SUT Journal of Mathematics
Vol. 43, No. 2 (2007), 201–213

Warped product CR-submanifolds in nearly Kaehler manifolds

Viqar Azam Khan, Khalid Ali Khan and Siraj-Uddin

(Received September 6, 2007)

Abstract. In this paper we study warped product CR-submanifolds in a nearly Kaehler manifold and extend the results of B.Y. Chen [7] concerning warped product CR-submanifolds in Kaehler manifolds to this more general setting.

AMS 2000 *Mathematics Subject Classification.* 53C40, 53B25, 53C25.

Key words and phrases. Nearly Kaehler, CR-submanifold, Warped product CR-submanifolds.

§1. Introduction

In [2], R.L. Bishop and B. O'Neill introduced the notion of warped product manifolds by homothetically warping the product metric of a manifold $B \times F$ on to the fibers $p \times F$ for each $p \in B$. The generalized product metric so obtained appears in differential geometric studies in a natural way. For instance, a surface of revolution is a warped product manifold. So far as it's applications are concerned, it has been shown that the warped product manifolds provide an excellent setting to model space-time around bodies with high gravitational field (cf. [16]). In view of this fact many research articles have recently appeared exploring existence or non existence of warped product submanifolds in known spaces. B.Y. Chen [7] initiated the investigations by showing that there doesn't exist warped product CR-submanifold in Kaehler manifolds. B. Sahin [20], extending the result of Chen, proved that there exist no semi-slant warped product submanifolds in a Kaehler manifold. However, many examples of CR-warped product submanifolds (obtained by reversing the two factors of the warped product CR-submanifold) of Kaehler manifolds are provided in [9] and [20]. In view of the interesting geometric features of nearly Kaehler manifolds and the non-existence of CR-product submanifolds in S^6 (cf. [22]), it is worthwhile to explore warped product CR and CR-warped product submanifolds in a nearly Kaehler manifold in general. In this pursuit, we have succeeded in extending the results of Chen to the setting of nearly Kaehler manifolds.

§2. Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with an almost complex structure J and Hermitian metric g i.e., for all $U,\ V \in T\bar{M}$

$$J^2 = -I, \qquad g(U,V) = g(JU,JV). \tag{2.1}$$

Further, let Ω be the fundamental 2-form associated to the Hermitian metric g on $\bar{M}$ i.e.,

$$\Omega(U,V) = g(JU,V). \tag{2.2}$$

The following is a useful relation exhibiting the relationship among Ω, $\bar{\nabla}J$ and the Nijenhuis tensor S of J (cf. [12]).

$$2g((\bar{\nabla}_U J)V,W) = d\Omega(U,V,W) - d\Omega(U,JV,JW) - g(U,S(V,JW)) \tag{2.3}$$

where the Nijenhuis tensor S of J is defined by

$$S(U,V) = [U,V] + J[JU,V] + J[U,JV] - [JU,JV]. \tag{2.4}$$

Let M be a submanifold of $\bar{M}$. For each $x \in M$, let $D_x = T_xM \cap JT_xM$ i.e., a maximal holomorphic subspace of the tangent space T_xM. If the dimension of D_x remains the same for each $x \in M$, and it defines a differentiable distribution D on M then M is said to be a *generic submanifold*. In addition, if the complementary distribution $D^\perp$ is totally real i.e., $JD^\perp \subseteq T^\perp M$, then M is said to be a *CR-submanifold* of M, where $T^\perp M$ denotes the normal bundle on M which admits the orthogonal direct decomposition

$$T^\perp M = JD^\perp \oplus \mu$$

It is straightforward to see that the orthogonal complementary distribution μ of $JD^\perp$ in $T^\perp M$ is an invariant subbundle of $T^\perp M$. Let $\bar{\nabla}$ be the Riemannian connection on $\bar{M}$. Let $\bar{\nabla}$ be the Levi-Civita connection on $T\bar{M}$ and ∇, $\nabla^\perp$ be the induced connections on the tangent bundle TM and the normal bundle $T^\perp M$ respectively. Further, if h and A_ξ denote the second fundamental form and the shape operator (corresponding to a normal vector field ξ) respectively then the Gauss and Weingarten formulae are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U,V), \tag{2.5}$$

$$\bar{\nabla}_U \xi = -A_\xi U + \nabla^\perp_U \xi \tag{2.6}$$

for each U, $V \in TM$. A_ξ and h are related as

$$g(A_\xi U, V) = g(h(U, V), \xi) \tag{2.7}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the induced Riemannian metric on M. The mean curvature vector H is given by

$$H = \sum_{i=1}^{n} h(e_i, e_i)$$

where n is the dimension of M and $\{e_1, \dots, e_n\}$ is a local orthonormal frame of vector fields on M. A submanifold M of $\bar{M}$ is said to be *totally geodesic submanifold* if $h(U, V) = 0$, for each $U, V \in TM$ and a submanifold is said to be *totally umbilical submanifold* if $h(U, V) = g(U, V)H$.

For U, $V \in TM$ and $\xi \in T^\perp M$, we decompose JU and $J\xi$ into tangential and normal parts as

$$JU = PU + FU, \tag{2.8}$$

$$J\xi = t\xi + f\xi. \tag{2.9}$$

Thus, P is a $(1, 1)$ tensor field on TM, F is a normal valued 1-form on TM, t is a tangential valued 1-form on $T^\perp M$ and f is a $(1, 1)$ tensor field on $T^\perp M$. Further, it is straightforward to observe that on a CR-submanifold M, $P(TM) \subseteq D$, $FD = 0$, $t(T^\perp M) = D^\perp$ and $f(T^\perp M) \subseteq \mu$.

The covariant derivatives of P, F, t and f are defined as

$$(\bar{\nabla}_U P)V = \nabla_U PV - P\nabla_U V, \tag{2.10}$$

$$(\bar{\nabla}_U F)V = \nabla^\perp_U FV - F\nabla_U V, \tag{2.11}$$

$$(\bar{\nabla}_U t)\xi = \nabla_U t\xi - t\nabla^\perp_U \xi, \tag{2.12}$$

$$(\bar{\nabla}_U f)\xi = \nabla^\perp_U f\xi - f\nabla^\perp_U \xi. \tag{2.13}$$

Furthermore, let $\mathcal{P}_U V$ and $\mathcal{Q}_U V$ denote respectively the tangential and the normal parts of $(\bar{\nabla}_U J)V$. Then by an easy computation, we obtain the following formulae

$$\mathcal{P}_U V = (\bar{\nabla}_U P)V - A_{FV}U - th(U, V), \tag{2.14}$$

$$\mathcal{Q}_U V = (\bar{\nabla}_U F)V + h(U, PV) - fh(U, V). \tag{2.15}$$

Similarly, for $\xi \in T^{\perp}M$, denoting the tangential and normal parts of $(\bar{\nabla}_U J)\xi$ by $\mathcal{P}_U\xi$ and $\mathcal{Q}_U\xi$, we find that

$$\mathcal{P}_U\xi = (\bar{\nabla}_U t)\xi + PA_\xi U - A_{f\xi}U, \tag{2.16}$$

$$\mathcal{Q}_U\xi = (\bar{\nabla}_U f)\xi + h(t\xi, U) + FA_\xi U. \tag{2.17}$$

It is straightforward to verify the following properties, which we enlist here for later use

p_1. (a) $\mathcal{P}_{U+V}W = \mathcal{P}_U W + \mathcal{P}_V W$, (b) $\mathcal{Q}_{U+V}W = \mathcal{Q}_U W + \mathcal{Q}_V W$.

p_2. (a) $\mathcal{P}_U(V + W) = \mathcal{P}_U V + \mathcal{P}_U W$, (b) $\mathcal{Q}_U(V + W) = \mathcal{Q}_U V + \mathcal{Q}_U W$.

p_3. (a) $g(\mathcal{P}_U V, W) = -g(V, \mathcal{P}_U W)$, (b) $g(\mathcal{Q}_U V, \xi) = -g(V, \mathcal{P}_U\xi)$.

p_4. $\mathcal{P}_U JV + \mathcal{Q}_U JV = -J(\mathcal{P}_U V + \mathcal{Q}_U V)$.

§3. Some basic results

A nearly Kaehler structure on an almost Hermitian manifold $\bar{M}$ is characterized by the condition

$$(\bar{\nabla}_U J)U = 0 \tag{3.1}$$

for each $U \in T\bar{M}$.

A typical example of a nearly Kaehler non-Kaehler manifold is the six dimensional sphere S^6. It has an almost complex structure J defined by the vector cross product in the space of purely imaginary Caley numbers which satisfies the condition $(\bar{\nabla}_U J)U = 0$. We recall this almost complex structure here for later use. Let C be the Cayley division algebra generated by $\{e_0 = 1, e_i, (1 \leq i \leq 7)\}$ over $\mathbb{R}$ and C_+ be the subspace of C consisting of all purely imaginary Cayley numbers. We may identify C_+ with a 7-dimensional Euclidean space $\mathbb{R}^7$ with the canonical inner product $g = (\, , \,)$. The automorphism group of C is the compact simple Lie-group G_2 and furthermore the inner product g is invariant under the action of G_2 and hence, the group G_2 may be considered as a subgroup of $SO(7)$. A vector cross product for vectors in $\mathbb{R}^7$ $(= C_+)$ is defined by

$$x \times y = (x, y)e_0 + xy, \quad \forall \; x, y \in C_+. \tag{3.2}$$

Then the multiplication table is given by

$e_j \times e_k =$

j/k	1	2	3	4	5	6	7
1	0	e_3	$-e_2$	e_5	$-e_4$	e_7	$-e_6$
2	$-e_3$	0	e_1	e_6	$-e_7$	$-e_4$	e_5
3	e_2	$-e_1$	0	$-e_7$	$-e_6$	e_5	e_4
4	$-e_5$	$-e_6$	e_7	0	e_1	e_2	$-e_3$
5	e_4	e_7	e_6	$-e_1$	0	$-e_3$	$-e_2$
6	$-e_7$	e_4	$-e_5$	$-e_2$	e_3	0	e_1
7	e_6	$-e_5$	$-e_4$	e_3	e_2	$-e_1$	0

Considering S^6 as $\{x \in C_+ : (x, x) = 1\}$, the almost complex structure J on S^6 is defined by

$$J_x(U) = x \times U \tag{3.3}$$

where $x \in S^6$ and $U \in T_xS^6$. The almost complex structure defined in (3.3) together with the induced metric on S^6 from g on $\mathbb{R}^7$ $(= C_+)$ gives rise to a nearly Kaehler structure on S^6 (cf. [11]).

On a submanifold M of a nearly Kaehler manifold $\bar{M}$, it follows from (3.1) that

$$\text{(a)}\ \ \mathcal{P}_UV + \mathcal{P}_VU = 0, \qquad \text{(b)}\ \ \mathcal{Q}_UV + \mathcal{Q}_VU = 0 \tag{3.4}$$

for each U, $V \in TM$.

So far as CR-submanifold of nearly Kaehler manifolds are concerned, we have

Theorem 3.1 [18]. *The holomorphic distribution D on a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ is integrable if and only if*

$$\mathcal{Q}_XY = 0 \quad \textit{and} \quad h(X, JY) = h(JX, Y)$$

for each X, $Y \in D$.

Theorem 3.2 [18]. *The totally real distribution $D^\perp$ on a CR-submanifold of a nearly Kaehler manifold is integrable if and only if*

$$g(\mathcal{P}_ZW, X) = 0,$$

or,

$$g(A_{JZ}W, X) = g(A_{JW}Z, X)$$

for each $Z, W \in D^\perp$ and $X \in D$.

A submanifold M of an almost Hermitian manifold is said to be a *CR-product submanifold* if M is locally a Riemannian product of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$ of $\bar{M}$. Thus, a CR-submanifolds M of an almost Hermitian manifold is a CR-product if and only if both the distributions D and $D^\perp$ on M are integrable and their leaves are totally geodesic in M. In other words, a CR-submanifold M is a CR-product in $\bar{M}$ if and only if $\nabla_U X \in D$ (or equivalently $\nabla_U Z \in D^\perp$) for each $U \in TM,\ X \in D$ and $Z \in D^\perp$. CR-submanifolds of Kaehler manifolds are studied extensively by B.Y. Chen (cf. [3], [4] and [5] etc.). He obtained conditions under which a CR-submanifold reduces to a CR-product. K.A. Khan et.al [18] extended some of these conditions to the setting of CR-submanifolds of nearly Kaehler manifolds. On the other hand, K. Sekigawa ([20], [21]) studied submanifolds of S^6. In particular, he showed that there do not exist CR-product submanifolds in S^6 and thus paved way to explore CR-warped product submanifolds in S^6. He did construct an example of a CR-warped product submanifold in S^6. However, N. Ejiri [10] provided a categorical answer to the more general problem by proving that there exist countably many immersions of $S^1 \times S^{n-1}$ into S^{n+1} such that the induced metric on it, is a warped product of constant scalar curvature $n(n-1)$.

Our aim, in the succeeding sections is to study the warped product submanifolds in nearly Kaehler manifolds. To begin the proceedings, we first recall the formal definition of a warped product manifold (cf. [2]).

Definition 3.1. Let (B, g_1) and (F, g_2) be two Riemannian manifolds with Riemannian metric g_1 and g_2 respectively and f a positive differentiable function on B. The warped product of B and F is the Riemannian manifold $B \times_f F = (B \times F, g)$, where

$$g = g_1 + f^2 g_2. \tag{3.5}$$

More explicitly, if U is tangent to $M = B \times_f F$ at (p, q), then

$$\|U\|^2 = \|d\pi_1 U\|^2 + f^2(p)\|d\pi_2 U\|^2$$

where π_i $(i = 1, 2)$ are the canonical projections of $B \times F$. onto B and F respectively and $d\pi_i{}'s$ are their differentials.

For a differentiable function ϕ on a manifold M, the grad(ϕ) is defined as

$$g(\text{grad}(\phi), U) = U\phi \tag{3.6}$$

for any vector field U tangent to M.

Bishop and O'Neill obtained the following basic result for warped product manifolds.

Theorem 3.3 [2]. *Let $M = B\times_f F$ be a warped product manifold. If $X, Y \in TB$ and $V, W \in TF$, then*

(i) $\nabla_X Y \in TB$,

(ii) $\nabla_X V = \nabla_V X = (\frac{Xf}{f})V$,

(iii) $\mathrm{nor}(\nabla_V W) = \frac{-g(V,W)}{f}\mathrm{grad}(f)$.

where $\mathrm{nor}(\nabla_V W)$ *is the component of* $\nabla_V W$ *in* TB *and* $\mathrm{grad}(f)$ *is the gradient of* f.

Let $M = B\times_f F$. If h_1 and h_2 are the second fundamental forms of the immersions of B and F respectively into M, then for any $X \in TB$ and Z, $W \in TF$,

$$g(h_2(Z,W),X) = -g(\nabla_X Z, W) \tag{3.7}$$

which on making use of Theorem 3.3 takes the form

$$g(h_2(Z,W),X) = -(X\ln f)g(Z,W).$$

As $h_2(Z,W) \in TB$, above equation on applying formula (3.6) yields

$$h_2(Z,W) = -g(Z,W)\mathrm{grad}(\ln f)$$

which by virtue of the formula (3.5), is written as

$$h_2(Z,W) = -f^2 g_2(Z,W)\mathrm{grad}(\ln f). \tag{3.8}$$

That shows that F is totally umbilical in M. Now, we may complete the statement of Theorem 3.3 by stating

Corollary 3.1 [2]. *On a warped product manifold $M = B\times_f F$,*

(i) *B is totally geodesic in M,*

(ii) *F is totally umbilical in M.*

If the manifolds N_T and $N_\perp$ are holomorphic and totally real submanifolds of an almost Hermitian manifold $\bar{M}$ respectively, then their warped products are

(a) $N_\perp \times_f N_T$,

(b) $N_T \times_f N_\perp$.

In the sequel, we call the warped product submanifolds of type (a) as warped product CR-submanifold and the warped products of type (b) as CR-warped product submanifold.

The notion of warped product manifolds was introduced as a natural generalization to Riemannian product of Riemannian manifolds. It is easy to observe that the warped product of two Riemannian manifolds is a Riemannian product if the warping function f is constant. In particular, the warped product submanifolds of type (a) and (b) reduce to CR-product submanifolds when the warping function f is a constant function.

§4. CR-submanifold as warped product submanifold in nearly Kaehler manifolds

Let $\bar{M}$ be a nearly Kaehler manifold and $M = N_\perp \times_f N_T$ be a warped product CR-submanifold of $\bar{M}$. By property (p_4), we have

$$\mathcal{P}_X JX + \mathcal{Q}_X JX = 0 \tag{4.1}$$

for each $X \in TN_T$.

The statement (ii) of the Theorem 3.3 can be restated as

$$\nabla_X Z = \nabla_Z X = (Z \ln f)X. \tag{4.2}$$

for each X, $Y \in TN_T$ and $Z \in TN^\perp$. Hence,

$$g(\nabla_X Z, X) = (Z \ln f)\|X\|^2 = g(\nabla_{JX} Z, JX). \tag{4.3}$$

On taking account of (2.5), (3.4) and (4.1), above equation can be written as

$$(Z \ln f)\|X\|^2 = g(JZ, h(X, JX)). \tag{4.4}$$

Replacing X by JX in (4.4), we get

$$(Z \ln f)\|X\|^2 = -g(JZ, h(X, JX)). \tag{4.5}$$

Thus from (4.4) and (4.5),

$$(Z \ln f)\|X\|^2 = 0. \tag{4.6}$$

If M is assumed to be a proper CR-submanifold, then $Z \ln f = 0$ i.e., M is simply a CR-product. In other words, the theorem of B.Y. Chen (cf. [7]) is extended to the setting of nearly Kaehler manifold as

Theorem 4.1. *There does not exist a proper warped product CR-submanifold* $N_\perp \times_f N_T$ *in nearly Kaehler manifolds.*

On the other hand existence of a CR-warped product submanifold of a nearly Kaehler manifold is ensured by K. Sekigawa, in view of the example he provided in [22]. We study some important differential geometric aspects of these submanifolds in this section. Moreover, as the example is relevant to the present study, we recall it at the end of the section.

Lemma 4.1. *Let* M *be a CR-warped product submanifold of a nearly Kaehler manifold* $\bar{M}$. *Then we have*

(i) $g(h(X,Y),JZ) = 0,$

(ii) $g(\nabla_Z X, W) = (X \ln f) g(Z,W) = g(h(JX,Z),JW)$

for each X, $Y \in TN_T$ *and* Z, $W \in TN_\perp$.

Proof. By Theorem 3.3, on M we have

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z.$$

Taking account of the above formula, (2.10) yields

$$(\bar{\nabla}_X P)Z = 0.$$

Now, using (2.14), we obtain

$$g(A_{FZ}X, Y) = -g(\mathcal{P}_X Z, Y).$$

The left hand side of the above equation is symmetric in X and Y whereas the right hand side is skew symmetric in X and Y . That proves

$$g(h(X,Y),JZ) = g(\mathcal{P}_X Z, Y) = 0.$$

The first equality in (ii) is an immediate consequence of Theorem 3.3 (ii). For the second equality, by Gauss formula, we may write

$$\begin{aligned} g(h(JX,Z),JW) &= g(\bar{\nabla}_Z JX, JW) \\ &= g(\mathcal{Q}_Z X, JW) + g(\nabla_Z X, W) \\ &= g(\mathcal{Q}_Z JX, W) + (X \ln f) g(Z,W) \\ &= -g(\mathcal{P}_Z W, JX) + (X \ln f) g(Z,W). \end{aligned}$$

The first term in the right hand side of the above equation is zero by virtue of Theorem 3.2 and thus the above equation reduces to

$$g(h(JX,Z),JW) = (X \ln f)g(Z,W),$$

which proves the statement (ii).

Theorem 4.2. *Let M be a CR-submanifold of a nearly Kaehler manifold $\bar{M}$ with integrable distributions D and $D^{\perp}$. Then M is locally a CR-warped product if and only if*

$$A_{JZ}X = -(JX\mu)Z \tag{4.7}$$

for each $X \in D$, $Z \in D^{\perp}$ and μ, a C^{∞}-function on M such that $W\mu = 0$ for each $W \in D^{\perp}$.

Proof. If M is a CR-warped product submanifold $N_T \times_f N_{\perp}$, then on applying Lemma 4.1, we obtain (4.7). In this case $\mu = \ln f$.

Conversely, suppose $A_{JZ}X = -(JX\mu)Z$, then

$$g(h(X,Y),JZ) = 0$$

i.e., $h(X,Y) \in \mu$, for each X, $Y \in D$. As D is assumed to be integrable, by Theorem 3.1, $\mathcal{Q}_X Y = 0$ and therefore by (2.15)

$$F\nabla_X Y = h(X,JY) - fh(X,Y).$$

It is easy to deduce from the above equation that $\nabla_X Y \in D$. That means, leaves of D are totally geodesic in M. Moreover,

$$\begin{aligned} g(\nabla_Z W, X) &= g(J\bar{\nabla}_Z W, JX) \\ &= -g(\mathcal{P}_Z W, JX) - g(A_{JW}Z, JX). \end{aligned}$$

The first term in the right hand side of the above equation vanishes in view of Theorem 3.2 and the second term on making use of (4.7) reduces to $-X\mu g(Z,W)$. Thus, we have

$$g(\nabla_Z W, X) = -X\mu g(Z,W). \tag{4.8}$$

Now, by Gauss formula

$$g(h^{\perp}(Z,W),X) = g(\nabla_Z W, X)$$

where $h^{\perp}$ denotes the second fundamental form of the immersion of $N_{\perp}$ into M. On using (4.8), the last equation gives

$$g(h^{\perp}(Z,W),X) = -X\mu g(Z,W)$$

which shows that each leaf $N_\perp$ of $D^\perp$ is totally umbilical in M. Moreover, the fact that $W\mu = 0$, for all $W \in D^\perp$, implies that the mean curvature vector on $N_\perp$ is parallel along $N_\perp$ i.e., each leaf of $D^\perp$ is an extrinsic sphere in M. Hence by virtue of the result of [15] which states that —*If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector sub bundles such that E_1 is spherical and it's orthogonal complement E_0 is auto parallel, then the manifold M is locally isometric to a warped product $M_0 \times_f M_1$*, we get that, M is locally a warped $N_T \times_f N_\perp$ of a holomorphic submanifold N_T and a totally real submanifold $N_\perp$ of M. Here N_T is a leaf of D and $N_\perp$ is a leaf of $D^\perp$ and f is a warping function.

Example 4.1. Let $\{e_0,\ e_i\ (1 \le i \le 7)\}$ be the canonical basis of the Cayley division algebra on $\mathbb{R}^8$ over $\mathbb{R}$ and $\mathbb{R}^7$ be the subspace of $\mathbb{R}^8$ generated by the purely imaginary Cayley numbers $e_i\ (1 \le i \le 7)$. Then

$$S^6 = \{y_1e_1 + y_2e_2 + \cdots + y_7e_7 :\ y_1^2 + y_2^2 + \cdots + y_7^2 = 1\} \subset \mathbb{R}^7$$

is a unit 6-sphere admitting a nearly Kaehler structure $(J, g, \bar{\nabla})$ as has been specified earlier. Now, suppose that

$$S^2 = \{y = (y_2, y_4, y_6) \in \mathbb{R}^3 :\ y_2^2 + y_4^2 + y_6^2 = 1\}$$

is a unit 2-sphere and

$$S^1 = \{z = e^{it},\ t \in \mathbb{R}\}$$

is a unit circle. Consider the mapping

$$\psi : S^2 \times S^1 \longrightarrow S^6$$

defined by

$$\begin{aligned}\psi(y, z) &= \psi\big((y_2, y_4, y_6), e^{it}\big)\\ &= (y_2 \cos t)e_2 - (y_2 \sin t)e_3 + (y_4 \cos 2t)e_4 + (y_4 \sin 2t)e_5\\ &\qquad + (y_6 \cos t)e_6 - (y_6 \sin t)e_7\end{aligned}$$

for $y = (y_2, y_4, y_6) \in S^2$ and $z = e^{it} \in S^1,\ t \in \mathbb{R}$. Then ψ gives rise to an isometric immersion from the warped product Riemannian manifold $S^2 \times_f S^1$ into S^6 (cf. [22]) where f is the function on S^2 which is given by the restriction of the function F on $\mathbb{R}^3$ defined as

$$F(y_2, y_4, y_6) = \sqrt{(1 + 3y_4^2)}.$$

References

[1] A. Bejancu, *CR-Submanifolds of a Kaehler manifold I*, Proc. Amer. Math. Soc., **69** (1978), 135–142.

[2] R. L. Bishop and B.O'Neill, *Manifolds of Negative curvature*, Trans. Amer. Math. Soc., **145** (1969), 1–49.

[3] B. Y. Chen, *CR-submanifolds of a Kaehler Manifold I*, J. Diff. Geom., **16** (1981), 305–323.

[4] B. Y. Chen, *CR-submanifolds of a Kaehler Manifold II*, J. Diff. Geom., **16** (1981), 493–509.

[5] B. Y. Chen, *Differential Geometry of Real Submanifolds in a Kaehler manifold*, Monatsh. Math., **91** (1981), 257–274.

[6] B. Y. Chen, *Slant Immersion*, Bull. Aust. Math. Soc., **41** (1990), 135–147.

[7] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler Manifolds*, Monatsh. Math., **133** (2001), 177–195.

[8] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler Manifolds II*, Monatsh. Math., **134** (2001), 103–119.

[9] B. Y. Chen, *Geometry of warped products as Riemannian submanifolds and related problems*, Soochow J. Math., **28** (2002), 125–156.

[10] N. Ejiri, *Some compact hypersurface of constant scalar curvature in a sphere*, J. Geom., **19** (1982), 197–199.

[11] T. Fukami and S. Ishihara, *Almost Hermitian Structure on* S^6, Tohoku Math. J., **7** (1955), 151–156.

[12] A. Gray, *Some examples of almost Hermitian manifolds*, Illinois J. Math., **10** (1966), 353–366.

[13] A. Gray, *Almost Complex submanifolds of Six Sphere*, Proc. Amer. Math. Soc., **20** (1969), 277–279.

[14] A. Gray, *Nearly Kaehler manifolds*, J. Diff. Geom., **4** (1970), 283–309.

[15] S. Hiepko, *Eine Inner Kennzeichungder verzerrten Produkte*, Math. Ann., **241** (1979), 209–215.

[16] S. T. Hong, *Warped products and black holes*, Nuovo Cim. J.B., **120** (2005), 1227–1234 (eprint: arXiv:gr-qc/0601023 v1).

[17] K. A. Khan, V. A. Khan and S. I. Husain, *On the Integrability of the distributions on a CR-submanifold*, Anal.Stiin. Ale Univ. Al-I Cuza Iasi, **38** (1992), 367–378.

[18] K. A. Khan, V. A. Khan and S. I. Husain, *Totally umbilical CR-submanifolds of a Nearly Kaehler manifold*, Geometraie Dedicata, **50** (1994), 47–51.

[19] N. Papaghiuc, *Semi-slant submanifolds of Kaehlerian manifold*, Ann. St. Univ. Iasi, tom. XL, S.I., **9** (f_1) (1994), 55–61.

[20] B. Sahin, *Non existence of warped product semi-slant submanifolds of Kaehler manifolds*, Geometriae Dedicata, **117** (2006), 195–202.

[21] K. Sekigawa, *Almost complex submanifolds of a 6-dimensional sphere*, Kodai Math. J., **6** (1983), 174–185.

[22] K. Sekigawa, *Some CR-submanifolds in* 6*-dimensional Sphere*, Tensor (N.S.), **41** (1984), 13–20.

Viqar Azam Khan
Department of Mathematics,College of Science, P.O. Box 80203,
King Abdul Aziz University, Jeddah-21589, K.S.A.
E-mail: viqarster@gmail.com

Khalid Ali Khan
School of Engineering and Logistics, Faculty of Technology,
Charles Darwin University, NT-0909, AUSTRALIA.
E-mail: khalid.mathematics@gmail.com

Siraj-Uddin
Department of Mathematics, Aligarh Muslim University,
Aligarh-202002, INDIA.
E-mail: siraj.ch@gmail.com