A NONLINEAR PARTIAL DIFFERENTIAL EQUATION RELATED WITH CERTAIN SPACES WITH GENERAL CONNECTIONS (III)

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Abstract. From a Minkowski-type metric on $\mathbb{R}^n$ satisfying the Einstein condition, we derived a nonlinear partial differential equation. We tried to get some numerical approximate solution with certain boundary conditions by the finite element method in [10]. Regarding the exact solution, we shall give a fundamental theorem, which tells us the above numerical approximate solution is not worth to using the word “approximate”.

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§ 1. Introduction

This work is a continuation of the previous papers [9] and [10] with the same title. Reflecting the results and the arguments in them, we shall try to obtain exact solutions of the fundamental nonlinear partial differential equation (1.3) in [10] for unknown $Q = Q(x,t) :

\begin{align}
(2Q-\varphi)x^3 \frac{\partial^2 Q}{\partial x^2} - (3Q-2\varphi)x t \frac{\partial^3 Q}{\partial x \partial t} + (Q-\varphi)t^2 \frac{\partial^2 Q}{\partial t^2} \\
+ ((2n-4)Q-n\varphi)x \frac{\partial Q}{\partial x} - ((n-4)Q-(n-2)\varphi)t \frac{\partial Q}{\partial t}
\end{align}

(\text{1.1})

\begin{align}
-\frac{1}{Q} \left( x \frac{\partial Q}{\partial x} - t \frac{\partial Q}{\partial t} \right) \left( 2(Q-\varphi)x \frac{\partial Q}{\partial x} - (Q-2\varphi)t \frac{\partial Q}{\partial t} \right)
\end{align}

+ 2(n-3)Q(1-Q) = 0,

where $\varphi = \varphi(x)$, $x = r/x_n$, $t = x_n$, is an auxiliary free function of $x$ such that

\begin{align}
P = \frac{x^2}{Q-\varphi}
\end{align}

(\text{1.2})

and the Minkowski-type pseudo-Riemannian metric :

\begin{align}
ds^2 = \frac{1}{(x_n)^2} \left( \frac{1}{Q} drdr + r^2 \sum_{a,b=1}^{n-1} h_{ab} du^a du^b - P dx_n dx_n \right)
\end{align}

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on $R^n = R^{n-1} \times R$ with the canonical coordinates, $\{x_1, \ldots, x_{n-1}, x_n\}, \ x_n > 0$, 
and let

$$d\sigma^2 = \sum_{\alpha, \beta=2}^{n-1} h_{\alpha \beta}(u) du^\alpha du^\beta$$

be the standard metric of the unit sphere $S^{n-2} : r^2 = 1$ in $R^{n-1}$, satisfies the Einstein condition:

$$\overline{R}_{ij} = \overline{g}_{ij},$$

where $\overline{R}_{ij}, \overline{g}_{ij}$ and $\overline{R}$ are the components of the Ricci tensor, the metric tensor and the scalar curvature of $d\overline{s}^2$, respectively.

The integral function $\varphi(x)$ becomes $1 - x^2$ for the space $Ml^n$ with the metric

$$d\overline{s}^2 = \frac{1}{x_n^2} \left( \sum_{\alpha=1}^{n-1} dx_\alpha dx_x - dx_n dx_x \right)$$

with $P = Q = 1$.

In the following, we shall investigate the special case:

$n = 4$ and $\varphi(x) = 1 - x^2$,

then the equation (1.1) becomes

$$\left(2Q-1+x^2\right) x^2 \frac{\partial^2 \overline{Q}}{\partial x^2} - (3Q-2+2x^2) x t \frac{\partial^3 \overline{Q}}{\partial x \partial t^2} + \left( Q - 1 + x^2 \right) t^2 \frac{\partial^2 \overline{Q}}{\partial t^2}$$

(1.3)

$$+ 4\left( Q - 1 + x^2 \right) x \frac{\partial \overline{Q}}{\partial x} + 2(1-x^2) t \frac{\partial \overline{Q}}{\partial t} - \frac{1}{Q} \left( x \frac{\partial \overline{Q}}{\partial x} - t \frac{\partial \overline{Q}}{\partial t} \right) x$$

$$+ \frac{1}{Q} \left( x \frac{\partial \overline{Q}}{\partial x} - t \frac{\partial \overline{Q}}{\partial t} \right) + 2 \overline{Q} (1 - Q) = 0.$$
§ 2. Certain change of independent variables \( x \) and \( t \)

Since \( x > 0 \) and \( t > 0 \), we put \( x = e^u \), \( t = e^v \). We can obtain

\[
\begin{align*}
\frac{\partial Q}{\partial u} &= x \frac{\partial Q}{\partial x}, \quad \frac{\partial^2 Q}{\partial u^2} = x^2 \frac{\partial^2 Q}{\partial x^2} + x \frac{\partial Q}{\partial x}, \\
\frac{\partial Q}{\partial v} &= t \frac{\partial Q}{\partial t}, \quad \frac{\partial^2 Q}{\partial v^2} = t^2 \frac{\partial^2 Q}{\partial t^2} + t \frac{\partial Q}{\partial t}, \\
\frac{\partial^3 Q}{\partial u \partial v} &= xt \frac{\partial^3 Q}{\partial x \partial t}.
\end{align*}
\]

We set \( X = 1 - x^2 \). Then the equation (1.3) can be written as

\[
(2Q - X) \left( \frac{\partial^3 Q}{\partial u \partial v} - (3Q - 2X) \frac{\partial^2 Q}{\partial u \partial v} + (Q - X) \left( \frac{\partial^2 Q}{\partial v^2} - \frac{\partial Q}{\partial v} \right) \right) \\
+ 4(Q - X) \frac{\partial Q}{\partial u} + 2X \frac{\partial Q}{\partial v} - \frac{1}{Q} \left( \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right) \left( 2(Q - X) \frac{\partial Q}{\partial u} - (Q - 2X) \frac{\partial Q}{\partial v} \right) \\
+ 2Q(1 - Q) = 0,
\]

that is

\[
\Theta := Q \left( 2 \frac{\partial^2 Q}{\partial u^2} - 3 \frac{\partial Q^2}{\partial u \partial v} + 3 \frac{\partial Q}{\partial u} \frac{\partial Q}{\partial v} + 2 \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right) - \left( \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right) \left( 2 \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right) \\
- X \left( \frac{\partial^2 Q}{\partial u^2} - 2 \frac{\partial^2 Q}{\partial u \partial v} + 3 \frac{\partial Q}{\partial u} - 3 \frac{\partial Q}{\partial v} - \frac{2}{Q} \left( \frac{\partial Q}{\partial u} - \frac{\partial Q}{\partial v} \right)^2 \right) \\
+ 2Q(1 - Q) = 0.
\]

Further we take the following change of variables

\[
\begin{align*}
\{u = -2 \xi + \eta, & \\
\{v = \xi - \eta, & \\
\{ \xi = -u - \eta, & \\
\{ \eta = -u - 2 \eta,
\end{align*}
\]

then we have

\[
\begin{align*}
\frac{\partial}{\partial \xi} &= -2 \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial \eta} = -2 \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \quad \frac{\partial}{\partial u} = - \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial v} = - \frac{\partial}{\partial \xi} - 2 \frac{\partial}{\partial \eta}.
\end{align*}
\]

Using these relations, (2.1) can be written as

\[
\Theta = -Q \left( \frac{\partial^2 Q}{\partial \xi^2} + \frac{\partial Q}{\partial \xi} \right) + \frac{\partial Q}{\partial \xi} \frac{\partial Q}{\partial \eta} - X \left( \frac{\partial^2 Q}{\partial \eta^2} + 3 \frac{\partial Q}{\partial \eta} - \frac{2}{Q} \frac{\partial Q}{\partial \eta} \right) + 2Q(1 - Q) = 0,
\]

from which we can see that if \( Q \) is constant, it must be \( 0 \) or \( 1 \). For any non constant solution \( Q \), let \( z = z(\xi, \eta) \) be an auxiliary function such that \( z(\xi, \eta) = \) constant gives a contour line of the function \( Q \). Then, we can consider \( Q \) as a function of \( z \) only.
Denoting the derivative with respect to $z$ by the notation prime "', we have
\[
\frac{\partial Q}{\partial \xi} = Q' \frac{\partial z}{\partial \xi}, \quad \frac{\partial Q}{\partial \eta} = Q' \frac{\partial z}{\partial \eta}, \quad \frac{\partial^2 Q}{\partial \xi^2} = Q' \frac{\partial^2 z}{\partial r^2} + Q'' \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \xi},
\]
\[
\frac{\partial^2 Q}{\partial \xi \partial \eta} = Q' \frac{\partial^2 z}{\partial \xi \partial \eta} + Q'' \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \eta} + Q'' \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \eta}.
\]
Hence (2.2) turns into the following equality
\[
(2.3) \quad \Theta = \left(-QQ'' + Q'Q'\right) \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \eta} - QQ' \left(\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial z}{\partial \xi} \frac{\partial z}{\partial \eta}\right) + 2Q(1-Q)
\]
\[
+ \left(-QQ'' + 2Q'Q'\right) \frac{1}{Q} X \frac{\partial z}{\partial \eta} \frac{\partial z}{\partial \eta} - Q'X \left(\frac{\partial^2 z}{\partial \eta^2} + 3 \frac{\partial z}{\partial \eta}\right) = 0.
\]
Now, we shall give a fundamental theorem as follows.

**Theorem 1.** Any non constant solutions of (1.3) are classified into the following two types:

**Type 1.** Their contour lines are the hyperbolic curves $xt = constant$ and $Q = 1 + ax^2 t^2$, $a = constant$.

**Type 2.** Their contour lines are $x = constant$.

**Proof.** For a non constant solution $Q$ of (1.3), let $z$ be a function of $(\xi, \eta)$ determining its contour lines. The arguments above implies from the left hand side of (2.3) that $\Theta$ must depend only on $z$. Looking over the expression of $\Theta$, we consider first the case (i) $\partial z/\partial \eta = 0$. Then we obtain
\[
\Theta = -QQ' \frac{\partial z}{\partial \xi} + 2Q(1-Q) = 0.
\]
Since $Q \neq 0$, we have
\[
\frac{\partial \log(Q-1)}{\partial \xi} = -2,
\]
from which we obtain
\[
Q = 1 + ae^{-2\xi} = 1 + ae^{2u+2v} = 1 + ax^2 t^2,
\]
where $a$ is an integral constant and $a \neq 0$.

Second, we consider the case (ii) $\partial z/\partial \eta \neq 0$. We may put
\[
\frac{\partial z}{\partial \eta} = \sqrt{X} \varphi(z), \quad \frac{\partial z}{\partial \xi} = \sqrt{X} \psi(z),
\]
where $\varphi(z)$ and $\psi(z)$ are suitable functions of $z$. Since we have
\[
X = 1 - x^2 = 1 - e^{2u} = 1 - e^{-4u+2v} \quad \text{and}
\]
\[
\frac{\partial X}{\partial \xi} = 4(1-X), \quad \frac{\partial X}{\partial \eta} = -2(1-X),
\]

\]
The functions $\varphi(z)$ and $\psi(z)$ must satisfy

\begin{equation}
\frac{1}{\sqrt{X}} \left( \frac{1}{X} \frac{d}{dz} \right) \varphi + \frac{1}{\sqrt{X}} \frac{d}{dz} \psi = \frac{1}{X} \left( \frac{1}{\sqrt{X}} \varphi + \varphi' \right).
\end{equation}

Using these relations, $\Theta$ can be written as

\begin{equation}
\Theta = (-QQ'' + Q'Q')\varphi \psi - QQ' \left( \frac{1}{\sqrt{X}} \psi + \varphi \psi' \right) + 2Q(1-Q)
\end{equation}

\begin{equation}
+ (-QQ'' + 2Q'Q') \frac{1}{Q} \varphi^2 - Q \left( \frac{1+2X}{\sqrt{X}} \varphi + \varphi' \right).
\end{equation}

In this case, it must be $\varphi \neq 0$. If $\psi = 0$, (2.4) implies $\varphi = 0$. Therefore we have $\psi \neq 0$. Now, setting $\sqrt{X} = \lambda$, from (2.4) we obtain the quartic equation on $\lambda$:

\[\psi \lambda^4 + (\varphi \psi' - \psi \varphi') \lambda^2 - (2\varphi + \psi) \lambda^2 + 2\varphi = 0,\]

which says that $X$ is a function of $z$ only. Therefore $e^{z''} = 1 - X$ is also dependent only on $z$. Hence we may put $z = -2\xi + \eta$ essentially. Then (2.3) turns into

\begin{equation}
\Theta = -2(-QQ'' + Q'Q') + 2Q(1-Q)
\end{equation}

\begin{equation}
+ (-QQ'' + 2Q'Q') \frac{1-e^{z''}}{Q} - 3Q'(1-e^{z''}) = 0.
\end{equation}

$z = \text{constant}$ means $u = \text{constant}$ and so $x = \text{constant}$. Q.E.D.

Note. From any boundary condition we can construct generally an approximate numerical solutions of (1.3) by the finite element method with suitable finite elements, but if the boundary conditions are not compatible with the situation in Theorem 1, the equation (1.3) can not have the solution with the same boundary conditions. Therefore the numerical approximate solution in [10] can not be considered as an approximation of a solution of (1.3) with the same boundary conditions (1.4).

§ 3. Solutions of Type 2 in Theorem 1

In this section we shall try to solve (2.5). We set $Q = y$, $u = -2\xi + \eta$, $X = 1 - e^{z''}$. The equation (2.5) can be written as follows.

\begin{equation}
\frac{dy}{du} = z, \quad 2(yz' - zz) + 2yz + 2y(1 - y) - (yz' - 2zz) \frac{X}{y} - 3Xz = 0,
\end{equation}
which is expressed as

\[(2y - X) \frac{dz}{du} = -2 \frac{y}{y} (y - X) z - (2y - 3X) z - 2y(1 - y). \tag{3.2}\]

Since we have \( \frac{dX}{du} = -2e^{2x} = -2(1 - X) \), for \( U = (2y - X) z \), we obtain from (3.1) and (3.2) that

\[U' = (2z + 2 - 2X) z + (2y - X) z' \]

\[= 2zz + 2(1 - X) z + \frac{2}{y} (y - X) z - (2y - 3X) z - 2y(1 - y) \]

\[= \frac{2y}{y} (2y - X) z - (2y - X - 2) z - 2y(1 - y) \]

\[= \frac{2z}{y} U - U + 2z - 2y(1 - y) = \frac{2z - y}{y} U + 2z - 2y(1 - y). \]

Then, we obtain

\[\left( \frac{e^y}{y^2} \right)' = \frac{y - 2z}{y^2} e^{yU} + \frac{1}{y^2} e^y \left( \frac{2z - y}{y} U + 2z - 2y(1 - y) \right) \]

\[= \frac{2}{y^2} e^y (z - y + y') = 2e^y \left( -\frac{1}{y} \right) - \frac{1}{y} + 1 \]

\[= -\left( \frac{1}{y} e^y \right)' + e^y, \]

hence by integration

\[\frac{1}{y^2} e^y U = \frac{1}{y^2} e^y (2y - X) z = \frac{2}{y} e^y + 2e^y + a. \]

That is

\[(3.3) \quad \frac{1}{y^2} \left( 2y - 1 + e^{2x} \right) \frac{dy}{dx} + \frac{2}{y} = 2(\frac{ae^{-x}}{x} + 1),\]

where \( a \) is an integral constant.

The next step is to solve the above differential equation of the first order on \( y \). To this end we change the independent variable \( u \) to \( x = e^u \). Then the equation (3.3) turns into the following equation as it can be easily seen:

\[(3.4) \quad x^2 \left( 2y - 1 + x^2 \right) \frac{dy}{dx} = 2y((a + x)y - x). \]

We want to solve this equation under the initial condition:

\[(3.5) \quad y|_{x=1} = 1, \]

from which we obtain

\[\frac{dy}{dx}|_{x=1} = a, \]

which gives the meaning of the integral constant \( a \).
Replacing the dependent variable $y$ by 

$$y_1 = y + \frac{x^2 - 1}{2},$$

the equation (3.4) becomes

$$2x^2 y_1 \left( \frac{dy_1}{dx} - x \right) = (2y_1 - x^2 + 1) \left( (a + x) \left( y_1 - \frac{x^2 - 1}{2} \right) - x \right)$$

$$= 2(a + x) y_1^2 - 2(x^3 + a(x^2 - 1)) y_1 + \frac{1}{2} (x^2 - 1)(x^3 + x + a(x^2 - 1)), $$

that is

$$\frac{dy_1}{dx} = \frac{x^2 - 1}{x^2} y_1 - \frac{a(x^2 - 1)}{x^2} + \frac{x^2 - 1}{4x^4 y_1} \left( x^3 + x + a(x^2 - 1) \right),$$

(3.6)

with the initial condition

(3.7) $$y_1 |_{x=1} = 1.$$  

Furthermore, putting

$$y_2 = \frac{1}{x} e^{\alpha x} y_1,$$

we obtain

$$\frac{dy_2}{dx} = -\frac{a(x^2 - 1)}{x^3} e^{\alpha x} + \frac{x^2 - 1}{4x^4 y_2} \left( x^3 + x + a(x^2 - 1) \right) e^{2\alpha x}$$

with $$y_2 |_{x=1} = e^\alpha,$$

which tells us that we could not obtain its solution in a closed form by means of elementary calculus. But we can show the existence of solutions as follows.

**Theorem 2.** The differential equation (3.6) has a solution near $x = 1$ for the initial condition (3.7), whose domain of existence depends on $a.$

**Proof.** Setting

(3.8) $$y_1 = \sum_{n=0}^\infty b_n (x-1)^n, \quad b_0 = 1, \quad b_1 = a + 1,$$

we substitute this into the expression:

$$x^2 y_1 \frac{dy_1}{dx} = (x + a)y_1^2 - a(x^2 - 1)y_1 + \frac{x^2 - 1}{4} (x^3 + x + a(x^2 - 1)).$$

By using

$$x + a = (x - 1) + a + 1; \quad x^2 - 1 = (x - 1)^2 + 2(x - 1),$$

$$\left( x^2 - 1 \right)(x^3 + x + a(x^2 - 1))$$

$$= (x - 1)^4 + (a + 5)(x - 1)^4 + 2(2a + 5)(x - 1)^4 + 2(2a + 5)(x - 1)^2 + 4(x - 1),$$

and comparing the like powers of $(x-1)$ of the both sides, we obtain
\[ b_2 = \frac{(a-1)^2}{2}, \quad b_3 = \frac{a}{6}(a^2 - 5a + 5) \]

\[ b_4 = \frac{a}{24}(a-2)(a^2 - 8a + 8) \]

\[ b_5 = \frac{a}{120}(a-1)(a^3 - 16a^2 + 78a - 64) \]

\[ b_6 = -\frac{a}{360}(a-1)(7a^4 - 70a^3 + 159a^2 + 83a - 48) \]

and for \( n > 5 \) inductively

\[ b_{n+1} = -\frac{1}{n+1}\left\{ ab_{n-3} + (n-1)b_{n-1} + (2n-1-a)b_n \right. \\
+ \sum_{p=1}^{n-2} b_p \left( (n-2-p)b_{n-1-p} + (2n-1-2p-a)b_{n-p} + (n+1-p)b_{n+1-p} \right) \}
\]

Looking over the expression of (3.6) and using the above results, we set

\[ f_n(x) = \sum_{n=0}^{6} b_n(x-1)^n \]

Suppose in the sequel that a series of functions \( f_2(x), f_3(x), \ldots, f_n(x) \) of \( x \) with the parameter \( a(\geq 0) \) is determined inductively as is shown as follows. We consider the differential equation on \( F \)

\[ \frac{dF}{dx} - \frac{2(x+a)}{x^2} F = \frac{1}{2x^2} (x^2 - 1)(x^3 + x + a(x^2 - 1)) - \frac{2a}{x^3} (x^2 - 1)f_n(x). \]

Solving this equation with the condition \( F(1) = 1 \), we put

\[ f_{n+1}(x) = \sqrt{F(x)}. \]

When \( a = 0 \), we obtain easily that

\[ f_n(x) = \frac{x^2 + 1}{2}, \quad n = 1, 2, 3, \ldots \]

In the following, we assume \( 0 < a \leq \frac{1}{2} \). From (3.11) we obtain

\[ F(x) = x^2 e^{-2alx} \int x \frac{1}{t^2} e^{2al} \left( \frac{1}{2t^2} \left( t^2 - 1 \right) + t + a \left( t^2 - 1 \right) \right) - \frac{2a}{t^2} \left( t^2 - 1 \right) f_n(t) \right) \right) \right) \right) dt + c x^2 e^{-2alx}, \quad c = \text{constant}. \]

By \( F(1) = 1 \), we obtain \( c = e^{2a} \). Hence we have the equality:

\[ F(x) = x^2 e^{2a(1-t^2)} \int x e^{2al} \left( \frac{1}{2t^2} \left( t^2 - 1 \right) + t + a \left( t^2 - 1 \right) \right) - \frac{2a}{t^2} \left( t^2 - 1 \right) f_n(t) \right) \right) \right) \right) \right) \right) dt. \]

Regarding the functions on the right hand side, we see
\( \frac{(1 - x^2)(x^3 + x - a(1 - x^2))}{x^4} = -a + \frac{x}{2} + ax^2 - ax^4 - x^4 \)

is decreasing in \( \frac{1}{2} < x < 1 \) and tends to 0 as \( x \to 1 \), since its derivative
\[
\frac{4a - 3x - 4ax^2 - x^5}{x^5} < 0 \quad \text{for} \quad \frac{1}{2} \leq x < 1,
\]
and \( \frac{1-x^2}{x^4} \) has the same property.

Now, we assume that
\( f_+(x) > 0 \), for \( \frac{1}{2} \leq x < 1 \).

Then, we obtain from (3.13)
\[
(3.14) \quad F(x) < x^2 e^{2a(1-x^2)} + x^2 e^{-2ax} \times e^{2a} \left( \frac{(1 - x^2)(x^3 + x - a(1 - x^2))(1 - x)}{2x^4} \right)
\]
\[
= x^2 e^{2a(1-x^2)} + \frac{1}{2x^2} (1 - x)(1 - x^2)(1 - x + a(1 - x^2)) \quad \text{for} \quad \frac{1}{2} \leq x < 1
\]
whose second term is decreasing in \( \frac{1}{2} < x < 1 \) and equal to \( \frac{3}{32} (5 - 6a) \) at \( x = \frac{1}{2} \), if
\[
0 < a \leq \frac{17}{39} = 0.43589\ldots
\]
In fact
\[
\left( \frac{1}{x^2} (1 - x)(1 - x^2)(x^3 + x - a(1 - x^2)) \right)'
\]
\[
= \left( x^4 - (1 - a)x^3 - ax^2 - 2ax - 1 + 2a + \frac{1}{x} + \frac{a}{x^2} \right)
\]
\[
= \frac{1-x}{x^3} (-4x^5 - (1 + 3a)x^4 - (1 + a)x^3 - (1 - a)x^2 - (1 - a)x + 2a)
\]
and
\[
-4x^5 - (1 + 3a)x^4 - (1 + a)x^3 - (1 - a)x^2 - (1 - a)x + 2a
\]
is decreasing in \( 0 < x < 1 \). They are equal to \( \frac{17}{16} + \frac{39}{16} a, -8 \) at \( x = \frac{1}{2}, 1 \), respectively.

Thus, we obtain
\[
(3.15) \quad F(x) < x^2 e^{2a(1-x^2)} + \frac{3}{32} (5 - 6a) < 1 + \frac{3}{32} (5 - 6a) = \frac{47 - 18a}{32}
\]
\[
\quad \text{for} \quad \frac{1}{2} \leq x < 1, \quad 0 < a \leq \frac{17}{39},
\]
and
\[
(3.16) \quad |f_{+1}(x)| < \sqrt{\frac{47 - 18a}{32}} < \frac{\sqrt{94}}{8} = 1.212\ldots
\]
Considering the approximating values of \( f_+(x) \), we may put
(3.17) \[0.24 < f_1(x) < 1\]

and

(3.18) \[f_n(x) < \frac{5}{4} = 1.25, \quad n = 2, 3, \ldots \quad \text{for} \quad \frac{1}{2} \leq x < 1, \quad 0 < a < \frac{1}{2},\]

which is assured for \(0 < a \leq \frac{17}{39}\) and \(n = 2\) by the above argument and for \(\frac{17}{39} < a \leq \frac{1}{2}\) by the numerical approximations.

For the sake of simplification we prepare some formulas. Denoting

\[I_m(x) = \int_x^1 \frac{e^{2at}}{t^m} \, dt \quad \text{and} \quad J_m(x) = \frac{e^{2atx}}{x^m} - e^{2a},\]

for integer \(m\), we have the following

**Lemma 1.** It holds the equalities:

\[l_{m+1} = \frac{m-1}{2a} I_m + \frac{1}{2a} J_{m+1},\]

and especially

\[I_4 = \frac{1}{2a} J_2 - \frac{1}{2a^3} J_1 + \frac{1}{4a^3} J_0 = \left(\frac{1}{2a^2} - \frac{1}{2a^3 x} + \frac{1}{4a^3} \right) e^{2atx},\]

\[I_3 = \frac{1}{2a} J_1 - \frac{1}{4a^2} J_0 = \left(\frac{1}{2a x} - \frac{1}{4a^2} \right) e^{2atx} - \left(\frac{1}{2a} - \frac{1}{4a^2} \right) e^{2a},\]

\[I_2 = \frac{1}{2a} J_0 = \frac{1}{2a} (e^{2atx} - e^{2a}),\]

\[I_1 = \frac{1}{2a} J_{-1} + \frac{1}{2a} I_0 = \frac{1}{2a} (xe^{2atx} - e^{2a}) + \frac{1}{2a} I_0, \quad I_{-1} = a I_0 - \frac{1}{2} J_{-2}.\]

**Proof.** For \(m \neq 1\), we have

\[I_m(x) = \left[\frac{e^{2atx}}{m-1 x^m}\right]_x^1 + \int_x^1 \frac{-2a e^{2at}}{(m-1)x^{m-1}} e^{2atx} \, dt\]

\[= \frac{1}{m-1} \left(\frac{e^{2atx}}{x^m} - e^{2a}\right) - \frac{2a}{m-1} I_{m+1}(x),\]

which implies the first formula. We can see that this holds for \(m = 1\). The others are easily obtained. \(\Box\)

Now, going back to (3.13) and the assumption \(f_s(x) > 0\) for \(\frac{1}{2} \leq x < 1\), we obtain
that is

\[ F(x) > \frac{x^2 e^{2a(1-\frac{1}{2}x)}}{x^2 e^{-2ax} x} \times \]

\[ \left[ \int_x^{e^{2a(1-\frac{1}{2}x)}} e^{2at} \left( \frac{1-t^2}{t^4} (t^2 + t - a(1-t^2)) - \frac{5}{2} a \frac{1-t^2}{t^4} \right) dt \right] \]

\[ = x^2 e^{2a(1-\frac{1}{2}x)} + x^2 e^{-2ax} \int_x^{e^{2a(1-\frac{1}{2}x)}} e^{2at} \left( -3a I_4 + \frac{1}{2} I_5 + \frac{7a}{2} I_2 - \frac{a}{2} I_0 - \frac{1}{2} I_1 \right) dt \]

\[ = x^2 e^{2a(1-\frac{1}{2}x)} + x^2 e^{-2ax} \left[ -3a \left( \frac{1}{2ax^2} - \frac{1}{2a^2 x} + \frac{1}{4a^3} \right) e^{2ax} \right. \]

\[ + 3a \left( -\frac{1}{2a} - \frac{1}{2a^2} + \frac{1}{4a^3} \right) e^{2ax} + \frac{1}{2} \left( \frac{1}{2ax^2} - \frac{1}{4ax^2} \right) e^{2ax} - \frac{1}{2} \left( \frac{1}{2a} - \frac{1}{4a^2} \right) e^{2ax} \]

\[ + \frac{7a}{2} \left( \frac{1}{2} e^{2ax} - \frac{7a}{2} e^{2ax} - \frac{a}{2} I_0 - \frac{1}{2} \left( -\frac{1}{2} x^2 e^{2ax} + \frac{1}{2} e^{2ax} + a I_0 \right) \right] \]

\[ = x^2 e^{2a(1-\frac{1}{2}x)} \left( \frac{1}{2} - \frac{7}{4a} + \frac{7}{8a^2} \right) - \frac{3}{2} + \frac{7}{4a} x + \left( \frac{7}{4} - \frac{7}{8a^2} \right) x^2 \]

\[ + \frac{1}{4} x^4 - ax^2 e^{-2ax} I_0(x), \]

that is

\[ (3.19) \quad F(x) > \Lambda(x,a) \quad \text{for} \quad \frac{1}{2} \leq x < 1, \]

where

\[ \Lambda(x,a) := x^2 e^{2a(1-\frac{1}{2}x)} \left( \frac{1}{2} - \frac{7}{4a} + \frac{7}{8a^2} \right) - \frac{3}{2} + \frac{7}{4a} x + \left( \frac{7}{4} - \frac{7}{8a^2} \right) x^2 \]

\[ + \frac{1}{4} x^4 - ax^2 e^{-2ax} I_0(x). \]

**Lemma 2.** For \( \frac{1}{2} \leq x < 1, \ 0 < a \leq \frac{1}{2} \), we have

\[ I_0(x) = \int_x^{e^{2ax}} e^{2at} dt < 1 - x + a \log \left( x \left( \frac{3-2a}{3x-2a} \right)^{3} \right) \]

**Proof.** We have for \( 0 < x < 3 \)

\[ e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \cdots < 1 + x + \frac{x^2}{2} \left( \frac{x}{3} \right)^2 + \cdots \]

\[ = 1 + x + \frac{3x^2}{2(3-x)} \]

and since \( 2a / t \leq 2 \) for \( 0 < a \leq \frac{1}{2} \) and \( \frac{1}{2} \leq t < 1 \), we obtain for \( \frac{1}{2} \leq x < 1 \) and

\[ 0 < a \leq \frac{1}{2} \]
Using this fact, we have

\[ A(x,a) > \Gamma(x,a), \]

where

\[ \Gamma(x,a) = x^2 e^{2a/(1-x)} \left( \frac{1}{2} - \frac{7}{4a} + \frac{7}{8a^2} \right) - \frac{3}{2} + \frac{7}{4a} x + \left( \frac{7}{4} - \frac{7}{16a^2} \right) x^2 + \frac{1}{4} x^4 \]

\[ -ae^{-2a/a} \left( x^2 - x^3 + ax^2 \log \left( x \left( \frac{3-2a}{3x-2a} \right) \right) \right) \]

We shall show that \( \Gamma(x,a) \) is increasing in \( \frac{1}{2} < x < 1 \) in the following.

First we have

\[
\frac{\partial \Gamma(x,a)}{\partial x} = e^{2a/(1-x)} \left( \frac{1-7/2a+7/4a^2}{2a-2ax+2a^2 \log \left( x \left( \frac{3-2a}{3x-2a} \right) \right)} \right) + \frac{7}{4a} x + \left( \frac{7}{2} - \frac{7}{4a^2} \right) x^2 + \frac{1}{4} x^4 
\]

\[ -ae^{2a/a} \left[ 2a - 2ax + 2a^2 \log \left( x \left( \frac{3-2a}{3x-2a} \right) \right) + 2x - 3x^2 \right] 
\]

\[ + 2ax \log \left( x \left( \frac{3-2a}{3x-2a} \right) \right) - ax \left( \frac{9}{3x-2a} - \frac{1}{x} \right) \]

\[ = e^{-2a/a} \left[ e^{2a/2} \left( 1 - \frac{7}{2a} + \frac{7}{4a^2} \right) - 2a \left( 1 + a \log \left( x \left( \frac{3-2a}{3x-2a} \right) \right) \right) \right] (x+a) \]

\[ + a^2 x + 3ax^2 + \frac{9a^2 x^2}{3x-2a} + \frac{7}{2} + \frac{7}{4a^2} x + x^3. \]

Furthermore, the derivative of the expression in the brackets of the last side with respect to \( x \) is

\[ e^{2a/2} \left( 1 - \frac{7}{2a} + \frac{7}{4a^2} \right) - 2a \left( 1 + a \log \left( x \left( \frac{3-2a}{3x-2a} \right) \right) \right) \]

\[ + (x+a) \left[ 2a^2 \left( \frac{9}{3x-2a} - \frac{1}{x} \right) \right] + a^3 + 6ax + \frac{9a^2}{3x-2a} \left( \frac{2}{x} - \frac{3}{3x-2a} \right) \]
\[
e^{2a} \left( 1 - \frac{7}{2a} + \frac{7}{4a^2} \right) - 1 - 2a + 7a - 2a \log \left( x \left( \frac{3 - 2a}{3x - 2a} \right)^{3} \right) = 10a \]
\[
\frac{27a + 30a^3}{3x - 2a} \cdot \frac{27a^2}{(3x - 2a)^2},
\]
which is positive for \( \frac{1}{2} \leq x < 1 \) and \( 0 < a \leq \frac{1}{2} \) by numerical approximations. This can be assured by elementary arguments omitted here. Making use of this property, we can prove the positiveness of \( \partial \Gamma(x,a) / \partial x \) for \( \frac{1}{2} \leq x < 1 \) and \( 0 < a \leq \frac{1}{2} \) by dividing this range into smaller ones. For example, for \( 0.5 \leq x \leq 0.6 \) we obtain the inequality
\[
\frac{\partial \Gamma(x,a)}{\partial x} > e^{-4a} \left[ e^{2a} \left( 1 - \frac{7}{2a} + \frac{7}{4a^2} \right) - a \left( \frac{6}{5} + 2a \right) \right] (1 + 3a \log (3 - 2a))
\[
+ a^2 (1 + 2a) \log \left( \frac{3 - 4a}{4} \right) + \frac{3}{2} + 3a^2 + \frac{9a^2}{6 - 8a} \left( \begin{array}{c}
\text{for} \ 0 < a < \frac{3}{8},
\text{for} \ 0 < a < \frac{1}{2},
\end{array} \right)
\]
\[
\left( \begin{array}{c}
8a^3 \left( \text{for} \ \frac{3}{8} \leq a < \frac{1}{2} \right) + \frac{7}{4a} \left( \frac{7}{2} - \frac{7}{4a^2} \right) \times \frac{3}{5} + \left( \frac{3}{2} \right)^{3/2},
\end{array} \right)
\]
since \( \log \frac{3x - 2a}{x} \) is increasing in \( 1/2 <? x < 1 \); \( \frac{x^2}{3x - 2a} \) is increasing in \( 1/2 < x < 1 \). when \( 0 < a < 3/8 \), and \( \frac{x^2}{3x - 2a} \) is decreasing in \( 1/2 < x < 4a/3 \) and increasing in \( 4a/3 < x < 1 \) when \( 3/8 < a < 1/2 \), and \( \frac{7}{4a} + \left( \frac{7}{2} - \frac{7}{4a^2} \right) x^3 \) is decreasing in \( 1/2 < x < 1 \). Therefore
\[
\frac{9a^2 x^2}{3x - 2a} \geq -\frac{9a^2}{6 - 8a} \text{ when } 0 < a < 3/8 \text{ and } \geq 8a^3 \text{ when } 3/8 < a < 1/2. \]
The above expression of \( a \) is positive by numerical approximations and also assured by elementary arguments omitted here. Omitting analogous results on the other ranges, we see that \( \Gamma(x,a) \) is increasing in \( 1/2 < x < 1 \) for \( 0 < a \leq 1/2 \).

Hence, we have for \( 1/2 \leq x < 1 \)
\[
\Gamma(x,a) \geq \Gamma \left( \frac{1}{2}, a \right) = \frac{1}{4} e^{-2a} \left( \frac{1}{2} - \frac{7}{4a} + \frac{7}{8a^2} \right) + \frac{3}{2} + \frac{7}{8a} + \frac{1}{4} \left( 4 - \frac{7}{8a^2} \right) + \frac{1}{64}
\]
\[
- ae^{-4a} \left( \frac{1}{4} - \frac{1}{8} + \frac{a}{4} \log \left( 4 \left( \frac{3 - 2a}{3 - 4a} \right)^{3} \right) \right)
\]
\[
= \frac{1}{4} e^{-2a} \left( \frac{1}{2} - \frac{7}{4a} + \frac{7}{8a^2} \right) - \frac{67}{64} + \frac{7}{8a} - \frac{7}{32a^2} - ae^{-4a} \left( \frac{1}{8} + \frac{a}{4} \log \left( 4 \left( \frac{3 - 2a}{3 - 4a} \right)^{3} \right) \right).
\]
whose right hand side is positive for $0 < a \leq 0.3$ and negative for $0.325 \leq a \leq 1/2$.

Next, we have for $0.6 = 3/5 < x < 1$

$$
\Gamma(x,a) \geq \Gamma \left( \frac{3}{5}, a \right) = \frac{9}{25} e^{-4a/5} \left( \frac{1}{2} - \frac{7}{4} \frac{x}{a^2} + \frac{7}{8a^2} \right) - \frac{3}{2} \frac{21}{20a} + \frac{9}{25} \left( \frac{7}{4} \frac{x}{a^2} - \frac{7}{8a^2} \right)
$$

$$
+ \frac{81}{2500} - ae^{-10a/11} \left( \frac{9}{25} - \frac{27}{125} + \frac{9a}{25} \log \left( \frac{75}{9} \frac{3 - 2a}{9 - 10a} \right) \right)
$$

$$
= \frac{9}{25} e^{-4a/5} \left( \frac{1}{2} - \frac{7}{4} \frac{x}{a^2} + \frac{7}{8a^2} \right) - \frac{1047}{1250} + \frac{21}{20a} - \frac{63}{200a^2}
$$

$$
- ae^{-10a/11} \left( \frac{18}{125} + \frac{9a}{25} \log \left( \frac{75}{9} \frac{3 - 2a}{9 - 10a} \right) \right),
$$

whose right hand side > $0.043172$ for $0 < a \leq 1/2$ and $>0.111897$ for $0 < a \leq 0.4 = 2/5$. Finally, for $0.8 = 4/5 \leq x < 1$ we obtain analogously that

$$
\Gamma(x,a) \geq \Gamma \left( \frac{4}{5}, a \right) = \frac{16}{25} e^{-4a/5} \left( \frac{1}{2} - \frac{7}{4} \frac{x}{a^2} + \frac{7}{8a^2} \right) - \frac{347}{1250} + \frac{7}{5a} - \frac{14}{25a^2}
$$

$$
- ae^{-5a/12} \left( \frac{16}{125} + \frac{16a}{25} \log \left( \frac{25(3 - 2a)}{6 - 5a} \right) \right),
$$

whose right hand side > $0.474459$ for $0 < a \leq 1/2$ and $> 0.511087$ for $0 < a \leq 0.4$. We can obtain these inequalities by numerical approximations and these processes are guaranteed by the following lemma.

**Lemma 3.**

$$
\frac{\partial \Gamma(x,a)}{\partial a} < 0 \quad \text{for} \quad \frac{1}{2} < x < 1 \quad \text{and} \quad 0 < a < \frac{1}{2}.
$$

**Proof.** From (3.22) we obtain

$$
\frac{\partial \Gamma(x,a)}{\partial a} = x^2 e^{2a(1-1/2)} \left( \left( x \frac{1}{x} \right) \left( \frac{7 - 7}{2a + 4a^2} \right) + \frac{7}{4a^2} \right)
$$

$$
- \frac{7}{4a^2} x^2 + \frac{7}{4a^3} x^2 - e^{-2a/5} \left( \frac{1}{x} \frac{2a}{x} \right) x^3 - x^3 + ax^2 \log \left( x \left( \frac{3 - 2a}{3x - 2a} \right) \right)
$$

$$
- ae^{-2a/5} \left( x^2 \log \left( x \left( \frac{3 - 2a}{3x - 2a} \right) \right) + \frac{6}{3 - 2a} + \frac{6}{3x - 2a} \right)
$$

$$
= x^2 e^{2a(1-1/2)} \left( \frac{1}{2a - 2a} - \frac{7}{4a^2} + \frac{7}{2a} - \frac{1}{x} \frac{7}{2a + 4a^2} \right) + \frac{7}{4a^2} x^2 + \frac{7}{4a^3} x^2
$$

$$
- e^{-2a/5} \left( x - 2a \right) x \left( x - 1 \right) + \frac{18a^2 x^2 (1 - x)}{(3 - 2a)(3x - 2a)} + 2ax(x - a) \log \left( x \left( \frac{3 - 2a}{3x - 2a} \right) \right).
$$
In the above expression, we can prove easily that
\[
(x - 2a)x(1 - x) + \frac{18a^2 x^2 (1 - x)}{(3 - 2a)(3x - 2a)} \quad \text{and} \quad x(x - a)\log\left(x\left(\frac{3 - 2a}{3x - 2a}\right)^3\right)
\]
are positive for \(\frac{1}{2} \leq x \leq 1\) and \(0 < a \leq \frac{1}{2}\). We shall show the first line is negative there.

In fact, multiplying it by \(a^2/x\) we have
\[
e^{2a(t-t')}\left(x\left(a^2 - \frac{7}{2}(a^2 - a + \frac{1}{2})\right) - a^3 + \frac{7}{2}(a^2 - a + \frac{1}{2}) + \frac{7}{4}(x - a)\right)
\]
\[
e^{2a(t-t')}\left(-a\left(a^2 - \frac{7}{2}a + \frac{7}{2}\right)(1 - x) - \frac{7}{4}(x - a)\right) + \frac{7}{4}(x - a)
\]
\[
= \frac{7}{4}(x - a)\left(1 - e^{-2a(t-t')}\right) - e^{-2a(t-t')}a\left(a^2 - \frac{7}{2}a + \frac{7}{2}\right)(1 - x).
\]

Setting \(t = \frac{2a(1 - x)}{x}\), we have \(0 < t < 2a < 1\) and
\[
e' - 1 = t + \frac{t^2}{2} + \frac{t^3}{6} + \cdots + t\left(\frac{1-t}{3}\right) = \frac{6t + t^2}{2(3-t)}
\]
\[
= \frac{2a(1 - x)(3x + a(1 - x))}{x(3x - 2a(1 - x))}.
\]

Using these facts, we have
\[
\frac{7}{4}(x - a)\left(e' - 1\right) - a\left(a^2 - \frac{7}{2}a + \frac{7}{2}\right)(1 - x)
\]
\[
< a\left(\frac{7(x - a)(3x + a(1 - x))}{2x(3x - 2a(1 - x))} - \left(a^2 - \frac{7}{2}a + \frac{7}{2}\right)\right)(1 - x),
\]

whose negativeness is derived from the inequality:
\[
7(x - a)((3 - a)x + a) - (2a^2 - 7a + 7)x((3 + 2a)x - 2a)
\]
\[
= 4a^2(2 - a)x^2 - a^2(7 - 4a)x - 7a^2
\]
\[
= a^2(4(2 - a)x^2 - (7 - 4a)x - 7) < 0,
\]
since \(x(3x - 2a(1 - x)) > 0\) for \(1/2 < x < 1\) and \(0 < a < 1/2\), and
\[
4(2 - a)x^2 - (7 - 4a)x - 7_{x=1/2} = a - \frac{17}{2} < 0
\]
\[
4(2 - a)x^2 - (7 - 4a)x - 7_{x=1} = -6, \quad 2 - a > 0.
\]

Hence, we obtained the inequality:
\[
\frac{\partial \Gamma(x,a)}{\partial a} < 0 \quad \text{for} \quad \frac{1}{2} < x < 1 \quad \text{and} \quad 0 < a < \frac{1}{2}.
\]

Q.E.D.
§ 4. Proof of Theorem 2

In this section, we shall complete the proof of Theorem 2. Considering the arguments and results in the previous sections, for a fixed \( x_0 \left( \frac{1}{2} \leq x_0 < 1 \right) \) and \( 0 < a \leq \frac{1}{2} \), we may suppose that

\[
0 < M_1 \leq f_i(x) \leq M_2 \quad \text{for} \quad x_0 \leq x \leq 1, \quad i = 1, 2, \ldots, n.
\]

Here we can put

\[
M_1 = \sqrt{1(x_0, a)}, \quad M_2 = \frac{\sqrt{5}}{2} = 1.180\ldots
\]

Then we have from (3.13) the recurrence formulas:

\[
f_{n+1}^2(x) = x^2 e^{2a(1-\sqrt{x})} + x^2 e^{-2a\sqrt{x}} \left[ \int_x^1 \frac{e^{2a\sqrt{t}}}{2t} \left( 1 - t^2 \right) \left( t^2 + t - a \left( 1 - t^2 \right) \right) dt \right] - 2a \int_x^1 \frac{e^{2a\sqrt{t}}}{t} \left( 1 - t^2 \right) f_n(t) dt
\]

and

\[
f_n^2(x) = x^2 e^{2a(1-\sqrt{x})} + x^2 e^{-2a\sqrt{x}} \left[ \int_x^1 \frac{e^{2a\sqrt{t}}}{2t} \left( 1 - t^2 \right) \left( t^2 + t - a \left( 1 - t^2 \right) \right) dt \right] - 2a \int_x^1 \frac{e^{2a\sqrt{t}}}{t} \left( 1 - t^2 \right) f_{n-1}(t) dt.
\]

From which we obtain

\[
f_{n+1}^2(x) - f_n^2(x) = -2ax^2 e^{-2a\sqrt{x}} \int_x^1 \frac{e^{2a\sqrt{t}}}{t} \left( 1 - t^2 \right) f_n(t) dt
\]

and

\[
\left| f_{n+1}(x) - f_n(x) \right| \leq \frac{2ax^2 e^{2a\sqrt{x}}}{M_1} \left[ \int_x^1 e^{2a\sqrt{t}} \left( 1 - t^2 \right) f_n(t) dt \right].
\]

Using the notation

\[
1f_a = \max_{x \geq 0} f_a(x),
\]

we obtain by (4.1)

\[
1f_{n+1}(x) - f_n(x) \leq \frac{a}{M_1} \left[ 1f_a - f_{n-1} \right] x^2 e^{-2a(1-\sqrt{1-x^2})} \left( I_4(x) - I_2(x) \right),
\]

where \( I_4(x) \) and \( I_2(x) \) are defined in Lemma 1, and

\[
I_4(x) - I_2(x) = \frac{1}{2a} \left( \frac{1}{2a^2} - \frac{1}{2} \right) + \frac{1-2a^2}{4a^3} - e^{2a} \left( \frac{1}{2a} - \frac{1}{2a^2} + \frac{1-2a^2}{4a^3} \right).
\]

Therefore the right hand side of the above inequality turns into
For \( t = \frac{2a(1-x)}{x} \), we have \( 0 < t < 2a < 1 \) and so

\[
e^{-t} = 1 - t + \frac{1}{2} t^2 + \frac{1}{6} t^3 + \ldots > 1 - t + \frac{1}{2} t^2 - \frac{1}{6} t^3.
\]

the expression in the brackets is less than

\[
\frac{1}{2} \frac{x}{2a} + \frac{1-2a^2}{4a^2} x^2 - \frac{1-2a}{4a^2} x^2 \left( 1 - \frac{2a(1-x)}{x} \right) + \frac{2a^2(1-x)^2}{x^2} - \frac{4a^3(1-x)^3}{3x^3}
\]

\[
= \frac{1}{2} \frac{x}{2a} + \frac{1-2a^2}{4a^2} x^2 - \frac{1-2a}{4a^2} x^2 + \frac{(1-2a)x(1-x)}{2a} \frac{(1-2a)(1-x)^2}{2} + \frac{a(1-2a)(1-x)^3}{3x}
\]

\[
= 2a^2 - a(1+2a)x + \frac{2}{3} a(1+a)x^2 + \frac{a(1-2a)}{3x}.
\]

Thus we obtain the inequality

\[
(4.2) \quad |f_{n+1}(x) - f_n(x)| < \frac{a}{M_1} \Phi(x,a) f_n - f_{n+1}^m,
\]

where

\[
(4.3) \quad \Phi(x,a) = 2a - (1+2a)x + \frac{2}{3}(1+a)x^2 + \frac{1-2a}{3x}.
\]

Now, regarding \( \Phi(x,a) \) we have

\[
\Phi\left(\frac{1}{2},a\right) = \frac{1}{3} - \frac{a}{6}, \quad \Phi(1,a) = 0.
\]

and

\[
\frac{\partial \Phi(x,a)}{\partial x} = -(1+2a) + \frac{4}{3}(1+a)x - \frac{1-2a}{3x^2}
\]

\[
= \frac{1}{3x^2}\left[4(1+a)x^3 - 3(1+2a)x^2 - (1-2a)\right] < 0
\]

for \( 0 < x < 1 \), because the cubic polynomial of \( x \):

\[
P(x) = 4(1+a)x^3 - 3(1+2a)x^2 - (1-2a)
\]

has the property as

\[
P(0) = -(1-2a) < 0, \quad P'(1) = 0,
\]

\[
P'(x) = 6x(2(1+a)x - (1+2a)),
\]

and
and hence we get
\[ P(x) < 0 \quad \text{for} \quad 0 < x < 1. \]
Consequently we see that
\[ \frac{\partial \Phi(x,a)}{\partial x} < 0 \quad \text{for} \quad 0 < x < 1 \]
and
\[ \frac{\partial \Phi(x,a)}{\partial x} |_{x=1} = 0. \]
Finally, we put
\[ |f_{n+1} - f_n|^M_x = |f_{n+1}(x) - f_n(x)| \quad \text{with} \quad x \leq x_1 < 1. \]
Then, we obtain by (4.2)
\[ |f_{n+1} - f_n|^M_x < \frac{a}{M_1} \Phi(x_1,a) |f_n - f_{n-1}|^M_x \]
\[ \leq \frac{a}{M_1} \Phi(x,a) |f_n - f_{n-1}|^M_x \]
since
\[ \Phi(x,a) \geq \Phi(x_1,a), \quad |f_n - f_{n-1}|^M_x \geq |f_n - f_{n-1}|^M_{x_1}. \]
Thus, we obtain the fundamental inequality:
\[ |f_{n+1} - f_n|^M_x < \frac{a}{M_1} \Phi(x,a) |f_n - f_{n-1}|^M_x \]
\[ \text{for} \quad \frac{1}{2} \leq x_0 \leq x < 1 \quad \text{and} \quad 0 < a < \frac{1}{2}, \]
from which we obtain
\[ \frac{|f_{n+1} - f_n|^M_x}{|f_n - f_{n-1}|^M_x} < \frac{a}{M_1} \Phi(x,a) < \frac{a}{M_1} \left( \frac{1}{3} - \frac{a}{6} \right). \]
Therefore if we take
\[ x_1(x_0 \leq x_1 < 1) \quad \text{for a small positive number} \quad \delta > 0 \quad \text{such that} \]
\[ \frac{a}{M_1} \Phi(x_1,a) \leq 1 - \delta \]
which is possible by the above mentioned property of \( \Phi(x,a) \), then we have
\[ \frac{|f_{n+1} - f_n|^M_x}{|f_n - f_{n-1}|^M_x} < 1 - \delta \quad x_1 \leq x < 1, \]
which tells us that \( \{f_n(x)\} \) must tend uniformly to a function \( f(x) \) satisfying the differential equation (3.6) in \( x_1 \leq x < 1 \).

**Q.E.D.**

**Supplement.** Regarding the title with the general connections of this paper, we mention the following remark. In [9] we treated the spaces \( H^*_m \) and \( MI^*_m \) with general connections on \( \mathbb{R}^n \) related with \( H^* \) and \( MI^* \) with the metric:
\[ ds^2 = \sum_{i=1}^{n} dx_i dx_i / (x_i)^2 \]

and

\[ ds^2 = \left( \sum_{a=1}^{n-1} dx_a dx_a - c^2 dx_n dx_n \right) / (x_n)^2 \]

respectively, on \( \mathbb{R}^n \). In a continuing paper, we shall construct spaces with general connections on \( \mathbb{R}^4 \) related with the space with the metric:

\[ ds^2 = \frac{1}{(x_4)^3} \left( \frac{1}{Q} dr dr + r^2 \sum_{a,b=2}^5 h_{ab} du^a du^b - P dx_4 dx_4 \right) \]

on \( \mathbb{R}^4 \), where \( Q \) is a solution in Theorem 1 and Theorem 2, and investigate the properties of their geodesics.

References


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