ON RESIDUES OF DIFFERENTIAL FORMS
OVER A FIELD OF CHARACTERISTIC \( p \)

Takeo OHI

(Received June 20, 1995)

Abstract. Let \( K \) be a function field over a field \( k \) of characteristic \( p > 0 \) and let \( R \) be a discrete valuation ring of \( K/k \). E. Kunz showed that if \( \omega \) is a closed differential form and \( \nu_R(\omega) \geq -1 \), then \( \text{res}_R(\omega) \) does not depend on the choice of parameter \( t = \{t_1, t_2, \cdots, t_n\} \).

In this paper, we investigate \( \text{res}_R, t(\omega) \) in the case where \( \nu_R(\omega) \geq -p^m + 1 \) for \( \omega \in \mathbb{Z}_m \).

AMS 1991 Mathematics Subject Classification. Primary 13N05, 12H05.

Key words and phrases. Differential form, residue, Cartier operator, function field.

§0. Introduction

Let \( K \) be a function field of \( n \) variables over a field \( k \) of characteristic \( p > 0 \) and let \( R \) be a discrete valuation ring of \( K/k \) such that the residue field \( D \) of \( R \) has transcendence degree \( n - 1 \) over \( k \). Y. Suzuki [3] proved the following Theorem A and Corollary B.

Theorem A. If \( \omega \) is a differential form in \( \mathbb{Z}_m \Omega^r(K/k) \) such that \( \nu_R(\omega) \geq -p^{m-1}, \) then \( \text{res}_R(\omega) \) is uniquely determined up to addition by differentials in \( B_{m-1}\Omega^{r-1}(D/k) \).

Corollary B. \( \text{res}_R : \mathbb{Z}_\infty \Omega^r(K/k) \rightarrow \mathbb{Z}_\infty \Omega^{r-1}(D/k)/B_\infty \Omega^{r-1}(D/k) \) is well defined. (for the definition, see section 1).

His method of proof is the following: First he proved the commutativity of residue map and Cartier operator. Secondly he proved that if \( \omega \in \mathbb{Z}_m \Omega^r(K/k) \) and \( \nu_R(\omega) \geq -p^{m-1}, \) then \( \nu_{R(m)}(C_K^{(m-1)}(\omega)) \geq -1 \) and \( C_K^{(m-1)}(\omega) \) is a closed differential, where \( C_K^{(m-1)} \) is an iterated Cartier operator. From two results
above and a result of E. Kunz (Exercise (1) in §17 of [1]), he proved Theorem A and Corollary B.

On the other hand, our main results are the following:

**Theorem 2.** If \( \omega \) is a differential form in \( Z_m \Omega(K/k) \) such that \( \nu_R(\omega) \geq -p^m + 1 \), then \( \text{res}_R(\omega) \) is uniquely determined up to addition by differentials in \( B_m \Omega(D/k) \).

**Corollary.** \( \text{res}_R : Z_\infty \Omega(K/k) \rightarrow Z_\infty \Omega(D/k) / B_\infty \Omega(D/k) \) is well defined.

Our method of proof is quite different from Suzuki’s method and our Theorem 2 and Suzuki’s Theorem A are independent to each other, that is, Theorem A does not imply Theorem 2 and vice versa. But both Theorem 2 and Theorem A imply the same Corollary.

An advantage of our result is in the following fact. The number \(-p^m + 1\) in our Theorem 2 is the best possible (see Example in §2).

### §1. Preliminaries

Throughout this paper, \( K \) will denote a function field of \( n \) variables over a field \( k \) of characteristic \( p > 0 \) and \( R \) a discrete valuation ring of rank one of \( K/k \) such that the residue field \( D \) of \( R \) has transcendence degree \( n - 1 \) over \( k \). Furthermore we always assume that \( K \) and \( D \) are separable over \( k \).

We choose \( n \) elements \( t_1, t_2, ..., t_n \) in \( R \) such that \( t_1R \) is the maximal ideal of \( R \) and such that \( \overline{t_2}, ..., \overline{t_n} \) is a \( p \)-basis of \( D/k \), where \( \pi \) denotes the canonical image in \( D \) of \( a \in R \). We will call such a family \( \ell = \{ t_1, t_2, ..., t_n \} \) a parameter of \((K/k, R)\). We put \( K_i = kK^{p^i}, R_i = kR^{p^i} \) and \( \pi^{(i)} = \overline{t}^{p^i} = \{ t_1^{p^i}, t_2^{p^i}, ..., t_n^{p^i} \} \) (\( i = 0, 1, 2, \cdots \)).

Let \( A \) be a \( G \)-algebra, where \( G \) and \( A \) are commutative rings, and let \( (\Omega(A/G), d_{A/G}) \) be the universal differential algebra of \( A/G \). Then we know that \( \Omega(A/G) = \bigoplus \Omega^r(A/G) \), \( \Omega^r(A/G) = \bigwedge^r \Omega^1(A/G) \) and \( \Omega^1(A/G) \) is the module of Kähler differentials of \( A/G \) (c.f. §3 in [1]). If there is no confusion, we simply write \( d, \Omega, \Omega(D) \) and \( \Omega(R) \) instead of \( d_{A/G}, \Omega(K/k), \Omega(D/k) \) and \( \Omega(R/k) \), respectively.

**Lemma 1.** Let \( \ell = \{ t_1, t_2, ..., t_n \} \) be a parameter of \((K/k, R)\). Then \( \ell = \{ t_1, t_2, ..., t_n \} \) is a \( p \)-basis of \( R/k \).

**Proof.** From the following exact sequence of vector spaces over \( D \)
is a local ring with the residue field $\mathbb{S}$ and that $D$ generates $\Omega^1$ separability of $kD$. We have $\dim \Omega^1(R)$ over $R$. On the other hand, since $\Omega^1 = \Omega^1(R) \otimes_R K$ has dimension $n$ over $K$, $\{dt_1, dt_2, ..., dt_n\}$ must form a basis of $\Omega^1(R)$ over $R$.

We will show that $kR^p[t_1, t_2, ..., t_n] = R$. Let $S = kR^p[t_1, t_2, ..., t_n]$. Then $S$ is a local ring with the residue field $kD/\mathfrak{p} \cong kD/\mathfrak{p}^n$. The completion of $R$, denoted by $\hat{R}$, is the completion of $R$ with respect to an ideal $\mathfrak{p}$ in $R$. The ring $\hat{R}$ is a complete discrete valuation ring of rank one with the residue field $\mathbb{S}$ and the quotient field $\mathbb{K}$.

By using the conditions that both $K/k$ and $D/k$ are separable, we observe that $kR^p$ is a discrete valuation ring of rank one with the residue field $kD/\mathfrak{p}$ and that $\mathfrak{p} = \{t_1^p, t_2^p, ..., t_n^p\}$ is a parameter of $(kK/k, kR^p)$. In fact, we have $K^p \otimes_k k = kK^p$ since $K^p/k$ is separable, and hence we also get $R^p \otimes_k k = kR^p$. Thus it follows that $kR^p/(t_i^p) = R^p/(t_i^p) \otimes_k k = D^p \otimes_k k$ and that $D^p \otimes_k k = kD^p$ by separability of $D/k$. Similarly, we observe that $\{t_1^p, ..., t_n^p\}$ is a $p$-basis of $kD^p/k$ and that $t_i^{(p)}$ is a parameter of $(K_i/k, R_i)$ for each $i$.

We will define a $k$-linear map of degree $-1$, $res_{R^p} : \Omega \longrightarrow \Omega(D)$. Let $\hat{R}$ be the completion of $R$. Then there exists a unique coefficient field $E = E_{t_1^p, ..., t_n^p}$ of $\hat{R}$ such that $\hat{R} = E[[t_1]]$ and $E \supset k(t_1, ..., t_n)$ (c.f. Th. 28.3 in [2]). The quotient field of $\hat{R}$ is the formal power series field $E((t_1))$ and $K$ can be regarded as a subfield of $E((t_1))$. Let $\omega$ be a differential form in $\Omega^r$ $(r \geq 1)$. Then $\omega$ is uniquely expressed in the form

$$\omega = \sum_{1 < i_1 < \cdots < i_r} g_{i_1 \cdots i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r} + \sum_{1 < i_2 < \cdots < i_r} h_{i_2 \cdots i_r} dt_1 \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r}$$

where $g_{i_1 \cdots i_r}, h_{i_2 \cdots i_r} \in K$. Let $h_{i_2 \cdots i_r} = \sum_k h_{i_2 \cdots i_r, k} t_1^k$ be the formal expression of $h_{i_2 \cdots i_r}$ in $\hat{K} = E((t_1))$. We define the residue of $\omega$ by

$$res_{R^p}(\omega) = \sum_{i_2 < \cdots < i_r} h_{i_2 \cdots i_r, -1} dt_{i_2} \wedge \cdots \wedge dt_{i_r}$$

where $\bar{a}$ is the canonical image of $a \in \hat{R}$ in $D$. Thus we can define the map $res_{R^p} : \Omega \longrightarrow \Omega(D)$ by linearity.
We observe that $\text{res}_{R,\mathcal{L}}$ has the following property
\[ \text{res}_{R,\mathcal{L}} \circ d + d_{D/k} \circ \text{res}_{R,\mathcal{L}} = 0. \]
It follows from this property that $\text{res}_{R,\mathcal{L}}$ maps closed differentials to closed ones and exact differentials to exact ones.

We will denote by $Z(\Omega) (= \text{ker } d)$, all of closed differentials in $\Omega$ and by $B(\Omega) (= \text{im } d)$, all of exact differentials in $\Omega$. If there is no confusion, we will write $Z, B$ instead of $Z(\Omega), B(\Omega)$, respectively. It follows that $Z$ is a graded $kK$-subalgebra of $\Omega$ with $Z_0 = kK$ and that $B$ is a two-sided homogeneous ideal of $Z$.

**Definition.** For a parameter $t = \{t_1, t_2, ..., t_n\}$ of $(K/k, R)$, we define the graded subalgebras $H_m(\mathcal{L})$ of $Z$ and $I_m(\mathcal{L})$ of $Z(\Omega(R))$ ($m = 1, 2, ...$) as follows:
\[
H_m(\mathcal{L}) := K_m[t_1^{p^m-1}dt_1, t_2^{p^m-1}dt_2, \ldots, t_n^{p^m-1}dt_n],
\]
\[
I_m(\mathcal{L}) := R_m[t_1^{p^m-1}dt_1, t_2^{p^m-1}dt_2, \ldots, t_n^{p^m-1}dt_n].
\]

We have by Exercise (6) in §5 of [1] that
\[ Z = B \bigoplus H_1(\mathcal{L}), \quad Z(\Omega(R)) = B(\Omega(R)) \bigoplus I_1(\mathcal{L}) \]
for every parameter $\mathcal{L}$ of $(K/k, R)$ (c.f. Lemma 1).

The Cartier operator $C_{K/k}$ (we denote it by $C$ if there is no confusion) is defined to be a surjective homomorphism of degree zero of graded $K_1$-algebra ($K_1 = kK$)
\[
C : Z \longrightarrow \Omega(K_1/k)
\]
such that $C(B) = 0$, $C(a) = a$ for any $a \in Z^0 = K_1$ and $C(t_i^{p^m-1}dt_i) = d_1t_i^p$ for each $i$, where $d_1$ is the differentiation of $\Omega(K_1/k)$ (Exercise (6) in §5 of [1]). It follows that $C$ induces an isomorphism of $H_1(\mathcal{L})$ on $\Omega(K_1/k)$, but $C$ does not depend on $R$ and a fortiori $C$ does not depend on $\mathcal{L}$. Similarly we can also define Cartier operators $C_{R/k}, C_{D/k}, C_{K_1/k}$ and $C_{R_1/k}$. We have by Lemma 2 of [3] that
\[
C_{D/k} \circ \text{res}_{R,\mathcal{L}} = \text{res}_{R_1,\mathcal{L}} \circ C
\]
for every parameter $\mathcal{L}$ of $(K/k, R)$.

The Cartier operators $C_{K_i/k} (= C_i)$ ($i = 0, 1, 2, ...$) define the subsets $B_m = B_m(\Omega)$ and $Z_m = Z_m(\Omega)$ of $\Omega$ inductively as follows: We first set $B_0(\mathcal{L}) = 0$,
$Z_0(\Omega_i) = \Omega_i$ for each $i$, where $\Omega_i = \Omega(K_i/k)$. We note that $C_0 = C$ and $\Omega_0 = \Omega$. Next we set, for every integer $m \geq 0$,

$$B_{m+1}(\Omega_i) = C^{-1}_i(B_m(\Omega_{i+1})), \quad Z_{m+1}(\Omega_i) = C^{-1}_i(Z_m(\Omega_{i+1})).$$

For example, $B_2 = B_2(\Omega)$ is obtained as follows: $B_1(\Omega_1) = C_1^{-1}(0)$ and $B_2(\Omega) = C_0^{-1}(B_1(\Omega_1)) = C_0^{-1}(C_1^{-1}(0))$.

We can easily see that $B_1 = B$, $Z_1 = Z$, and $0 = Z_0 = \cdots = Z_1 = \cdots Z_m = Z_0 = \Omega$.

It follows that $Z_m$ ($m \geq 0$) is a graded $K_m$-subalgebra of $\Omega$ and that $B_m$ is a two-sided homogeneous ideal of $Z_m$ such that $Z_m/B_m \cong \Omega$. Furthermore, we set $Z_\infty = \bigcap_{m=1}^{\infty} Z_m$ and $B_\infty = \bigcup_{m=1}^{\infty} B_m$.

Let $t = \{t_1, t_2, \ldots, t_n\}$ be a parameter of $(K/k, R)$. Then for every element $\omega$ of $\Omega$, we define $\nu_R(\omega)$ as follows:

$$\nu_R(\omega) = \max \{s \in Z | t_s^* \omega \in \Omega(R)\}.$$ 

If $\omega \in \Omega^0 = K$, then $\nu_R(\omega)$ is the valuation value of $\omega$ such that $\nu_R(t_1) = 1$.

We note that $\nu_R(\omega)$ is dependent on $R$ but not dependent on the parameter $t$.

Furthermore we fix a special basis of $\Omega$ over $K$ for the parameter $t$ named $\Lambda$:

$$\Lambda = \{dt_{i_1} \wedge dt_{i_2} \wedge \cdots \wedge dt_{i_r} | 0 \leq r \leq n, \ 1 \leq i_1 < \cdots < i_r \leq n \}$$

(when $r = 0$, $dt_{i_1} \wedge \cdots \wedge dt_{i_r}$ means 1). Then $\Lambda$ is also a basis of $\Omega(R)$ over $R$.

Furthermore we see that an element $\omega = \sum a_{i_1, \ldots, i_r} dt_{i_1} \wedge \cdots \wedge dt_{i_r}$ belongs to $\Omega(R)$ if and only if all $a_{i_1, \ldots, i_r}$ belong to $R$.

**Lemma 2.** For any parameter $t$ of $(K/k, R)$ and for any natural number $m$,

$$Z_m = B_m \bigoplus H_m(t)$$

as $K_m$-modules (additive groups or $k$-modules).

**Proof.** We shall prove this by induction on $m$, it holding for $m = 1$ (Exercise (6) in §5 in [1]). We assume it for $m-1 (m \geq 2)$. By using the assumption of induction to the case of the parameter $t^p$ of $(K_1/k, R_1)$, we have that

$$Z_{m-1}(\Omega_1) = B_{m-1}(\Omega_1) \bigoplus H_{m-1}(t^p).$$
where $H_{m-1}(p) = kK_1^{-p-1}[(t_1^p)^{p-1-1}dt_1, (t_2^p)^{p-1-1}dt_2, \ldots, (t_n^p)^{p-1-1}dt_n]$. $\mathbf{Since} K_m = kK_1^{-p-1}$ and $t_j^p = (t_j^p)^{p-1-1}dt_j$, it follows that $C(H_m(t)) = H_{m-1}(p)$. By the definition of $Z_m$ and $B_m$, 

$$Z_m = C^{-1}(Z_{m-1}(\Omega_1)) \text{ and } B_m = C^{-1}(B_{m-1}(\Omega_1)).$$

If $\omega \in B_m \cap H_m(t)$, then $C(\omega) \in B_{m-1}(\Omega_1) \cap H_{m-1}(p) = (0)$; hence $\omega \in \ker C \cap H_m(t) \subset B_1 \cap H_1(t) = (0)$.

It holds that $Z_m \supset B_m + H_m(t)$. Conversely, we will prove that $Z_m \subset B_n + H_m(t)$. Let $\omega \in Z_m$. Then $C(\omega) = x + y$ for some $x \in B_{m-1}(\Omega_1)$ and $y \in H_{m-1}(p)$. Since $C$ is surjective, there exist $\alpha \in B_n$ and $\beta \in H_m(t)$ such that $C(\alpha) = x$ and $C(\beta) = y$. Hence $\omega - \alpha - \beta \in \ker C = B = B_1 \subset B_m$. Thus $Z_m = B_m + H_m(t)$.

Let $t$ be a parameter of $(K/k, R)$. Any element $a \neq 0$ of $K$ can be uniquely expressed in the form

$$a = \sum \alpha_{s_1, \ldots, s_n} t_1^{s_1} t_2^{s_2} \cdots t_n^{s_n}, \quad \alpha_{s_1, \ldots, s_n} \in kK^p,$$

where $s_i$ runs over $\{0, 1, \ldots, p - 1\}$ for each $i$. Then we have the following lemma.

**Lemma 3.** $\nu_R(a) = \min_{s_1, \ldots, s_n} (\nu_R(\alpha_{s_1, \ldots, s_n} t_1^{s_1} \cdots t_n^{s_n})).$

**Proof.** It is easy to see that $\nu_R(\alpha_{s_1, \ldots, s_n})$ are multiples of $p$, $\nu_R(t_i) = 0$ for $i \geq 2$ and $\nu_R(t_1) = 1$. Therefore the values of valuation $\nu_R$ of the following $p$ elements are distinct to each other except $\infty = \infty$;

$$\sum \alpha_{0, s_2, \ldots, s_n} t_2^{s_2} \cdots t_n^{s_n} (\sum \alpha_{1, s_2, \ldots, s_n} t_2^{s_2} \cdots t_n^{s_n}) t_1, \ldots, (\sum \alpha_{p-1, s_2, \ldots, s_n} t_2^{s_2} \cdots t_n^{s_n}) t_1^{p-1}.$$ 

Therefore it holds that

$$\nu_R(a) = \min_{i=0,1,\ldots,p-1} (\nu_R(\sum \alpha_{i, s_2, \ldots, s_n} t_1^i t_2^{s_2} \cdots t_n^{s_n})).$$

Let $\min_{s_2, \ldots, s_n} (\nu_R(\alpha_{i, s_2, \ldots, s_n})) = r_i p \quad (r_i \in Z)$ and $\alpha_{i, s_2, \ldots, s_n} = t_1^{r_1} \alpha'_{i, s_2, \ldots, s_n}$ ($\alpha'_{i, s_2, \ldots, s_n} \in kK^p$) for each $i$. Then $\sum \alpha'_{i, s_2, \ldots, s_n} t_2^{s_2} \cdots t_n^{s_n}$ is an element of $R$ and its image $\sum \alpha'_{i, s_2, \ldots, s_n} t_2^{s_2} \cdots t_n^{s_n}$ in $D$ is not zero, because $\{t_2, \ldots, t_n\}$ is a $p$-basis of $D/kD^p$ and at least one of the elements $\{\alpha'_{i, s_2, \ldots, s_n}\}$ is not zero. Therefore, for each $i$, it holds that

$$\nu_R(\sum \alpha_{i, s_2, \ldots, s_n} t_1^i t_2^{s_2} \cdots t_n^{s_n}) = \min_{s_2, \ldots, s_n} (\nu_R(\alpha_{i, s_2, \ldots, s_n} t_1^i t_2^{s_2} \cdots t_n^{s_n})).$$
This completes the proof.

**Lemma 4.** Let $\alpha, \beta \in H_1(\ell)$. If $\nu_R(\alpha) = \nu_R(\beta)$, then $\nu_{kR^p}(C(\alpha)) = \nu_{kR^p}(C(\beta))$.

**Proof.** Since $\alpha, \beta \in H_1(\ell) = kK^p[t_1^{p-1}dt_1, \cdots, t_n^{p-1}dt_n]$, it follows that $\nu_R(\alpha) = \nu_R(\beta) = mp$, or $\nu_R(\alpha) = \nu_R(\beta) = mp + p - 1$, for some integer $m$. Since $kR^p$ is a discrete valuation ring with a prime element $t_i^p$, $\nu_{kR^p}(t_i^p) = 1$ and since $C(t_i^{p-1}dt_i) = dt_i^p$ for each $i$, we obtain that $\nu_{kR^p}(C(\alpha)) = \nu_{kR^p}(C(\beta)) = m$.

§2. Main theorems

Let $\omega$ be an element of $Z_m$ $(m \geq 1)$. Then we have by Lemma 2 that $\omega$ is uniquely expressed in the form $\omega_1 + \omega_2$, where $\omega_1 \in B_m$, $\omega_2 \in H_m(\ell)$.

**Theorem 1.** Let $\omega, \omega_1$, and $\omega_2$ be as above. Then we have $\nu_R(\omega) = \min(\nu_R(\omega_1), \nu_R(\omega_2))$.

**Proof.** If $\nu_R(\omega_1) \neq \nu_R(\omega_2)$, then we have $\nu_R(\omega) = \min(\nu_R(\omega_1), \nu_R(\omega_2))$. Therefore we may assume that $\nu_R(\omega_1) = \nu_R(\omega_2) = s$. Then it is enough to show that $\nu_R(\omega) = s$. We prove this by induction on $m$.

First we prove the case of $m = 1$. Using the base $\Lambda$ of $\Omega$ over $K$, we can express $\omega_1$ and $\omega_2$ as follows:

$\omega_1 = \cdots + xdt_{i_1} \land \cdots \land dt_{i_r} + \cdots$

$\omega_2 = \cdots + ydt_{j_1} \land \cdots \land dt_{j_r} + \cdots$.

In the case $\nu_R(x) = \nu_R(y) = s$, it will be enough to show $\nu_R(x + y) = s$. Since $\omega_2 \in H_1(\ell)$, $y$ is of the form $\alpha t_{i_1}^{p-1} \cdots t_{i_r}^{p-1}$ $(\alpha \in kK^p)$. Since $\omega_1 \in B_1$, $\omega_1 = d\omega_0$ for some $\omega_0 \in \Omega$. Since any element $a$ of $K$ is uniquely written in the form

$$a = \sum_{i_1, \cdots, i_r = 0}^{p-1} \alpha_{i_1 \cdots i_r} t_{i_1}^{i_1} \cdots t_{i_r}^{i_r} \quad (\alpha_{i_1 \cdots i_r} \in kK^p),$$

hence the definition of $da$, the definition of $d\omega_0$ and Lemma 3 show that $\nu_R(x + y) = s$.

Next we assume that this theorem is true for $1, 2, \cdots, m - 1$ $(m \geq 2)$. We may assume that $\nu_R(\omega_1) = \nu_R(\omega_2) = s$. Since $B_m \subset Z_{m-1}$, it follows that $B_m = B_m \cap Z_{m-1} = B_m \cap (B_{m-1} + H_{m-1}) = B_{m-1} + B_m \cap H_{m-1}$ (direct sum). Therefore $\omega_1 \in B_m$ is uniquely written in the form $\omega_{11} + \omega_{12}$, where $\omega_{11} \in B_{m-1}$ and $\omega_{12} \in B_m \cap H_{m-1}$. Since $\omega_{11} \in B_{m-1}$ and $\omega_{12} \in H_{m-1}$, we get by the assumption of induction that $s = \nu_R(\omega_1) = \min(\nu_R(\omega_{11}), \nu_R(\omega_{12}))$. 

RESIDUES OF DIFFERENTIAL FORMS 109
Case I. \( \nu_R(\omega_{11}) = s \). It is easy to see that \( \omega_{12} + \omega_2 \in H_{m-1} + H_m = H_{m-1} \) and \( \nu_R(\omega_{11} + \omega_2) \geq s \). Since \( \omega_{11} \in B_{m-1} \), we get by the assumption of induction on \( m \) that

\[
\nu_R(\omega) = \nu_R(\omega_{11} + (\omega_{12} + \omega_2)) = \min(\nu_R(\omega_{11}), \nu_R(\omega_{12} + \omega_2)) = s.
\]

Case II. \( \nu_R(\omega_{12}) = s \) and \( \nu_R(\omega_{11}) > s \). In this case, we have that \( \omega_{12} \in B_m \cap H_{m-1} \subset H_1 \), \( \omega_2 \in H_m \subset H_1 \) and \( \nu_R(\omega_{12}) = \nu_R(\omega_2) = s \), where \( s = mp \), or \( s = mp + p - 1 \) for some integer \( m \) (see Lemma 4). By Lemma 4, \( \nu_{kR}(C(\omega_{12})) = \nu_{kR}(C(\omega_2)) = m \). On the other hand, since \( \omega_{12} \in B_m \) and \( \omega_2 \in H_m \), we have \( C(\omega_{12}) \in B_{m-1}(\Omega_1) \) and \( C(\omega_2) \in H_{m-1}(\Omega_1) \). By the assumption of induction on \( m \), we get that

\[
\nu_{kR}(C(\omega_{12} + \omega_2)) = \nu_{kR}(C(\omega_2) + 2) = \min(\nu_{kR}(C(\omega_{12})), \nu_{kR}(C(\omega_2))) = m.
\]

It then follows that \( \nu_R(\omega_{12} + \omega_2) = mp \) or \( mp + p - 1 \) (c.f. Lemma 4). Furthermore one can observe that \( \nu_R(\omega_{12} + \omega_2) = s \) (c.f. Lemma 3). Since \( \nu_R(\omega_{11}) > s \), we get that \( \nu_R(\omega) = \nu_R(\omega_{11} + \omega_{12} + \omega_2) = s \), as desired.

**Theorem 2.** Let \( \omega \) be an element of \( Z_m \) such that \( \nu_R(\omega) \geq -p^m + 1 \). Let \( \xi = \{t_1, \ldots, t_n\} \) and \( \eta = \{u_1, \ldots, u_n\} \) be two parameters of \( (K/k, R) \). Then \( \text{res}_{R,\xi}(\omega) - \text{res}_{R,\eta}(\omega) \) is an element of \( B_m\Omega(D/k) \). In other words, \( \text{res}_{R,\xi}(\omega) \) is uniquely determined by \( R \) up to addition by differentials in \( B_m\Omega(D/k) \).

**Proof.** By Lemma 2 we have \( \omega = \omega_1 + \omega_2 \), where \( \omega_1 \in B_m \) and \( \omega_2 \in H_m(\xi) \). Theorem 1 says that \( \nu_R(\omega_2) \geq -p^m + 1 \). On the other hand, since \( H_m(\xi) = K_m[t_1^{p^m-1}dt_1, \ldots, t_n^{p^m-1}dt_n] \), we get \( \nu_R(\omega_2) \equiv 0 \pmod{p^m} \). Hence it follows that \( \nu_R(\omega_2) \geq -1 \). From E. Kunz (Exercise (1) in §17 of [1]), we have \( \text{res}_{R,\xi}(\omega_2) = \text{res}_{R,\eta}(\omega_2) \). Since both \( \text{res}_{R,\xi} \) and \( \text{res}_{R,\eta} \) map \( B_m \) to \( B_m\Omega(D/k) \), we get that

\[
\text{res}_{R,\xi}(\omega) - \text{res}_{R,\eta}(\omega) = \text{res}_{R,\xi}(\omega_1) - \text{res}_{R,\eta}(\omega_1) \in B_m\Omega(D/k).
\]

From this theorem, we can define the residue map \( \text{res}_R \), which is independent from the choice of a parameter \( \xi \).

**Corollary.** \( \text{res}_R : Z_\infty \to Z_\infty\Omega(D/k)/B_\infty\Omega(D/k) \) is well defined.

We will show an example which asserts that the number \(-p^m + 1\) in Theorem 2 is the best possible. In fact, we can find a function field \( K/k \), a
valuation ring $R$ of $K/k$, two parameters $t$ and $u$ of $(K/k, R)$ and a differential form $\omega \in Z_m$ such that $\nu_t(\omega) = -p^m$, $\text{res}_{R,t}(\omega) = 0$ and such that $\text{res}_{R,u}(\omega) \notin Z_{m+1}(\Omega(D))$. So the difference $\text{res}_{R,t}(\omega) - \text{res}_{R,u}(\omega)$ does not belong to $B_1(\Omega(D))$ because $B_1(\Omega(D)) \subseteq Z_{m+1}(\Omega(D))$.

Example. Let $K = k(x, y, z)$ be the rational function field of 3 variables $x, y, z$ over $k$ and let $R = k(y, z)[x]/(x)$. Then $t = \{x, y, z\}$ is a parameter of $(K/k, R)$. If we set $y_1 = y - x$, then $\mathfrak{u} = \{x, y_1, z\}$ is also a parameter of $(K/k, R)$. We note that $R = k(y_1, z)[x]/(x)$ and that $\bar{R} = k(y, z)[x]/(x)$.

Let $\omega = (x^{-1}y)^p z^{p-1} dy \wedge dz$. It follows that $\omega \in H_m(t) \subseteq Z_m$ and $\text{res}_{R,t}(\omega) = 0$. On the other hand, $\text{res}_{R,u}(\omega) = y_1^p z^{p-1} dz$.

From this, it follows that $\text{res}_{R,u}(\omega) = \overline{y_1^p z^{p-1}} d\overline{z}$ and $C_{D/k}^m (\overline{y_1^p z^{p-1}} d\overline{z}) = \overline{y_1^p} d\overline{z}^m$, where $C_{D/k}^m = C_{D_{m-1}/k} \circ \cdots \circ C_{D/k} (D_i = kD^p)$.

Since $\{\overline{y_1^p}, \overline{z}^m\}$ is a p-basis of $D_m/k$, we have $d(\overline{y_1^p} d\overline{z}^m) = d\overline{y_1^p} \wedge d\overline{z}^m \neq 0$.

Thus we get that $\overline{y_1^p} d\overline{z}^m \notin \mathcal{Z}(D_m)$ and $\text{res}_{R,u}(\omega) \notin Z_{m+1}(\Omega(D))$.

References


Takeo Ohi
Faculty of Science, Science University of Tokyo
26 Wakamiya-cho, Shinjuku-ku, Tokyo 162, Japan