SUT Journal of Mathematics
Vol. 46, No. 1 (2010), 79–91

Equitable irregular edge-weighting of graphs

I. Sahul Hamid and S. Ashok Kumar

(Received October 5, 2009; Revised July 12, 2010)

Abstract. Given a graph $G = (V, E)$, an k-edge-weighting is a map $\phi : E(G) \to \{1, 2, 3, \dots, k\}$, where k is a positive integer. For a vertex v of G, let $S_\phi(v)$ denote the sum of edge-weights appearing on the edges incident at v under the edge-weighting ϕ. An k-edge-weighting of G is said to be equitable irregular if $|S_\phi(u) - S_\phi(v)| \leq 1$, for every pair of adjacent vertices u and v in G. A graph G is said to be equitable irregular if G admits an equitable irregular edge-weighting. If G is equitable irregular then the equitable irregular strength of G is defined to be the smallest positive integer k such that there is a k-edge-weighting of G, and is denoted by $S_e(G)$. In this paper we initiate a study of this new non-proper edge weighting of graphs.

AMS 2000 *Mathematics Subject Classification.* 05C

Key words and phrases. Equitable irregular edge-weighting, Equitable irregular strength.

§1. Introduction

By a graph $G = (V, E)$, we mean a finite, undirected graph with neither loops nor multiple edges. For graph theoretic terminology we refer to Chartrand and Lesniak [3]. All graphs in this paper are assumed to be connected and non-trivial.

The study of graph labeling is one of the fastest growing areas within graph theory which has been extensively studied. A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Graph labelings were first introduced in the late 1960s. In the intervening years dozens of graph labelings have been studied in over 800 papers. A detailed survey of graph labeling is given in [6].

The labeling we deal with this paper is basically an edge-labeling, that is, assigning label to the edges of the graph. We prefer the term edge-weighting instead of edge-labeling. The class of edge-weighting problems can roughly be split into two parts. The first are the proper edge-weightings, that is, edge-weightings of G where any two incident edges must get different labels, which is just the classical edge-coloring problem. The second are the non-proper edge-weightings, that is edge-weightings where we do not require that incident edges get different labels.

The first question about non-proper edge-weightings is the irregularity strength of a graph. This question, introduced by Chartrand *et al.* [2], asks for the smallest k such that there is a k-edge-weighting ϕ of a given graph G such that for any two vertices x and y we have $S_\phi(x) \neq S_\phi(y)$, where $S_\phi(v)$ of a vertex v denotes the sum of edge weights appearing on the edges incident at v under the edge-weighting ϕ, called weighted degree of v. In other words, the weighted degrees of the vertices of G are all distinct. This graph parameter is called the irregularity strength of G and is denoted by $S(G)$ and the k-edge-weighting is called irregular edge-weighting. For a survey of known results and many open questions about the irregularity strength, see Lehel [8]. For many results which were not mentioned in the survey, see [1], [4], [5].

Motivated by the results on irregularity strength, Karonski *et al.* [7] propose the study of non-proper edge-weightings where they require that only adjacent vertices have distinct weighted degrees and called such edge-weighting as chromatic irregular edge-weighting. Thus in an irregular edge-weighting of a graph G, the weighted degrees of the vertices of G are all admit, whereas in a chromatic irregular edge-weighting only adjacent vertices have different weighted degrees. In this sequence we introduce the notion of equitable irregular edge-weighting, where we demand that the weighted degrees of any two adjacent vertices differ by at most one and we call a graph admitting an equitable edge-weighting as an equitable irregular graph. In this paper we initiate a study of this new non-proper edge-weighting of a graph.

§2. Classes of Equitable Irregular Graphs

In this section we give some families of equitable irregular graphs. We first formally define the notion of equitable irregular edge-weighting and equitable irregular strength of a graph.

Definition 2.1. Given a graph $G = (V, E)$, an k-edge-weighting is a map $\phi : E(G) \to \{1, 2, 3, \ldots, k\}$, where k is a positive integer. For a vertex v of G, let $S_\phi(v)$ denote the sum of edge-weights appearing on the edges incident at v, under the edge-weighting ϕ. An k-edge-weighting of G is said to be equitable irregular if $|S_\phi(u) - S_\phi(v)| \leq 1$, for every pair of adjacent vertices u and v in G.

A graph G is said to be equitable irregular if G admits an equitable irregular edge-weighting. If G is equitable irregular then the equitable irregular strength of G is defined to be the smallest positive integer k such that there is a k-edge-weighting of G and is denoted by $S_e(G)$.

Example 2.2. *(i) Obviously, if G is either regular or semi regular (that is, a graph in which degree of any vertex is either k or $k+1$ for some integer $k > 0$) then G is equitable irregular and $S_e(G) = 1$.*

(ii) The grid graph $P_m \times P_n$ is equitable irregular, because the edge-weighting of $P_m \times P_n$ obtained by assigning one to all of whose edges is an equitable irregular edge-weighting and consequently $S_e(P_m \times P_n) = 1$.

(iii) Consider the graph G given in Figure 1. If ϕ is any k-edge-weighting of G with $S_\phi(u_1u_2) = a$, then $S_\phi(u_1) \geq a + 2$ because $\phi(u_1u_3)$ and $\phi(u_1u_4)$ are at least one, whereas $S_\phi(u_2) = a$ so that $|S_\phi(u_1) - S_\phi(u_2)| \geq 2$. Hence G has no equitable irregular edge-weighting and hence G is not equitable irregular.

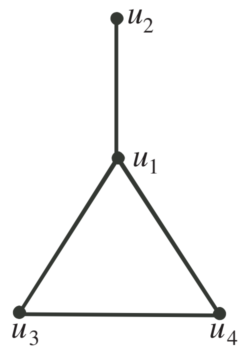

Figure 1:

Theorem 2.3. *The wheel W_n on n vertices is equitable irregular.*

Proof. Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_i : 1 \leq i \leq n-1\} \cup \{v_iv_{i+1} : 1 \leq i \leq n-2\} \cup \{v_1v_{n-1}\}$. Define an edge-weighting ϕ of W_n as follows.

Let $\phi(v_0v_i) = 1$ for all $i = 1, 2, \ldots, n-1$

$$\phi(v_iv_{i+1}) = \left\lceil \frac{n-3}{2} \right\rceil, \quad \text{for all } i = 1, 2, \ldots, n-2,$$

$$\text{and} \qquad \phi(v_1v_{n-1}) = \left\lceil \frac{n-3}{2} \right\rceil.$$

Then $S_\phi(v_0) = \sum_{i=1}^{n-1} \phi(v_0 v_i) = n-1$ and for all $i = 1, 2, \ldots, n-1$,

$$\begin{aligned} S_\phi(v_i) &= \phi(v_0 v_i) + \phi(v_{i-1} v_i) + \phi(v_i v_{i+1}) \\ &= 1 + 2\left\lceil \frac{n-3}{2} \right\rceil \\ &= \begin{cases} n-2 & \text{if } n \text{ is odd} \\ n-1 & \text{if } n \text{ is even} \end{cases} \end{aligned}$$

Thus, for any two adjacent vertices of W_n, the weighted degrees differ by at most one so that ϕ is an equitable edge-weighting of W_n and hence W_n is equitable irregular. □

A *triangular cactus* is a connected graph all of whose blocks are triangles. A *triangular snake* S_n is a triangular cactus whose block-cut point-graph is a path. That is a triangular snake is obtained from a path $(u_1, u_2, \ldots, u_{n+1})$ by joining u_i and u_{i+1} to a new vertex v_i for $i = 1, 2, \ldots, n$.

Theorem 2.4. *The snake graph S_n is equitable irregular for all n.*

Proof. Consider the edge-weighting ϕ of S_n defined as follows. For all $i = 1, 2, \ldots, n$, let

$$\begin{aligned} &\phi(u_i u_{i+1}) = 1 \\ &\phi(u_i v_i) = n + 1 - i \\ \text{and} \quad &\phi(u_{i+1} v_i) = i \end{aligned}$$

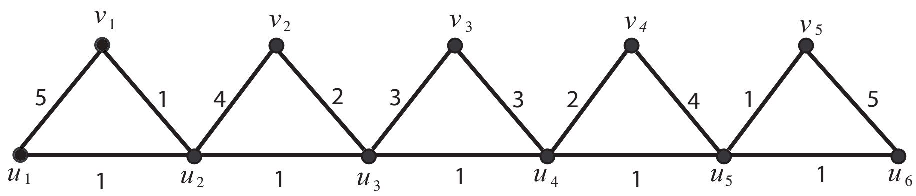

Figure 2:

(For the snake S_5 the labeling ϕ is illustrated in Figure 2). Then for all $i = 1, 2, \ldots, n$, we have $S_\phi(v_i) = \phi(u_i v_i) + \phi(u_{i+1} v_i) = n+1$ and for all $i = 2, 3, \ldots, n$, we have $S_\phi(u_i) = \phi(u_i v_{i-1}) + \phi(u_i + v_i) + \phi(u_{i-1} u_i) + \phi(u_i u_{i+1}) = n+2$. Also, $S_\phi(u_1) = \phi(u_1 v_1) + \phi(u_1 u_2) = n+1$ and $S_\phi(u_{n+1}) = \phi(u_n u_{n+1}) + \phi(u_{n+1} u_n) = n+1$. Hence ϕ is an equitable irregular edge-weighting of S_n so that the snake S_n is equitable irregular. □

For a given graph G, we define G^∇ to be the graph obtained from G by attaching a triangle at each vertex G. That is, for a graph G if $V(G) =$

$\{v_1, v_2, v_3, \ldots, v_n\}$, then G^∇ is obtained from G by introducing two vertices x_i and y_i for each i, where $1 \le i \le n$, and joining v_i to the both x_i and y_i, and also joining the vertices x_i and y_i.

Theorem 2.5. *For any graph G, the graph G^∇ is equitable irregular.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$. Then G^∇ is the graph with $V(G^\nabla) = \{v_1, x_1, y_1, v_2, x_2, y_2, \ldots, v_n, x_n, y_n\}$ and $E(G^\nabla) = E(G) \cup \{x_iy_i : 1 \le i \le n\} \cup \{v_ix_i : 1 \le i \le n\} \cup \{v_iy_i : 1 \le i \le n\}$. Define an edge-weighting ϕ of G^∇ as follows. For all $i = 1, 2, \ldots, n$, let

$$\phi(v_ix_i) = \phi(v_iy_i) = \left\lfloor \frac{\Delta(G) - \deg\, v_i + 2}{2} \right\rfloor$$

$$\phi(x_iy_i) = \left\lceil \frac{\Delta(G) + \deg\, v_i}{2} \right\rceil.$$

Further assign the label one to all the edges of G. For the given graph G, the graph G^∇ and its edge-weighting ϕ are illustrated in Figure 3.

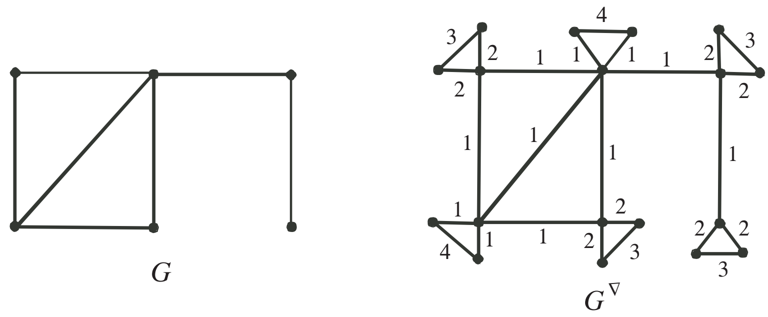

Figure 3:

Now, for all $i = 1, 2, \ldots, n$, we have,

$$\begin{aligned} S_\phi(v_i) &= \deg\, v_i + 2\left\lfloor \frac{\Delta(G) - \deg\, v_i + 2}{2} \right\rfloor \\ &= \begin{cases} \Delta(G) + 1 & \text{if } \Delta(G) - \deg\, v_i + 2 \text{ is odd} \\ \Delta(G) + 2 & \text{if } \Delta(G) - \deg\, v_i + 2 \text{ is even} \end{cases} \end{aligned}$$

$$\text{and} \qquad S_\phi(x_i) = S_\phi(y_i) = \left\lfloor \frac{\Delta(G) - \deg\, v_i + 2}{2} \right\rfloor + \left\lceil \frac{\Delta(G) + \deg\, v_i}{2} \right\rceil = \Delta(G) + 1.$$

Thus, the weighted degrees of any two adjacent vertices of G^∇ differ by at most one so that G^∇ is equitable irregular. $\square$

For a graph G, the cartesian product $G \times K_2$ is called the *prism* of G. That is, the prism of a graph G is the graph obtained by taking two copies of G and joining each vertex of one copy of G with the same vertex in the other copy of G.

Theorem 2.6. *The prism of any graph is equitable irregular.*

Proof. If G is the given graph, we define an edge-weighting ϕ of the prism $G \times K_2$ as follows. Assign the weight one to all the edges of G (in both copies). Also, if e is an edge of $G \times K_2$ which joins a vertex v of one copy of G with the same vertex in the other copy of G, define $\phi(e) = \Delta(G) - \deg_G v + 1$. Then, for any vertex u in the prism, we have $S_\phi(u) = \deg_G u + \Delta(G) - \deg_G u + 1 = \Delta(G) + 1$ so that ϕ is an equitable irregular edge-weighting of the prism. A graph G and its prism along with the equitable irregular edge-weighting ϕ are given in Figure 4. □

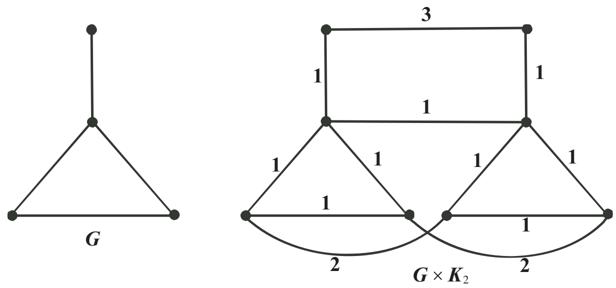

Figure 4:

An *n-fan graph* is defined to be a graph obtained by attaching n number cycles, say $C_{k_1}, C_{k_2}, \ldots, C_{k_n}$ of length $k_1, k_2, \ldots, k_n$ respectively at a vertex and is denoted by $G(C_{k_1}, C_{k_2}, \ldots, C_{k_n})$. Here the cycles of the n-fan graph are called *leaves* and leaves of even length are called *even leaves*. A 3-fan graph is given in Figure 5.

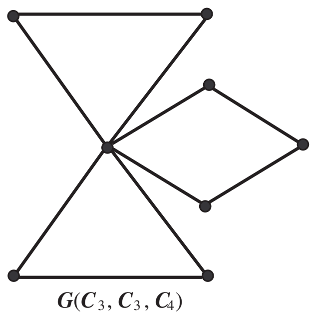

Figure 5:

Theorem 2.7. *The n-fan graph G is equitable irregular if and only if all of whose even leaves are of length at least $4(n-1)$.*

Proof. Suppose all the leaves of the n-fan graph $G(C_{k_1}, C_{k_2}, \ldots, C_{k_n})$ have length at least $4(n-1)$. We have to define an equitable irregular edge-weighting of the n-fan graph. In order to obtain such an equitable irregular edge-weighting it is sufficient if we say that how an arbitrary leaf must be labeled. Now, let us consider a leaf C_{k_i}, of the n-fan graph.

If the length of C_{k_i} is odd, then assign the weights 1 and $2n-2$ to the edges of the leaf C_{k_i} alternatively starting from the edge of this leaf incident at v, where v is the centre vertex of the n-fan graph.

Now, suppose the length of the leaf C_{k_i} is even and let its length be $4(n-1)+r$ where $r \geq 0$. For our convenience let us denote the vertices of the leaf C_{k_i} as in Figure 6.

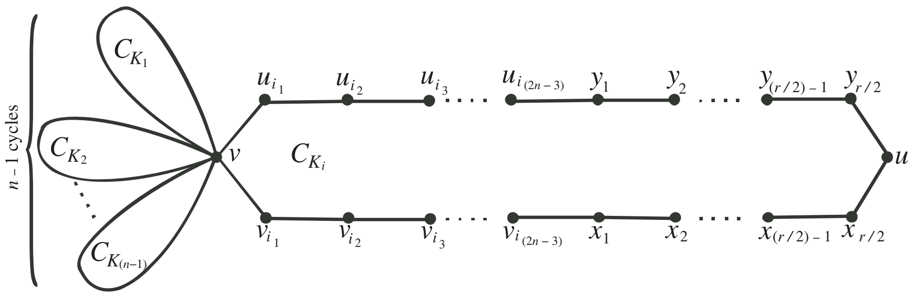

Figure 6:

We now assign weights to the edges of C_{k_i} as follows. Let $\phi(vv_{i_1}) = \phi(vu_{i_1}) = 1$. Also for all $j = 1, 2, \ldots, n-2$,

$$\begin{aligned} \text{let} \quad & \phi(v_{i_{2j-1}}v_{i_{2j}}) = 2n-(j+1) \\ & \phi(v_{i_{2j}}v_{i_{2j+1}}) = j+1 \\ & \phi(u_{i_{2j-1}}u_{i_{2j}}) = 2n-(j+1) \\ \text{and} \quad & \phi(u_{i_{2j}}u_{i_{2j+1}}) = j+1. \end{aligned}$$

Further assign the weight n to all the remaining edges of C_{k_i}. Hence, for all $j = 1, 2, \ldots, n-2$, we have

$$\begin{aligned} S_\phi(v_{i_{2j}}) &= \phi(v_{i_{2j-1}}v_{i_{2j}}) + \phi(v_{i_{2j}}v_{i_{2j+1}}) \\ &= 2n-j-1+j+1 = 2n, \\ S_\phi(v_{i_{2j+1}}) &= \phi(v_{i_{2j}}v_{i_{2j+1}}) + \phi(v_{i_{2j+1}}v_{i_{2j+2}}) \\ &= j+1+2n-j-2 = 2n-1, \\ \text{and} \quad S_\phi(v_{i_1}) &= \phi(vv_{i_1}) + \phi(v_{i_1}v_{i_2}) = 2n-1. \end{aligned}$$

Thus, the degree weights for any two adjacent vertices of C_{k_i} differ by at most one, so that the n-fan graph G is equitable irregular.

Suppose the n-fan graph G is equitable irregular. Then G has an equitable irregular edge-weighting ϕ. We now wish to prove that the length of any even leaf of G is at least $4(n-1)$. Let C be an arbitrary leaf of even length say $l = 2r$, where $r \geq 1$. Let $C = (v, x_1, x_2, \ldots, x_{r-1}, u, y_{r-1}, \ldots, y_1, v)$. Now, let us denote the edge weights of the edges vx_1 and vy_1 under the labeling ϕ as a_1 and a_2 respectively. Also let l_i and l'_i denote the edge weights of the edges y_iy_{i+1} and x_ix_{i+1} respectively for all $i = 1, 2, \ldots, r-1$.

Now, there are $2n-2$ edges (other than vx_1 and vy_1) incident at the vertex v and also the weight of each of those edges is at least one. Hence $S_\phi(v) \geq \phi(vx_1) + \phi(vy_1) + 2n - 2 = a_1 + a_2 + 2n - 2$. Since $\phi(vy_1) = a_2$ and $|S_\phi(v) - S_\phi(y_1)| \leq 1$, it follows that $l_1 = a_1 + k_1$, for some k_1 with $k_1 \in \{2n-3, 2n-2, 2n-1\}$. Similarly, since $S_\phi(y_1) = \phi(vy_1) + \phi(y_1y_2) = a_2 + l_1 = a_1 + a_2 + k_1$ and $|S_\phi(y_1) - S_\phi(y_2)| \leq 1$, it follows that $l_2 = a_2 + k_2$ for some k_2 with $k_2 \in \{-1, 0, 1\}$. Proceeding like this we have for all $i = 1, 2, \ldots, r-1$,

$$l_i = \begin{cases} a_1 + \sum_{j=1}^{\frac{i+1}{2}} k_{2j-1} & \text{if } i \text{ is odd} \\ a_2 + \sum_{j=1}^{\frac{i}{2}} k_{2j} & \text{if } i \text{ is even,} \end{cases}$$

$$\text{and} \qquad l'_i = \begin{cases} a_2 + \sum_{j=1}^{\frac{i+1}{2}} k'_{2j-1} & \text{if } i \text{ is odd} \\ a_1 + \sum_{j=1}^{\frac{i}{2}} k'_{2j} & \text{if } i \text{ is even,} \end{cases}$$

where $k_j, k'_j \in \{-1, 0, 1\}$, for all $j = 2, 3, \ldots, r-1$. We now consider following two cases.

Case (i) r is even

Now, $S_\phi(u) = l_{r-1} + l'_{r-1}$ and $S_\phi(x_{r-1}) = l'_{r-2} + l'_{r-1}$. Also, since ϕ is equitable irregular, we have $S_\phi(u) - S_\phi(x_{r-1}) \leq 1$, so that $l_{r-1} - l'_{r-2} \leq 1$

$$\text{Hence} \qquad \left(a_1 + \sum_{j=1}^{\frac{r}{2}} k_{2j-1}\right) - \left(a_1 + \sum_{j=1}^{\frac{r}{2}-1} k'_{2j}\right) \leq 1$$

$$\Rightarrow \sum_{j=1}^{\frac{r}{2}} k_{2j-1} - \sum_{j=1}^{\frac{r}{2}-1} k'_{2j} \leq 1$$

$$\Rightarrow \sum_{j=1}^{\frac{r}{2}} k_{2j-1} \leq 1 + \sum_{j=1}^{\frac{r}{2}-1} k'_{2j} \leq 1 + \frac{r}{2} - 1 = \frac{r}{2}. \qquad \text{(i)}$$

Also, since $k_1 \geq 2n-3$ and $k_i \geq -1$, for each $i \geq 2$, we have

$$\text{(ii)} \qquad \sum_{j=1}^{\frac{r}{2}} k_{2j-1} \geq 2n-3-\frac{r}{2}+1 = 2n-2-\frac{r}{2}.$$

Using (i) and (ii) we get $2r \geq 4n-4$ so that $l \geq 4(n-1)$.

Case (ii) r is odd

Now, $S_\phi(u) = l_{r-1}+l'_{r-1}$ and $S_\phi(y_{r-1}) = l_{r-1}+l_{r-2}$. Also, since ϕ is equitable irregular, we have $S_\phi(y_{r-1}) - S_\phi(u) \leq 1$, so that $l_{r-2} - l'_{r-1} \leq 1$.

$$\text{Hence} \qquad \left(a_1 + \sum_{j=1}^{\frac{r-1}{2}} k_{2j-1}\right) - \left(a_1 + \sum_{j=1}^{\frac{r-1}{2}} k'_{2j}\right) \leq 1$$

$$\text{(iii)} \qquad \Rightarrow \sum_{j=1}^{\frac{r-1}{2}} k_{2j-1} \leq 1 + a_1 - \sum_{j=1}^{\frac{r-1}{2}} k'_{2j} \leq 1 + \frac{r}{2} - \frac{1}{2} = \frac{r}{2} + \frac{1}{2}.$$

$$\text{(iv)} \qquad \text{Also,} \sum_{j=1}^{\frac{r-1}{2}} k_{2j-1} \geq 2n-3+\left(\frac{r-1}{2}-1\right)(-1).$$

Using (iii) and (iv) we get $2r \geq 4n-4$ so that $l \geq 4(n-1)$.

In either cases all even leaves of G are of the length at least $4(n-1)$. □

Theorem 2.8. *The complete bipartite graph $K_{2,n}$ is equitable irregular if and only if $n \leq 3$.*

Proof. Obviously $K_{2,n}$ is equitable irregular if $n \leq 3$. Suppose $n \geq 4$. Let (X, Y) be the bipartition of $K_{2,n}$, where $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Suppose ϕ is any k-edge-weighting of $K_{2,n}$. Let $\phi(x_1 y_i) = a_i, 1 \leq i \leq n$. Then $S_\phi(x_1) = a_1 + a_2 + a_3 + \cdots + a_n$. Hence $\phi(x_2 y_i) \geq \left(\sum_{j=1, j\neq i}^{n} a_j\right) - 1$, for all $i = 1, 2, \ldots, n$. Thus

$$\begin{aligned} S_\phi(x_2) &= \sum_{i=1}^{n} \phi(x_2 y_i) \\ &\geq \sum_{i=1}^{n} \left(\sum_{j=1, j\neq i}^{n} a_j\right) - n \\ &= (n-1)\sum_{i=1}^{n} a_i - n \\ &= (n-1)S_\phi(x_1) - n. \end{aligned}$$

Also, $S_\phi(y_1) \leq S_\phi(x_1)+1$. Hence $S_\phi(x_2)-S_\phi(y_1) \geq (n-1)S_\phi(x_1)-n-S_\phi(x_1)-1 = (n-2)S_\phi(x_1)-n-1$. Thus $|S_\phi(x_2)-S_\phi(y_1)| \geq (n-2)S_\phi(x_1)-n-1$. But $|S_\phi(x_2)-S_\phi(y_1)| \leq 1$ so that $(n-2)S_\phi(x_1)-n-1 \leq 1$ and hence $S_\phi(x_1) \leq \frac{n+2}{n-2}$. Since $S_\phi(x_1) \geq n$, it follows that $n \leq \frac{n+2}{n-2}$ which is a contradiction when $n \geq 4$. $\square$

Theorem 2.9. *The complete bipartite graph $K_{m,n}$ is equitable irregular if and only if $|m-n| \leq 1$.*

Proof. We may assume $m \leq n$. Let $G = K_{m,n}$, and let X and Y be the partite sets of G having cardinalities m and n, respectively. It is clear that if $n \leq m+1$, then G is equitable irregular. Conversely, suppose that G has an equitable irregular edge-weighting ϕ. Let r denote the minimum of $S_\phi(y)$ as y ranges over Y. Then $\sum_{e\in E(G)} \phi(e) = \sum_{y\in Y} S_\phi(y) \geq rn$. On the other hand, since ϕ is equitable irregular, $S_\phi(x) \leq r+1$ for all $x \in X$, and hence $\sum_{e\in E(G)} \phi(e) = \sum_{x\in X} S_\phi(x) \leq (r+1)m$. Consequently $rn \leq (r+1)m$. Note that $r \geq m$ because $\phi(e) \geq 1$ for all $e \in E(G)$. Therefore $n \leq m(r+1)/r \leq m+1$, as desired. $\square$

§3. Properties of Equitable Irregular Graphs

In this section, we present some properties of equitable irregular graphs. A vertex of degree one is called *pendant vertex* and a vertex which is adjacent to a pendant vertex is called a *support vertex*.

Theorem 3.1. *If G is equitable irregular, then every support vertex in G (if any) is of degree two.*

Proof. Suppose there exists a support vertex v of degree greater than two in G. Let u be a pendant vertex adjacent to v. Now, if ϕ is a k-edge-weighting of G with $\phi(uv) = a$ then $S_\phi(u) = a$ and since $\deg v \geq 3$ it follows that $S_\phi(v) = a+2$ so that $|S_\phi(u) - S_\phi(v)| \geq 2$. Hence G has no equitable irregular k-edge-weighting. $\square$

Remark 3.2. The converse of the above theorem is not true. That is a graph G in which every support vertex is of degree two need not be equitable irregular. For example, in the graph G given in Figure 7 the support vertex is of degree two, whereas G is not equitable irregular.

We now present a lower bound for the equitable irregular strength of a graph G, which is useful in determining the value of $S_e(G)$. For this purpose, we define the term $\mu(G)$ as follows. For a vertex $x \in V(G)$, let $\mu_x = \min\{\deg y : yx \in E(G)\}$, and then define $\mu(G) = \min\{\mu_x : x \in V(G) \text{ with } \deg x = \Delta\}$.

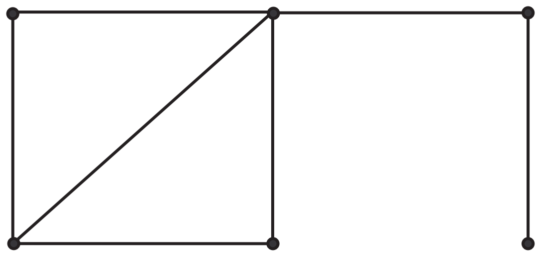

Figure 7:

Theorem 3.3. *If G is equitable irregular, then $S_e(G) \geq \frac{\Delta-2}{\mu(G)-1}$.*

Proof. Let ϕ be any k-edge-weighting of G. Let x be a vertex of degree Δ with $\mu(G) = \mu_x$ and let y be a neighbour of x with $\mu_x = \deg y$. Let $\phi(xy) = a$. Now, if the weight of each of the edges incident at y other then xy is less then $\frac{\Delta-2}{\mu(G)-1}$, then $S_\phi(y) < \Delta - 2 + a$. Also $S_\phi(x) \geq \Delta - 1 + a$ and hence $S_\phi(x) - S_\phi(y) > \Delta - 1 + a - \Delta + 2 - a$ so that $|S_\phi(x) - S_\phi(y)| \geq 2$. Thus if ϕ is any equitable irregular edge-weighting of G, then at least one of the edges incident at y has weight at least $\frac{\Delta-2}{\mu(G)-1}$ under the edge-weighting ϕ and hence $S_e(G) \geq \frac{\Delta-2}{\mu(G)-1}$. □

Corollary 3.4. *For the wheel W_n on n vertices, $S_e(W_n) = \lceil \frac{n-3}{2} \rceil$*

Proof. By considering the equitable irregular labeling of W_n as in Theorem 2.3, we have $S_\phi(W_n) \leq \lceil \frac{n-3}{2} \rceil$. Now, it follows from Theorem 3.3 that $S_\phi(W_n) \geq \lceil \frac{n-3}{2} \rceil$. □

Corollary 3.5. *For the n-fan graph $G(C_{k_1}, C_{k_2}, \ldots, C_{k_n}), S_e(G) = 2(n-1)$.*

Proof. By considering the equitable irregular labeling of n-fan graph $G(C_{k_1}, C_{k_2}, \ldots, C_{k_n})$ as in Theorem 2.7, we have $S_\phi(G) = 2(n-1)$. Now, it follows from Theorem 3.3 that $S_\phi(G) = 2(n-1)$. □

Corollary 3.6. *For any graph G, we have $S_e(G^\nabla) = \Delta(G)$.*

Proof. Consider the equitable irregular edge-weighting ϕ of G^∇ as in Theorem 2.5. Then $S_e(G^\nabla) \leq \Delta(G)$. Now, since $\Delta(G^\nabla) = \Delta(G) + 2$ and $\mu(G^\nabla) = 2$ it follows from Theorem 3.3 that $S_e(G^\nabla) \geq \Delta(G)$. □

Remark 3.7. By the virtue of Theorem 2.5 and Theorem 2.6, one can observe that any graph G (whether or not it is equitable irregular) can be embedded as an induced subgraph of an equitable irregular graph H which shows the impossibility of obtaining a forbidden subgraph characterization for an equitable irregular graph. Here the graph H is called an embedding of G; and infact this embedding is not unique, as we have seen that, for any graph G, the graph G^{∇} and the prism $G \times K_2$ are equitable irregular. Also, it is always possible to embed any graph as an induced subgraph of a regular graph which is equitable irregular so that we have another way of embedding a graph in a equitable irregular graph. Thus, at this point we have three types of embeddings; however these are not the only such embedding. For example, the graph H given in Figure 8 is an embedding of the graph G; but H is neither of the above three types and infact H is an embedding of G with minimum order. In this connection, the following problem naturally arises.

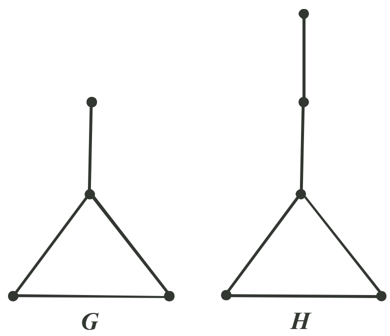

Figure 8:

Problem 3.8. *If G is non-equitable irregular, what is the minimum order of an equitable irregular graph H having G as an induced subgraph?*

Conclusion and Scope

In this paper we have introduced a new edge labeling namely equitable irregular edge-weighting and have initiated a study of this labeling. However, there is a wide scope for further research on this topic. Here we present some directions for further research.

(A) In Theorem 3.1, we have given a necessary condition for a graph to be equitable irregular. However, we have no other tool so far to say that whether or not a graph is equitable irregular. Hence, obtaining a necessary or sufficient condition for being equitable irregular is worthy trying.

(B) Since a complete graph is equitable irregular, it is always possible to make a graph equitable irregular by adding edges within the graph and so one can naturally look for the minimum number of edges to be added in order to make a graph equitable irregular.

(C) It seems to us that the problem of characterizing trees which are equitable irregular would be very interesting.

(D) The effect of removal of an edge from a graph on a parameter is of some practical importance. One can observe that removal of an edge from a graph G can make it either equitable irregular or non-equitable irregular. Hence, let us call an edge e of G a *critical edge* if $G-e$ becomes equitable irregular when G is not and vice versa. Now, we can initiate a study on critical edges of a graph.

(E) Characterize equitable irregular graphs G for which $S_e(G) = \frac{\Delta(G)-2}{\mu(G)-1}$.

References

[1] D. Amar, Irregularity strength of regular graphs of large degree. Combinatorics and algorithms (Jerusalem, 1988), *Discr. Math.*, **114** (1993), no 1 – 3, 9 – 17.

[2] G. Chartrand, M. S. Jacobson, J. Lehel, O. R. Oellermann, S. Ruiv and F. Saba, Irregular networks, *Congr. Numer.*, **64** (1988), 197 – 210.

[3] G. Chartrand and Lesniak, Graphs and Digraphs, Fourth Edition, CRC Press, Boca Raton (2005).

[4] G. Elbert, J. Hemmeter, F. Lazebnik and A. J. Woldar, On the number of irregular assignments on a graph, *Discr. Math.*, **93** (1991), 131 – 142

[5] D. K. Garnick and D. H. Dinitz, Heuristic algorithms for finding irregularity strength of graphs, *J. Combrin. Math., Combrin. Comput.*, **8** (1990), 195 – 208.

[6] J. A. Gallian, A dynamic survey of graph labeling, *Electron. J. Combinator.*, **15** (2008), 190.

[7] M. Karonski, T. Luczak and A. Thomson, Edge weights and vertex colours. *J. Combinator. Theory B*, **91** (2004), 151 – 157.

[8] J. Lehel, Facts and questions on degree irregular assignments, *Graph Theory, Combinator., Appl.*, vol. 2, Kalamazoo, **M1** (1988) 265 – 781; Wiley, New York (1991).

Department of Mathematics
The Madura College, Madurai, INDIA
E-mail: sahulmat@yahoo.co.in