j+1}^{l} \mathcal{Q}_k(n_k, \mathcal{D})u_j = v_j, \; j = 1, 2, \ldots, l-1, \tag{1.6}$$

where v_j, $j = 1, 2, \ldots, l-1$, are solutions of

$$\prod_{i=1}^{l-j} \mathcal{Q}_i(n_i, \mathcal{D})v_j = 0, \tag{1.7}$$

then $u_0, u_1, \ldots, u_{l-1}$ are l particular solutions of the equation (1.1).

Proof. Let u_0 and u_j, $j = 1, 2, \ldots, l-1$ be solutions of (1.5) and (1.6), respectively. Then

$$\mathcal{P}(n, \mathcal{D})u_0 = \left(\prod_{i=1}^{l-1} \mathcal{Q}_i(n_i, \mathcal{D})\right) \mathcal{Q}_l(n_l, \mathcal{D})u_0 = 0,$$

and for $j = 1, 2, \ldots, l-1$,

$$\begin{aligned} \mathcal{P}(n, \mathcal{D})u_j &= \left(\prod_{i=1}^{l-j} \mathcal{Q}_i(n_i, \mathcal{D})\right) \left(\prod_{k=l-j+1}^{l} \mathcal{Q}_k(n_k, \mathcal{D})\right) u_j \\ &= \prod_{i=1}^{l-j} \mathcal{Q}_i(n_i, \mathcal{D})v_j = 0, \end{aligned}$$

where the use of (1.6) and (1.7) has been made. □

Expanding (1.3) leads to the relations between unknown functions $\mathcal{L}_{ij}$ of the differential operators $\mathcal{Q}_i(n_i, \mathcal{D})$ and the known functions g_k of the original differential operator $\mathcal{P}(n, \mathcal{D})$.

In the framework of this work, our study is restricted to the second order linear differential operator

$$\mathcal{P}(2,\mathcal{D}) = g_2(x)\mathcal{D}^2 + g_1(x)\mathcal{D} + g_0(x). \tag{1.8}$$

Provided the factorized form

$$\mathcal{P}(2,\mathcal{D}) = \mathcal{Q}_1(1,\mathcal{D})\mathcal{Q}_2(1,\mathcal{D}) = (\mathcal{L}_{11}\mathcal{D} + \mathcal{L}_{10})(\mathcal{L}_{21}\mathcal{D} + \mathcal{L}_{20}), \tag{1.9}$$

the functions $\mathcal{L}_{ij}$ satisfy the following algebraic and differential equations (1.10)-(1.12):

$$\mathcal{L}_{11}\mathcal{L}_{21} = g_2, \tag{1.10}$$
$$\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20} = g_1, \tag{1.11}$$
$$\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x = g_0. \tag{1.12}$$

Finally, two particular solutions u_0 and u_1 of the equation associated with the operator $\mathcal{P}(2,\mathcal{D})$ defined by (1.8) can be obtained by solving the following differential equations:

$$\mathcal{Q}_2(1,\mathcal{D})u_0(x) := \mathcal{L}_{21}(x)u_0'(x) + \mathcal{L}_{20}(x)u_0(x) = 0, \tag{1.13}$$
$$\mathcal{Q}_1(1,\mathcal{D})v_1(x) := \mathcal{L}_{11}(x)v_1'(x) + \mathcal{L}_{10}(x)v_1(x) = 0, \tag{1.14}$$
$$\mathcal{Q}_2(1,\mathcal{D})u_1(x) := \mathcal{L}_{21}(x)u_1'(x) + \mathcal{L}_{20}(x)u_1(x) = v_1(x). \tag{1.15}$$

Every first order right factor of (1.9) leads to a hyperexponential solution [19], u_0, of the differential equation associated with (1.8) which can be written in terms of exponential functions. Another solution, u_1, of the same equation is obtained with the functions u_0 and v_1, solutions of (1.13) and (1.14), respectively, as follows:

$$u_1(x) = u_0(x)\int \frac{v_1(x)}{u_0(x)\,\mathcal{L}_{21}(x)}\,dx.$$

Now, we probe various classes of factorizable second order linear ODEs with rational coefficients. Dealing with the second order linear differential operator (1.8), where, for analysis convenience, we define

$$g_2(x) := P_p(x) = \sum_{i=1}^{p}\sigma_i x^i, \qquad g_1(x) := Q_q(x) = \sum_{j=1}^{q}\gamma_j x^j, \tag{1.16}$$

$$g_0(x) := R_r(x) = \sum_{l=1}^{r}\rho_l x^l, \qquad p,\, q,\, r \in \mathbb{N},\;\; \sigma_i,\, \gamma_j,\, \rho_l \in \mathbb{R}, \tag{1.17}$$

one can deduce from (1.10)-(1.12) the following three types of second order linear ODEs:

$$P_{k+1}(x)\,u''(x) + Q_{k+1}(x)\,u'(x) + R_k(x)\,u(x) = 0, \quad k \in \mathbb{N}; \tag{1.18}$$

$$(1.19)\quad P_k(x)\,u''(x)+Q_{k+h+1}(x)\,u'(x)+R_{k+h}(x)\,u(x)=0,\quad (k,\,h)\in\mathbb{N}^\star\times\mathbb{N};$$

$$(1.20)\qquad P_{k+2}(x)\,u''(x)+Q_{k+1}(x)\,u'(x)+R_k(x)\,u(x)=0,\quad k\in\mathbb{N}.$$

Depending on the relations between the degrees p, q, r of the polynomial functions g_i, $(i=0,1,2)$, these ODEs can be factorized into the form (1.9). They are worth something as they contain a large class of relevant second order linear ODEs of mathematical physics such as the equations of Heun, Heine and Wangerin, which will be treated in the sequel.

Recall that, by the fundamental theorem of algebra, the polynomial P_p can be put in the form: $P_p(x)=a_p\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}$, where a_p, λ_i are complex numbers such that $\lambda_i\neq\lambda_j$ for $i\neq j$ and $a_p\neq 0$; p_0, m_i are positive integers such that $p_0\le p$ and $\sum_{i=1}^{p_0}m_i=p$. In what follows, without loss of generality, we set $a_p=1$. Besides, using the Euclidean division and the partial fraction expansion theorem in the set of rational functions with complex coefficients $\mathbb{C}[X]$,

$$(1.21)\qquad \frac{\sum_{j=0}^{q}\gamma_j x^j}{\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}}=E(x)+\sum_{i=1}^{p_0}\sum_{j=1}^{m_i}\frac{\mu_{i,j}}{(x-\lambda_i)^j},$$

where $\mu_{i,j}$ are complex numbers; $E(x)$ is a nonzero polynomial of degree $q-p$ if $q\ge p$ and $E(x)=0$ if $q<p$. There results that equation (1.8) together with (1.16) and (1.17) can be transformed into the following canonical form:

$$(1.22)\quad u''(x)+\left(E(x)+\sum_{i=1}^{p_0}\sum_{j=1}^{m_i}\frac{\mu_{i,j}}{(x-\lambda_i)^j}\right)u'(x)+\frac{\sum_{l=0}^{r}\rho_l x^l}{\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}}\,u(x)=0.$$

Remark 1. *Böcher equations*

$$(1.23)\qquad u''(x)+\left(\sum_{j=1}^{n}\frac{\epsilon_j}{x-\lambda_j}\right)u'(x)+\frac{\sum_{l=0}^{r}\rho_l x^l}{\prod_{i=1}^{n}(x-\lambda_i)^{m_i}}\,u(x)=0,$$

where $n,\,r,\,m_i\in\mathbb{N}$, $\epsilon_j,\,a_i,\,\rho_l\in\mathbb{C}$, $\lambda_i\neq\lambda_j$ *for* $i\neq j$, *are particular cases of (1.22) with* $E(x)=0$.

§2. Classes of factorizable equations of the first type

In this section, we investigate the classes of factorizable second order linear ODEs of the type

$$(2.1)\qquad P_{k+1}(x)\,u''(x)+Q_{k+1}(x)\,u'(x)+R_k(x)\,u(x)=0,\qquad k\in\mathbb{N}$$

explicitly written as

$$\left(\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}\right)u''(x)+\left(\sum_{j=0}^{k+1}\gamma_j x^j\right)u'(x)+\left(\sum_{l=0}^{k}\rho_l x^l\right)u(x)=0, \tag{2.2}$$

where $\sum_{i=1}^{p_0} m_i = k+1$, or, equivalently, in the canonical form: $(E_0 \neq 0)$

$$u''(x)+\left(E_0+\sum_{i=1}^{p_0}\sum_{j=1}^{m_i}\frac{\mu_{i,j}}{(x-\lambda_i)^j}\right)u'(x)+\frac{\sum_{l=0}^{k}\rho_l x^l}{\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}}u(x)=0. \tag{2.3}$$

Proposition 2. ***(Necessary condition for the factorization of (2.1))*** *Let equation (2.1) be factorizable into the form (1.9). Then, the degrees of the polynomials $\mathcal{L}_{ij}$ satisfy the following relations:*

$$\deg\mathcal{L}_{11}+\deg\mathcal{L}_{21}=k+1 \quad \textit{and} \tag{2.4}$$

$$\begin{cases}\deg\mathcal{L}_{10}=p\\ \deg\mathcal{L}_{20}=k-p, \qquad 0\le p\le k \quad \textit{or}\end{cases} \tag{2.5}$$

$$\begin{cases}\deg\mathcal{L}_{20}=k+1-p, \qquad 1\le p\le k+1\\ \deg\mathcal{L}_{10}=j, \qquad 0\le j\le p-1,\end{cases} \tag{2.6}$$

where $\quad p=\deg\mathcal{L}_{11}.$

Proof. The system (1.10)-(1.12) becomes:

$$\mathcal{L}_{11}\mathcal{L}_{21} = P_{k+1}, \tag{2.7}$$
$$\mathcal{L}_{10}\mathcal{L}_{21}+\mathcal{L}_{11}(\mathcal{L}_{21})_x+\mathcal{L}_{11}\mathcal{L}_{20} = Q_{k+1}, \tag{2.8}$$
$$\mathcal{L}_{10}\mathcal{L}_{20}+\mathcal{L}_{11}(\mathcal{L}_{20})_x = R_k. \tag{2.9}$$

The identification of both sides of the equation (2.7) yields:

$$\deg\,(\mathcal{L}_{11}\,\mathcal{L}_{21})=\deg\,(P_{k+1})$$

which implies

$$\deg\,(\mathcal{L}_{11})+\deg\,(\mathcal{L}_{21})=k+1. \tag{2.10}$$

Since $p=\deg\mathcal{L}_{11}$, we have from the relation (2.10):

$$\deg\,(\mathcal{L}_{21})=k+1-p. \tag{2.11}$$

From the equation (2.8), we can write:

$$\deg\,(\mathcal{L}_{10}\mathcal{L}_{21}+\mathcal{L}_{11}(\mathcal{L}_{21})_x+\mathcal{L}_{11}\mathcal{L}_{20})=\deg\,(Q_{k+1})$$

giving

$$\begin{aligned}
k+1 &= \max\{\deg\,(\mathcal{L}_{10}\mathcal{L}_{21}),\deg\,(\mathcal{L}_{11}(\mathcal{L}_{21})_x),\deg\,(\mathcal{L}_{11}\mathcal{L}_{20})\}\\
&= \max\{\deg\,(\mathcal{L}_{10})+\deg\,(\mathcal{L}_{21}),\deg\,(\mathcal{L}_{11})+\deg\,((\mathcal{L}_{21})_x),\\
&\quad \deg\,(\mathcal{L}_{11})+\deg\,(\mathcal{L}_{20})\}\\
&= \max\{\deg\,(\mathcal{L}_{10})+\deg\,(\mathcal{L}_{21}),\deg\,(\mathcal{L}_{11})+[\deg\,(\mathcal{L}_{21})-1],\\
&\quad \deg\,(\mathcal{L}_{11})+\deg\,(\mathcal{L}_{20})\}.
\end{aligned}\tag{2.12}$$

The substitution of (2.11) into (2.13) gives:

$$\begin{aligned}
k+1 &= \max\{\deg\,(\mathcal{L}_{10})+k+1-p,p+[(k+1-p)-1],p+\deg\,(\mathcal{L}_{20})\}\\
&= \max\{\deg\,(\mathcal{L}_{10})+k+1-p,k,p+\deg\,(\mathcal{L}_{20})\}\\
&= \max\{\deg\,(\mathcal{L}_{10})+k+1-p,p+\deg\,(\mathcal{L}_{20})\}\equiv m_1.
\end{aligned}$$

Besides, the identification of both sides of the equation (2.9) allows to write:

$$\deg\,(\mathcal{L}_{10}\mathcal{L}_{20}+\mathcal{L}_{11}(\mathcal{L}_{20})_x)=\deg\,(R_k)$$

or equivalently

$$\begin{aligned}
k &= \max\{\deg\,(\mathcal{L}_{10}\mathcal{L}_{20}),\deg\,(\mathcal{L}_{11}(\mathcal{L}_{20})_x)\}\\
&= \max\{\deg\,(\mathcal{L}_{10})+\deg\,(\mathcal{L}_{20}),\deg\,(\mathcal{L}_{11})+\deg\,((\mathcal{L}_{20})_x)\}\\
&= \max\{\deg\,(\mathcal{L}_{10})+\deg\,(\mathcal{L}_{20}),p+[\deg\,(\mathcal{L}_{20})-1]\}\equiv m_2.
\end{aligned}\tag{2.13}$$

- If $m_1=\deg\,(\mathcal{L}_{10})+k+1-p$ then $\deg\,(\mathcal{L}_{10})=p$ and

  $$m_2=\max\{p+\deg\,(\mathcal{L}_{20}),p+\deg\,(\mathcal{L}_{20})-1\}=p+\deg\,(\mathcal{L}_{20})$$

  which gives, taking into account (2.14), $\deg\,(\mathcal{L}_{20})=k-p$.

- If $m_1=p+\deg\,(\mathcal{L}_{20})$ then $\deg\,(\mathcal{L}_{20})=k+1-p$ and

  $$\begin{aligned}
  m_2 &= \max\{\deg\,(\mathcal{L}_{10})+k+1-p,p+[(k+1-p)-1]\}\\
  &= \max\{\deg\,(\mathcal{L}_{10})+k+1-p,k\}
  \end{aligned}$$

  which gives, taking into account (2.14),
  $\deg\,(\mathcal{L}_{10})+k+1-p\le k$, i.e. $\deg\,(\mathcal{L}_{10})=j,\quad 0\le j\le p-1.$

□

The polynomials $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ are characterized by $(j+1)+(k+1-p+1) = k+j-p+3$ constants, $0 \leq j \leq p-1$. The results of the Proposition 3 are determined in the case where $j = p-1$ because all these constants can be obtained by solving a system of linear algebraic equations coming from the identification of all coefficients of polynomials in the equation (2.8) only. After substitution of polynomials $\mathcal{L}_{11}$, $\mathcal{L}_{10}$, $\mathcal{L}_{21}$, $\mathcal{L}_{20}$ determined by equations (2.7) and (2.8) into the equation (2.9), a simple identification of coefficients gives a set of relations expressing the ρ_l as functions of the constants λ_i, E_0 and $\mu_{i,j}$. These relations can be easily computed using a symbolic computational software, for instance Maple. The two following situations are worthy of attention:

(i) the first order equation associated with the left factor of (1.9) admits the solution v_1 given by:

$$v_1(x) = e^{-E_0\, x} \quad \text{if} \quad p = 0,$$

$$v_1(x) = e^{-E_0\, x} \prod_{n=1}^{q} (x-\lambda_{i_n})^{-\mu_{i_n,1}} \exp\left(\sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n,j+1}}{(x-\lambda_{i_n})^j} \right), \tag{2.14}$$

if $1 \leq p \leq k+1$, while the first order equation of the right factor of (1.9) admits the solution u_0 :

$$u_0(x) = \prod_{n=1}^{p_0-q} (x-\lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp\left(\sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i} \right) \tag{2.15}$$

which is a particular solution of equation (2.3).

(ii) the first order equation associated with the left factor of (1.9) admits the solution v_1 given by:

$$v_1(x) = \prod_{n=1}^{q} (x-\lambda_{i_n})^{-\mu_{i_n,1}} \exp\left(\sum_{j=1}^{m_{i_n}-1} \frac{1}{j} \frac{\mu_{i_n,j+1}}{(x-\lambda_{i_n})^j} \right) \tag{2.16}$$

while the first order equation corresponding to the right factor of (1.9) generates the solution u_0 given by:

$$u_0(x) = e^{-E_0 x} \prod_{n=1}^{p_0-q} (x-\lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp\left(\sum_{i=1}^{m_{j_n}-1} \frac{1}{i} \frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i} \right), \tag{2.17}$$

which is a particular solution of equation (2.3).

Here $\mu_{i_n,l},\, \mu_{j_n,l} \in \{\mu_{1,1},\, \ldots,\, \mu_{p_0,m_{p_0}}\}$, $p = \sum_{l=1}^{q} m_{i_l},\ 1 \le q \le p_0$; $m_{i_l},\, m_{j_l} \in \{m_1,\, \ldots,\, m_{p_0}\}$; $\lambda_{i_n} \neq \lambda_{j_n},\, \lambda_{i_n},\, \lambda_{j_n} \in \{\lambda_1,\, \ldots,\, \lambda_{p_0}\}$.

Proposition 3. ***(Sufficient condition for the factorization of (2.3))*** *Consider the equation (2.3) and assume that the polynomial*

$$R_k(x) = \sum_{l=0}^{k} \rho_l x^l \tag{2.18}$$

satisfies the relation

$$R_k(x) = \mathcal{L}_{10}(x)\mathcal{L}_{20}(x) + \mathcal{L}_{11}(x)(\mathcal{L}_{20})_x(x) \tag{2.19}$$

with

$$\begin{cases} \mathcal{L}_{11}(x) = 1 & \textit{if} \quad p = 0 \\ \mathcal{L}_{11}(x) = \prod_{n=1}^{q}(x - \lambda_{i_n})^{m_{i_n}} & \textit{if} \quad 1 \le p \le k+1, \end{cases} \tag{2.20}$$

and $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ explicitly given by one of the two following situations:

(i)

$$\mathcal{L}_{10}(x) = E_0 \quad \textit{if} \quad p = 0, \tag{2.21}$$

$$\begin{aligned} \mathcal{L}_{10}(x) \;\; &= \;\; \sum_{n=1}^{q}(x - \lambda_{i_n})^{m_{i_n}-1}\left[\mu_{i_n,1} + \sum_{j=1}^{m_{i_n}-1} \frac{\mu_{i_n,j+1}}{(x - \lambda_{i_n})^j}\right] \prod_{\substack{l=1 \\ l \neq n}}^{q}(x - \lambda_{i_l})^{m_{i_l}} \\ &\quad + E_0 \prod_{n=1}^{q}(x - \lambda_{i_n})^{m_{i_n}} \quad \textit{if} \quad 1 \le p \le k+1; \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{20}(x) \;\; &= \;\; \sum_{n=1}^{p_0-q}(x - \lambda_{j_n})^{m_{j_n}-1}\left[(\mu_{j_n,1} - m_{j_n})\right. \\ &\quad \left. + \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x - \lambda_{j_n})^i}\right] \prod_{\substack{l=1 \\ l \neq n}}^{p_0-q}(x - \lambda_{j_l})^{m_{j_l}}; \end{aligned} \tag{2.22}$$

(ii)

$$\mathcal{L}_{10}(x) = \sum_{n=1}^{q}(x - \lambda_{i_n})^{m_{i_n}-1}\left[\mu_{i_n,1} + \sum_{i=1}^{m_{i_n}-1} \frac{\mu_{i_n,i+1}}{(x - \lambda_{i_n})^i}\right] \prod_{\substack{l=1 \\ l \neq n}}^{q}(x - \lambda_{i_l})^{m_{i_l}};$$

$$\begin{aligned}\mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}-1}\left[(\mu_{j_n,1}-m_{j_n}) \right.\\ &\quad \left.+\sum_{i=1}^{m_{j_n}-1}\frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i}\right]\prod_{\substack{l=1\\ l\neq n}}^{p_0-q}(x-\lambda_{j_l})^{m_{j_l}}\\ &\quad +E_0\prod_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}} \quad \text{if} \quad 1\leq p\leq k+1,\end{aligned}$$

where $\mu_{i_n,l}$, $\mu_{j_n,l}\in\{\mu_{1,1},\ldots,\mu_{p_0,m_{p_0}}\}$, $p=\sum_{l=1}^{q} m_{i_l}$, $1\leq q\leq p_0$; m_{i_l}, $m_{j_l}\in\{m_1,\ldots,m_{p_0}\}$; $\lambda_{i_n}\neq\lambda_{j_n}$, λ_{i_n}, $\lambda_{j_n}\in\{\lambda_1,\ldots,\lambda_{p_0}\}$.

Then, the second order differential operator governing the equation (2.3) can be written in the form (1.9) where

$$\mathcal{L}_{21}(x)=\prod_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}} \tag{2.23}$$

is such that

$$\mathcal{L}_{11}(x)\,\mathcal{L}_{21}(x)=\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}. \tag{2.24}$$

Proof. Given the expressions of $\mathcal{L}_{11}$ and $\mathcal{L}_{21}$ from Proposition 3, then $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ can be explicitly determined using (1.13) and (1.14) as follows:

$$\mathcal{L}_{10}(x) = -\mathcal{L}_{11}(x)\frac{v_1'(x)}{v_1(x)} \tag{2.25}$$

$$\mathcal{L}_{20}(x) = -\mathcal{L}_{21}(x)\frac{u_0'(x)}{u_0(x)}. \tag{2.26}$$

□

Example 1. *Consider the confluent Heun equation [13, 18]*

$$u''(x)+\left(E_0+\frac{\mu_{1,1}}{x-\lambda_1}+\frac{\mu_{2,1}}{x-\lambda_2}\right)u'(x)+\frac{\rho_0+\rho_1 x}{(x-\lambda_1)(x-\lambda_2)}u(x)=0, \tag{2.27}$$

where E_0, λ_1, λ_2, $\mu_{1,1}$, $\mu_{2,1}$, ρ_0, ρ_1 *are constants such that* $E_0\neq 0$ *and* $\lambda_1\neq\lambda_2$. *We distinguish here the following three formal factorisable classes:*

(i) *First class,* $\rho_1=E_0\mu_{1,1}+E_0\mu_{2,1}-2E_0$, $\quad\rho_0=\mu_{1,1}+\mu_{2,1}-2-E_0\mu_{1,1}\lambda_2-E_0\mu_{2,1}\lambda_1+E_0\lambda_1+E_0\lambda_2$:

$$\mathcal{L}_{11}(x)=1,\qquad \mathcal{L}_{21}(x)=(x-\lambda_1)(x-\lambda_2),$$
$$\mathcal{L}_{10}(x)=h_0,\qquad \mathcal{L}_{20}(x)=k_0+k_1\,x,$$

$$h_0 = E_0, \quad k_0 = -\mu_{1,1}\lambda_2 - \mu_{2,1}\lambda_1 + \lambda_1 + \lambda_2, \quad k_1 = \mu_{1,1} + \mu_{2,1} - 2.$$

Two particular solutions emerge, given by

$$\begin{aligned} u_0(x) &= (x-\lambda_1)^{1-\mu_{1,1}}\,(x-\lambda_2)^{1-\mu_{2,1}}, \\ u_1(x) &= u_0(x)\int (x-\lambda_1)^{\mu_{1,1}-2}(x-\lambda_2)^{\mu_{2,1}-2}\,e^{-E_0 x}\,dx. \end{aligned}$$

(ii) *Second class,* $\rho_0 = -E_0\lambda_1 - E_0\mu_{1,1}\lambda_2 + \mu_{1,1}\mu_{2,1} - \mu_{1,1}, \quad \rho_1 = E_0 + E_0\mu_{1,1}$:

$$\begin{aligned} \mathcal{L}_{11}(x) &= (x-\lambda_1), \qquad \mathcal{L}_{21}(x) = (x-\lambda_2), \\ \mathcal{L}_{10}(x) &= h_0, \qquad \mathcal{L}_{20}(x) = k_0 + k_1\,x, \end{aligned}$$

$$h_0 = \mu_{1,1}, \quad k_0 = -E_0\lambda_2 + \mu_{2,1} - 1, \quad k_1 = E_0.$$

There exist the following two particular solutions:

$$\begin{aligned} u_0(x) &= (x-\lambda_2)^{1-\mu_{2,1}}\,e^{-E_0\,x}, \\ u_1(x) &= u_0(x)\int (x-\lambda_2)^{\mu_{2,1}-2}(x-\lambda_1)^{-\mu_{1,1}}\,e^{E_0 x}\,dx. \end{aligned}$$

(iii) *Third class,* $\rho_0 = E_0\lambda_1 - E_0\mu_{2,1}\lambda_1 - \mu_{1,1} + \mu_{1,1}\mu_{2,1}, \quad \rho_1 = -E_0 + E_0\mu_{2,1}$:

$$\begin{aligned} \mathcal{L}_{11}(x) &= (x-\lambda_1), \qquad \mathcal{L}_{21}(x) = (x-\lambda_2), \\ \mathcal{L}_{10}(x) &= h_0 + h_1 x, \qquad \mathcal{L}_{20}(x) = k_0, \end{aligned}$$

$$h_0 = -E_0\lambda_1 + \mu_{1,1}, \quad k_0 = -1 + \mu_{2,1}, \quad h_1 = E_0.$$

Two particular solutions of the corresponding equation (2.27) are given by

$$\begin{aligned} u_0(x) &= (x-\lambda_2)^{1-\mu_{2,1}}, \\ u_1(x) &= u_0(x)\int (x-\lambda_2)^{\mu_{2,1}-2}(x-\lambda_1)^{-\mu_{1,1}}\,e^{-E_0 x}\,dx. \end{aligned}$$

Example 2. *Consider the following second order linear ODE*

(2.28)
$$u''(x) + \left(E_0 + \frac{\mu_{1,1}}{x-\lambda_1} + \frac{\mu_{1,2}}{(x-\lambda_1)^2} + \frac{\mu_{1,3}}{(x-\lambda_1)^3}\right) u'(x) + \frac{\rho_0+\rho_1 x+\rho_2 x^2}{(x-\lambda_1)^3}\,u(x) = 0,$$

where E_0, λ_1, $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{1,3}$, ρ_0, ρ_1, ρ_2 *are constants such that* $E_0 \neq 0$. *When* $\mu_{1,3} = 0$, *(2.28) is reduced to the double confluent Heun equation [13, 18].*

Then, the equation (2.28) admits a unique formal factorizable class characterized by:

$$\begin{aligned}\rho_0 &= -2\mu_{1,1}\lambda_1 + 6\lambda_1 + \mu_{1,2} - 3E_0\lambda_1^2 + E_0\mu_{1,1}\lambda_1^2 + E_0\mu_{1,3} - E_0\mu_{1,2}\lambda_1,\\ \rho_1 &= -6 + 2\mu_{1,1} - 2E_0\mu_{1,1}\lambda_1 + 6E_0\lambda_1 + E_0\mu_{1,2},\\ \rho_2 &= -3E_0 + E_0\mu_{1,1};\end{aligned}$$

$$\mathcal{L}_{11}(x) = 1, \qquad \mathcal{L}_{21}(x) = (x-\lambda_1)^3,$$
$$\mathcal{L}_{10}(x) = h_0, \qquad \mathcal{L}_{20}(x) = k_0 + k_1\,x + k_2\,x^2,$$

$$k_1 = -2\mu_{1,1}\lambda_1 + 6\lambda_1 + \mu_{1,2}, \qquad h_0 = E_0,$$
$$k_0 = -3\lambda_1^2 + \mu_{1,1}\lambda_1^2 + \mu_{1,3} - \mu_{1,2}\lambda_1, \qquad k_2 = -3 + \mu_{1,1}.$$

Two particular solutions of the related equation (2.28) are given by

$$\begin{aligned}u_0(x) &= (x-\lambda_1)^{3-\mu_{1,1}}\, e^{\frac{\mu_{1,2}}{x-\lambda_1}+\frac{1}{2}\frac{\mu_{1,3}}{(x-\lambda_1)^2}},\\ u_1(x) &= u_0(x)\int (x-\lambda_1)^{\mu_{1,1}-6}\, e^{-\frac{\mu_{1,2}}{x-\lambda_1}-\frac{1}{2}\frac{\mu_{1,3}}{(x-\lambda_1)^2}}\, e^{-E_0x}\,dx.\end{aligned}$$

§3. Classes of factorizable equations of the second type

In this section, we examine the classes of factorizable second order linear ODEs of the type

$$P_k(x)\,u''(x) + Q_{k+h+1}(x)\,u'(x) + R_{k+h}(x)\,u(x) = 0, \quad (k,\,h) \in \mathbb{N}^\star \times \mathbb{N} \tag{3.1}$$

explicitly written as

$$\left(\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}\right) u''(x) + \left(\sum_{j=1}^{k+h+1}\gamma_j x^j\right) u'(x) + \left(\sum_{l=1}^{k+h}\rho_l x^l\right) u(x) = 0, \tag{3.2}$$

where $\sum_{i=1}^{p_0} m_i = k$, or, equivalently, in the canonical form: $b_{h+1} \neq 0$

$$u''(x) + \left(\sum_{j=0}^{h+1} b_j x^j + \sum_{i=1}^{p_0}\sum_{j=1}^{m_i}\frac{\mu_{i,j}}{(x-\lambda_i)^j}\right) u'(x) + \frac{\sum_{l=0}^{k+h}\rho_l x^l}{\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}}\,u(x) = 0. \tag{3.3}$$

Proposition 4. ***(Necessary condition for the factorization of (3.1))*** *Let equation (3.1) be decomposable into the form (1.9). Then, the degrees of polynomials $\mathcal{L}_{ij}$ satisfy the following relations:*

$$\deg\mathcal{L}_{11} + \deg\mathcal{L}_{21} = k \quad \textit{and} \tag{3.4}$$

$$(3.5)\qquad \begin{cases} \deg \mathcal{L}_{10} = h + p + 1 \\ \deg \mathcal{L}_{20} = k - p - 1, \qquad 0 \le p \le k - 1 \quad \textit{or} \end{cases}$$

$$(3.6)\qquad \begin{cases} \deg \mathcal{L}_{20} = k + h + 1 - p, \qquad 1 \le p \le k \\ \deg \mathcal{L}_{10} = j, \qquad 0 \le j \le p - 1, \end{cases}$$

where $\qquad p = \deg \mathcal{L}_{11}$.

Proof. The system (1.10)-(1.12) becomes:

$$\begin{aligned} (3.7)\qquad && \mathcal{L}_{11}\mathcal{L}_{21} &= P_k, \\ (3.8)\qquad && \mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20} &= Q_{k+h+1}, \\ (3.9)\qquad && \mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x &= R_{k+h}. \end{aligned}$$

The identification of both sides of the equation (3.7) yields:

$$\deg\,(\mathcal{L}_{11}\,\mathcal{L}_{21}) = \deg\,(P_k)$$

which implies

$$(3.10)\qquad \deg\,(\mathcal{L}_{11}) + \deg\,(\mathcal{L}_{21}) = k.$$

Since $p = \deg \mathcal{L}_{11}$ we have from the relation (3.10):

$$(3.11)\qquad \deg\,(\mathcal{L}_{21}) = k - p.$$

From the equation (3.8), we can deduce:

$$\deg\,(\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20}) = \deg\,(Q_{k+h+1})$$

which implies

$$\begin{aligned} k + h + 1 &= \max\{\deg\,(\mathcal{L}_{10}\mathcal{L}_{21}), \deg\,(\mathcal{L}_{11}(\mathcal{L}_{21})_x), \deg\,(\mathcal{L}_{11}\mathcal{L}_{20})\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{21}), \deg\,(\mathcal{L}_{11}) + \deg\,((\mathcal{L}_{21})_x), \\ (3.12)\qquad &\quad \deg\,(\mathcal{L}_{11}) + \deg\,(\mathcal{L}_{20})\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{21}), \deg\,(\mathcal{L}_{11}) + [\deg\,(\mathcal{L}_{21}) - 1], \\ &\quad \deg\,(\mathcal{L}_{11}) + \deg\,(\mathcal{L}_{20})\}. \end{aligned}$$

The substitution of (3.11) into (3.13) gives:

$$\begin{aligned} k + h + 1 &= \max\{\deg\,(\mathcal{L}_{10}) + k - p, p + [(k - p) - 1], p + \deg\,(\mathcal{L}_{20})\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + k - p, k - 1, p + \deg\,(\mathcal{L}_{20})\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + k - p, p + \deg\,(\mathcal{L}_{20})\} \equiv m_1. \end{aligned}$$

Besides, the identification of both sides of the equation (3.9) yields:

$$\deg\,(\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x) = \deg\,(R_{k+h})$$

which implies

$$\begin{aligned} k+h &= \max\{\deg\,(\mathcal{L}_{10}\mathcal{L}_{20})\,, \deg\,(\mathcal{L}_{11}(\mathcal{L}_{20})_x)\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{20})\,, \deg\,(\mathcal{L}_{11}) + \deg\,((\mathcal{L}_{20})_x)\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{20})\,, p + \deg\,[(\mathcal{L}_{20}) - 1]\} \equiv m_2. \end{aligned}$$

- If $m_1 = \deg\,(\mathcal{L}_{10}) + k - p$ then $\deg\,(\mathcal{L}_{10}) = h + p + 1$ and

$$\begin{aligned} m_2 &= \max\{h + p + 1 + \deg\,(\mathcal{L}_{20})\,, p + \deg\,(\mathcal{L}_{20}) - 1\} \\ &= h + p + 1 + \deg\,(\mathcal{L}_{20}) \end{aligned}$$

  which gives, taking into account (3.3), $\deg\,(\mathcal{L}_{20}) = k - p - 1$.

- If $m_1 = p + \deg\,(\mathcal{L}_{20})$ then $\deg\,(\mathcal{L}_{20}) = k + h + 1 - p$ and

$$\begin{aligned} m_2 &= \max\{\deg\,(\mathcal{L}_{10}) + k + h + 1 - p, p + [(k + h + 1 - p) - 1]\} \\ &= \max\{\deg\,(\mathcal{L}_{10}) + k + h + 1 - p, k + h\} \end{aligned}$$

  which implies, taking into account (3.3),
  $\deg\,(\mathcal{L}_{10}) + k + h + 1 - p \leq k + h$, i.e. $\deg\,(\mathcal{L}_{10}) = j, \quad 0 \leq j \leq p - 1$.

□

The polynomials $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ are characterized by $(j+1)+(k+h+1-p+1) = k + h + j - p + 3$ constants, $0 \leq j \leq p - 1$. The results of the Proposition 5 are determined in the case where $j = p - 1$ because all these constants can be obtained by solving a system of linear algebraic equations coming from the identification of all coefficients of polynomials in the equation (3.8) only. After substitution of polynomials $\mathcal{L}_{11}$, $\mathcal{L}_{10}$, $\mathcal{L}_{21}$, $\mathcal{L}_{20}$ determined by equations (3.7) and (3.8) into the equation (3.9), a simple identification of coefficients gives a set of relations expressing the ρ_l as functions of the constants λ_i, b_j and $\mu_{i,j}$. As in the previous case, these relations can be also easily computed using a symbolic computational software, for instance Maple. There follow two possibilities:

(i) the corresponding first order left factor of (1.9) admits the solution v_1 given by:

$$v_1(x) = e^{-\left(\sum_{j=0}^{h+1} \frac{b_j}{j+1} x^{j+1}\right)} \quad \text{if} \quad p = 0,$$

$$v_1(x) \;=\; e^{-\left(\sum_{j=0}^{h+1}\frac{b_j}{j+1}x^{j+1}\right)}\prod_{n=1}^{q}(x-\lambda_{i_n})^{-\mu_{i_n,1}}\exp\left(\sum_{j=1}^{m_{i_n}-1}\frac{1}{j}\frac{\mu_{i_n,j+1}}{(x-\lambda_{i_n})^j}\right)$$

if $\quad 1\le p\le k,$

while the first order right factor of (1.9) admits the solution u_0 given by:

$$u_0(x) \;=\; \prod_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}}\exp\left(\sum_{i=1}^{m_{j_n}-1}\frac{1}{i}\frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i}\right)$$

which is a particular solution of equation (3.3).

(ii) the corresponding first order left factor of (1.9) admits the solution v_1 given by:

$$v_1(x)=1 \quad \text{if} \quad p=0,$$

$$v_1(x)=\prod_{n=1}^{q}(x-\lambda_{i_n})^{-\mu_{i_n,1}}\exp\left(\sum_{j=1}^{m_{i_n}-1}\frac{1}{j}\frac{\mu_{i_n,j+1}}{(x-\lambda_{i_n})^j}\right) \quad \text{if} \quad 1\le p\le k,$$

while the first order right factor of (1.9) admits the solution u_0 given by:

$$u_0(x)=e^{-\left(\sum_{j=0}^{h+1}\frac{b_j}{j+1}x^{j+1}\right)}\prod_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}}\exp\left(\sum_{i=1}^{m_{j_n}-1}\frac{1}{i}\frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i}\right),$$

which is a particular solution of the equation (3.3).

In all these expressions, $\mu_{i_n,l},\,\mu_{j_n,l}\in\{\mu_{1,1},\,\dots,\,\mu_{p_0,m_{p_0}}\}$, $p=\sum_{l=1}^{q}m_{i_l}$, $1\le q\le p_0$; $m_{i_l},\,m_{j_l}\in\{m_1,\,\dots,\,m_{p_0}\}$; $\lambda_{i_n}\ne\lambda_{j_n}$, $\lambda_{i_n},\,\lambda_{j_n}\in\{\lambda_1,\,\dots,\,\lambda_{p_0}\}$.

Proposition 5. ***(Sufficient condition for the factorization of (3.3))*** *Consider the equation (3.3) and assume that the polynomial*

$$R_{k+h}(x)=\sum_{l=0}^{k+h}\rho_l x^l \tag{3.13}$$

satisfies the relation

$$R_{k+h}(x)=\mathcal{L}_{10}(x)\mathcal{L}_{20}(x)+\mathcal{L}_{11}(x)(\mathcal{L}_{20})_x(x) \tag{3.14}$$

with

$$\begin{cases}\mathcal{L}_{11}(x)=1 & \textit{if}\quad p=0,\\ \mathcal{L}_{11}(x)=\prod_{n=1}^{q}(x-\lambda_{i_n})^{m_{i_n}} & \textit{if}\quad 1\le p\le k,\end{cases}$$

and $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ explicitly given by one of the two following situations:

(i)

$$\mathcal{L}_{10}(x) = \sum_{j=0}^{h+1} b_j x^j \quad \textit{if} \quad p = 0,$$

$$\begin{aligned}
\mathcal{L}_{10}(x) &= \sum_{n=1}^{q} (x-\lambda_{i_n})^{m_{i_n}-1} \left[\mu_{i_n,1} + \sum_{j=1}^{m_{i_n}-1} \frac{\mu_{i_n,j+1}}{(x-\lambda_{i_n})^j} \right] \prod_{\substack{l=1\\ l\neq n}}^{q} (x-\lambda_{i_l})^{m_{i_l}} \\
&+ \left(\sum_{j=0}^{h+1} b_j x^j \right) \prod_{n=1}^{q} (x-\lambda_{i_n})^{m_{i_n}} \quad \textit{if} \quad 1 \le p \le k,
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q} (x-\lambda_{j_n})^{m_{j_n}-1} \left[(\mu_{j_n,1} - m_{j_n}) \right. \\
&\left. + \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i} \right] \prod_{\substack{l=1\\ l\neq n}}^{p_0-q} (x-\lambda_{j_l})^{m_{j_l}};
\end{aligned}$$

(ii)

$$\begin{aligned}
\mathcal{L}_{10}(x) &= 0 \quad \textit{if} \quad p = 0, \\
\mathcal{L}_{10}(x) &= \sum_{n=1}^{q} (x-\lambda_{i_n})^{m_{i_n}-1} \left[\mu_{i_n,1} + \sum_{i=1}^{m_{i_n}-1} \frac{\mu_{i_n,i+1}}{(x-\lambda_{i_n})^i} \right] \prod_{\substack{l=1\\ l\neq n}}^{q} (x-\lambda_{i_l})^{m_{i_l}} \\
&\quad \textit{if} \quad 1 \le p \le k;
\end{aligned}$$

$$\begin{aligned}
\mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q} (x-\lambda_{j_n})^{m_{j_n}-1} \left[(\mu_{j_n,1} - m_{j_n}) \right. \\
&\left. + \sum_{i=1}^{m_{j_n}-1} \frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i} \right] \prod_{\substack{l=1\\ l\neq n}}^{p_0-q} (x-\lambda_{j_l})^{m_{j_l}} \\
&+ \left(\sum_{j=0}^{h+1} b_j x^j \right) \prod_{n=1}^{p_0-q} (x-\lambda_{j_n})^{m_{j_n}},
\end{aligned}$$

where $\mu_{i_n,l},\, \mu_{j_n,l} \in \{\mu_{1,1},\, \ldots,\, \mu_{p_0,m_{p_0}}\}$, $p = \sum_{l=1}^{q} m_{i_l},\ 1 \le q \le p_0;$ $m_{i_l},\, m_{j_l} \in \{m_1,\, \ldots,\, m_{p_0}\};\ \lambda_{i_n} \neq \lambda_{j_n},\, \lambda_{i_n},\, \lambda_{j_n} \in \{\lambda_1,\, \ldots,\, \lambda_{p_0}\}.$

Then, the equation (3.3) can be written in the form (1.9) where

$$\mathcal{L}_{21}(x) = \prod_{n=1}^{p_0-q} (x-\lambda_{j_n})^{m_{j_n}} \tag{3.15}$$

is such that

$$\mathcal{L}_{11}(x)\,\mathcal{L}_{21}(x) = \prod_{i=1}^{p_0} (x-\lambda_i)^{m_i}. \tag{3.16}$$

Proof. It is similar to that of the Proposition 3. □

Example 3. *Consider the biconfluent Heun equation [13, 18]*

$$u''(x) + \left(b_0 + b_1 x + \frac{\mu_{1,1}}{x-\lambda_1}\right) u'(x) + \frac{\rho_0 + \rho_1 x}{x-\lambda_1}\, u(x) = 0, \tag{3.17}$$

where b_0, b_1, λ_1, $\mu_{1,1}$, ρ_0, ρ_1 *are constants such that* $b_1 \neq 0$. *Then, the equation (3.17) gives two formal factorizable classes:*

(i) *First class,* $\rho_0 = 2b_1, \quad \rho_1 = b_0 - b_1\lambda_1$:

$$\begin{array}{ll} \mathcal{L}_{11}(x) = 1, & \mathcal{L}_{21}(x) = (x-\lambda_1), \\ \mathcal{L}_{10}(x) = 0, & \mathcal{L}_{20}(x) = k_0 + k_1\,x + k_2\,x^2, \end{array}$$

$$k_0 = \mu_{1,1} - b_0\lambda_1 - 1, \quad k_1 = b_0 - b_1\lambda_1, \quad k_2 = b_1.$$

Two particular solutions of the equation (3.17) are given by

$$\begin{array}{rcl} u_0(x) & = & (x-\lambda_1)^{1-\mu_{1,1}}\, e^{-b_0\,x-\frac{1}{2}b_1\,x^2}, \\ u_1(x) & = & u_0(x)\displaystyle\int (x-\lambda_1)^{\mu_{1,1}-2}\, e^{b_0\,x+\frac{1}{2}b_1\,x^2}\, dx. \end{array}$$

(ii) *Second class,* $\rho_0 = b_0(\mu_{1,1}-1), \quad \rho_1 = b_1(\mu_{1,1}-1)$:

$$\begin{array}{rl} \mathcal{L}_{11}(x) = 1, & \mathcal{L}_{21}(x) = (x-\lambda_1), \\ \mathcal{L}_{10}(x) = h_0 + h_1\,x, & \mathcal{L}_{20}(x) = k_0, \end{array}$$

$$h_0 = b_0, \quad h_1 = b_1, \quad k_0 = \mu_{1,1} - 1.$$

Two particular solutions of the equation (3.17) are provided by

$$\begin{array}{rcl} u_0(x) & = & (x-\lambda_1)^{1-\mu_{1,1}}, \\ u_1(x) & = & u_0(x)\displaystyle\int (x-\lambda_1)^{\mu_{1,1}-2}\, e^{-b_0\,x-\frac{1}{2}b_1\,x^2}\, dx. \end{array}$$

§4. Classes of factorizable equations of third type

In this section, we deal with the classes of factorizable second order linear ODEs of the type

$$P_{k+2}(x)\,u''(x)+Q_{k+1}(x)\,u'(x)+R_k(x)\,u(x)=0,\quad k\in\mathbb{N} \tag{4.1}$$

explicitly written as

$$\left(\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}\right)u''(x)+\left(\sum_{j=0}^{k+1}\gamma_j x^j\right)u'(x)+\left(\sum_{l=0}^{k}\rho_l x^l\right)u(x)=0, \tag{4.2}$$

where $\sum_{i=1}^{p_0} m_i=k+2$, or, equivalently, in the canonical form:

$$u''(x)+\left(\sum_{i=1}^{p_0}\sum_{j=1}^{m_i}\frac{\mu_{i,j}}{(x-\lambda_i)^j}\right)u'(x)+\frac{\sum_{l=0}^{k}\rho_l x^l}{\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}}\,u(x)=0. \tag{4.3}$$

Proposition 6. ***(Factorizability necessary condition of (4.1))***
Let equation (4.1) be decomposable into the form (1.9). Then, the degrees of polynomials $\mathcal{L}_{ij}$ satisfy the following relations:

$$\deg\mathcal{L}_{11}+\deg\mathcal{L}_{21}=k+2\quad\textit{and} \tag{4.4}$$

$$\begin{cases}\deg\mathcal{L}_{20}=k-p+1, & 1\le p\le k+1\\ \deg\mathcal{L}_{10}=j, & 0\le j\le p-1,\end{cases} \tag{4.5}$$

where $\qquad p=\deg\mathcal{L}_{11}.$

Proof. The system (1.10)-(1.12) becomes:

$$\mathcal{L}_{11}\mathcal{L}_{21} = P_{k+2}, \tag{4.6}$$

$$\mathcal{L}_{10}\mathcal{L}_{21}+\mathcal{L}_{11}(\mathcal{L}_{21})_x+\mathcal{L}_{11}\mathcal{L}_{20} = Q_{k+1}, \tag{4.7}$$

$$\mathcal{L}_{10}\mathcal{L}_{20}+\mathcal{L}_{11}(\mathcal{L}_{20})_x = R_k. \tag{4.8}$$

The identification of both sides of the equation (4.6) yields:

$$\deg\,(\mathcal{L}_{11}\,\mathcal{L}_{21})=\deg\,(P_{k+2})$$

which implies

$$\deg\,(\mathcal{L}_{11})+\deg\,(\mathcal{L}_{21})=k+2. \tag{4.9}$$

Since $p=\deg\mathcal{L}_{11}$, we get from the relation (4.9):

$$\deg\,(\mathcal{L}_{21})=k+2-p. \tag{4.10}$$

The equation (4.7) also allows to write:

$$\deg\,(\mathcal{L}_{10}\mathcal{L}_{21} + \mathcal{L}_{11}(\mathcal{L}_{21})_x + \mathcal{L}_{11}\mathcal{L}_{20}) = \deg\,(Q_{k+1})$$

which implies

$$\begin{aligned} k+1 &= \max\left\{\deg\,(\mathcal{L}_{10}\mathcal{L}_{21})\,, \deg\,(\mathcal{L}_{11}(\mathcal{L}_{21})_x)\,, \deg\,(\mathcal{L}_{11}\mathcal{L}_{20})\right\} \\ &= \max\left\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{21})\,, \deg\,(\mathcal{L}_{11}) + \deg\,((\mathcal{L}_{21})_x)\,,\right. \\ &\quad \left.\deg\,(\mathcal{L}_{11}) + \deg\,(\mathcal{L}_{20})\right\} \\ &= \max\left\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{21})\,, \deg\,(\mathcal{L}_{11}) + [\deg\,(\mathcal{L}_{21}) - 1]\,,\right. \\ &\quad \left.\deg\,(\mathcal{L}_{11}) + \deg\,(\mathcal{L}_{20})\right\}. \end{aligned} \tag{4.11}$$

The substitution of (4.10) into (4.12) gives:

$$\begin{aligned} k+1 &= \max\left\{\deg\,(\mathcal{L}_{10}) + k + 2 - p, p + [(k+2-p) - 1], p + \deg\,(\mathcal{L}_{20})\right\} \\ &= \max\left\{\deg\,(\mathcal{L}_{10}) + k + 2 - p, k+1, p + \deg\,(\mathcal{L}_{20})\right\}. \end{aligned}$$

Therefore,

$$\begin{cases} \deg\,(\mathcal{L}_{10}) + k + 2 - p \le k+1 \\ p + \deg\,(\mathcal{L}_{20}) \le k+1, \end{cases}$$

that is

$$\begin{cases} \deg\,(\mathcal{L}_{10}) = j, & 0 \le j \le p-1 \\ \deg\,(\mathcal{L}_{20}) = i, & 0 \le i \le k+1-p. \end{cases} \tag{4.12}$$

Besides, the identification of both sides of the equation (4.8) leads to:

$$\deg\,(\mathcal{L}_{10}\mathcal{L}_{20} + \mathcal{L}_{11}(\mathcal{L}_{20})_x) = \deg\,(R_k)$$

which implies

$$\begin{aligned} k &= \max\left\{\deg\,(\mathcal{L}_{10}\mathcal{L}_{20})\,, \deg\,(\mathcal{L}_{11}(\mathcal{L}_{20})_x)\right\} \\ &= \max\left\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{20})\,, \deg\,(\mathcal{L}_{11}) + \deg\,((\mathcal{L}_{20})_x)\right\} \\ &= \max\left\{\deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{20})\,, p + [\deg\,(\mathcal{L}_{20}) - 1]\right\} \equiv m. \end{aligned}$$

- If $m = \deg\,(\mathcal{L}_{10}) + \deg\,(\mathcal{L}_{20})$ then $\deg\,(\mathcal{L}_{20}) = k - j$ which yields by the first equality of (4.12) $k+1-p \le k-j \le k$. Therefore, by the second equality of (4.12) we must have $\deg\,(\mathcal{L}_{20}) = k+1-p$.
- If $m = p + [\deg\,(\mathcal{L}_{20}) - 1]$ then $\deg\,(\mathcal{L}_{20}) = k+1-p$.

□

The polynomials $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ are characterized by $(j+1)+(k+1-p+1) = k+j-p+3$ constants, $0 \leq j \leq p-1$. The results of Proposition 7 are determined in the case where $j = p-1$ because all these constants can be obtained by solving a system of linear algebraic equations coming from the identification of all coefficients of polynomials in the equation (4.7) only. After substitution of polynomials $\mathcal{L}_{11}$, $\mathcal{L}_{10}$, $\mathcal{L}_{21}$, $\mathcal{L}_{20}$ determined by equations (4.6) and (4.7) into the equation (4.8), a simple identification of coefficients gives a set of relations expressing the ρ_l as functions of the constants λ_i and $\mu_{i,j}$. For each of such relations, the corresponding first order left factor of (1.9) admits the solution v_1 given by:

$$\begin{aligned} v_1(x) &= 1 \quad \text{if} \quad p=0, \\ v_1(x) &= \prod_{n=1}^{q}(x-\lambda_{i_n})^{-\mu_{i_n,1}} \exp\left(\sum_{j=1}^{m_{i_n}-1} \frac{1}{j}\frac{\mu_{i_n,j+1}}{(x-\lambda_{i_n})^j}\right) \quad \text{if} \quad 1 \leq p \leq k+1, \end{aligned}$$

while the first order right factor of (1.9) possesses the solution u_0 given by:

$$u_0(x) = \prod_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}-\mu_{j_n,1}} \exp\left(\sum_{i=1}^{m_{j_n}-1} \frac{1}{i}\frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i}\right)$$

which is a particular solution of equation (4.3). $\mu_{i_n,l}$, $\mu_{j_n,l} \in \{\mu_{1,1}, \ldots, \mu_{p_0,m_{p_0}}\}$, $p = \sum_{l=1}^{q} m_{i_l}$, $1 \leq q \leq p_0$; $m_{i_l}, m_{j_l} \in \{m_1, \ldots, m_{p_0}\}$; $\lambda_{i_n} \neq \lambda_{j_n}$, $\lambda_{i_n}, \lambda_{j_n} \in \{\lambda_1, \ldots, \lambda_{p_0}\}$.

Proposition 7. ***(Sufficient condition for the factorization of (4.3))*** *Consider the equation (4.3) and assume that the polynomial*

$$R_k(x) = \sum_{l=0}^{k} \rho_l x^l \tag{4.13}$$

satisfies the relation

$$R_k(x) = \mathcal{L}_{10}(x)\mathcal{L}_{20}(x) + \mathcal{L}_{11}(x)(\mathcal{L}_{20})_x(x) \tag{4.14}$$

with

$$\begin{cases} \mathcal{L}_{11}(x) = 1 & \text{if} \quad p=0, \\ \mathcal{L}_{11}(x) = \prod_{n=1}^{q}(x-\lambda_{i_n})^{m_{i_n}} & \text{if} \quad 1 \leq p \leq k+1, \end{cases} \tag{4.15}$$

and $\mathcal{L}_{10}$ and $\mathcal{L}_{20}$ explicitly given by

$$\begin{aligned} \mathcal{L}_{10}(x) &= 0 \quad \text{if} \quad p=0, \\ \mathcal{L}_{10}(x) &= \sum_{n=1}^{q}(x-\lambda_{i_n})^{m_{i_n}-1}\left[\mu_{i_n,1} + \sum_{i=1}^{m_{i_n}-1}\frac{\mu_{i_n,i+1}}{(x-\lambda_{i_n})^i}\right]\prod_{\substack{l=1\\ l\neq n}}^{q}(x-\lambda_{i_l})^{m_{i_l}} \\ &\quad \text{if} \quad 1 \leq p \leq k+1; \end{aligned}$$

$$\begin{aligned}\mathcal{L}_{20}(x) &= \sum_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}-1}\left[(\mu_{j_n,1}-m_{j_n})\right.\\ &\left. + \sum_{i=1}^{m_{j_n}-1}\frac{\mu_{j_n,i+1}}{(x-\lambda_{j_n})^i}\right]\prod_{\substack{l=1\\ l\neq n}}^{p_0-q}(x-\lambda_{j_l})^{m_{j_l}}.\end{aligned}$$

where $\mu_{i_n,l},\ \mu_{j_n,l}\in\{\mu_{1,1},\,\ldots,\,\mu_{p_0,m_{p_0}}\}$, $p=\sum_{l=1}^{q} m_{i_l},\ 1\le q\le p_0;\ m_{i_l},\,m_{j_l}\in\{m_1,\,\ldots,\,m_{p_0}\};\ \lambda_{i_n}\neq\lambda_{j_n},\ \lambda_{i_n},\,\lambda_{j_n}\in\{\lambda_1,\,\ldots,\,\lambda_{p_0}\}$.

Then, the equation (2.3) can be written in the form (1.9) where

$$\mathcal{L}_{21}(x)=\prod_{n=1}^{p_0-q}(x-\lambda_{j_n})^{m_{j_n}} \tag{4.16}$$

is such that

$$\mathcal{L}_{11}(x)\,\mathcal{L}_{21}(x)=\prod_{i=1}^{p_0}(x-\lambda_i)^{m_i}. \tag{4.17}$$

Proof. It is similar to that of the Proposition 3. □

Example 4. *Consider the following second order linear ODE*

$$\begin{aligned}u''(x)+\left(\tfrac{\mu_{1,1}}{x-\lambda_1}+\tfrac{\mu_{1,2}}{(x-\lambda_1)^2}+\tfrac{\mu_{2,1}}{x-\lambda_2}+\tfrac{\mu_{3,1}}{x-\lambda_3}\right)u'(x)\\ +\tfrac{\rho_0+\rho_1x+\rho_2x^2}{(x-\lambda_1)^2(x-\lambda_2)(x-\lambda_3)}\,u(x)=0,\end{aligned} \tag{4.18}$$

where $\mu_{1,1},\ \mu_{1,2},\ \mu_{2,1},\ \mu_{3,1},\ \lambda_1,\ \lambda_2,\ \lambda_3,\ \rho_0,\ \rho_1,\ \rho_2$ *are constants such that* $\lambda_i\neq\lambda_j$ *for* $i\neq j$. *When* $\mu_{1,2}=0$, $\mu_{1,1}=1$ *and* $\mu_{2,1}=\mu_{3,1}=\frac{1}{2}$, *(4.18) is an extension of the Wangerin's equation [2]. Then,*

(i) One of the factorizable classes is characterized by

$$\rho_2 = \mu_{2,1}-2+\mu_{3,1}+\mu_{1,1}\mu_{2,1}-2\mu_{1,1}+\mu_{3,1}\mu_{1,1},$$

$$\begin{aligned}\rho_0 &= \mu_{2,1}\lambda_1^2-2\lambda_1^2+\mu_{3,1}\lambda_1^2-\mu_{1,1}\lambda_1\lambda_3+\mu_{1,1}\lambda_1\mu_{2,1}\lambda_3-\mu_{1,1}\lambda_1\lambda_2\\ &+\mu_{3,1}\mu_{1,1}\lambda_1\lambda_2+\mu_{1,2}\lambda_3-\mu_{1,2}\mu_{2,1}\lambda_3+\mu_{1,2}\lambda_2-\mu_{3,1}\mu_{1,2}\lambda_2,\end{aligned}$$

$$\begin{aligned}\rho_1 &= -2\mu_{2,1}\lambda_1+4\lambda_1-2\mu_{3,1}\lambda_1-\mu_{1,1}\lambda_1\mu_{2,1}+2\mu_{1,1}\lambda_1\\ &-\mu_{3,1}\mu_{1,1}\lambda_1+\mu_{1,2}\mu_{2,1}-2\mu_{1,2}+\mu_{3,1}\mu_{1,2}+\mu_{1,1}\lambda_3\\ &-\mu_{1,1}\mu_{2,1}\lambda_3+\mu_{1,1}\lambda_2-\mu_{3,1}\mu_{1,1}\lambda_2:\end{aligned}$$

$$\mathcal{L}_{11}(x) = (x-\lambda_1)^2, \qquad \mathcal{L}_{21}(x) = (x-\lambda_2)\,(x-\lambda_3),$$
$$\mathcal{L}_{10}(x) = h_0 + h_1\,x, \qquad \mathcal{L}_{20}(x) = k_0 + k_1\,x,$$

$$h_1 = \mu_{1,1}, \qquad h_0 = -\mu_{1,1}\lambda_1 + \mu_{1,2},$$
$$k_1 = \mu_{2,1} - 2 + \mu_{3,1}, \qquad k_0 = \lambda_3 - \mu_{2,1}\lambda_3 + \lambda_2 - \mu_{3,1}\lambda_2.$$

Two particular solutions of equation (4.18) are given by

$$\begin{aligned} u_0(x) &= (x-\lambda_3)^{1-\mu_{3,1}}\,(x-\lambda_2)^{1-\mu_{2,1}},\\ u_1(x) &= u_0(x)\int (x-\lambda_3)^{\mu_{3,1}-2}\,(x-\lambda_2)^{\mu_{2,1}-2}(x-\lambda_1)^{-\mu_{1,1}}\,e^{\frac{\mu_{1,2}}{x-\lambda_1}}\,dx. \end{aligned}$$

(ii) *Another factorizable class is given by*

$$\rho_2 = -2 + \mu_{1,1} - 2\mu_{2,1} + \mu_{1,1}\mu_{2,1} - 2\mu_{3,1} + \mu_{3,1}\mu_{1,1},$$

$$\begin{aligned} \rho_0 = {}& -2\lambda_2\lambda_3 + \mu_{1,1}\lambda_2\lambda_3 - 2\mu_{3,1}\lambda_1\lambda_2 - \mu_{3,1}\mu_{1,2}\lambda_2 + \mu_{3,1}\mu_{1,1}\lambda_1\lambda_2\\ & -2\mu_{2,1}\lambda_1\lambda_3 - \mu_{1,2}\mu_{2,1}\lambda_3 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_3, \end{aligned}$$

$$\begin{aligned} \rho_1 = {}& 2\lambda_2 - \mu_{1,1}\lambda_2 + 2\lambda_3 - \mu_{1,1}\lambda_3 + 2\mu_{3,1}\lambda_2 - \mu_{3,1}\mu_{1,1}\lambda_2\\ & +2\mu_{2,1}\lambda_3 - \mu_{1,1}\mu_{2,1}\lambda_3 + 2\mu_{2,1}\lambda_1 + \mu_{1,2}\mu_{2,1} - \mu_{1,1}\lambda_1\mu_{2,1}\\ & +2\mu_{3,1}\lambda_1 + \mu_{3,1}\mu_{1,2]} - \mu_{3,1}\mu_{1,1}\lambda_1 : \end{aligned}$$

$$\mathcal{L}_{11}(x) = (x-\lambda_2)\,(x-\lambda_3), \qquad \mathcal{L}_{21}(x) = (x-\lambda_1)^2,$$
$$\mathcal{L}_{10}(x) = h_0 + h_1\,x, \qquad \mathcal{L}_{20}(x) = k_0 + k_1\,x,$$

$$h_1 = \mu_{2,1} + \mu_{3,1}, \qquad h_0 = -\mu_{3,1}\lambda_2 - \mu_{2,1}\lambda_3,$$
$$k_1 = -2 + \mu_{1,1}, \qquad k_0 = 2\lambda_1 + \mu_{1,2} - \mu_{1,1}\lambda_1.$$

Two particular solutions of the equation (4.18) can be written as:

$$\begin{aligned} u_0(x) &= (x-\lambda_1)^{2-\mu_{1,1}}\,e^{\frac{\mu_{1,2}}{x-\lambda_1}},\\ u_1(x) &= u_0(x)\int (x-\lambda_1)^{\mu_{1,1}-4}\,(x-\lambda_2)^{-\mu_{2,1}}(x-\lambda_3)^{-\mu_{3,1}}\,e^{-\frac{\mu_{1,2}}{x-\lambda_1}}\,dx. \end{aligned}$$

Example 5. *Consider the following second order linear ODE*

$$\begin{aligned} u''(x) + \left(\tfrac{\mu_{1,1}}{x-\lambda_1} + \tfrac{\mu_{1,2}}{(x-\lambda_1)^2} + \tfrac{\mu_{2,1}}{x-\lambda_2} + \tfrac{\mu_{2,2}}{(x-\lambda_2)^2} + \tfrac{\mu_{3,1}}{x-\lambda_3}\right) u'(x)&\\ + \tfrac{\rho_0+\rho_1 x+\rho_2 x^2+\rho_3 x^3}{(x-\lambda_1)^2(x-\lambda_2)^2(x-\lambda_3)}\,u(x) = 0,& \end{aligned} \tag{4.19}$$

where $\mu_{1,1}$, $\mu_{1,2}$, $\mu_{2,1}$, $\mu_{2,2}$, $\mu_{3,1}$, λ_1, λ_2, λ_3, ρ_0, ρ_1, ρ_2, ρ_3 *are constants such that* $\lambda_i \neq \lambda_j$ *for* $i \neq j$. *When* $\mu_{1,2} = \mu_{2,2} = \mu_{2,1} = 0$, $\mu_{1,1} = 1$ *and* $\mu_{3,1} = \frac{1}{2}$, *(4.19) is an extension of the Heine's equation [2]. Then, one of the factorizable classes is characterized by*

$$\rho_3 = -6 + 2\mu_{3,1} + 2\mu_{2,1} - 3\mu_{1,1} + \mu_{3,1}\mu_{1,1} + \mu_{1,1}\mu_{2,1},$$

$$\begin{aligned}\rho_0 &= -2\mu_{3,1}\lambda_1^2\lambda_2 + 4\lambda_1^2\lambda_2 + 2\lambda_1^2\lambda_3 - \mu_{2,1}\lambda_1^2\lambda_2 + \mu_{2,2}\lambda_1^2 - \mu_{2,1}\lambda_1^2\lambda_3 \\ &+ 2\mu_{1,1}\lambda_1\lambda_2\lambda_3 + \mu_{1,1}\lambda_1\lambda_2^2 - \mu_{3,1}\mu_{1,1}\lambda_1\lambda_2^2 + \mu_{1,1}\lambda_1\mu_{2,2}\lambda_3 - \mu_{1,1}\lambda_1\mu_{2,1}\lambda_2\lambda_3 \\ &- 2\mu_{1,2}\lambda_2\lambda_3 - \mu_{1,2}\lambda_2^2 + \mu_{3,1}\mu_{1,2}\lambda_2^2 - \mu_{1,2}\mu_{2,2}\lambda_3 + \mu_{1,2}\mu_{2,1}\lambda_2\lambda_3,\end{aligned}$$

$$\begin{aligned}\rho_1 &= -2\mu_{1,1}\lambda_1\lambda_3 - 4\mu_{1,1}\lambda_1\lambda_2 - 2\mu_{1,1}\lambda_2\lambda_3 + 2\mu_{2,1}\lambda_1\lambda_3 + 2\mu_{2,1}\lambda_1\lambda_2 - 4\lambda_1\lambda_3 \\ &- 8\lambda_1\lambda_2 + 4\mu_{3,1}\lambda_1\lambda_2 + 4\mu_{1,2}\lambda_2 + 2\mu_{1,2}\lambda_3 + 2\mu_{3,1}\lambda_1^2 + 2\mu_{2,1}\lambda_1^2 - \mu_{1,1}\lambda_2^2 \\ &- 2\mu_{2,2}\lambda_1 - 2\mu_{3,1}\mu_{1,2}\lambda_2 + \mu_{3,1}\mu_{1,1}\lambda_2^2 - 6\lambda_1^2 + \mu_{1,2}\mu_{2,2} + 2\mu_{3,1}\mu_{1,1}\lambda_1\lambda_2 \\ &- \mu_{1,1}\mu_{2,2}\lambda_3 - \mu_{1,2}\mu_{2,1}\lambda_3 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_2 + \mu_{1,1}\lambda_1\mu_{2,1}\lambda_3 + \mu_{1,1}\mu_{2,1}\lambda_2\lambda_3 \\ &- \mu_{1,1}\lambda_1\mu_{2,2} - \mu_{1,2}\mu_{2,1}\lambda_2,\end{aligned}$$

$$\begin{aligned}\rho_2 &= 12\lambda_1 - 4\mu_{3,1}\lambda_1 - 4\mu_{2,1}\lambda_1 - 2\mu_{3,1}\lambda_2 + 4\lambda_2 + 2\lambda_3 - \mu_{2,1}\lambda_2 + \mu_{2,2} \\ &- \mu_{2,1}\lambda_3 + 3\mu_{1,1}\lambda_1 - \mu_{3,1}\mu_{1,1}\lambda_1 - \mu_{1,1}\lambda_1\mu_{2,1} - 3\mu_{1,2} + \mu_{3,1}\mu_{1,2} + \mu_{1,2}\mu_{2,1} \\ &- 2\mu_{3,1}\mu_{1,1}\lambda_2 + 4\mu_{1,1}\lambda_2 + 2\mu_{1,1}\lambda_3 - \mu_{1,1}\mu_{2,1}\lambda_2 + \mu_{1,1}\mu_{2,2} - \mu_{1,1}\mu_{2,1}\lambda_3 :\end{aligned}$$

$$\begin{aligned}\mathcal{L}_{11}(x) &= (x-\lambda_1)^2, \qquad \mathcal{L}_{21}(x) = (x-\lambda_2)^2\,(x-\lambda_3), \\ \mathcal{L}_{10}(x) &= h_0 + h_1\,x, \qquad \mathcal{L}_{20}(x) = k_0 + k_1\,x + k_2\,x^2,\end{aligned}$$

$$k_2 = -3 + \mu_{3,1} + \mu_{2,1}, \quad h_1 = \mu_{1,1}, \quad h_0 = -\mu_{1,1}\lambda_1 + \mu_{1,2},$$

$$\begin{aligned}k_1 &= -2\mu_{3,1}\lambda_2 + 4\lambda_2 + 2\lambda_3 - \mu_{2,1}\lambda_2 + \mu_{2,2} - \mu_{2,1}\lambda_3, \\ k_0 &= -2\lambda_2\lambda_3 - \lambda_2^2 + \mu_{3,1}\lambda_2^2 - \mu_{2,2}\lambda_3 + \mu_{2,1}\lambda_2\lambda_3.\end{aligned}$$

Two particular solutions of equation (4.19) are given by

$$\begin{aligned}u_0(x) &= (x-\lambda_3)^{1-\mu_{3,1}}\,(x-\lambda_2)^{2-\mu_{2,1}}\,e^{\frac{\mu_{2,2}}{x-\lambda_2}}, \\ u_1(x) &= u_0(x)\int (x-\lambda_3)^{\mu_{3,1}-2}\,(x-\lambda_2)^{\mu_{2,1}-4}(x-\lambda_1)^{-\mu_{1,1}}\,e^{-\frac{\mu_{2,2}}{x-\lambda_2}}\,e^{\frac{\mu_{1,2}}{x-\lambda_1}}\,dx.\end{aligned}$$

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