A CELLULAR SIMPLEX WITH PRESCRIBED NUMBERS OF POINTS IN REGIONS DETERMINED BY ITS FACETS

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Abstract. Let $P$ be a finite set of points in the 3-dimensional Euclidean space $\mathbb{R}^3$ in general position. For $x_0, x_1, x_2, x_3 \in P$, let $H^+(x_0; x_1, x_2, x_3)$ (resp. $H^-(x_0, x_1, x_2, x_3)$) denote the open half space containing $x_0$ (resp. not containing $x_0$) and bounded by the plane containing $x_1, x_2, x_3$. Further let

$$P(x_0; x_1, x_2, x_3) := P \cap H^+(x_1; x_0, x_2, x_3) \cap H^+(x_2; x_0, x_1, x_3) \cap H^+(x_3; x_0, x_1, x_3).$$

In this paper, we show the following statement: if $|P| \geq 4$, and if $k_1, k_2, k_3, k_4$ are integers with $k_1 + k_2 + k_3 + k_4 = |P| - 4, 0 \leq k_1, k_2, k_3, k_4 \leq \frac{|P|-2}{3}$ and $k_1 + k_2 \leq \frac{|P|-2}{3}$, then for any $p_1, p_2 \in P (p_1 \neq p_2)$, there exist $q_1, q_2 \in P$ such that the convex hull of $\{p_1, p_2, q_1, q_2\}$ is a 3-simplex (tetrahedron) containing no point of $P$ in its interior and such that

$$|P(p_1; p_2, q_1, q_2)| \leq k_1 \leq P \cap H^-(p_1; p_2, q_1, q_2),$$
$$|P(p_1; q_1, q_2)| \leq k_2 \leq P \cap H^-(p_1, q_1, q_2),$$
$$|P(q_1; q_2, p_1, p_2)| \leq k_3 \leq P \cap H^-(q_1; q_2, p_1, p_2),$$
$$|P(q_1; q_1, p_1, p_2)| \leq k_4 \leq P \cap H^-(q_1; q_1, p_1, p_2).$$

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§1. Introduction.

For a subset $V$ of the $d$-dimensional Euclidean space $\mathbb{R}^d$, let $\text{conv}(V)$ denote the convex hull of $V$, and let $\text{aff}(V)$ denote the affine flat spanned by $V$. For $d + 1$ points $x_0, x_2, \cdots, x_d$ not lying in the same (affine) $(d - 1)$-flat in $\mathbb{R}^d$, 155
let $H^+(x_0; x_1, \ldots, x_d)$ (resp. $H^-(x_0; x_1, \ldots, x_d)$) denote the open half-space which is bounded by $\text{aff}\{x_1, \ldots, x_d\}$ and contains $x$ (resp. does not contain $x$). Now let $P$ be a fixed set of points in $\mathbb{R}^d$. We say that $P$ is in general position if no $d + 1$ points of $P$ lie in the same $(d - 1)$-flat. For $d + 1$ points $x_0, x_1, \ldots, x_d$ not lying in the same $(d - 1)$-flat, let

$$P(x_0; x_1, \ldots, x_d) := P \cap \bigcap_{1 \leq i \leq d} H^+(x_i; x_0, x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d).$$

If a subset $V$ of $\mathbb{R}^3$ contains no point of $P$ in its interior, $V$ is said to be vacuum. Further, following Kupitz[2], we call a polyhedron $D$ cellular if $D$ is vacuum and all vertices of $D$ are points of $P$. In this paper, we show the following theorem as a 3-dimensional version of Lemma 3 in [1]:

**Theorem 1.** Let $P$ be a finite set of points in $\mathbb{R}^3$ in general position. Suppose that $|P| \geq 4$, and let $k_1, k_2, k_3, k_4$ be integers such that $k_1 + k_2 + k_3 + k_4 = |P| - 4$, $0 \leq k_1, k_2, k_3, k_4 \leq \frac{|P|-2}{2}$ and $k_1 + k_2 \leq \frac{|P|-2}{2}$. Further let $p_1, p_2$ be specified points of $P$ with $p_1 \neq p_2$. Then there exist two points $q_1, q_2$ of $P$ such that $\text{conv}\{(p_1, p_2, q_1, q_2)\}$ is a cellular 3-simplex and the following inequalities hold:

1. $|P(p_1; p_2, q_1, q_2)| \leq k_1 \leq |P \cap H^-(p_1; p_2, q_1, q_2)|$,
2. $|P(p_2; p_1, q_1, q_2)| \leq k_2 \leq |P \cap H^-(p_2; p_1, q_1, q_2)|$,
3. $|P(q_1; q_2, p_1, p_2)| \leq k_3 \leq |P \cap H^-(q_1; q_2, p_1, p_2)|$,
4. $|P(q_2; q_1, p_1, p_2)| \leq k_4 \leq |P \cap H^-(q_2; q_1, p_1, p_2)|$.

§2. Proof of Theorem 1.

Let $P, k_1, k_2, k_3, k_4, p_1, p_2$ be as in Theorem 1. Let $\pi: \mathbb{R}^3 \to \mathbb{R}^2$ be the orthogonal projection in the direction of $\overrightarrow{p_1p_2}$. We use the following result in the plane case (a slight modification of Claim 1 in [1]):

**Proposition 1.** Let $P'$ be a finite set of points in $\mathbb{R}^2$, and let $r'_0$ be a specified point of $P'$. Suppose that $|P'| \geq 3$ and any line passing through $r'_0$ contains at most one point of $P'$ other than $r'_0$. Let $k'_1, k'_2, k'_3$ be integers satisfying $0 \leq k'_1, k'_2, k'_3 \leq \frac{|P'|-1}{2}$ and $k'_1 + k'_2 + k'_3 = |P'|-3$. Then there exist $x' \in \mathbb{R}^2 - P$ and $r'_1, r'_2 \in P' - \{r'_0\}$ such that

$$P' = \{r'_0, r'_1, r'_2\} \cup P'(r'_0', x', r'_1') \cup P'(r'_0; r'_1, r'_2) \cup P'(r'_0; r'_2, x')$$

and

$$|P'(r'_0; x', r'_1')| = k'_1, \quad |P'(r'_0; r'_1, r'_2)| = k'_2, \quad |P'(r'_0; r'_2, x')| = k'_3.$$
Proposition 1 is essentially the same as Claim 1 in [1], so we omit the proof. Since $k_1 + k_2 \leq \frac{|P|-2}{2}$, we can apply Proposition 1 to $\pi(P) = \{\pi(p) \mid p \in P\}$ with $r'_0 = \pi(p_1) = \pi(p_2)$ and $k'_1 = k_3$, $k'_2 = k_1 + k_2$, $k'_3 = k_4$. Let $x', r'_1, r'_2$ be as in the conclusion of the Proposition 1. We use the same technique as in the proof of Lemma 3 in [1]. Let $l_0$ be a line passing through $r'_0$ and $x'$. Take $s'_1, s'_2 \in P(r'_0; r'_1, r'_2) \cup \{r'_1, r'_2\}$ so that for $i = 1, 2$, $s'_i$ lies in the same side of $l_0$ as $r'_i$, and
\begin{align}
(2.5) \quad \text{the line segment } s'_1 s'_2 \text{ is an edge of conv}(P(r'_0; r'_1, r'_2) \cup \{r'_1, r'_2\}) \\
\quad \text{satisfying conv}\{\{r'_0, s'_1, s'_2\}\cap H^-(r'_0; s'_1, s'_2) = \emptyset.}
\end{align}

Now we return to $\mathbb{R}^3$. Let $x, r_i, s_i$ ($i = 1, 2$) be the points of $P$ such that $x, \pi(r_i) = r'_i, \pi(s_i) = s'_i$, respectively. Let
\begin{align}
K_1 &:= H^+(x; r_1, p_1, p_2) \cap H^+(r_1; x, p_1, p_2), \\
K_2 &:= H^+(r_1; r_2, p_1, p_2) \cap H^+(r_2; r_1, p_1, p_2), \\
K_3 &:= H^+(r_2; x, p_1, p_2) \cap H^+(x; r_2, p_1, p_2).
\end{align}

Then the conclusion of Proposition 1 implies that $K_i \cap K_j = \emptyset$ if $i \neq j$, and
\begin{align}
(2.6) \quad |P \cap K_1| = k'_1 &= k_3, \\
(2.7) \quad |P \cap K_2| = k'_2 &= k_1 + k_2, \\
(2.8) \quad |P \cap K_3| = k'_3 &= k_4.
\end{align}

Let $H_0 := \pi^{-1}(l_0)$ and let $S = (P \cap K_2) \cup \{r_1, r_2\}$. By (2.5),
\[
\Delta := H^+(r'_1; r'_0, r'_2) \cap H^+(r'_2; r'_0, r'_1) \cap H^+(r'_0; r'_1, r'_2)
\]
is vacuum. Since $K_2 \cap H^+(p_1; p_2, s_1, s_2) \cap H^+(p_2; p_1, s_1, s_2) \subset \pi^{-1}(\Delta)$, this implies that $S \cap H^+(p_1; p_2, s_1, s_2) \cap H^+(p_2; p_1, s_1, s_2) = \emptyset$. Thus by (2.7), $|S \cap H^+(p_2; p_1, s_1, s_2)| \leq k_1 + 2$ or $|S \cap H^+(p_1; p_2, s_1, s_2)| \leq k_2 + 2$ holds. By symmetry, we may assume
\begin{align}
(2.9) \quad |S \cap H^+(p_2; p_1, s_1, s_2)| \leq k_1 + 2.
\end{align}

For a plane $H$ and a point $x \notin H$, let $H^+(x)$ (resp. $\bar{H}^+(x)$) denote the open (resp. closed) half-space which is bounded by $H$ and contains $x$, and let $H^-(x)$ (resp. $\bar{H}^-(x)$) denote the open (resp. closed) half-space which is bounded by $H$ and does not contain $x$. Let $H_1$ be a plane containing $p_1$ such that
\begin{align}
(2.10) \quad |S \cap \bar{H}_1^+(p_2)| &= k_1 + 2, \\
(2.11) \quad S \cap \bar{H}_1^+(p_2) \cap H^+_0(r_i) \neq \emptyset \quad \text{for } i = 1, 2.
\end{align}
Note that by (2.9), there exists a plane satisfying (2.10) and (2.11). We choose \( H_3 \) so that the angle between \( H_0 \cap H_1 \cap K_2 \) and \( p_1p_2 \) is as small as possible. Take \( q_1, q_2 \) so that

\[
(2.12) \quad q_1 \in S \cap H_1^+(p_2) \cap H_0^+(r_1),
\]

\[
(2.13) \quad q_2 \in S \cap H_1^+(p_2) \cap H_0^-(r_2),
\]

and

\[
(2.14) \quad \triangle p_2q_1q_2 \text{ is a facet of } \text{conv}\left((S \cup \{p_2\}) \cap H_1^+(p_2)\right) \text{ satisfying}
\]

\[
\text{conv}\left(\{p_1, p_2, q_1, q_2\}\right) \cap H^-(p_1; p_2, q_1, q_2) = \emptyset.
\]

By (2.14), \( \text{conv}\left(\{p_1, p_2, q_1, q_2\}\right) \) is vacuum. We now proceed to verify the inequalities in the conclusion of Theorem 1. By (2.12) and (2.13),

\[
P(q_1; q_2, p_1, p_2) \subseteq P \cap K_1 \subseteq P \cap H^-(q_1; q_2, p_1, p_2)
\]

and

\[
P(q_2; q_1, p_1, p_2) \subseteq P \cap K_3 \subseteq P \cap H^-(q_2; q_1, p_1, p_2)
\]

hold, and hence (2.6), (2.8) imply (1.3), (1.4), respectively. Similarly by (2.14),

\[
P(p_1; p_2, q_1, q_2) \subseteq S \cap H_1^+(p_2) - \{q_1, q_2\} \subseteq P \cap H^-(p_1; p_2, q_1, q_2)
\]

holds, and hence (2.10) implies (1.1). Further, it also follows from the choice of \( q_1, q_2 \) that

\[
P(p_2; p_1, q_1, q_2) \subseteq S \cap H_1^-(p_2).
\]

Since

\[
|S \cap H_1^-(p_2)| = (k_1 + k_2 + 2) - (k_1 + 2) = k_2
\]

by (2.7) and (2.10), this immediately implies the first inequality in (1.2).

We are now left with the verification of the second inequality in (1.2). Suppose

\[
|P \cap H^-(p_2; p_1, q_1, q_2)| < k_2.
\]

Then clearly

\[
(2.15) \quad |S \cap H^-(p_2; p_1, q_1, q_2)| < k_2.
\]

On the other hand, by (2.7) and (2.9),

\[
(2.16) \quad |S \cap H^-(p_2; p_1, s_1, s_2)| \geq (k_1 + k_2 + 2) - (k_1 + 2) = k_2
\]

holds. Let \( y, z \) be the intersection points of the line passing through \( s_1, s_2 \) and \( \text{aff}(\{p_1, p_2, r_1\}), \text{aff}(\{p_1, p_2, r_2\}) \), respectively. Then (2.15) and (2.16) imply that \( S \cap H^-(p_2; p_1, s_1, s_2) \not\subseteq S \cap H^-(p_2; p_1, q_1, q_2) \), which implies that at least one of \( y, z \) belongs to \( H^+(p_2; p_1, q_1, q_2) \). We may assume

\[
(2.17) \quad y \in H^+(p_2; p_1, q_1, q_2)
\]

without loss of generality. We now show the existence of a plane containing \( p_0 \) which gives rise to a contradiction to the choice of \( H_1 \). Toward this end, we divide the situation into two cases according to the location of \( q_2 \).
Case 1 $q_2 = s_2$ or $q_2 \in H^+(p_2; p_1, s_1, s_2)$

In this case,

$$S \cap H^-(p_2; p_1, y, q_2) \supseteq S \cap H^-(p_2; p_1, s_1, s_2)$$

holds and hence by (2.16),

$$(2.18) \quad |S \cap H^-(p_2; p_1, y, q_2)| \geq |S \cap H^-(p_2; p_1, s_1, s_2)| \geq k_2.$$

Let $l_1$ be the line passing through $p_1, q_2$, and let $H$ be a (movable) plane containing $l_1$. If we gradually rotate $H$ with $l_1$ as the axis, the value of $|S \cap H^-(p_2)|$ changes by one at each moment when $H$ hits a point of $P$. Therefore by (2.15) and (2.18), there exists $H_2 \in l_1 \cup H^+(p_2, p_1, q_1, q_2) \cup H^+(p_2, p_1, y, q_2)$ such that $l_1 \in H_2$ and $|S \cap H_2^{-}(p_2)| = k_2$, or equivalently, $|S \cap H_2^{+}(p_2)| = k_1 + 2$. Now to get a contradiction, we let $K_2' := H^+(q_1; q_2, p_1, p_2) \cap H^+(q_2; q_1, p_1, p_2)$ (note that by (2.12) and (2.13), $H_0$ intersects with $K_2'$). Then by (2.17), it is easy to see that

$$H_0 \cap K_2' \cap H_2^+(p_2) \subset H_0 \cap K_2' \cap H^+(p_2; p_1, q_1, q_2)$$

$$\subseteq H_0 \cap K_2' \cap H_1^+(p_2),$$

which yields a contradiction to the minimality of the angle between $H_0 \cap H_1 \cap K_2$ and $\overline{p_1 p_2}$.

Case 2 $q_2 \in H^-(p_2; p_1, s_1, s_2)$

If (2.18) holds, a contradiction can be derived in the same way as in Case 2. Thus we may assume

$$(2.19) \quad |S \cap H^-(p_2; p_1, y, q_2)| < k_2.$$ 

Let $l_2$ be the line passing through $p_1, y$. Then again as in Case 1, (2.16) and (2.19) imply that we can find a plane $H_3 \in l_2 \cup H^+(p_2, p_1, y, q_2) \cup H^+(p_2, p_1, s_1, s_2)$ such that $l_2 \in H_3$ and $|S \cap H_3^+(p_2)| = k_1 + 2$ by considering the rotation of a plane containing $l_2$ with $l_2$ as the axis. Thus again it is easy to see that

$$H_0 \cap K_2' \cap H_3^+(p_2) \subset H_0 \cap K_2' \cap H^+(p_2; p_1, q_1, q_2)$$

$$\subseteq H_0 \cap K_2' \cap H_1^+(p_2),$$

which yields a contradiction. \qed
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References


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