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Existence of constant sign solutions for the p-Laplacian problems in the resonant case with respect to Fučík spectrum

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Abstract. We consider the following the p-Laplacian equation in a bounded domain Ω:

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We treat the case of nonlinearity term f satisfying the following conditions

$$f(x,t) = \begin{cases} a_0 t_+^{p-1} - b_0 t_-^{p-1} + o(|t|^{p-1}) & \text{at } 0, \\ a t_+^{p-1} - b t_-^{p-1} + o(|t|^{p-1}) & \text{at } \infty, \end{cases}$$

for constants a_0, b_0, a and b. We prove the existence of a positive solution or a negative solution in the case of $(a_0 - \lambda_1)(a - \lambda_1) = 0$ or $(b_0 - \lambda_1)(b - \lambda_1) = 0$ respectively, where λ_1 is the first eigenvalue of $-\Delta_p$.

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§1. Introduction and statements of results

1.1. Introduction

In this paper, we consider the equation

$$\text{(P)} \qquad \begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < \infty$, $\Omega \subset \mathbb{R}^N$ is a bounded domain, Δ_p denotes the p-Laplacian defined by $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$. Our purpose is to show the existence of constant sign solutions to (P). Here we say that $u \in W_0^{1,p}(\Omega)$ is a (weak) positive (resp. negative) solution of (P) if $u(x) > 0$ (resp. $u(x) < 0$) a.e. $x \in \Omega$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \varphi \, dx = \int_\Omega f(x,u)\varphi \, dx$$

holds for any $\varphi \in W_0^{1,p}(\Omega)$.

We will treat f satisfying $f(x,0) = 0$ a.e. $x \in \Omega$ and

(1.1)
$$f(x,t) = \begin{cases} a_0 t_+^{p-1} - b_0 t_-^{p-1} + o(|t|^{p-1}) & \text{as } |t| \to 0, \text{ uniformly in a.e. } x \in \Omega, \\ a t_+^{p-1} - b t_-^{p-1} + o(|t|^{p-1}) & \text{as } |t| \to \infty, \text{ uniformly in a.e. } x \in \Omega, \end{cases}$$

where $t_\pm = \max\{\pm t, 0\}$ and a_0, a, b_0 and b are some real constants. Thus, we consider the case where (P) has a trivial solution $u = 0$.

Equation (P) in the case of $f(x,t) = at_+^{p-1} - bt_-^{p-1}$ (where a, $b \in \mathbb{R}$) has been considered by Fučík [6]($p = 2$) and by many authors (*cf.* [3], [2], [4]). The set Σ_p of the points $(a,b) \in \mathbb{R}^2$ for which the equation

$$-\Delta_p u = au_+^{p-1} - bu_-^{p-1}, \quad u \in W_0^{1,p}(\Omega) \tag{1.2}$$

has a non-trivial weak solution is called Fučík spectrum of the p-Laplacian on $W_0^{1,p}(\Omega)$ ($1 < p < \infty$) ([2]). In the case of $a = b = \lambda \in \mathbb{R}$, the equation (1.2) reads $-\Delta_p u = \lambda |u|^{p-2}u$. Hence (λ,λ) belongs to Σ_p if and only if λ is an *eigenvalue* of $-\Delta_p$, i.e., there exists a non-zero weak solution $u \in W_0^{1,p}(\Omega)$ to $-\Delta_p u = \lambda|u|^{p-2}u$. The set of all eigenvalues of $-\Delta_p$ is, as usual, denoted by $\sigma(-\Delta_p)$. It is well known that the first eigenvalue λ_1 of $-\Delta_p$ is positive, simple, and has a positive eigenfunction $\varphi_1 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \cap C^1(\Omega)$ with $\int_\Omega \varphi_1^p \, dx = 1$ (see [7, Proposition 1.5.19]). Therefore, Σ_p contains the lines $\{\lambda_1\} \times \mathbb{R}$ and $\mathbb{R} \times \{\lambda_1\}$ since φ_1 or $-\varphi_1$ becomes a solution to (1.2) with $(a,b) = (\lambda_1, b)$ or (a, λ_1), respectively. Furthermore, [2] gave a Lipschitz continuous curve contained in Σ_p which is called the first nontrivial curve $\mathscr{C}$. This result was proved by applying the mountain pass theorem to the functional defined on a manifold in $W_0^{1,p}(\Omega)$ (see [2] for details).

Many authors treated equation (P) for the nonlinear term f like (1.1) especially in the non-resonant case ($(a_0,b_0) \notin \Sigma_p$ and $(a,b) \notin \Sigma_p$) (*cf.* [4], [8], [10], [11], [14], [19], [20]). In the so-called resonant case where $(a,b) \in \Sigma_p$ or $(a_0,b_0) \in \Sigma_p$, there are a few existence results (*cf.* [9], [10], [11] where $a = b = \lambda_1$) and the present author obtained existence results of non-trivial solutions to (P) in [14], [15], [16] and [17], including both in resonant cases and non-resonant cases.

As for constant-sign solutions, [4] showed the existence of a positive (resp. negative) solution to (P) under the condition $(a_0 - \lambda_1)(a - \lambda_1) < 0$ (resp.

$(b_0 - \lambda_1)(b - \lambda_1) < 0$). However, the results of [4] does not cover several cases where (a_0, b_0) or (a, b) belongs to Σ_p (that is, *resonant case*).

Thus, the purpose of the present paper is to show the existence of a positive solution or negative solution for (P) in the case of $(a_0 - \lambda_1)(a - \lambda_1) = 0$ or $(b_0 - \lambda_1)(b - \lambda_1) = 0$, respectively (containing possibly "*doubly resonant*" case).

1.2. Statements of results

In this paper, we assume that the nonlinear term f satisfies the following assumption (F):

(F) f is a Carathéodory function on $\Omega \times \mathbb{R}$ with $f(x, 0) = 0$ for a.e. $x \in \Omega$ and satisfies the following conditions for some constants a_0, b_0, a, $b \in \mathbb{R}$ and a positive constant C_0:

$$
\tag{1.3} f(x,u) = \begin{cases} a_0 u_+^{p-1} - b_0 u_-^{p-1} + g_0(x,u), \\ a u_+^{p-1} - b u_-^{p-1} + g(x,u), \end{cases}
$$

$$
\begin{aligned}
&g_0(x,t) = o(|t|^{p-1}) \quad \text{as } |t| \to 0, \text{ uniformly in a.e. } x \in \Omega, \\
&g(x,t) = o(|t|^{p-1}) \quad \text{as } |t| \to \infty, \text{ uniformly in a.e. } x \in \Omega, \\
&|f(x,t)| \le C_0 |t|^{p-1} \quad \text{for every } t \in \mathbb{R}, \text{ a.e. } x \in \Omega.
\end{aligned}
$$

Setting $G(x,u) := \int_0^u g(x,s)\,ds$ and $G_0(x,u) := \int_0^u g_0(x,s)\,ds$ for the nonlinear terms g and g_0 in (1.3), we can now state relevant conditions on $g(x,u)$ or $g_0(x,u)$, which are not necessarily simultaneously assumed in our results.

$(G{+}{+})$ $pG(x,t) - g(x,t)t \to +\infty$ as $t \to +\infty$, uniformly in a.e. $x \in \Omega$,

$(G{-}{+})$ $pG(x,t) - g(x,t)t \to +\infty$ as $t \to -\infty$, uniformly in a.e. $x \in \Omega$.

$(G{+}{-})$ $pG(x,t) - g(x,t)t \to -\infty$ as $t \to +\infty$, uniformly in a.e. $x \in \Omega$.

$(G{-}{-})$ $pG(x,t) - g(x,t)t \to -\infty$ as $t \to -\infty$, uniformly in a.e. $x \in \Omega$.

$(G_0{+}{+})$ there exist a $\delta > 0$ and a measurable subset Ω' of Ω with $\mu(\Omega') > 0$ such that

$$
\begin{aligned}
&G_0(x,t) \ge 0 \quad \text{for } 0 \le t \le \delta, \text{ a.e. } x \in \Omega, \\
&G_0(x,t) > 0 \quad \text{for } 0 < t \le \delta, \text{ a.e. } x \in \Omega',
\end{aligned}
$$

where $\mu(\Omega')$ denotes the Lebesgue measure of Ω'.

(G_0-+) there exist a $\delta > 0$ and a measurable subset Ω' of Ω with $\mu(\Omega') > 0$ such that

$$G_0(x,t) \geq 0 \quad \text{for } -\delta \leq t \leq 0, \text{ a.e. } x \in \Omega,$$
$$G_0(x,t) > 0 \quad \text{for } -\delta \leq t < 0, \text{ a.e. } x \in \Omega'.$$

(G_0+-) there exist positive constants δ, C and $q \in (p, p^*)$ such that

$$G_0(x,t) \leq -C|t|^q \quad \text{for } 0 \leq t \leq \delta, \text{ a.e. } x \in \Omega,$$

where $p^* = pN/(N-p)$ if $p < N$, $p^* = +\infty$ if $p \geq N$.

(G_0--) there exist positive constants δ, C and $q \in (p, p^*)$ (p^* is the number defined just above) such that

$$G_0(x,t) \leq -C|t|^q \quad \text{for } -\delta \leq t \leq 0, \text{ a.e. } x \in \Omega.$$

Now we can state our results.

Theorem 1 *Assume that f satisfies (F) for some constants a_0, b_0, a, $b \in \mathbb{R}$ and a positive constant C_0. Then, if one of the following conditions holds,* (P) *has at least one positive solution.*

(i) $a = \lambda_1 < a_0$ *and* $(G+-)$;

(ii) $a = \lambda_1 > a_0$ *and* $(G++)$;

(iii) $a < \lambda_1 = a_0$ *and* (G_0++);

(iv) $a > \lambda_1 = a_0$ *and* (G_0+-);

(v) $a = a_0 = \lambda_1$, $(G+-)$ *and* (G_0++);

(vi) $a = a_0 = \lambda_1$, $(G++)$ *and* (G_0+-).

Theorem 2 *Assume that f satisfies (F) for some constants a_0, b_0, a, $b \in \mathbb{R}$ and a positive constant C_0. Then, if one of the following conditions holds,* (P) *has at least one negative solution.*

(i) $b = \lambda_1 < b_0$ *and* $(G--)$;

(ii) $b = \lambda_1 > b_0$ *and* $(G-+)$;

(iii) $b < \lambda_1 = b_0$ *and* (G_0-+);

(iv) $b > \lambda_1 = b_0$ *and* (G_0--);

(v) $b = b_0 = \lambda_1$, $(G{-}{-})$ *and* $(G_0{-}{+})$;

(vi) $b = b_0 = \lambda_1$, $(G{-}{+})$ *and* $(G_0{-}{-})$.

We remark that many nonlinearities satisfy assumptions above, for example, $g(x,u) = \pm|u|^{q-2}u$ near infinity $(1 \le q < p)$ and $g_0(x,u) = \pm|u|^{r-2}u$ near zero $(p < r < p^*)$.

1.3. Notation and the structure of the paper

In what follows, we set $X = W_0^{1,p}(\Omega)$ with norm $\|u\| = \left(\int_\Omega |\nabla u|^p\,dx\right)^{1/p}$ and define two functionals I^+ and I^- on X by

$$I^{\pm}(u) := \int_\Omega |\nabla u|^p\,dx - p\int_\Omega F_{\pm}(x,u)\,dx.$$

where

$$f_{\pm}(x,u) := \begin{cases} f(x,u) & \text{if } \pm u > 0, \\ 0 & \text{if } \pm u \le 0, \end{cases} \qquad F_{\pm}(x,u) := \int_0^u f_{\pm}(x,s)\,ds\ .$$

For the sake of brevity, we use the notation $I^\pm$ to denote either I^+ or I^-. $f_\pm$ or $F_\pm$ should be understood in the same way.

Moreover, $\|u\|_q$ *denotes the* L^q *norm of* $u \in L^q(\Omega)$ $(1 \le q \le \infty)$. Note that X is uniformly convex since we have assumed $1 < p < \infty$.

Remark 3 *Under condition* (F)*, it is well known that* $I^\pm$ *are* C^1 *functionals and non-trivial critical points of* I^+ *and* I^- *correspond to (weak) positive solutions and negative solutions of equation* (P)*, respectively. Indeed, let* u *be a critical point of* I^-*. Noting that* $0 = \langle (I^-)'(u), u_+\rangle = p\|u_+\|^p$*, we have* $u \le 0$*, hence* u *is a non-positive weak solution to* $-\Delta_p u = f(x,u)$*. Therefore,* u *belongs to* $L^\infty(\Omega) \cap C^1(\Omega)$ (*cf.* [1], [5])*. Moreover, we have* $u < 0$ *or* $u \equiv 0$ *in* Ω *by Harnack inequality* (*cf.* [18])*. Thus,* u *is a negative solution of* $-\Delta_p u = f(x,u)$ *in* Ω *if* $u \not\equiv 0$*. Similarly, if* u *is a non-trivial critical point of* I^+*, then* $u > 0$ *in* Ω *holds.*

Firstly, in the next section, we prepare several results for our proofs. In Section 3, we can obtain a non-trivial critical point of I^+ (resp. I^-) under each conditions in Theorem 1 (resp. Theorem 2), whence follows the existence of a positive (resp. negative) solution for (P), respectively.

§2. Preliminaries

2.1. The Cerami condition

It is well known that the Palais–Smale condition and the Cerami condition imply the compactness of a critical set at any level $c \in \mathbb{R}$, and they play an important role in minimax argument. Here, we recall the definition of the Cerami condition.

Definition 4 *A C^1 functional J on a Banach space E is said to satisfy the Cerami condition at $c \in \mathbb{R}$ if any sequence $\{u_n\} \subset E$ satisfying*

$$J(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|\,)\,\|J'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty$$

has a convergent subsequence. We say that J satisfies the Cerami condition if J satisfies the Cerami condition at any $c \in \mathbb{R}$.

We note that the Cerami condition is weaker than the usual Palais–Smale condition.

Now we introduce assumption $(g0)$ for the nonlinear term g in (1.3) to prepare the results concerning the Cerami condition.

$(g0)$ g is a Carathéodory function on $\Omega \times \mathbb{R}$ such that $|g(x,t)| \le C(1+|t|^{p-1})$ for every $t \in \mathbb{R}$, a.e. $x \in \Omega$ and $g(x,t) = o(|t|^{p-1})$ as $|t| \to \infty$ uniformly in a.e. $x \in \Omega$, where C is a positive constant.

For a, $b \in \mathbb{R}$ and a nonlinear term g satisfying $(g0)$, we define two C^1 functionals on X as follows:

$$I^+_{(a,0)}(u) = \|u\|^p - a\|u_+\|_p^p - p\int_\Omega G_+(x,u)\,dx, \tag{2.1}$$

$$I^-_{(0,b)}(u) = \|u\|^p - b\|u_-\|_p^p - p\int_\Omega G_-(x,u)\,dx, \tag{2.2}$$

where

$$g_\pm(x,u) := \begin{cases} g(x,u) & \text{if } \pm u > 0, \\ 0 & \text{if } \pm u \le 0, \end{cases} \qquad G_\pm(x,u) := \int_0^u g_\pm(x,s)\,ds\ .$$

Then, the following result has been obtained concerning the Cerami condition or the Palais-Smale condition on the above two functionals.

Lemma 5 ([16, Lemma 16]) *Let g satisfy $(g0)$. Then the following assertions hold:*

(i) *if $a \ne \lambda_1$, then $I^+_{(a,0)}$ satisfies the Palais–Smale condition;*

(ii) *if* $b \neq \lambda_1$, *then* $I^-_{(0,b)}$ *satisfies the Palais–Smale condition;*

(iii) *if* g *satisfies* $(G{+}{+})$ *or* $(G{+}{-})$ (*resp.* $(G{-}{+})$ *or* $(G{-}{-})$), *then* $I^+_{(a,0)}$ (*resp.* $I^-_{(0,b)}$) *satisfies the Cerami condition for every* $a, b \in \mathbb{R}$.

2.2. The boundedness of a Cerami sequence

Under condition $(g0)$, we define C^1 functional $I_{(a,b)}$ on X by

$$I_{(a,b)}(u) = \int_\Omega |\nabla u|^p \, dx - a\int_\Omega u_+^p \, dx - b\int_\Omega u_-^p \, dx - p\int_\Omega G(x,u)\, dx \tag{2.3}$$

for a and $b \in \mathbb{R}$. Here, we recall the following results to prove the boundedness of a Cerami sequence.

Lemma 6 ([16, Lemma 13]) *We assume that* g *satisfies* $(g0)$. *Let* $I_{(a,b)}$ *be the functional defined by* (2.3) *for* a, $b \in \mathbb{R}$ *and suppose that* $\{u_n\} \subset X$ *satisfy*

$$\|u_n\| \to \infty \quad \text{and} \quad \|I'_{(a,b)}(u_n)\|_{X^*}/\|u_n\|^{p-1} \to 0 \qquad \text{as } n \to \infty.$$

Then, $\{u_n/\|u_n\|\}$ *has a subsequence converging to some* $v_0 \in X$ *which is a non-trivial solution of*

$$-\Delta_p u = a u_+^{p-1} - b u_-^{p-1} \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Using above result, we can prove the following lemma (see [16, Lemma 19] for the proof).

Lemma 7 ([16, Lemma 19]) *Assume that* g *satisfies* $(g0)$ *and* $(G{-}\,{-})$ (*resp.* $(G{+}{-})$). *Moreover, let* $\{u_n\} \subset X$ *satisfy*

$$\lim_{n\to\infty} \|u_n\| \, \| \left(I^-_{(0,\lambda_1 - 1/n)} \right)'(u_n)\|_{X^*} = 0 \quad \textit{and} \quad \sup_n I^-_{(0,\lambda_1-1/n)}(u_n) < +\infty,$$

$$\left(\textit{resp.} \ \lim_{n\to\infty} \|u_n\| \, \| \left(I^+_{(\lambda_1 - 1/n,0)} \right)'(u_n)\|_{X^*} = 0 \quad \textit{and} \quad \sup_n I^+_{(\lambda_1-1/n,0)}(u_n) < +\infty \right),$$

where $I^-_{(0,\lambda_1-1/n)}$ *and* $I^+_{(\lambda_1-1/n,0)}$ *are functionals defined by* (2.2) *and* (2.1) *with the nonlinear term* g, *respectively. Then,* $\{u_n\}$ *is bounded in* X.

The following lemma can be shown by a similar argument as in the proof of Lemma 7. Here, we give a sketch of the proof for readers' convenience.

Lemma 8 *Assume that g satisfies $(g0)$ and $(G{+}{+})$ (resp. $(G{-}{+})$). Moreover, let $\{u_n\} \subset X$ satisfy*

$$\lim_{n\to\infty} \|u_n\| \, \| \left(I^+_{(\lambda_1+1/n,0)} \right)' (u_n)\|_{X^*} = 0 \quad \text{and} \quad \inf_n I^+_{(\lambda_1+1/n,0)}(u_n) > -\infty,$$

$$\left(\text{resp.} \ \lim_{n\to\infty} \|u_n\| \, \| \left(I^-_{(0,\lambda_1+1/n)} \right)' (u_n)\|_{X^*} = 0 \quad \text{and} \quad \inf_n I^-_{(0,\lambda_1+1/n)}(u_n) > -\infty \right),$$

where $I^-_{(0,\lambda_1+1/n)}$ and $I^+_{(\lambda_1+1/n,0)}$ are functionals defined by (2.2) and (2.1) with the nonlinear term g, respectively. Then, $\{u_n\}$ is bounded in X.

Proof. We prove only the case where g satisfies $(g0)$ and $(G{+}{+})$ because another case is shown by a similar argument. Throughout this proof, we write $I^+_n = I^+_{(\lambda_1+1/n,0)}$ for $n \in \mathbb{N}$ to simplify the notation.

We prove the boundedness of $\{u_n\}$ by contradiction. Thus, supposing that $\{u_n\}$ is not bounded, by taking a subsequence, we may assume that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $v_n = u_n/\|u_n\|$, we may suppose that there exists a $v \in X$ such that

$$v_n \rightharpoonup v \quad \text{in } X \quad \text{and hence} \quad v_n \to v \quad \text{in } L^p$$

and $v_n(x) \to v(x)$ for a.e. $x \in \Omega$ as $n \to \infty$.

Since g_+ also satisfies $(g0)$ and

$$\| \left(I^+_{(\lambda_1,0)} \right)' (u_n) \|_{X^*} \le \|(I^+_n)'(u_n)\|_{X^*} + \frac{p}{\lambda_1 n}\|u_{n+}\|^{p-1}$$

holds, Lemma 6 implies that v_n strongly converges to v being a non-trivial solution of $-\Delta_p u = \lambda_1 u_+^{p-1}$ in Ω, $u = 0$ on $\partial\Omega$. This yields that $v = \varphi_1/\|\varphi_1\|$ because λ_1 is simple. Hence $u_n(x) \to +\infty$ for a.e. $x \in \Omega$.

Now let us note the inequality

$$\begin{aligned} o(1) - \inf_m I^+_m(u_m) &= \frac{1}{p}\langle (I^+_n)'(u_n), u_n\rangle - \inf_m I^+_m(u_m) \\ &\ge \frac{1}{p}\langle (I^+_n)'(u_n), u_n\rangle - I^+_n(u_n) \\ &= \int_\Omega pG_+(x,u_n) - g_+(x,u_n)u_n \, dx. \end{aligned} \tag{2.4}$$

On the other hand, by $(g0)$ and $(G{+}{+})$, we have

$$\operatorname{ess.inf} \{ pG_+(x,t) - g_+(x,t)t \, ; \, x \in \Omega, t \in \mathbb{R}\} > -\infty$$

and hence by $(G{+}{+})$ and $u_n(x) \to +\infty$ for a.e. $x \in \Omega$,

$$\liminf_{n\to\infty} \int_\Omega pG_+(x,u_n) - g_+(x,u_n)u_n \, dx = +\infty$$

by Fatou's lemma. This gives a contradiction to (2.4). ∎

2.3. Some key results

In this subsection, we prepare several results for the proofs of Theorems 1 and 2. At first, we state the following result concerning the mountain pass argument.

Lemma 9 *Let f satisfy (F) and assume that $a_0 = \lambda_1$ and (G_0+-) hold. Then, there exists a positive constant δ_0 satisfying*

$$\inf_{\|u\|=\delta_0} I^+(u) > 0,$$

where I^+ is the functional defined in section 1.3.

Proof. From (F) and (G_0+-), there exist $C_1 > 0$, $C_2 > 0$ and $p < q < r \leq p^*$ such that

$$G_0(x,u) \leq -C_1 u^q + C_2 u^r \quad \text{for every } u \geq 0, \text{ a.e. } x \in \Omega.$$

Therefore,

$$I^+(u) \geq \|u_-\|^p + \|u_+\|^p - \lambda_1 \|u_+\|_p^p + pC_1\|u_+\|_q^q - pC_2\|u_+\|_r^r \tag{2.5}$$

holds for every $u \in X$. In addition, we can get positive constants C_3 and C_4 satisfying

$$\|u\|_p \leq C_3\|u\|_q \quad \text{and} \quad \|u\|_r \leq C_4\|u\| \quad \text{for every } u \in X \tag{2.6}$$

by Höder's inequality and the continuity of the inclusion by X into $L^r(\Omega)$, respectively.

For every $u \in X$ with $\lambda_2\|u_+\|_p^p \leq \|u_+\|^p$ (where λ_2 is the second eigenvalue of $-\Delta_p$), we can get the following inequality

$$I^+(u) \geq \|u_-\|^p + \|u_+\|^p \left(1 - \lambda_1/\lambda_2 - pC_2C_4^r\|u_+\|^{r-p}\right)$$

by (2.5) and (2.6). Because of $\lambda_2 > \lambda_1$ and $p < r$, there exist positive constants δ_1 and C_5 such that

$$I^+(u) \geq \|u_-\|^p + C_5\|u_+\|^p \geq \min\{1, C_5\}\|u\|^p \tag{2.7}$$

for every $u \in X$ provided $\lambda_2\|u_+\|_p^p \leq \|u_+\|^p \leq \delta_1^p$.

Next, let $u \in X$ satisfy $\lambda_2\|u_+\|_p^p > \|u_+\|^p$. Then, noting the inequality

$$\|u_+\|_q^q \geq (\|u_+\|_p/C_3)^q > (\|u_+\|/(C_3\lambda_2^{1/p}))^q,$$

we obtain

$$I^+(u) \geq \|u_-\|^p + \|u_+\|^q \left(\frac{pC_1}{C_3^q \lambda_2^{q/p}} - pC_2C_4^r\|u_+\|^{r-q} \right)$$

by (2.5), (2.6) and $\|u_+\|^p \geq \lambda_1\|u_+\|_p^p$, and hence there exist $\delta_2 \in (0, 1]$ and $C_6 > 0$ such that

$$I^+(u) \geq \|u_-\|^p + C_6\|u_+\|^q \geq \min\{1, C_6\}\|u\|^q \tag{2.8}$$

for every $u \in X$ provided $\|u_+\| \leq \delta_2$ and $\lambda_2\|u_+\|_p^p > \|u_+\|^p$.

Put $\delta_0 = \min\{\delta_1, \delta_2\} > 0$. Then, the inequalities (2.7) and (2.8) imply

$$I^+(u) \geq \min\{1, C_5, C_6\}\|u\|^q = \min\{1, C_5, C_6\}\delta_0^q > 0$$

for every $u \in X$ with $\|u\| = \delta_0$. ∎

Because the following lemma concerning I^- defined in section 1.3 can be shown by a similar argument as for Lemma 9, we omit the proof here.

Lemma 10 *Let f satisfy (F) and we assume that $b_0 = \lambda_1$ and $(G_0{-}{-})$ hold. Then, there exists a positive constant δ_0 satisfying*

$$\inf_{\|u\|=\delta_0} I^-(u) > 0.$$

A similar result to the following proposition can be found as in [16, Proposition 18]. Here, we sketch the proof for readers' convenience.

Proposition 11 *Assume that f satisfies (F) with $a = \lambda_1$* (resp. $b = \lambda_1$) *and* $(G{+}{-})$ (resp. $(G{-}{-})$)*. Then, I^+* (resp. I^-) *has a global minimium.*

Proof. At first, we consider I^+. Let us set

$$I_n^+(u) = I^+_{(\lambda_1 - 1/n, 0)}(u) = I^+(u) + \frac{1}{n}\|u_+\|_p^p$$

for $u \in X$ and $n \in \mathbb{N}$ to simplify the notation.

For each $n \in \mathbb{N}$, it is easy to see that I_n^+ is bounded from below on X since $\int_\Omega G_+(x, u)\,dx = o(\|u_+\|_p^p)$ as $\|u_+\|_p^p \to \infty$ and $\|u\|^p \geq \lambda_1\|u\|_p^p$ for every $u \in X$. Moreover, let us note that I_n^+ satisfies the Palais–Smale condition for every $n \in \mathbb{N}$ by Lemma 5. Therefore, by a standard argument ([13, Theorem 4.2]) and by the Palais–Smale condition, for every $n \in \mathbb{N}$, there exists a $u_n \in X$ such that

$$\|(I_n^+)'(u_n)\|_{X^*} = 0 \quad \text{and} \quad I_n^+(u_n) = \inf_X I_n^+ \leq I_n^+(0) = 0.$$

Since g satisfies $(G+-)$, by Lemma 7, $\{u_n\}$ is a bounded sequence in X, and hence we may assume that there exists a $u_0 \in X$ such that

$$u_n \rightharpoonup u_0 \quad \text{in } X \quad \text{and} \quad u_n \to u_0 \quad \text{in } L^p$$

by taking a subsequence. Furthermore, for every $w \in X$ and $n \in \mathbb{N}$,

$$I^+(u_n) \le I_n^+(u_n) \le I_n^+(w) = I^+(w) + \frac{1}{n}\|w_+\|_p^p$$

holds (where we use the fact that u_n is a global minimizer of I_n^+ in the second inequality). By taking the limit inferior with respect to n in the above inequality, $I^+(u_0) \le I^+(w)$ holds for every $w \in X$ since I^+ is weakly sequentially lower semi-continuous. This shows that u_0 is a global minimum point of I^+.

Next, we consider I^-. By using $I^-_{(0,\lambda_1-1/n)}$ (see (2.2) for the definition) instead of $I^+_{(\lambda_1-1/n,0)}$, we can obtain a bounded sequence $\{u_n\}$ such that u_n is a global minimum point of $I^-_{(0,\lambda_1-1/n)}$ for each n. Because Lemma 7 gives the boundedness of $\{u_n\}$, we may assume that $\{u_n\}$ weakly converges to some $u_0 \in X$, by choosing a subsequence. Then, by the same argument as that for I^+, we can prove that u_0 is a global minimizer of I^-. ∎

§3. Proofs of Theorems

3.1. Proof of Theorem 1

Now, we start to prove Theorem 1.

Proof of Theorem 1. Case (i) $a = \lambda_1 < a_0$ and $(G+-)$ hold: In this case, we note that I^+ has a global minimum point $u_0 \in X$ by Proposition 11. So, we shall prove that $\inf_X I^+$ is negative to obtain $u_0 \neq 0$.

From (F), for any ε and r satisfying $0 < \varepsilon < (a_0 - \lambda_1)/p$ and $r > p$, there exists a $C > 0$ such that

$$G_0(x,u) \ge -\varepsilon|u|^p - C|u|^r \quad \text{for every } u \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

Thus, we have for $t > 0$

$$\begin{aligned} I^+(t\varphi_1) &\le t^p \left(\|\varphi_1\|^p - a_0\|\varphi_1\|_p^p + \varepsilon p\|\varphi_1\|_p^p + pCt^{r-p}\|\varphi_1\|_r^r \right) \\ &= t^p \left(\lambda_1 - a_0 + \varepsilon p + pCt^{r-p}\|\varphi_1\|_r^r \right). \end{aligned}$$

Because $\lambda_1 - a_0 + \varepsilon p < 0$ and $r > p$, this inequality shows that $I^+(t\varphi_1) < 0$ for sufficiently small $t > 0$, and hence $I^+(u_0) = \inf_X I^+ < 0$. Therefore, (P) has a positive solution (see Remark 3).

Case(ii) $a = \lambda_1 > a_0$ and $(G{+}{+})$ hold: In this case, by applying the mountain pass theorem to

$$I^+_{-n}(u) := I^+(u) - \frac{1}{n}\|u_+\|_p^p = I^+_{(\lambda_1+1/n,0)}(u) \quad \text{for } u \in X$$

(see (2.1) for the definition of $I^+_{(\lambda_1+1/n,0)}$ with g), we shall construct a Cerami sequence for I^+.

Since $\int_\Omega G_{0+}(x,u)\,dx = o(\|u_+\|^p)$ as $\|u_+\| \to 0$, we have $I^+(u) \geq \|u_-\|^p + (1 - a_0/\lambda_1)\|u_+\|^p - o(\|u_+\|^p)$ as $\|u_+\| \to 0$. Thus, there exists a positive constant δ_0 satisfying

$$\alpha := \inf\{I^+(u)\,;\, \|u\| = \delta_0\,\} > 0$$

since $a_0 < \lambda_1$.

On the other hand, noting that for each $n \in \mathbb{N}$

$$I^+_{-n}(t\varphi_1) = \int_\Omega G(x,t\varphi_1)\,dx - \frac{t^p}{n} = o(t^p) - \frac{t^p}{n} \quad \text{as } t \to +\infty,$$

we obtain a $T_n > \delta_0/\|\varphi_1\|$ such that $I^+_{-n}(T_n\varphi_1) < 0$. Define

$$\Gamma_n := \{\,\gamma \in C([0,1],X)\,;\, \gamma(0) = 0,\ \gamma(1) = T_n\varphi_1\,\}$$

and

$$c_n := \inf_{\gamma\in\Gamma_n}\max_{t\in[0,1]} I^+_{-n}(\gamma(t))$$

for $n \in \mathbb{N}$. Let us note that $\delta_0 < \|T_n\varphi_1\|$ and

$$\inf\{I^+_{-n}(u)\,;\, \|u\| = \delta_0\} \geq \inf\{I^+(u)\,;\, \|u\| = \delta_0\,\} - \frac{\delta_0^p}{n\lambda_1} = \alpha - \frac{\delta_0^p}{n\lambda_1},$$

and so $\inf\{I^+_{-n}(u)\,;\, \|u\| = \delta_0\} > 0$ for $n > \delta_0^p/(\alpha\lambda_1)$. Hence, by the mountain pass theorem, for each $n > \delta_0^p/(\alpha\lambda_1)$, we have that c_n is a critical value of I^+_{-n} since I^+_{-n} satisfies the Palais–Smale condition by Lemma 5 (note $\lambda_1 + 1/n \neq \lambda_1$). Therefore, there exists a $u_n \in X$ such that

$$(I^+_{-n})'(u_n) = 0 \quad \text{and} \quad I^+_{-n}(u_n) = c_n \geq \inf\{I^+_{-n}(u)\,;\, \|u\| = \delta_0\} \geq \alpha - \frac{\delta_0^p}{n\lambda_1}.$$

Because $\{u_n\}$ is bounded in X by Lemma 8 (note $I^+_{-n} = I^+_{(\lambda_1+1/n,0)}$), we may assume that there exists a $u_0 \in X$ such that u_n weakly converges to u_0 in X by taking a subsequence. Also, by choosing a subsequence again, we may suppose that $\{c_n\}$ is a convergent sequence since $c_n \in [0, I(u_n)]$ and I is bounded on any bounded subsets of X.

Furthermore, the following inequality

$$\|(I^+)'(u_n)\|_{X^*} = \|(I^+)'(u_n) - (I^+_{-n})'(u_n)\|_{X^*} \le \frac{p}{n\lambda_1}\|u_{n_+}\|^{p-1}$$

shows $\|(I^+)'(u_n)\|_{X^*} \to 0$ as $n \to \infty$. Thus, $\{u_n\}$ is a bounded Palais–Smale sequence of I^+, that is to say that $\{u_n\}$ is a Cerami sequence of I^+. Since I^+ satisfies the Cerami condition by Lemma 5, u_n strongly converges to a critical point u_0 of I^+.

In addition, the following inequality

$$I^+(u_n) = I^+_{-n}(u_n) + \frac{1}{n}\|u_{n+}\|_p^p \ge c_n \ge \alpha - \frac{\delta_0^p}{n\lambda_1}$$

implies $I^+(u_0) \ge \lim_{n\to\infty} c_n \ge \alpha > 0$, and hence u_0 is a non-trivial critical point of I^+.

Case(iii) $a < \lambda_1 = a_0$ and $(G_0{+}{+})$ hold: From (F), we have $\int_\Omega G_+(x,u)\,dx = o(\|u_+\|_p^p)$ as $\|u_+\|_p^p \to \infty$. Hence, the following inequality

$$\begin{aligned} I^+(u) &= \|u\|^p - a\|u_+\|_p^p - p\int_\Omega G_+(x,u)\,dx \\ &\ge \|u_-\|^p + (1 - \frac{a}{\lambda_1})\|u_+\|^p - o(\|u_+\|_p^p) \quad \text{as } \|u_+\|_p^p \to \infty \end{aligned}$$

and $a < \lambda_1$ show that I^+ is coercive and bounded from below on X. Moreover, it is easy to see that I^+ is weakly lower semi-continuous. It follows from the standard argument (*cf.* [13, Theorem 1.1]) that I^+ has a global minimum point.

On the other hand, for $t > 0$ such that $\|t\varphi_1\|_\infty \le \delta$ where δ is a positive constant described in $(G_0{+}{+})$, we obtain

$$I^+(t\varphi_1) = -p\int_\Omega G_0(x, t\varphi_1)\,dx < 0,$$

and hence $\inf_X I^+ < 0$. Therefore, I^+ has a non-trivial critical point u_0 satisfying $I^+(u_0) = \min_X I^+ < 0$.

Case(iv) $a > \lambda_1 = a_0$ and $(G_0{+}{-})$ hold: It follows from Lemma 9 that there exists a $\delta_0 > 0$ satisfying $\inf\{I^+(u)\,;\, \|u\| = \delta_0\} > 0$. On the other hand, we have for $t > 0$

$$I^+(t\varphi_1) = (\lambda_1 - a)t^p\|\varphi_1\|_p^p - o(t^p) \to -\infty \quad \text{as } t \to +\infty$$

by $\lambda_1 - a < 0$ and $\int_\Omega G_+(x, t\varphi_1)\,dx = o(t^p)$ as $t \to +\infty$. Thus, we can choose a positive constant T such that $T > \delta_0/\|\varphi_1\|$ and $I^+(T\varphi_1) < 0$. So, we define

$$\Gamma := \{\, \gamma \in C([0,1], X)\,;\, \gamma(0) = 0,\ \gamma(1) = T\varphi_1 \,\}$$

and

$$c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I^+(\gamma(t)).$$

Then, by mountain pass theorem, c is a critical value of I^+ with

$$c \geq \inf\{I^+(u)\,;\, \|u\| = \delta_0\,\} > 0$$

because $I^+(= I^+_{(a,0)})$ satisfies the Palais–Smale condition by Lemma 5. So, I^+ has a non-trivial critical point.

Case(v) $a = a_0 = \lambda_1$, $(G{+}{-})$ and $(G_0{+}{+})$ hold: In this case, we note that I^+ has a global minimum point by Proposition 11. Hence, we shall show that the minimum value of I^+ is negative.

Let δ be a positive constant described in $(G_0{+}{+})$. For $t > 0$ with $\|t\varphi_1\|_\infty \leq \delta$, we get $I^+(t\varphi_1) = -p\int_\Omega G_0(x, t\varphi_1)\,dx < 0$, which implies that $\inf_X I^+ < 0$ holds, and so I^+ has a non-trivial critical point.

Case(vi) $a = a_0 = \lambda_1$, $(G{+}{+})$ and $(G_0{+}{-})$ hold: Recall the definition of the approximate functional I^+_{-n} setting in case (ii) as follows:

$$I^+_{-n}(u) := I^+(u) - \frac{1}{n}\|u_+\|_p^p = I^+_{(\lambda_1+1/n,0)}(u) \quad \text{for } u \in X$$

Let δ_0 be a positive constant obtained by Lemma 9, that is, δ_0 satisfies

$$\alpha := \inf\{I^+(u)\,;\, \|u\| = \delta_0\,\} > 0.$$

By the same argument as in case (ii), we can obtain a $u_n \in X$ for each $n > \delta_0^p/(\alpha\lambda_1)$ such that

$$(I^+_{-n})'(u_n) = 0 \quad \text{and} \quad I^+_{-n}(u_n) \geq \inf\{I^+_{-n}(u)\,;\, \|u\| = \delta_0\} \geq \alpha - \frac{\delta_0^p}{n\lambda_1}. \tag{3.1}$$

Furthermore, it can be shown that there exists a subsequence of $\{u_n\}$ (we write this subsequence again by $\{u_n\}$) that is a Cerami sequence at some level $c \in \mathbb{R}$ by the same argument as in case (ii) by Lemma 8. Since I^+ satisfies the Cerami condition by Lemma 5, $\{u_n\}$ has a subsequence strongly converging to some critical point u_0 of I^+. By taking a limit with respect to n in (3.1), we have $I^+(u_0) \geq \alpha > 0$, and hence u_0 is a non-trivial critical point of I^+. ∎

3.2. Proof of Theorem 2

Next, we start to prove Theorem 2 which can be shown by a similar argument to Theorem 1. We give only a sketch of the proof.

Proof of Theorem 2. Case(i) $b = \lambda_1 < b_0$ and $(G{-}{-})$ hold: In this case, it follows from Proposition 11 that I^- has a global minimizer. On the other hand, because we have for $t > 0$

$$I^-(-t\varphi_1) = t^p(\lambda_1 - b_0) - p\int_\Omega G_0(x, -t\varphi_1)\,dx$$

and $\int_\Omega G_0(x, -t\varphi_1)\,dx = o(t^p)$ as $t \to +0$ by (F), $\min_X I^- < 0$ holds (note $\lambda_1 < b_0$). Hence I^- has a non-trivial critical point corresponding to a negative solution of (P) (see Remark 3).

Case(ii) $b = \lambda_1 > b_0$ and $(G{-}{+})$ hold: We shall construct a bounded Palais–Smale sequence for I^- by using the approximate functional I_n^- defined as follows:

$$I_n^-(u) := I^-(u) - \frac{1}{n}\|u_-\|_p^p = I^-_{(0,\lambda_1+1/n)}(u) \quad \text{for } u \in X,\ n \in \mathbb{N}$$

(see (2.2) for the definition of $I^-_{(0,\lambda_1+1/n)}$ with g).

From $\int_\Omega G_{0-}(x,u)\,dx = o(\|u_-\|^p)$ as $\|u_-\| \to 0$ and $b_0 < \lambda_1$, we can obtain a positive constant δ_0 satisfying $\alpha := \{I^-(u)\,;\, \|u\| = \delta_0\,\} > 0$. Then, by applying the mountain pass theorem to I_n^- (note that for each n, we have $I_n^-(-t\varphi_1) \to -\infty$ as $t \to \infty$), we can get a Palais–Smale sequence $\{u_n\}$ such that

$$I^-(u_n) = I_n^-(u_n) + \frac{1}{n}\|u_{n-}\|_p^p \geq \alpha - \frac{\delta_0^p}{n\lambda_1} \tag{3.2}$$

for $n > \delta_0^p/(\alpha\lambda_1)$ and we have that $\{u_n\}$ is bounded by Lemma 8 (see the proof of Theorem 1 (ii) for details). Since I^- satisfies the Cerami condition by Lemma 5, we may assume, by taking a subsequence, that u_n strongly converges to some critical point u_0 of I^-. In addition, by taking $n \to \infty$ in (3.2), we have $I^-(u_0) \geq \alpha > 0$ and so u_0 is a non-trivial critical point of I^-.

Case(iii) $b < \lambda_1 = b_0$ and $(G_0{-}{+})$ hold: From $b < \lambda_1$ and $\int_\Omega G_-(x,u)\,dx = o(\|u_-\|_p^p)$ as $\|u_-\|_p \to \infty$, we can easily show that I^- is coercive and bounded from below on X. Because I^- is weakly lower semi-continuous, I^- has a global minimum point (*cf.* [13, Theorem 1.1]). Let δ be a positive constant as in $(G_0{-}{+})$ and let $t > 0$ satisfy $\|t\varphi_1\|_\infty \leq \delta$. Then $I^-(-t\varphi_1) = -p\int_\Omega G_0(x,-t\varphi_1)\,dx < 0$ holds, whence the minimum value of I^- is negative, that is, the global minimum point of I^- is a non-trivial critical point.

Case(iv) $b > \lambda_1 = b_0$ and $(G_0{-}{-})$ hold: Let δ_0 be a positive constant obtained in Lemma 10, that is, δ_0 is a number such that $\inf\{I^-(u)\,;\, \|u\| = \delta_0\,\} > 0$ holds. Because it follows from $b > \lambda_1$ and (F) that $I^-(-t\varphi_1) \to -\infty$ as $t \to \infty$, there exists a $T > 0$ such that $T > \delta_0/\|\varphi_1\|$ and $I^-(-T\varphi_1) < 0$. Since I^- satisfies the Palais–Smale condition by Lemma 5, we can obtain a critical value c of I^- with $c \geq \inf\{I^-(u)\,;\,\|u\| = \delta_0\} > 0$ by the mountain pass theorem (see the proof of case (iv) in Theorem 1 for details).

Case(v) $b = b_0 = \lambda_1$, $(G--)$ and (G_0-+) hold: In this case, we already get a global minimum point of I^- by Proposition 11. Furthermore, if we take a $t > 0$ satisfying $\|t\varphi_1\|_\infty \leq \delta$ where δ is a positive constant described in (G_0-+), then we have $I^-(-t\varphi_1) = -p\int_\Omega G_0(x, -t\varphi_1)\,dx < 0$. Hence, the minimum value of I^- is negative, and so I^- has a non-trivial critical point.

Case(vi) $b = b_0 = \lambda_1$, $(G-+)$ and (G_0--) hold: Let δ_0 be a constant as in Lemma 10, that is, $\alpha := \inf\{I^-(u)\,;\,\|u\| = \delta_0\} > 0$. Recall the definition of the approximate function I_n^- introducing in case (ii) as follows:

$$I_n^-(u) := I^-(u) - \frac{1}{n}\|u_-\|_p^p = I^-_{(0,\lambda_1+1/n)}(u) \quad \text{for } u \in X,\ n \in \mathbb{N}.$$

Then, for each $n \in \mathbb{N}$ there exists a number $T_n > 0$ satisfying $\|T_n\varphi_1\| > \delta_0$ and $I_n^-(-T_n\varphi_1) < 0$ by (F). Therefore, we can construct a *bounded* Palais–Smale sequence $\{u_n\}$ for I^- such that

$$I^-(u_n) = I_n^-(u_n) + \frac{1}{n}\|u_{n-}\|_p^p \geq \alpha - \frac{\delta_0^p}{n\lambda_1} \quad \text{for } n > \frac{\delta_0^p}{\alpha\lambda_1} \tag{3.3}$$

by applying the mountain pass theorem to I_n^- and by Lemma 8 (see the proof of case (vi) or (ii) in Theorem 1 for details). Since I^- satisfies the Cerami condition by Lemma 5 and $\{I^-(u_n)\}$ is bounded by the boundedness of $\{u_n\}$, we may assume that u_n strongly converges to some critical point u_0 of I^- by choosing a subsequence. In addition, by taking $n \to \infty$ in (3.3), we have $I^-(u_0) \geq \alpha > 0$ and hence u_0 is a non-trivial critical point of I^-. ∎

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