SUT Journal of Mathematics
Vol. 33, No. 1 (1997), 115–119

A REMARK ON TIME-DEPENDENT GINZBURG-LANDAU EQUATIONS

Akihito Unai

(Received May 12, 1997)

Abstract. The purpose of this paper is to show the existence of unique global C^1-solutions to the time-dependent complex Ginzburg-Landau equation. We regard the equation as a genuinely nonlinear equation and simultaneously as a semilinear equation.

AMS 1991 *Mathematics Subject Classification.* Primary 47H20, Secondary 34G20.

Key words and phrases. Ginzburg-Landau equations, global C^1-solutions, nonlinear evolution equations, semilinear evolution equations.

1. Introduction

In this paper we consider the generalized complex Ginzburg-Landau equation (see e.g. Temam [**3**])

$$\frac{\partial u}{\partial t} - (\lambda + i\alpha)\Delta u + (\kappa + i\beta)|u|^{p-1}u - \gamma u = 0, \quad (x,t) \in \Omega \times \mathbf{R}_+, \tag{1}$$

where Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$, $i = \sqrt{-1}$ and u is a complex-valued unknown function. The equation will be supplemented with the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on} \quad \partial\Omega \times \mathbf{R}_+, \tag{2-a}$$

or the homogeneous Neumann boundary condition

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \partial\Omega \times \mathbf{R}_+, \tag{2-b}$$

where ν is the unit outward normal on $\partial\Omega$, and the initial value of u:

$$u(x,0) = u_0(x), \quad x \in \Omega. \tag{3}$$

Recently in [**4**], we proved that the initial-boundary value problem (1)$-$(3) has a unique strong global solution in $X := L^2(\Omega; \mathbf{C})$ under some conditions on the exponent $p > 1$ and the real parameters λ, κ, α, β, γ. We used the theory of nonlinear semi-groups in [**4**]. Therefore the solution $u(t)$ to the problem (1)$-$(3) exists globally, but there is no guarantee that $u(t)$ is differentiable for any $t \in [0, \infty)$ ($u(t)$ is differentiable for almost every $t \in [0, \infty)$).

The purpose of this paper is to show the differentiability for any $t \in [0, \infty)$ of the solution $u(t)$ to the problem (1)$-$(3) under additional restrictions on the exponent p and the dimension n.

The basic idea is that we regard (1) as a genuinely nonlinear equation and simultaneously as a semilinear equation (Lipschitz perturbations of linear equations). As mentioned above, we can obtain a unique *global* strong solution by the theory of nonlinear semi-groups. On the other hand, we can prove the existence of a unique local C^1-solution (*continuously differentiable* solution) by the theory of semilinear equations. Hence, by combining these two facts, namely the *global* existence by the theory of nonlinear semi-groups and the *continuous differentiability* by the theory of semilinear equations, we can prove the existence of a unique global C^1-solution to the problem (1)$-$(3).

2. The Main Result and Proof

For the abstract setting we define three operators A_1, B, A in the complex Hilbert space $X := L^2(\Omega; \mathbf{C})$ with norm and inner product denoted by $\|\cdot\|$ and $(\cdot, \cdot)$, respectively:

$$\begin{aligned}
D(A_1) &:= H_0^1(\Omega; \mathbf{C}) \cap H^2(\Omega; \mathbf{C}) \quad \text{(in case of (2-a))}, \\
D(A_1) &:= \left\{ u \in H^2(\Omega; \mathbf{C});\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\} \quad \text{(in case of (2-b))}, \\
A_1 u &:= -\Delta u \quad \text{for} \quad u \in D(A_1), \\
D(B) &:= \left\{ u \in X;\ |u|^{p-1} u \in X \right\} = L^{2p}(\Omega; \mathbf{C}), \\
Bu &:= |u|^{p-1} u \quad \text{for} \quad u \in D(B), \\
D(A) &:= D(A_1) \cap D(B), \\
Au &:= (\lambda + i\alpha) A_1 u + (\kappa + i\beta) Bu - \gamma u \quad \text{for } u \in D(A),
\end{aligned}$$

where $H^2(\Omega; \mathbf{C})$ and $H_0^1(\Omega; \mathbf{C})$ are the usual Sobolev Hilbert spaces.

The problem (1)$-$(3) is now equivalent to the following initial value problem for the abstract evolution equation

$$\frac{d}{dt} u(t) + Au(t) = 0, \qquad t \geq 0, \tag{4}$$
$$u(0) = u_0.$$

For convenience we quote the existence theorem from [**4**]. It is summarized as follows:

Theorem A ([4]). *Let* $\lambda > 0$, $\kappa > 0$, $p > 1$, $\dfrac{|\beta|}{\kappa} \leq \dfrac{2\sqrt{p}}{p-1}$, $\lambda\kappa + \alpha\beta > 0$. *Then for any* $T > 0$ *and* $u_0 \in D(A)$ *there exists a unique strong solution* $u(t)$ $(t \in [0, T])$ *to the problem* (4) *such that*
(a) $u(t) \in D(A)$ *for* $t \in [0, T]$.
(b) $u(t)$ *is Lipschitz continuous for* $t \in [0, T]$.
(c) $u(t)$ *is strongly differentiable for almost every* $t \in [0, T]$ *and satisfies* (4).
(d) $Au(t)$ *is weakly continuous for* $t \in [0, T]$ (see [**5,** Theorem 31.A]).

At the same time we can regard (1)$-$(3) as a semilinear evolution equation

$$\frac{d}{dt}u(t) + (\lambda + i\alpha)A_1 u(t) = -(\kappa + i\beta)Bu(t) + \gamma u(t), \quad t \geq 0, \tag{5}$$
$$u(0) = u_0.$$

Let $n = 1, 2, 3$. Then $H^2(\Omega; \mathbf{C})$ is embedded in $L^\infty(\Omega; \mathbf{C})$, and therefore $D(A_1) \subset D(B)$ (consequently, $D(A) = D(A_1) \cap D(B) = D(A_1)$). Since the function $f(s) = |s|^{p-1}s$ $\;(p \geq 3)$ is three times continuously differentiable, we can see that the operator B is locally Lipschitz continuous on $D(A_1)$ with graph norm $\|\cdot\|_{D(A_1)}$. Hence applying general theory of semilinear equations (see e.g. Ôtani [**1,** Theorem B] or Pazy [**2,** Remark after Theorem 6.1.7]), we have

Theorem B. *Let* $n = 1, 2, 3$. *Assume that* $\lambda > 0$ *and* $p \geq 3$. *Then for any* $u_0 \in D(A_1)$ *there exists* T_m $(0 < T_m \leq \infty)$ *such that the problem* (5) *has a unique* C^1*-solution* $u(\cdot) \in \mathbf{C}^1\big([0, T_m) : X\big) \cap \mathbf{C}\big([0, T_m) : D(A_1)\big) \cap \mathbf{C}\big([0, T_m) : D(A)\big)$. *Furthermore, if* $T_m < \infty$ *then* $\lim_{t \uparrow T_m} \big(\|u(t)\| + \|A_1 u(t)\|\big) = \infty$.

As a combination of Theorem A and Theorem B, our theorem is stated as follows:

Theorem. *Let* $n = 1, 2, 3$. *Assume that* $\lambda > 0$, $\kappa > 0$, $p \geq 3$, $\dfrac{|\beta|}{\kappa} \leq \dfrac{2\sqrt{p}}{p-1}$, *and* $\lambda\kappa + \alpha\beta > 0$. *Then for any* $u_0 \in D(A)$ *the problem* (4) *(or* (5)*) has a unique global* C^1*-solution* $u(t)$ *such that*

$$u(\cdot) \in C^1\big([0, \infty); X\big) \cap C\big([0, \infty); D(A)\big) \cap C\big([0, \infty); D(A_1)\big).$$

Proof. Under the assumption of our Theorem we can simultaneously apply Theorem A and Theorem B to the poblem (4) (or (5)). Hence it is easy to see that the solution obtained by Theorem A coincides with the one obtained by Theorem B in the common time interval $[0,\, T_m)$. Using the properties (a)-(d) (especially (d)) of the solution $u(t)$ $(0 \le t < \infty)$ in Theorem A, we shall prove that $T_m = \infty$. To this end, it suffices by Theorem B to show that if $T_m < \infty$, then

$$\sup_{0 \le t < T_m} \big(\|u(t)\| + \|A_1 u(t)\|\big) < \infty. \tag{6}$$

We know that $u(t)$ satisfies the following equation

$$\frac{d}{dt}u(t) - (\lambda + i\alpha)\Delta u(t) + (\kappa + i\beta)|u(t)|^{p-1}u(t) - \gamma u(t) = 0, \quad 0 \le t < T_m.$$

Dividing by $(\lambda + i\alpha)$ and taking inner product with $|u|^{p-1}u$, we have

$$\begin{aligned} \frac{1}{\lambda + i\alpha}\Big(\frac{d}{dt}u(t),\ |u(t)|^{p-1}u(t)\Big) &- \big(\Delta u(t),\ |u(t)|^{p-1}u(t)\big) \\ &+ \frac{\kappa + i\beta}{\lambda + i\alpha}\|u(t)\|_{L^{2p}}^{2p} - \frac{\gamma}{\lambda + i\alpha}\int_\Omega |u(t)|^{p+1}\, dx = 0. \end{aligned} \tag{7}$$

Integration by parts yields

$$\begin{aligned} -\mathrm{Re}\big(\Delta u(t),\ |u(t)|^{p-1}u(t)\big) &= \int_\Omega |u(t)|^{p-1}|\nabla u(t)|^2\, dx \\ &+ (p-1)\int_\Omega |u(t)|^{p-3}\sum_{j=1}^{n}\{\mathrm{Re}(\overline{u(t)}\cdot \frac{\partial}{\partial x_j}u(t))\}^2\, dx \ge 0, \end{aligned} \tag{8}$$

where $\mathrm{Re}(\cdot)$ and $\overline{u(t)}$ mean the real part of $(\cdot)$ and the complex conjugate of $u(t)$, respectively. In view of (7), (8) we obtain for any $\varepsilon > 0$

$$\begin{aligned} \frac{\lambda\kappa + \alpha\beta}{\lambda^2 + \alpha^2}\|u(t)\|_{L^{2p}}^{2p} &\le \left|\frac{1}{\lambda + i\alpha}\right| \int_\Omega \Big|\frac{d}{dt}u(t)\Big| \cdot |u(t)|^p dx + \frac{\lambda\gamma}{\lambda^2 + \alpha^2}\int_\Omega |u(t)|^{p+1} dx \\ &\le \frac{1}{\sqrt{\lambda^2 + \alpha^2}}\int_\Omega \left(\frac{1}{4\varepsilon}\Big|\frac{d}{dt}u(t)\Big|^2 + \varepsilon |u(t)|^{2p}\right) dx \\ &+ \frac{\lambda|\gamma|}{\lambda^2 + \alpha^2}\int_\Omega \left(\varepsilon|u(t)|^{2p} + \frac{1}{4\varepsilon}|u(t)|^2\right) dx. \end{aligned}$$

Thus we have

$$\begin{aligned} &\frac{1}{\lambda^2 + \alpha^2}\big\{(\lambda\kappa + \alpha\beta) - \varepsilon(\sqrt{\lambda^2 + \alpha^2} + \lambda|\gamma|)\big\}\|u(t)\|_{L^{2p}}^{2p} \\ &\le \frac{1}{4\varepsilon}\frac{1}{\sqrt{\lambda^2 + \alpha^2}}\Big\|\frac{d}{dt}u(t)\Big\|^2 + \frac{1}{4\varepsilon}\frac{\lambda|\gamma|}{\lambda^2 + \alpha^2}\|u(t)\|^2 \\ &= \frac{1}{4\varepsilon}\left(\frac{1}{\sqrt{\lambda^2 + \alpha^2}}\big\|Au(t)\big\|^2 + \frac{\lambda|\gamma|}{\lambda^2 + \alpha^2}\|u(t)\|^2\right) \end{aligned} \tag{9}$$

for $0 \leq t < T_m$. Choose $\varepsilon > 0$ small enough in such a way that the left hand side of (9) is positive. We know from Theorem A (d) that $\{\|Au(t)\|;\ t \in [0, T)\}$ is bounded for evry $T > 0$. In particular, it follows that

$$\sup_{0 \leq t < T_m} \|Au(t)\| < \infty. \tag{10}$$

Moreover, it is not difficult to see that

$$\sup_{0 \leq t < T_m} \|u(t)\| \leq e^{\gamma T_m} \|u_0\| < \infty. \tag{11}$$

From (9), (10), (11) we have

$$\sup_{0 \leq t < T_m} \|u(t)\|_{L^{2p}} < \infty.$$

Finally, in view of the definition of the operator A, we obtain (6). This completes the proof of Theorem. □

Acknowledgement

The author would like to thank the referee for his helpful comments and suggestions.

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Akihito Unai
Department of Applied Mathematics, Science University of Tokyo
1-3 Kagurazaka, Shinjuku-ku, Tokyo 162, Japan