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A periodic projective bimodule resolution of an algebra associated with a cyclic quiver and a separable algebra, and the Hochschild cohomology ring

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Abstract. Let Δ be a separable algebra over a commutative ring R and $f(x)$ a monic polynomial over the center of Δ. We deal with the R-algebra $\Lambda = \Delta\Gamma/(f(X^s))$, where $\Delta\Gamma$ is the path algebra of the cyclic quiver Γ with s vertices and s arrows, and X is the sum of all arrows. We show that Λ has a periodic projective bimodule resolution of period 2. Moreover, by using the resolution, we describe the structure of the Hochschild cohomology ring of Λ by means of the Yoneda product.

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§1. Introduction

The Hochschild cohomology rings of path algebras of an oriented cyclic quiver with relations have been studied by some authors. Let A be the algebra $K\Gamma/(h(X))$ over a commutative ring K, where $K\Gamma$ is the path algebra of the oriented cyclic quiver Γ with s vertices and s arrows, $h(x)$ is a monic polynomial over K and X is the sum of all arrows in $K\Gamma$. If K is a field and $h(x) = x^k$ for an integer $k \geq 2$, then $A = K\Gamma/(X^k)$ is a basic self-injective Nakayama algebra and the Hochschild cohomology ring of the algebra is determined by Erdmann and Holm [EH]. Also, if $s = 1$, then A is equal to $K[x]/(h(x))$ and the structure of the Hochschild cohomology ring of A is described by Holm [H]. Furthermore, if $s \geq 2$ and $h(x) = f(x^s)$ with a monic polynomial $f(x)$ over K, then the Hochschild cohomology ring of $A = K\Gamma/(f(X^s))$ is determined by Furuya and Sanada [FS].

On the other hand, $\Delta\Gamma/(X^s - \alpha)$, a path algebra over a noncommutative ring Δ with a relation, is isomorphic to a subalgebra $B = \Delta[E_{11}, E_{22}, \dots, E_{ss}, C]$ of the full matrix ring $M_s(\Delta)$ (see Lemma 6.1). We are interested in the Hochschild cohomology for a class of matrix algebras including the above B and basic hereditary orders which we studied in [SS]. Thus we will consider a general case that the coefficient rings of path algebras are noncommutative.

In this paper, we deal with the algebra $\Lambda = \Delta\Gamma/(f(X^s))$ over R, where Δ is a separable algebra over a commutative ring R, which is finitely generated projective as an R-module, and $f(x)$ a monic polynomial over the center of Δ. Using methods similar to [FS] and [SS], we show that the R-algebra Λ has a periodic projective bimodule resolution of period 2 and calculate the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda)$ of Λ by means of the Yoneda product. We note that if $\Delta = R$ then the same results for $s = 1$ and $s \geq 2$ have been given in [H] by the cup product and in [FS] by the Yoneda product, respectively.

The content of the paper is as follows. In Section 2, we give the definitions and the notation. Then we have some Λ^e-projective modules which are direct summands of $\Lambda \otimes_R \Lambda$ and are used to give the resolution of Λ, where Λ^e denotes the enveloping algebra of Λ. In Section 3, by using the Λ^e-projective modules, we construct a periodic Λ^e-projective resolution of period 2 of Λ (Theorem 3.2). In Section 4, we compute the Hochschild cohomology groups of Λ. The complex which is obtained by the Λ^e-projective resolution and is used to give the Hochschild cohomology groups of Λ has a difference between the case $s \geq 2$ and the case $s = 1$. Hence, we deal with the case $s \geq 2$ in Section 4.2 (Theorem 4.4) and the case $s = 1$ in Section 4.3 (Theorem 4.5). In Section 5, we describe the structure of the Hochschild cohomology ring of Λ by means of the Yoneda product. We deal with the case $s \geq 2$ in Section 5.1 (Theorems 5.2 and 5.4) and the case $s = 1$ in Section 5.2 (Theorems 5.11 and 5.13). In Section 6, we give some applications (Propositions 6.2 and 6.3). We remark that if $\Delta = R$ then the results of Propositions 6.2 and 6.3 coincide with [KSS, Theorem 1.1] and [H, Theorem 7.1], respectively.

§2. Preliminaries

Let Δ be an algebra over a commutative ring R, s a positive integer and Γ the oriented cyclic quiver with s vertices $e_1, e_2, \dots, e_s$ and s arrows $a_1, a_2, \dots, a_s$ such that a_i starts at e_i and ends at e_{i+1}. We consider the *path algebra* $\Delta\Gamma := \Delta \otimes_R R\Gamma$ over R, where $R\Gamma$ is the path algebra of Γ over R. Hence $a_i = e_{i+1}a_ie_i$ holds for each $1 \leq i \leq s$, where the subscripts i of e_i are considered to be modulo s. We put $X = a_1 + a_2 + \cdots + a_s$ and $f(x) = x^n + z_{n-1}x^{n-1} + \cdots + z_1x + z_0 \in Z(\Delta)[x]$, where $f(x)$ is a monic polynomial over the center $Z(\Delta)$ of Δ. Note that $Xe_i = e_{i+1}X$ for $1 \leq i \leq s$. In this paper, we deal with the R-algebra $\Lambda := \Delta\Gamma/(f(X^s))$, where $(f(X^s))$ is the two-sided ideal of $\Delta\Gamma$ generated by $f(X^s)$. Note that $f(X^s)$ is an element of $Z(\Delta\Gamma)$, so $(f(X^s)) = f(X^s)\Delta\Gamma$. Thus we have $\Lambda = \bigoplus_{i=1}^{s} \bigoplus_{k=0}^{ns-1} \Delta X^k e_i$ and $\mathrm{rank}_\Delta \Lambda = ns^2$. We identify Λ with $\Delta[x]/(f(x))$ in the case $s = 1$.

Throughout the paper, we denote $\otimes_R$ by $\otimes$ and the enveloping algebra $\Lambda \otimes \Lambda^\circ$ of Λ by Λ^e. We assume that Δ is a separable R-algebra which is projective as an R-module from now on. Then Δ is a finitely generated R-module. If $s = 1$ and $n = 1$ then $\Lambda = \Delta$ has trivial cohomology, so we assume $n \geq 2$ in the case $s = 1$.

It is well known that Δ is a separable R-algebra if and only if there exist $(x_\nu)_{1 \leq \nu \leq m}$ and $(y_\nu)_{1 \leq \nu \leq m}$ in Δ such that

$$\sum_{\nu=1}^{m} x_\nu y_\nu = 1 \tag{2.1}$$

and

$$\sum_{\nu=1}^{m} (ax_\nu) \otimes {y_\nu}^\circ = \sum_{\nu=1}^{m} x_\nu \otimes (y_\nu a)^\circ \qquad \text{for all } a \in \Delta. \tag{2.2}$$

We set $\delta^e = \sum_{\nu=1}^{m} x_\nu \otimes y_\nu^\circ \in \Delta^e$, which is called a separating idempotent for Δ (cf. [P]). Note that $\delta^e \delta^e = \delta^e$ and $\delta^e \Delta := \{\sum_{\nu=1}^{m} x_\nu a y_\nu \mid a \in \Delta\} = Z(\Delta)$. We regard elements in Δ as elements in Λ by the natural embedding $\Delta \to \Lambda$. Since there exists the left Λ^e-isomorphism $\Lambda^e \xrightarrow{\sim} \Lambda \otimes \Lambda$; $a \otimes b^\circ \mapsto a \otimes b$, if we denote the image of δ^e by δ, i.e., $\delta = \sum_{\nu=1}^{m} x_\nu \otimes y_\nu \in \Lambda \otimes \Lambda$, then

$$a\delta = \delta a \qquad \text{for all } a \in \Delta \tag{2.3}$$

holds by (2.2). Moreover, since $(e_i \otimes e_j^\circ)\delta^e$ is an idempotent for Λ^e, we have that $\Lambda^e\big((e_i \otimes e_j^\circ)\delta^e\big)$ is a left Λ^e-projective module for each $1 \leq i, j \leq s$, hence we can define the following left Λ^e-projective modules which are direct summands of $\Lambda \otimes \Lambda$:

$$P_0 = \bigoplus_{i=1}^{s} \Lambda e_i \delta e_i \Lambda, \quad P_1 = \bigoplus_{i=1}^{s} \Lambda e_{i+1} \delta e_i \Lambda.$$

Note that $P_0 = P_1 = \Lambda\delta\Lambda$ in the case $s = 1$.

§3. A periodic Λ^e-projective resolution of Λ

In this section, we will construct a periodic Λ^e-projective resolution of period 2 of Λ by using the left Λ^e-projective modules P_0 and P_1 defined in Section 2.

Lemma 3.1. *There exist the left Λ^e-homomorphisms $\phi : P_1 \to P_0$ and $\kappa : \Lambda \to P_1$ which satisfy the following*:

$$\begin{aligned}\phi(e_{i+1}\delta e_i) &= e_{i+1}(X\delta - \delta X)e_i,\\ \kappa(e_i) &= e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_i\end{aligned}$$

for $1 \le i \le s$, where we set $z_n = 1$.

Proof. We define the left Λ^e-homomorphism $\widetilde{\phi} : \Lambda\otimes\Lambda \to \Lambda\otimes\Lambda$ by $\widetilde{\phi}(1\otimes 1) = X\delta - \delta X$. Then, by (2.1), (2.3) and $Xe_i = e_{i+1}X$ for $1 \le i \le s$, we have

$$\begin{aligned}\widetilde{\phi}(e_{i+1}\delta e_i) &= \big((e_{i+1}\otimes e_i^\circ)\delta^e\big)\widetilde{\phi}(1\otimes 1) = \big((e_{i+1}\otimes e_i^\circ)\delta^e\big)(X\delta - \delta X)\\ &= (e_{i+1}\otimes e_i^\circ)\sum_{\nu=1}^{m}(Xx_\nu\delta y_\nu - x_\nu\delta y_\nu X)\\ &= (e_{i+1}\otimes e_i^\circ)\left(X\delta\left(\sum_{\nu=1}^{m} x_\nu y_\nu\right) - \left(\sum_{\nu=1}^{m} x_\nu y_\nu\right)\delta X\right)\\ &= e_{i+1}(X\delta - \delta X)e_i \in P_0.\end{aligned}$$

Hence, if we set $\widetilde{\phi}|_{P_1} = \phi$ then ϕ is the desired left Λ^e-homomorphism.

Next, we define the left Λ-homomorphism $\kappa : \Lambda = \bigoplus_{i=1}^{s}\Lambda e_i \to P_1$ by

$$\kappa(e_i) = e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_i,$$

since $X^k e_i = e_{i+k}X^k$ holds for $1 \le i \le s$ and $k \ge 0$. We will show that κ is a right Λ-homomorphism. First, note that $\kappa(e_ie_j) = \kappa(e_i)e_j$ for $1 \le i, j \le s$. Second, by (2.3), we have

$$\begin{aligned}\kappa(e_iX) - \kappa(e_i)X &= X\kappa(e_{i-1}) - \kappa(e_i)X\\ &= Xe_{i-1}\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_{i-1}\\ &\qquad - e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_iX\\ &= e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^{l+1}\delta X^{js-l-1}\right)\right)e_{i-1}\end{aligned}$$

$$
\begin{aligned}
&- e_i \left(\sum_{j=1}^{n} z_j \left(\sum_{l=0}^{js-1} X^l \delta X^{js-l} \right) \right) e_{i-1} \\
&= e_i \left(\sum_{j=1}^{n} z_j (X^{js}\delta - \delta X^{js}) \right) e_{i-1} \\
&= e_i \left(\left(\sum_{j=1}^{n} z_j X^{js} \right) \delta - \delta \left(\sum_{j=1}^{n} z_j X^{js} \right) \right) e_{i-1} \\
&= e_i\big((-z_0)\delta - \delta(-z_0)\big)e_{i-1} = e_i(-z_0\delta + z_0\delta)e_{i-1} = 0.
\end{aligned}
$$

Hence, $\kappa(e_iX) = \kappa(e_i)X$ holds. Finally, we show that $\kappa(e_i\lambda) = \kappa(e_i)\lambda$ for all $\lambda \in \Lambda$. Note that $\kappa(ae_i) = a\kappa(e_i) = \kappa(e_i)a$ for all $a \in \Delta$, since $z_1, z_2, \dots, z_{n-1}, z_n$ are elements of $Z(\Delta)$. If we set $\lambda = \sum_{j=1}^{s} \sum_{k=0}^{ns-1} a_{jk} X^k e_j \in \Lambda$ ($a_{jk} \in \Delta$) then it follows that

$$
\begin{aligned}
\kappa(e_i)\lambda = \kappa(e_i)e_i\lambda = \kappa(e_i) \sum_{j=1}^{s} \sum_{k=0}^{ns-1} a_{jk} X^k e_{i-k} e_j &= \sum_{j=1}^{s} \sum_{k=0}^{ns-1} \kappa(a_{jk} e_i) X^k e_{i-k} e_j \\
= \kappa \left(\sum_{j=1}^{s} \sum_{k=0}^{ns-1} a_{jk} e_i X^k e_{i-k} e_j \right) &= \kappa \left(e_i \left(\sum_{j=1}^{s} \sum_{k=0}^{ns-1} a_{jk} X^k e_j \right) \right) = \kappa(e_i\lambda).
\end{aligned}
$$

This completes the proof of the lemma. □

Theorem 3.2. *There exists the following exact sequence of left Λ^e-modules which is (Λ, Δ)-split:*

$$
0 \longrightarrow \Lambda \xrightarrow{\kappa} P_1 \xrightarrow{\phi} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0, \tag{3.1}
$$

where $\pi : P_0 \to \Lambda$ is the multiplication map. Hence we have the periodic left Λ^e-projective resolution of period 2:

$$
\cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0, \tag{3.2}
$$

where d_1 and d_0 are left Λ^e-homomorphisms given by

$$
\begin{aligned}
d_1(e_{i+1}\delta e_i) &= \phi(e_{i+1}\delta e_i) = e_{i+1}(X\delta - \delta X)e_i, \\
d_0(e_i\delta e_i) &= (\kappa\pi)(e_i\delta e_i) = e_i \left(\sum_{j=1}^{n} z_j \left(\sum_{l=0}^{js-1} X^l \delta X^{js-l-1} \right) \right) e_i
\end{aligned}
$$

for $1 \le i \le s$.

To prove Theorem 3.2, we prepare the following lemmas.

Lemma 3.3. *The sequence* (3.1) *is a complex of left* Λ^e*-modules.*

Proof. Since $\pi(\delta) = \sum_{\nu=1}^{m} x_\nu y_\nu = 1$, we have

$$(\pi\phi)(e_{i+1}\delta e_i) = \pi(e_{i+1}(X\delta - \delta X)e_i) = e_{i+1}(X - X)e_i = 0$$

and

$$\begin{aligned}
(\phi\kappa)(e_i) &= \phi\left(e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_i\right)\\
&= \phi\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l e_{i-l}\delta e_{i-l-1}X^{js-l-1}\right)\right)\\
&= \sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l e_{i-l}(X\delta - \delta X)e_{i-l-1}X^{js-l-1}\right)\\
&= e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1}\bigl(X^{l+1}\delta X^{js-l-1} - X^l\delta X^{js-l}\bigr)\right)\right)e_i\\
&= e_i\left(\sum_{j=1}^{n} z_j(X^{js}\delta - \delta X^{js})\right)e_i\\
&= e_i\left(\left(\sum_{j=1}^{n} z_jX^{js}\right)\delta - \delta\left(\sum_{j=1}^{n} z_jX^{js}\right)\right)e_i\\
&= e_i\bigl((-z_0)\delta - \delta(-z_0)\bigr)e_i = 0
\end{aligned}$$

for $1 \le i \le s$. This completes the proof of the lemma. $\square$

Lemma 3.4. *There exist the* (Λ, Δ)*-homomorphisms* $h_{-1} : \Lambda \to P_0$, $h_0 : P_0 \to P_1$ *and* $h_1 : P_1 \to \Lambda$ *which satisfy the following*:

$$h_{-1}(1) = \sum_{j=1}^{s} e_j\delta e_j,$$

$$h_0(e_i\delta e_iX^k) = \begin{cases} 0 & \text{if } k = 0,\\ -e_i\left(\displaystyle\sum_{j=0}^{k-1} X^j\delta X^{k-j-1}\right)e_{i-k} & \text{if } 1 \le k \le ns-1,\end{cases}$$

$$h_1(e_{i+1}\delta e_i X^k) = \begin{cases} 0 & \text{if } 0 \le k \le ns-2, \\ e_{i+1} & \text{if } k = ns-1, \end{cases}$$

for $1 \le i \le s$, *where we denote a left* Λ- *and right* Δ-*homomorphism by a* (Λ, Δ)-*homomorphism. Then* $\{h_{-1}, h_0, h_1\}$ *is a contracting homotopy of* (3.1).

Proof. If we define the left Λ-homomorphism $h_{-1} : \Lambda \to P_0$ by $h_{-1}(1) = \sum_{j=1}^{s} e_j\delta e_j$, then it is clear that h_{-1} is a (Λ, Δ)-homomorphism by (2.3). Next, since $X^k e_i = e_{i+k} X^k$ holds for $1 \le i \le s$ and $k \ge 0$, we define the (Λ, Δ)-homomorphisms $\widetilde{h}_0 : \Lambda \otimes \Lambda \to P_1$ and $\widetilde{h}_1 : \Lambda \otimes \Lambda \to \Lambda$ by

$$\widetilde{h}_0(1 \otimes e_i X^k) = \begin{cases} 0 & \text{if } k = 0, \\ -\left(\displaystyle\sum_{j=0}^{k-1} X^j \delta X^{k-j-1}\right) e_{i-k} & \text{if } 1 \le k \le ns-1, \end{cases}$$

$$\widetilde{h}_1(1 \otimes e_i X^k) = \begin{cases} 0 & \text{if } 0 \le k \le ns-2, \\ e_{i+1} & \text{if } k = ns-1, \end{cases}$$

for $1 \le i \le s$. If we set $\widetilde{h}_0|_{P_0} = h_0$ and $\widetilde{h}_1|_{P_1} = h_1$, then it easily follows that h_0 and h_1 are the desired (Λ, Δ)-homomorphisms by (2.1) and (2.3).

(1) $\pi h_{-1} = \mathrm{id}_\Lambda$; For all $\lambda \in \Lambda$, we have

$$(\pi h_{-1})(\lambda) = \pi\left(\lambda\left(\sum_{j=1}^{s} e_j\delta e_j\right)\right) = \lambda\left(\sum_{j=1}^{s} e_j\right) = \lambda.$$

Hence we get the desired equation.

(2) $h_{-1}\pi + \phi h_0 = \mathrm{id}_{P_0}$;

(a) Case $k = 0$: For $1 \le i \le s$, we have

$$(h_{-1}\pi + \phi h_0)(e_i\delta e_i) = h_{-1}(e_i) + \phi(0) = e_i\left(\sum_{j=1}^{s} e_j\delta e_j\right) = e_i\delta e_i.$$

(b) Case $1 \le k \le ns-1$: For $1 \le i \le s$, we have

$$\begin{aligned}&(h_{-1}\pi + \phi h_0)(e_i\delta e_i X^k) \\ &\quad = h_{-1}(e_i X^k) - \phi\left(e_i\left(\sum_{j=0}^{k-1} X^j \delta X^{k-j-1} e_{i-k}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= e_i X^k \left(\sum_{j=1}^{s} e_j \delta e_j \right) - e_i \left(\sum_{j=0}^{k-1} X^j (X\delta - \delta X) X^{k-j-1} \right) e_{i-k} \\
&= X^k e_{i-k} \delta e_{i-k} - e_i \left(\sum_{j=0}^{k-1} (X^{j+1} \delta X^{k-j-1} - X^j \delta X^{k-j}) \right) e_{i-k} \\
&= e_i X^k \delta e_{i-k} - e_i (X^k \delta - \delta X^k) e_{i-k} = e_i \delta e_i X^k.
\end{aligned}
$$

Hence we get the desired equation.

(3) $h_0\phi + \kappa h_1 = \mathrm{id}_{P_1}$;

(a) Case $k = 0$: For $1 \le i \le s$, we have

$$
\begin{aligned}
(h_0\phi + \kappa h_1)(e_{i+1}\delta e_i) &= h_0\big(e_{i+1}(X\delta - \delta X)e_i\big) + \kappa(0) \\
&= h_0(X e_i \delta e_i - e_{i+1}\delta e_{i+1} X) = e_{i+1}\delta e_i.
\end{aligned}
$$

(b) Case $1 \le k \le ns - 2$: For $1 \le i \le s$, we have

$$
\begin{aligned}
&(h_0\phi + \kappa h_1)(e_{i+1}\delta e_i X^k) \\
&\quad = h_0\big(e_{i+1}(X\delta - \delta X)e_i X^k\big) + \kappa(0) = h_0(X e_i \delta e_i X^k - e_{i+1}\delta e_{i+1} X^{k+1}) \\
&\quad = -X e_i \left(\sum_{j=0}^{k-1} X^j \delta X^{k-j-1} \right) e_{i-k} + e_{i+1} \left(\sum_{j=0}^{k} X^j \delta X^{k-j} \right) e_{i-k} \\
&\quad = -e_{i+1} \left(\sum_{j=0}^{k-1} X^{j+1} \delta X^{k-j-1} \right) e_{i-k} + e_{i+1} \left(\sum_{j=0}^{k} X^j \delta X^{k-j} \right) e_{i-k} \\
&\quad = e_{i+1}\delta X^k e_{i-k} = e_{i+1}\delta e_i X^k.
\end{aligned}
$$

(c) Case $k = ns - 1$: For $1 \le i \le s$, we have

$$
\begin{aligned}
&(h_0\phi + \kappa h_1)(e_{i+1}\delta e_i X^{ns-1}) \\
&= h_0\big(e_{i+1}(X\delta - \delta X)e_i X^{ns-1}\big) + \kappa(e_{i+1}) \\
&= h_0(X e_i \delta e_i X^{ns-1} - e_{i+1}\delta e_{i+1} X^{ns}) + \kappa(e_{i+1}) \\
&= -X e_i \left(\sum_{j=0}^{ns-2} X^j \delta X^{ns-j-2} \right) e_{i+1} \\
&\qquad + h_0 \left(e_{i+1}\delta e_{i+1} \left(\sum_{j=0}^{n-1} z_j X^{js} \right) \right) + \kappa(e_{i+1}) \\
&= -e_{i+1} \left(\sum_{j=0}^{ns-2} X^{j+1} \delta X^{ns-j-2} \right) e_{i+1} + \sum_{j=0}^{n-1} z_j h_0(e_{i+1}\delta e_{i+1} X^{js}) + \kappa(e_{i+1})
\end{aligned}
$$

$$= -e_{i+1}\left(\sum_{j=0}^{ns-2} X^{j+1}\delta X^{ns-j-2}\right)e_{i+1} - \sum_{j=1}^{n-1} z_j e_{i+1}\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)e_{i+1}$$
$$+ e_{i+1}\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_{i+1}$$
$$= -e_{i+1}\left(\sum_{j=0}^{ns-2} X^{j+1}\delta X^{ns-j-2}\right)e_{i+1} + e_{i+1}\left(\sum_{l=0}^{ns-1} X^l\delta X^{ns-l-1}\right)e_{i+1}$$
$$= e_{i+1}\delta X^{ns-1}e_{i+1} = e_{i+1}\delta e_i X^{ns-1}.$$

Hence we get the desired equation.

(4) $h_1\kappa = \mathrm{id}_\Lambda$; For $1 \le i \le s$, we have

$$(h_1\kappa)(e_i) = h_1\left(e_i\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_i\right)$$
$$= h_1\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l e_{i-l}\delta e_{i-l-1}X^{js-l-1}\right)\right)$$
$$= h_1(e_i\delta e_{i-1}X^{ns-1}) = e_i.$$

Hence we get the desired equation.

These complete the proof of the lemma. □

Proof of Theorem 3.2. We have the exact sequence (3.1) of left Λ^e-modules which is (Λ, Δ)-split by means of Lemmas 3.3 and 3.4. Then the latter statement is clear. □

§4. The Hochschild cohomology groups of Λ

In this section, we compute the Hochschild cohomology group $\mathrm{HH}^t(\Lambda) := \mathrm{Ext}^t_{\Lambda^e}(\Lambda, \Lambda)$ of Λ for each $t \ge 0$ by means of the projective Λ^e-resolution (3.2). We regard $\mathrm{HH}^t(\Lambda)$ as a $Z(\Lambda)$-module. Since the resolution (3.2) is periodic of period 2, we have a $Z(\Lambda)$-isomorphism $\mathrm{HH}^{i+2}(\Lambda) \simeq \mathrm{HH}^i(\Lambda)$ for each $i \ge 1$. Therefore, it suffices to compute $\mathrm{HH}^t(\Lambda)$ for $t = 0, 1, 2$.

4.1. Some lemmas

In this subsection, we give some lemmas in order to calculate the Hochschild cohomology groups of Λ.

Lemma 4.1. *We have* $Z(\Delta\Gamma) = Z(\Delta)[X^s]$*. Also we have*

$$Z(\Lambda) = \big(Z(\Delta)[X^s] + (f(X^s))\big)/(f(X^s)) \simeq Z(\Delta)[X^s]/\big(Z(\Delta)[X^s] \cap (f(X^s))\big)$$

as rings, where $Z(\Delta)[X^s] \cap (f(X^s))$ *is equal to the ideal of* $Z(\Delta)[X^s]$ *generated by* $f(X^s)$*. So we have* $Z(\Lambda) \simeq Z(\Delta)[x]/(f(x))$ *as rings.*

Proof. First, we will show $Z(\Delta\Gamma) = Z(\Delta)[X^s]$. Let

$$y = \sum_{i=1}^{s} \sum_{j=0}^{N} b_{i,j} X^j e_i \in Z(\Delta\Gamma), \quad \text{where } b_{i,j} \in \Delta \text{ and } N \geq 0.$$

Then we have

$$y = \sum_{i=1}^{s} \sum_{l=0}^{q} b_{i,ls} X^{ls} e_i, \quad \text{where } N = sq + r \text{ and } 0 \leq r \leq s-1,$$

since $ye_p = ye_pe_p = e_pye_p$ for $1 \leq p \leq s$. Next, we have $b_{1,ls} = b_{2,ls} = \cdots = b_{s,ls}$, since $y(Xe_p) = (Xe_p)y$ for $1 \leq p \leq s$. So it follows that

$$y = \sum_{i=1}^{s} \sum_{l=0}^{q} b_{1,ls} X^{ls} e_i = \sum_{l=0}^{q} b_{1,ls} X^{ls} \in \Delta[X^s].$$

Moreover, we have $b_{1,ls} \in Z(\Delta)$ for $0 \leq l \leq q$, since $ay = ya$ for all $a \in \Delta$. Hence $Z(\Delta\Gamma) \subset Z(\Delta)[X^s]$ holds. The converse inclusion follows from the fact that $X^s \in Z(\Delta\Gamma)$ and $Z(\Delta) \subset Z(\Delta\Gamma)$. Therefore we have the desired equation.

Second, we will show $Z(\Lambda) = \big(Z(\Delta)[X^s] + (f(X^s))\big)/(f(X^s))$. Let

$$y = \sum_{i=1}^{s} \sum_{j=0}^{ns-1} b_{i,j} X^j e_i \in Z(\Lambda), \quad \text{where } b_{i,j} \in \Delta.$$

By similar calculation, we have

$$y = \sum_{l=0}^{n-1} b_{1,ls} X^{ls} \in \big(\Delta[X^s] + (f(X^s))\big)/(f(X^s)),$$

hence $Z(\Lambda) \subset \big(Z(\Delta)[X^s] + (f(X^s))\big)/(f(X^s))$. The converse inclusion follows from the fact that $X^s \in Z(\Lambda)$ and $\big(Z(\Delta) + (f(X^s))\big)/(f(X^s)) \subset Z(\Lambda)$. Therefore we have the desired equation. It is clear that the ring isomorphism

$$\big(Z(\Delta)[X^s] + (f(X^s))\big)/(f(X^s)) \simeq Z(\Delta)[X^s]/\big(Z(\Delta)[X^s] \cap (f(X^s))\big)$$

exists.

Third, let I be the ideal of $Z(\Delta)[X^s]$ generated by $f(X^s)$. We will show $I = Z(\Delta)[X^s] \cap (f(X^s))$. Since $f(X^s) \in Z(\Delta\Gamma)$, we set

$$y = f(X^s)v \in Z(\Delta)[X^s] \cap (f(X^s)), \quad \text{where } v \in \Delta\Gamma.$$

Then we have $yu = uy$ for all $u \in \Delta\Gamma$, hence it follows that $f(X^s)(vu - uv) = 0$. Now we will show that $f(X^s)$ is not a zero divisor in $\Delta\Gamma$. Let

$$0 \neq w = \sum_{i=1}^{s} \sum_{j=0}^{N} b_{i,j} X^j e_i \in \Delta\Gamma, \quad \text{where } b_{i,j} \in \Delta \text{ and } N \geq 0,$$

i.e., $b_{i_0,N} \neq 0$ for some $1 \leq i_0 \leq s$. If $f(X^s)w = 0$, then $b_{i_0,N} = 0$ since $f(X^s)we_{i_0} = 0$. This contradicts the assumption. So $f(X^s)$ is not a zero divisor. Hence we have $vu = uv$ for all $u \in \Delta\Gamma$, i.e., $v \in Z(\Delta\Gamma) = Z(\Delta)[X^s]$. Therefore $y = f(X^s)v \in I$, so $Z(\Delta)[X^s] \cap (f(X^s)) \subset I$. The converse inclusion follows from $f(X^s) \in Z(\Delta)[X^s]$. Hence we have $I = Z(\Delta)[X^s] \cap (f(X^s))$ as required.

Finally, we will show $Z(\Lambda) \simeq Z(\Delta)[x]/(f(x))$ as rings. It is clear that the map

$$Z(\Delta)[X^s]/I \longrightarrow Z(\Delta)[x]/(f(x)); \quad X^s \longmapsto x$$

is a ring isomorphism. Therefore we have the ring isomorphism as required. This completes the proof of the lemma. □

By this lemma, we also regard $\mathrm{HH}^t(\Lambda)$ as a $Z(\Delta)[x]/(f(x))$-module for $t \geq 0$.

Lemma 4.2. *We have* $e_{i+k}\Lambda e_i = \big(\Delta[X^s]X^k e_i + (f(X^s))\big)/(f(X^s))$ *for* $1 \leq i \leq s$ *and* $0 \leq k \leq s-1$. *Moreover, we have* $\delta^e(e_{i+k}\Lambda e_i) = Z(\Lambda)X^k e_i$ *which is a free* $Z(\Lambda)$*-module of rank* 1.

Proof. For $0 \leq k \leq s-1$ and $1 \leq i \leq s$, let

$$y = \sum_{p=1}^{s} \sum_{j=0}^{ns-1} b_{p,j} X^j e_p \in e_{i+k}\Lambda e_i, \quad \text{where } b_{p,j} \in \Delta.$$

Then we have

$$\begin{aligned} y = e_{i+k} y e_i &= \sum_{j=0}^{ns-1} b_{i,j} X^j e_{i+k-j} e_i \\ &= \sum_{l=0}^{n-1} b_{i,k+ls} X^{k+ls} e_i \in \big(\Delta[X^s] X^k e_i + (f(X^s))\big)/(f(X^s)), \end{aligned}$$

hence $e_{i+k}\Lambda e_i \subset \big(\Delta[X^s]X^k e_i + (f(X^s))\big)/(f(X^s))$. It is clear that the converse inclusion holds. Moreover we have

$$\begin{aligned} \delta^e(e_{i+k}\Lambda e_i) &= \delta^e\big(\Delta[X^s]X^k e_i + (f(X^s))\big)/(f(X^s)) \\ &= \big((\delta^e\Delta)[X^s]X^k e_i + (f(X^s))\big)/(f(X^s)) \\ &= \big(Z(\Delta)[X^s]X^k e_i + (f(X^s))\big)/(f(X^s)) \\ &= Z(\Lambda)X^k e_i \end{aligned}$$

by Lemma 4.1. We will show that $Z(\Lambda)X^k e_i$ is a free $Z(\Lambda)$-module of rank 1. Let $z = \sum_{l=0}^{n-1} b_l X^{ls} \in Z(\Lambda)$ where $b_l \in \Delta$. If $zX^k e_i = 0$, then we have $b_l = 0$ for $0 \le l \le n-1$, hence $z = 0$ follows. $\square$

By this lemma, for $1 \le i \le s$ and $0 \le k \le s-1$, there exist the following $Z(\Lambda)$-isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{\Lambda^e}(\Lambda e_{i+k}\delta e_i \Lambda, \Lambda) &\xrightarrow{\sim} \big((e_{i+k} \otimes e_i^\circ)\delta^e\big)\Lambda = Z(\Lambda)X^k e_i; \\ \phi &\longmapsto \phi(e_{i+k}\delta e_i), \end{aligned}$$

since $(e_{i+k}\otimes e_i^\circ)\delta^e$ are idempotents in Λ^e, where we regard $\mathrm{Hom}_{\Lambda^e}(\Lambda e_{i+k}\delta e_i\Lambda, \Lambda)$ as $Z(\Lambda)$-modules by setting

$$(z\phi)(y) := z(\phi(y))$$

for $z \in Z(\Lambda)$, $\phi \in \mathrm{Hom}_{\Lambda^e}(\Lambda e_{i+k}\delta e_i \Lambda, \Lambda)$ and $y \in \Lambda e_{i+k}\delta e_i \Lambda$. Note that the inverse maps of the above isomorphisms are

$$\begin{aligned} \mathit{\Phi}_{i,k} : \big((e_{i+k} \otimes e_i^\circ)\delta^e\big)\Lambda &\longrightarrow \mathrm{Hom}_{\Lambda^e}(\Lambda e_{i+k}\delta e_i\Lambda, \Lambda); \\ \big((e_{i+k} \otimes e_i^\circ)\delta^e\big)\lambda &\longmapsto \Big(e_{i+k}\delta e_i \mapsto \big((e_{i+k} \otimes e_i^\circ)\delta^e\big)\lambda\Big) \end{aligned}$$

respectively. By means of these isomorphisms, we have the following $Z(\Lambda)$-isomorphisms:

$$\begin{array}{rlcccc} u_0 : & \mathrm{Hom}_{\Lambda^e}(P_0, \Lambda) & \xrightarrow{\sim} & \displaystyle\bigoplus_{i=1}^{s} \mathrm{Hom}_{\Lambda^e}(\Lambda e_i \delta e_i \Lambda, \Lambda) & \xrightarrow{\sim} & \displaystyle\bigoplus_{i=1}^{s} Z(\Lambda)e_i; \\ & \phi & \longmapsto & (\phi_i)_i & \longmapsto & \displaystyle\sum_i \phi_i(e_i\delta e_i) \end{array}$$

for $s \geq 1$, and

$$\begin{array}{rccccc} u_1: & \mathrm{Hom}_{\Lambda^e}(P_1,\Lambda) & \xrightarrow{\sim} & \displaystyle\bigoplus_{i=1}^{s}\mathrm{Hom}_{\Lambda^e}(\Lambda e_{i+1}\delta e_i\Lambda,\Lambda) & \xrightarrow{\sim} & \displaystyle\bigoplus_{i=1}^{s} Z(\Lambda)Xe_i; \\ & \psi & \longmapsto & (\psi_i)_i & \longmapsto & \displaystyle\sum_i \psi_i(e_{i+1}\delta e_i) \end{array}$$

for $s \geq 2$, where we set $\phi_i = \phi|_{\Lambda e_i \delta e_i \Lambda}$ and $\psi_i = \psi|_{\Lambda e_{i+1}\delta e_i \Lambda}$.

4.2. The Hochschild cohomology groups of Λ in the case $s \geq 2$

In this subsection, we assume that $s \geq 2$. By means of the resolution (3.2) and Lemma 4.2, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathrm{Hom}_{\Lambda^e}(P_0,\Lambda) & \xrightarrow{d_1^{\#}} & \mathrm{Hom}_{\Lambda^e}(P_1,\Lambda) & \xrightarrow{d_0^{\#}} & \mathrm{Hom}_{\Lambda^e}(P_0,\Lambda) & \xrightarrow{d_1^{\#}} & \cdots \\ \wr\downarrow u_0 & & \wr\downarrow u_1 & & \wr\downarrow u_0 & & \\ 0 \longrightarrow \displaystyle\bigoplus_{i=1}^{s} Z(\Lambda)e_i & \xrightarrow{d_1^{*}} & \displaystyle\bigoplus_{i=1}^{s} Z(\Lambda)Xe_i & \xrightarrow{d_0^{*}} & \displaystyle\bigoplus_{i=1}^{s} Z(\Lambda)e_i & \xrightarrow{d_1^{*}} & \cdots, \end{array}$$

where we set $d_1^{\#} = \mathrm{Hom}_{\Lambda^e}(d_1,\Lambda)$, $d_0^{\#} = \mathrm{Hom}_{\Lambda^e}(d_0,\Lambda)$, $d_1^* = u_1 d_1^{\#} u_0^{-1}$ and $d_0^* = u_0 d_0^{\#} u_1^{-1}$. The inverse maps of u_0 and u_1 are given by the following:

$$u_0^{-1}(\lambda e_i)(e_j\delta e_j) = \begin{cases} \Phi_{i,0}(\lambda e_i) = \lambda e_i & \text{if } j = i, \\ 0 & \text{if } j \neq i, \end{cases} \tag{4.1}$$

$$u_1^{-1}(\lambda X e_i)(e_{j+1}\delta e_j) = \begin{cases} \Phi_{i,1}(\lambda X e_i) = \lambda X e_i & \text{if } j = i, \\ 0 & \text{if } j \neq i \end{cases} \tag{4.2}$$

for $\lambda \in Z(\Lambda)$ and $1 \leq i, j \leq s$.

Lemma 4.3. *In the case $s \geq 2$, we have*

$$d_1^*(\lambda e_i) = \lambda X(e_i - e_{i-1}),$$
$$d_0^*(\lambda X e_i) = \lambda X^s f'(X^s)$$

for $\lambda \in Z(\Lambda)$ and $1 \leq i \leq s$, where $f'(x)$ denotes the derivative of $f(x)$.

Proof. Let $\lambda \in Z(\Delta)$ and $1 \leq i \leq s$. Then, by (4.1), we have

$$d_1^*(\lambda e_i) = (u_1 d_1^{\#})\left(u_0^{-1}(\lambda e_i)\right) = u_1\left(u_0^{-1}(\lambda e_i) d_1\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{s} \left(u_0^{-1}(\lambda e_i)d_1\right)(e_{j+1}\delta e_j) \\
&= \sum_{j=1}^{s} u_0^{-1}(\lambda e_i)\big(e_{j+1}(X\delta - \delta X)e_j\big) \\
&= \sum_{j=1}^{s} u_0^{-1}(\lambda e_i)(Xe_j\delta e_j - e_{j+1}\delta e_{j+1}X) \\
&= Xu_0^{-1}(\lambda e_i)(e_i\delta e_i) - u_0^{-1}(\lambda e_i)(e_i\delta e_i)X \\
&= X\lambda e_i - \lambda e_i X = \lambda X(e_i - e_{i-1}).
\end{aligned}
$$

We also have

$$
\begin{aligned}
d_0^*(\lambda X e_i) &= (u_0 d_0^{\#})\left(u_1^{-1}(\lambda X e_i)\right) = u_0\left(u_1^{-1}(\lambda X e_i)d_0\right) \\
&= \sum_{k=1}^{s} \left(u_1^{-1}(\lambda X e_i)d_0\right)(e_k\delta e_k) \\
&= \sum_{k=1}^{s} u_1^{-1}(\lambda X e_i)\left(e_k\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l\delta X^{js-l-1}\right)\right)e_k\right) \\
&= \sum_{k=1}^{s} u_1^{-1}(\lambda X e_i)\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l e_{k-l}\delta e_{k-l-1}X^{js-l-1}\right)\right) \\
&= \sum_{k=1}^{s}\left(\sum_{j=1}^{n} z_j\left(\sum_{l=0}^{js-1} X^l u_1^{-1}(\lambda X e_i)(e_{k-l}\delta e_{k-l-1})X^{js-l-1}\right)\right) \\
&= \sum_{k=1}^{s}\sum_{j=1}^{n} z_j\left(\sum_{\substack{0\le l\le js-1\\ \text{s.t. } i\equiv k-l-1 \pmod{s}}} X^l(\lambda X e_i)X^{js-l-1}\right) \\
&= \lambda\sum_{k=1}^{s}\sum_{j=1}^{n} z_j\left(\sum_{\substack{0\le l\le js-1\\ \text{s.t. } i\equiv k-l-1 \pmod{s}}} X^{js}e_k\right) = \lambda\sum_{k=1}^{s}\sum_{j=1}^{n} z_j(jX^{js}e_k) \\
&= \lambda X^s\left(\sum_{j=1}^{n} jz_jX^{(j-1)s}\right)\left(\sum_{k=1}^{s} e_k\right) = \lambda X^s f'(X^s),
\end{aligned}
$$

by means of (4.2). $\square$

The results of Lemmas 4.1, 4.2 and 4.3 are similar to those of [FS, Lemmas

2.1, 2.2 and 2.3]. Thus the following theorem is easily shown by a similar proof to that given in [FS, Theorem 2 and Corollary 2.4], so we omit the details.

Theorem 4.4. *In the case $s \geq 2$, there exist the following isomorphisms of $Z(\Delta)[x]/(f(x))$-modules*:

$$\mathrm{HH}^t(\Lambda) \simeq \begin{cases} Z(\Delta)[x]/(f(x)) & \text{if } t = 0, \\ \mathrm{Ann}_{Z(\Delta)[x]/(f(x))}(xf'(x)) & \text{if } t \text{ is odd}, \\ Z(\Delta)[x]/(xf'(x), f(x)) & \text{if } t \text{ is even}. \end{cases}$$

Moreover, if $Z(\Delta)$ is a field then $\mathrm{HH}^t(\Lambda) \simeq Z(\Delta)[x]/(xf'(x), f(x))$ for $t \geq 1$.

4.3. The Hochschild cohomology groups of Λ in the case $s = 1$

In this subsection, we assume that $s = 1$ (i.e., $\Lambda = \Delta[x]/(f(x))$) and $n \geq 2$. In this case, we recall that $P_0 = P_1 = \Lambda\delta\Lambda$. By Theorem 3.2, we have the periodic left Λ^e-projective resolution:

$$\cdots \xrightarrow{d_0} \Lambda\delta\Lambda \xrightarrow{d_1} \Lambda\delta\Lambda \xrightarrow{d_0} \Lambda\delta\Lambda \xrightarrow{d_1} \Lambda\delta\Lambda \xrightarrow{\pi} \Lambda \longrightarrow 0, \tag{4.3}$$

where π is the multiplication map, and d_1, d_0 are the left Λ^e-homomorphisms given by

$$d_1(\delta) = x\delta - \delta x, \quad d_0(\delta) = \sum_{j=1}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right),$$

since X is identified with x. So, by Lemma 4.2, we have the following commutative diagram:

$$\begin{array}{ccccccc}
0 \to \mathrm{Hom}_{\Lambda^e}(\Lambda\delta\Lambda, \Lambda) & \xrightarrow{d_1^{\#}} & \mathrm{Hom}_{\Lambda^e}(\Lambda\delta\Lambda, \Lambda) & \xrightarrow{d_0^{\#}} & \mathrm{Hom}_{\Lambda^e}(\Lambda\delta\Lambda, \Lambda) & \xrightarrow{d_1^{\#}} & \cdots \\
\wr\downarrow u_0 & & \wr\downarrow u_0 & & \wr\downarrow u_0 & & \\
0 \to \quad Z(\Lambda) & \xrightarrow{d_1^*} & Z(\Lambda) & \xrightarrow{d_0^*} & Z(\Lambda) & \xrightarrow{d_1^*} & \cdots,
\end{array}$$

where we set $d_1^{\#} = \mathrm{Hom}_{\Lambda^e}(d_1, \Lambda)$, $d_0^{\#} = \mathrm{Hom}_{\Lambda^e}(d_0, \Lambda)$, $d_1^* = u_0 d_1^{\#} u_0^{-1}$ and $d_0^* = u_0 d_0^{\#} u_0^{-1}$. Since

$$u_0 : \mathrm{Hom}_{\Lambda^e}(\Lambda\delta\Lambda, \Lambda) \xrightarrow{\sim} Z(\Lambda); \quad \phi \longmapsto \phi(\delta)$$

and $u_0^{-1}(\lambda)(\delta) = \lambda$ for all $\lambda \in Z(\Lambda)$, we have $d_1^* = 0$ and $d_0^*(\lambda) = \lambda f'(x)$. Therefore the following theorem follows.

Theorem 4.5. *In the case $s = 1$, i.e., $\Lambda = \Delta[x]/(f(x))$, there exist the following isomorphisms of $Z(\Lambda)$-modules*:

$$\mathrm{HH}^t(\Lambda) \simeq \begin{cases} Z(\Lambda) = Z(\Delta)[x]/(f(x)) & \text{if } t = 0, \\ \mathrm{Ann}_{Z(\Lambda)}(f'(x)) = \mathrm{Ann}_{Z(\Delta)[x]/(f(x))}(f'(x)) & \text{if } t \text{ is odd}, \\ Z(\Lambda)/(f'(x)) \simeq Z(\Delta)[x]/(f'(x), f(x)) & \text{if } t \text{ is even}. \end{cases}$$

Moreover, if $Z(\Delta)$ is a field then $\mathrm{HH}^t(\Lambda) \simeq Z(\Delta)[x]/(f'(x), f(x))$ for $t \geq 1$.

§5. The Hochschild cohomology ring of Λ

In this section, we determine the ring structures of the even Hochschild cohomology ring $\mathrm{HH}^{\mathrm{ev}}(\Lambda) := \bigoplus_{i\geq 0} \mathrm{HH}^{2i}(\Lambda)$ of Λ and the Hochschild cohomology ring $\mathrm{HH}^*(\Lambda) := \bigoplus_{t\geq 0} \mathrm{HH}^t(\Lambda)$ of Λ, where the multiplication is given by the Yoneda product $\times$ (cf. [FS, Section 3]). We deal with the case $s \geq 2$ in Section 5.1 and the case $s = 1$ in Section 5.2.

5.1. The Hochschild cohomology ring of Λ in the case $s \geq 2$

In this subsection except Remark 5.5, we assume that $s \geq 2$. The following results in this subsection are easily shown by similar proofs to those given in [FS]. Therefore, we will describe the results only and omit the detailed proof.

Proposition 5.1. *There exists the following isomorphism of $Z(\Delta)$-algebras*:

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda) \simeq Z(\Delta)[u, w]/(f(u), uf'(u)w),$$

where $\deg u = 0$ *and* $\deg w = 2$.

Proof. By using Theorem 4.4, we can prove the proposition by similar arguments to [FS, Proposition 3.2]. □

We consider the case $f'(x) = 0$. Then we identify $\mathrm{HH}^t(\Lambda)$ with $Z(\Delta)[x]/(f(x))$ for $t \geq 0$, by Theorem 4.4.

Theorem 5.2. *Let $Z(\Delta)$ be an integral domain,* $\mathrm{char}\, Z(\Delta) = p > 0$ *and $f(x) \in Z(\Delta)[x]$ a monic polynomial with $f'(x) = 0$, so we set $f(x) = \sum_{j=0}^{n_0} z_{jp} x^{jp}$ for some positive integer n_0.*

(i) *If $p = 2$, then we have the following isomorphism of $Z(\Delta)$-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w] \Bigg/ \left(f(u), v^2 - \left(\sum_{0\leq j\leq n_0 \ s.t.\ j \text{ is odd}} z_{2j} u^{2j} \right) w \right),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

(ii) *If* $p \neq 2$, *then we have the following isomorphism of* $Z(\Delta)$*-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(f(u), v^2),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

Proof. We can prove the theorem by similar arguments to [FS, Theorem 3]. □

Now we consider the case $f'(x) \neq 0$. So, from now on, we assume that $f'(x) \neq 0$ in this subsection except Remark 5.5. We treat the elementary case $f(x) = g^k(x)$ with a monic irreducible polynomial $g(x) \in Z(\Delta)[x]$ and a positive integer k. Then, since $0 \neq f'(x) = kg'(x)g^{k-1}(x)$, it follows that $\mathrm{char}\, Z(\Delta) \nmid k$.

First, we consider the case $g(x) = x$. In this case, we note that if $\Delta = R$ is a field then the ring structure of $\mathrm{HH}^*(\Lambda)$ is determined in [EH, Proposition 5.6].

Proposition 5.3. *Let* $f(x) = x^k$ *with a positive integer* k *and* $f'(x) \neq 0$. *Then we have the following isomorphism of* $Z(\Delta)$*-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(u^k, v^2),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

Proof. By Theorem 4.4, we identify $\mathrm{HH}^t(\Lambda)$ with $Z(\Delta)[x]/(x^k) = Z(\Lambda)$ for $t \geq 0$. Let $u = x + (x^k) \in \mathrm{HH}^0(\Lambda)$, $v = 1 + (x^k) \in \mathrm{HH}^1(\Lambda)$ and $w = 1 + (x^k) \in \mathrm{HH}^2(\Lambda)$. Since we have the results which are similar to [FS, Lemmas 3.1, 3.3 and 3.4], the following follows. For $i \geq 0$, $\mathrm{HH}^{2i}(\Lambda)$ is the $Z(\Lambda)$-module generated by w^i and $\mathrm{HH}^{2i+1}(\Lambda)$ is the $Z(\Lambda)$-module generated by $w^i v$. We obtain the relation $u^k = 0$ in degree 0. We also obtain the relation $v^2 = 0$ in degree 2. Indeed, if $k = 1$ then the relation is clear, and if $k \geq 2$ then we have $v^2 = \sum_{j=2}^{k} z_j \left(\sum_{l=1}^{j-1} l\right) x^j + (x^k) = \left(\sum_{l=1}^{k-1} l\right) x^k + (x^k) = 0$. Therefore we get the desired isomorphism. □

Second, we consider the case $g(x) \neq x$ and $Z(\Delta)$ is a unique factorization domain. Then we have

$$\begin{aligned}\mathrm{HH}^1(\Lambda) &= \mathrm{Ann}_{Z(\Delta)[x]/(g^k(x))}(xkg'(x)g^{k-1}(x)) = (g(x))/(g^k(x)),\\ \mathrm{HH}^2(\Lambda) &= Z(\Delta)[x]/(g^k(x), xkg'(x)g^{k-1}(x))\end{aligned}$$

for $k \geq 1$. If $k = 1$ then $\mathrm{HH}^1(\Lambda) = 0$, and hence the Hochschild cohomology ring of Λ has been calculated by Proposition 5.1.

Theorem 5.4. *Let $Z(\Delta)$ be a unique factorization domain, $p = \operatorname{char} Z(\Delta) \geq 0$ and $f(x) = g^k(x) = \sum_{j=0}^{n} z_j x^j \in Z(\Delta)[x]$ with $f'(x) \neq 0$, where $g(x) \in Z(\Delta)[x]$ is monic irreducible, $g(x) \neq x$ and $k \geq 2$.*

(i) *If $p = 2$, then there exists the following isomorphism of $Z(\Delta)$-algebras:*

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/I,$$

where I is the ideal of $Z(\Delta)[u, v, w]$ generated by

$$g^k(u), g^{k-1}(u)v, v^2 - g^2(u)\left(\sum_{\substack{0 \leq j \leq n \\ s.t.\ j \equiv 2\ or\ 3 \pmod 4}} z_j u^j\right) w, kug^{k-1}(u)g'(u)w,$$

and $\deg u = 0$, $\deg v = 1$, $\deg w = 2$.

(ii) *If $p \neq 2$ (including the case $p = 0$), then there exists the following isomorphism of $Z(\Delta)$-algebras:*

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(g^k(u), g^{k-1}(u)v, v^2, kug^{k-1}(u)g'(u)w),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

Proof. We can prove the theorem by similar arguments to [FS, Theorem 4]. □

Remark 5.5. Suppose that $Z(\Delta)$ is a field and $s \geq 1$. Let $f(x) = g_1^{k_1}(x) \cdots g_l^{k_l}(x)$ be a factorization of $f(x)$ into irreducible factors in $Z(\Delta)[x]$. Since the result of [FS, Lemma 3.6] holds in the case $s \geq 1$, we have the following decomposition of $Z(\Delta)$-algebras:

$$\begin{aligned}\Lambda = \Delta\Gamma/(f(X^s)) &\simeq \Delta \otimes_{Z(\Delta)} \big(Z(\Delta)\Gamma/(f(X^s))\big) \\ &\simeq \Delta \otimes_{Z(\Delta)} \big(Z(\Delta)\Gamma/(g_1^{k_1}(X^s)) \oplus \cdots \oplus Z(\Delta)\Gamma/(g_l^{k_l}(X^s))\big) \\ &\simeq \Delta\Gamma/(g_1^{k_1}(X^s)) \oplus \cdots \oplus \Delta\Gamma/(g_l^{k_l}(X^s)).\end{aligned}$$

Then there exists the following isomorphism of $Z(\Delta)$-algebras:

$$\mathrm{HH}^*\big(\Delta\Gamma/(f(X^s))\big) \simeq \mathrm{HH}^*\big(\Delta\Gamma/(g_1^{k_1}(X^s))\big) \oplus \cdots \oplus \mathrm{HH}^*\big(\Delta\Gamma/(g_l^{k_l}(X^s))\big).$$

Hence, it suffices to consider the case $f(x) = g^k(x)$ for an irreducible polynomial $g(x) \in Z(\Delta)[x]$ and a positive integer k in order to determine the ring structure of $\mathrm{HH}^*(\Lambda)$.

5.2. The Hochschild cohomology ring of Λ in the case $s = 1$

In this subsection, we assume that $s = 1$ (i.e., $\Lambda = \Delta[x]/(f(x))$) and $n \geq 2$. Note that the isomorphisms of Theorem 4.5 are given explicitly as follows:

$$\begin{aligned} Z(\Delta)[x]/(f(x)) &\xrightarrow{\sim} \mathrm{HH}^0(\Lambda); \quad q(x) + (f(x)) \longmapsto \phi, \\ \mathrm{Ann}_{Z(\Delta)[x]/(f(x))}(f'(x)) &\xrightarrow{\sim} \mathrm{HH}^1(\Lambda); \quad q(x) + (f(x)) \longmapsto \phi, \\ Z(\Delta)[x]/(f'(x), f(x)) &\xrightarrow{\sim} \mathrm{HH}^2(\Lambda); \quad q(x) + (f'(x), f(x)) \longmapsto \phi + \mathrm{Im}\, d_0^{\#}, \end{aligned}$$

where $\phi : \Lambda\delta\Lambda \to \Lambda$ is the Λ^e-homomorphism given by $\phi(\delta) = q(x) + (f(x))$. Thus we will identify

$$\begin{aligned} &\mathrm{HH}^0(\Lambda) = Z(\Delta)[x]/(f(x)), \quad \mathrm{HH}^1(\Lambda) = \mathrm{Ann}_{Z(\Delta)[x]/(f(x))}(f'(x)) \\ &\text{and} \quad \mathrm{HH}^2(\Lambda) = Z(\Delta)[x]/(f'(x), f(x)) \end{aligned}$$

by these isomorphisms.

We denote the resolution (4.3) by

$$\cdots \xrightarrow{d_4} P_3 \xrightarrow{d_3} P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} \Lambda \longrightarrow 0,$$

where $P_i = P_0 = \Lambda\delta\Lambda$, $d_{2i} = d_0$ and $d_{2i+1} = d_1$ for $i \geq 1$. Let w be the coset in $\mathrm{HH}^2(\Lambda)$ with $1 \in Z(\Delta)[x]$: $w = 1 + (f'(x), f(x)) \in \mathrm{HH}^2(\Lambda)$. Then w is represented by the multiplication map $\pi : P_2 (= P_0) \to \Lambda$. In this subsection, we will use w in the meaning above.

Lemma 5.6. *If $Q = q(x) + (f(x)) \in \mathrm{HH}^0(\Lambda)$, where $q(x) \in Z(\Delta)[x]$, then we have $Q \times w = q(x) + (f'(x), f(x)) \in \mathrm{HH}^2(\Lambda)$. Also, we have $w \times w = 1 + (f'(x), f(x)) \in \mathrm{HH}^4(\Lambda)$. Hence $\mathrm{HH}^{2i}(\Lambda)$ is the $Z(\Lambda)$-module generated by $w^i \in \mathrm{HH}^{2i}(\Lambda)$ for $i \geq 1$.*

Proof. The element $Q = q(x) + (f(x)) \in \mathrm{HH}^0(\Lambda)$ where $q(x) \in Z(\Delta)[x]$ is represented by the Λ^e-homomorphism $\phi : P_0 \to \Lambda$ given by $\phi(\delta) = q(x) + (f(x))$.

First, we compute the product $Q \times w \in \mathrm{HH}^2(\Lambda)$. It is clear that $\mathrm{id}_{\Lambda\delta\Lambda} : P_2 \to P_0$ is a lifting of $\pi : P_2 \to \Lambda$. Hence $Q \times w$ is the element in $\mathrm{HH}^2(\Lambda)$ represented by $\phi : P_2 \to \Lambda$. Therefore we have $Q \times w = q(x) + (f'(x), f(x)) \in \mathrm{HH}^2(\Lambda)$.

Second, we compute the product $w \times w \in \mathrm{HH}^4(\Lambda)$. It is clear that $\mathrm{id}_{\Lambda\delta\Lambda} : P_2 \to P_0, P_3 \to P_1, P_4 \to P_2$ are liftings of $\pi : P_2 \to \Lambda$. Hence $w \times w$ is the element in $\mathrm{HH}^4(\Lambda)$ represented by $\pi : P_4 \to \Lambda$. Therefore we have $w \times w = 1 + (f'(x), f(x)) \in \mathrm{HH}^4(\Lambda)$. □

By this Lemma, we have the structure of the even Hochschild cohomology ring of Λ.

Proposition 5.7. *There exists the following isomorphism of $Z(\Delta)$-algebras*:

$$\mathrm{HH}^{\mathrm{ev}}(\Lambda) \simeq Z(\Delta)[u,w]/(f(u), f'(u)w),$$

where $\deg u = 0$ *and* $\deg w = 2$.

Proof. Let $u = x + (f(x)) \in Z(\Delta)[x]/(f(x)) = \mathrm{HH}^0(\Lambda)$. Then we have the relation $f(u) = 0$ in degree 0. Moreover, by Lemma 5.6, $\mathrm{HH}^{2i}(\Lambda)$ is the $\mathrm{HH}^0(\Lambda)$-module generated by w^i and there is the relation $f'(u)w^i = 0$ in degree $2i$ for $i \geq 1$. Therefore we have the desired isomorphisms of $Z(\Delta)$-algebras. □

Now we calculate the Yoneda product in odd degree.

Lemma 5.8. *If* $Q_0 = q_0(x) + (f(x)) \in \mathrm{HH}^0(\Lambda)$ *where* $q_0(x) \in Z(\Delta)[x]$ *and* $Q_1 = q_1(x) + (f(x)) \in \mathrm{HH}^1(\Lambda)$ *where* $q_1(x)$ *is an element in* $Z(\Delta)[x]$ *such that* $f'(x)q_1(x) \in (f(x))$, *then we have* $Q_0 \times Q_1 = q_0(x)q_1(x) + (f(x)) \in \mathrm{HH}^1(\Lambda)$. *Also, we have* $Q_1 \times w = q_1(x) + (f(x)) \in \mathrm{HH}^3(\Lambda)$.

Proof. The elements Q_0 and Q_1 are represented by the Λ^e-homomorphisms $\phi_0 : P_0 \to \Lambda$ and $\phi_1 : P_1 \to \Lambda$ given by $\phi_0(\delta) = q_0(x) + (f(x))$ and $\phi_1(\delta) = q_1(x) + (f(x))$, respectively. Then the Λ^e-homomorphism $\sigma : P_1 \to P_0$ given by $\sigma(\delta) = \delta q_1(x)$ is a lifting of ϕ_1 and $\phi_0\sigma : P_1 \to \Lambda$ satisfies $(\phi_0\sigma)(\delta) = q_0(x)q_1(x) + (f(x))$. Therefore we have $Q_0 \times Q_1 = q_0(x)q_1(x) + (f(x))$.

Next we compute $Q_1 \times w$. It is clear that $\mathrm{id}_{\Lambda\delta\Lambda} : P_2 \to P_0$, $P_3 \to P_1$ are liftings of of $\pi : P_2 \to \Lambda$. Hence $Q_1 \times w$ is the element in $\mathrm{HH}^3(\Lambda)$ represented by $\phi_1 : P_3 \to \Lambda$. Therefore we have $Q_1 \times w = q_1(x) + (f(x)) \in \mathrm{HH}^3(\Lambda)$. □

Lemma 5.9. *If* $Q = q(x) + (f(x))$, $\tilde{Q} = \tilde{q}(x) + (f(x)) \in \mathrm{HH}^1(\Lambda)$ *where* $q(x)$, $\tilde{q}(x)$ *are elements in* $Z(\Delta)[x]$ *such that* $f'(x)q(x)$, $f'(x)\tilde{q}(x) \in (f(x))$, *then we have*

$$Q \times \tilde{Q} = q(x)\tilde{q}(x) \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} l \right) x^{j-2} + (f'(x), f(x)).$$

Proof. The elements Q and $\tilde{Q}$ are represented by the Λ^e-homomorphisms $\phi : P_1 \to \Lambda$ and $\tilde{\phi} : P_1 \to \Lambda$ given by $\phi(\delta) = q(x) + (f(x))$ and $\tilde{\phi}(\delta) = \tilde{q}(x) + (f(x))$ respectively. It is clear that the Λ^e-homomorphism $\sigma_0 : P_1 \to P_0$ given by $\sigma_0(\delta) = \delta\tilde{q}(x)$ is a lifting of $\tilde{\phi} : P_1 \to \Lambda$. Define the Λ^e-homomorphism $\sigma_1 : P_2 \to P_1$ by

$$\sigma_1(\delta) = \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} \sum_{k=0}^{l-1} x^k \delta x^{j-k-2} \right) \tilde{q}(x).$$

Then we have that σ_1 is a lifting of $\tilde{\phi}$, i.e., $\sigma_0 d_0 = \sigma_0 d_2 = d_1 \sigma_1$. Indeed, by means of the equation $f'(x)\tilde{q}(x) = 0$ in Λ, we can calculate as follows. First, note that

$$(\sigma_0 d_0)(\delta) = \sigma_0 \left(\sum_{j=1}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \right) = \sum_{j=1}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \tilde{q}(x).$$

We also have

$$\begin{aligned}
(d_1 \sigma_1)(\delta) &= d_1 \left(\sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} \sum_{k=0}^{l-1} x^k \delta x^{j-k-2} \right) \tilde{q}(x) \right) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} \sum_{k=0}^{l-1} (x^{k+1} \delta x^{j-k-2} - x^k \delta x^{j-k-1}) \right) \tilde{q}(x) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} (x^l \delta x^{j-l-1} - \delta x^{j-1}) \right) \tilde{q}(x) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} x^l \delta x^{j-l-1} - (j-1) \delta x^{j-1} \right) \tilde{q}(x) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} - j \delta x^{j-1} \right) \tilde{q}(x) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \tilde{q}(x) - \delta \left(\sum_{j=2}^{n} j z_j x^{j-1} \right) \tilde{q}(x) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \tilde{q}(x) + \delta z_1 \tilde{q}(x) \\
&= \sum_{j=1}^{n} z_j \left(\sum_{l=0}^{j-1} x^l \delta x^{j-l-1} \right) \tilde{q}(x).
\end{aligned}$$

Hence $\sigma_0 d_0 = d_1 \sigma_1$ holds, so σ_1 is a lifting of $\tilde{\phi} : P_1 \to \Lambda$. Then, we have

$$\begin{aligned}
(\phi \sigma_1)(\delta) &= \phi \left(\sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} \sum_{k=0}^{l-1} x^k \delta x^{j-k-2} \right) \tilde{q}(x) \right) \\
&= \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} \sum_{k=0}^{l-1} x^k q(x) x^{j-k-2} \right) \tilde{q}(x)
\end{aligned}$$

$$= \sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} l \right) x^{j-2} q(x)\tilde{q}(x).$$

This completes the proof of the lemma. □

From now on, let $Z(\Delta)$ be an integral domain in this subsection.

We consider the case $f'(x) = 0$, that is, $\operatorname{char} Z(\Delta) = p > 0$ and $f(x) = \sum_{j=0}^{n_0} z_{jp} x^{jp}$ for some positive integer n_0. Then, by Theorem 4.5, we identify $\mathrm{HH}^t(\Lambda)$ with $Z(\Delta)[x]/(f(x))$ for $t \geq 0$.

Lemma 5.10. *Let $Z(\Delta)$ be an integral domain, $\operatorname{char} Z(\Delta) = p > 0$ and $f(x) \in Z(\Delta)[x]$ a monic polynomial with $f'(x) = 0$, i.e., $f(x) = \sum_{j=0}^{n_0} z_{jp} x^{jp}$ for some positive integer n_0. If i and k are odd, then we have*

$$Q \times \tilde{Q} = \begin{cases} q(x)\tilde{q}(x) \left(\displaystyle\sum_{1 \leq j \leq n_0 \textit{ s.t. } j \textit{ is odd}} z_{2j} x^{2j-2} \right) + (f(x)) & \textit{if } p = 2, \\ 0 & \textit{if } p \neq 2, \end{cases}$$

for $Q = q(x) + (f(x)) \in \mathrm{HH}^i(\Lambda)$ and $\tilde{Q} = \tilde{q}(x) + (f(x)) \in \mathrm{HH}^k(\Lambda)$ where $q(x), \tilde{q}(x) \in Z(\Delta)[x]$.

Proof. For $Q = q(x) + (f(x)) \in \mathrm{HH}^i(\Lambda)$ and $\tilde{Q} = \tilde{q}(x) + (f(x)) \in \mathrm{HH}^k(\Lambda)$ where $q(x)$ and $\tilde{q}(x)$ are in $Z(\Delta)[x]$, by Lemma 5.9, we have

$$Q \times \tilde{Q} = q(x)\tilde{q}(x) \sum_{j=1}^{n_0} z_{jp} \left(\sum_{l=1}^{jp-1} l \right) x^{jp-2} + (f(x)).$$

If $p = 2$, then we have $Q \times \tilde{Q} = q(x)\tilde{q}(x) \left(\sum_{\substack{1 \leq j \leq n_0 \\ \text{s.t. } j \text{ is odd}}} z_{2j} x^{2j-2} \right) + (f(x))$, since

$$\sum_{l=1}^{2j-1} l \equiv \begin{cases} 0 \pmod 2 & \text{if } j \text{ is even}, \\ 1 \pmod 2 & \text{if } j \text{ is odd}. \end{cases}$$

If $p \neq 2$, then we have $Q \times \tilde{Q} = 0$, since $\sum_{l=1}^{jp-1} l \equiv 0 \pmod p$ for all $j \geq 1$. □

Theorem 5.11. *Let $Z(\Delta)$ be an integral domain, $\operatorname{char} Z(\Delta) = p > 0$ and $f(x) \in Z(\Delta)[x]$ a monic polynomial with $f'(x) = 0$, i.e., $f(x) = \sum_{j=0}^{n_0} z_{jp} x^{jp}$ for some positive integer n_0.*

(i) *If* $p = 2$, *then there exists the following isomorphism of* $Z(\Delta)$*-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w] \Big/ \left(f(u), v^2 - \left(\sum_{\substack{1 \le j \le n_0 \\ s.t.\ j \text{ is odd}}} z_{2j} u^{2j-2} \right) w \right),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

(ii) *If* $p \neq 2$, *then there exists the following isomorphism of* $Z(\Delta)$*-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(f(u), v^2),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

Proof. Let $u = x + (f(x)) \in \mathrm{HH}^0(\Lambda)$, $v = 1 + (f(x)) \in \mathrm{HH}^1(\Lambda)$ and $w = 1 + (f(x)) \in \mathrm{HH}^2(\Lambda)$. By Lemmas 5.6 and 5.8, $\mathrm{HH}^{2i+1}(\Lambda)$ is the $Z(\Lambda)$-module generated by $w^i v$ for $i \geq 0$. If $p \neq 2$, then we obtain the relation $v^2 = 0$ in degree 2 by Lemma 5.10. If $p = 2$, then $v \times v$ is the coset in $\mathrm{HH}^2(\Lambda)$ represented by $\sum_{\substack{1 \le j \le n_0 \\ \text{s.t. } j \text{ is odd}}} z_{2j} x^{2j-2} \in Z(\Delta)[x]$ by Lemma 5.10, so we have the relation $v^2 - \sum_{\substack{1 \le j \le n_0 \\ \text{s.t. } j \text{ is odd}}} z_{2j} u^{2j-2} w = 0$ in degree 2. Therefore we have the desired isomorphisms. $\square$

Next we consider the case $f'(x) \neq 0$. So, from now on, we assume that $f'(x) \neq 0$ and $Z(\Delta)$ is a unique factorization domain in this subsection. We treat the elementary case $f(x) = g^k(x)$ with a monic irreducible polynomial $g(x) \in Z(\Delta)[x]$ and $k \geq 1$. Then, since $0 \neq f'(x) = kg'(x)g^{k-1}(x)$, it follows that $\mathrm{char}\, Z(\Delta) \nmid k$. By Theorem 4.5, we also have

$$\begin{aligned}
\mathrm{HH}^1(\Lambda) &= \mathrm{Ann}_{Z(\Delta)[x]/(g^k(x))}(kg'(x)g^{k-1}(x)) = (g(x))/(g^k(x)), \\
\mathrm{HH}^2(\Lambda) &= Z(\Delta)[x]/(g^k(x), kg'(x)g^{k-1}(x)).
\end{aligned}$$

If $k = 1$ then $\mathrm{HH}^1(\Lambda) = 0$, and hence the Hochschild cohomology ring of Λ has been calculated by Proposition 5.7. So we assume $k \geq 2$.

Lemma 5.12. *Let* $Z(\Delta)$ *be a unique factorization domain,* $p = \mathrm{char}\, Z(\Delta) \geq 0$ *and* $f(x) = g^k(x) = \sum_{j=0}^n z_j x^j \in Z(\Delta)[x]$ *with* $f'(x) \neq 0$, *where* $g(x) \in Z(\Delta)[x]$ *is monic irreducible and* $k \geq 2$. *If* i *and* t *are odd, then we have*

$$Q \times \tilde{Q} = \begin{cases} q(x)\tilde{q}(x)g^2(x) \left(\displaystyle\sum_{\substack{2 \le j \le n \\ s.t.\ j \equiv 2 \text{ or } 3 \pmod 4}} z_j x^{j-2} \right) + (f(x), f'(x)) & \text{if } p = 2, \\ 0 & \text{if } p \neq 2, \end{cases}$$

for $Q = q(x)g(x) + (f(x)) \in \mathrm{HH}^i(\Lambda)$ *and* $\tilde{Q} = \tilde{q}(x)g(x) + (f(x)) \in \mathrm{HH}^t(\Lambda)$ *where* $q(x), \tilde{q}(x) \in Z(\Delta)[x]$.

Proof. By Lemma 5.9, we have

$$Q \times \tilde{Q} = q(x)\tilde{q}(x)g^2(x)\sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} l\right) x^{j-2} + (f(x), f'(x)).$$

If $p = 2$, then we have

$$Q \times \tilde{Q} = q(x)\tilde{q}(x)g^2(x)\left(\sum_{\substack{2\le j\le n\\ \text{s.t. } j\equiv 2 \text{ or } 3 \pmod 4}} z_j x^{j-2}\right) + (f(x), f'(x)),$$

since

$$\sum_{l=1}^{j-1} l \equiv \begin{cases} 0 \pmod 2 & \text{if } j \equiv 0 \text{ or } 1 \pmod 4, \\ 1 \pmod 2 & \text{if } j \equiv 2 \text{ or } 3 \pmod 4. \end{cases}$$

If $p \neq 2$, then

$$\begin{aligned}\sum_{j=2}^{n} z_j \left(\sum_{l=1}^{j-1} l\right) x^{j-2} &= \sum_{j=2}^{n} z_j \frac{j(j-1)}{2} x^{j-2} = \frac{1}{2}\sum_{j=2}^{n} j(j-1) z_j x^{j-2} \\ &= \frac{1}{2} f''(x) = \frac{1}{2} k g^{k-2}(x)\left((k-1)(g'(x))^2 + g(x)g''(x)\right),\end{aligned}$$

so we have $Q \times \tilde{Q} = 0$. $\square$

Theorem 5.13. *Let* $Z(\Delta)$ *be a unique factorization domain,* $p = \operatorname{char} Z(\Delta) \geq 0$ *and* $f(x) = g^k(x) = \sum_{j=0}^{n} z_j x^j \in Z(\Delta)[x]$ *with* $f'(x) \neq 0$*, where* $g(x) \in Z(\Delta)[x]$ *is monic irreducible and* $k \geq 2$.

(i) *If* $p = 2$*, then there exists the following isomorphism of* $Z(\Delta)$*-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/I,$$

where I *is the ideal of* $Z(\Delta)[u, v, w]$ *generated by*

$$g^k(u), g^{k-1}(u)v, v^2 - g^2(u)\left(\sum_{\substack{2\le j\le n\\ s.t.\ j\equiv 2 \ or\ 3 \pmod 4}} z_j u^{j-2}\right) w, k g^{k-1}(u) g'(u) w,$$

and $\deg u = 0$*,* $\deg v = 1$*,* $\deg w = 2$.

(ii) *If* $p \neq 2$ (*including the case* $p = 0$), *then there exists the following isomorphism of* $Z(\Delta)$*-algebras*:

$$\mathrm{HH}^*(\Lambda) \simeq Z(\Delta)[u, v, w]/(g^k(u), g^{k-1}(u)v, v^2, kg^{k-1}(u)g'(u)w),$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

Proof. Let $u = x + (g^k(x)) \in \mathrm{HH}^0(\Lambda)$, $v = g(x) + (g^k(x)) \in \mathrm{HH}^1(\Lambda)$ and $w = 1+(g^k(x), kg^{k-1}(x)g'(x)) \in \mathrm{HH}^2(\Lambda)$. Then we have the relation $g^k(u) = 0$ in degree 0. By Lemma 5.6, for $i \geq 1$, $\mathrm{HH}^{2i}(\Lambda)$ is the $Z(\Lambda)$-module generated by w^i, and we have the relation $kg^{k-1}(u)g'(u)w = 0$ in degree 2. Moreover, by Lemmas 5.6 and 5.8, for $i \geq 0$, $\mathrm{HH}^{2i+1}(\Lambda)$ is the $Z(\Lambda)$-module generated by vw^i, and we have the relation $g^{k-1}(u)v = 0$ in degree 1.

If $p \neq 2$, then by Lemma 5.12 we have the relation $v^2 = 0$ in degree 2. If $p = 2$, then by Lemma 5.12 $v \times v$ is the coset in $\mathrm{HH}^2(\Lambda)$ represented by $g^2(x)\left(\sum_{\substack{2\leq j\leq n \\ \text{s.t. } j\equiv 2 \text{ or } 3 \,(\mathrm{mod}\, 4)}} z_j x^{j-2}\right)$. So we have the relation

$$v^2 - g^2(u)\left(\sum_{\substack{2\leq j\leq n \\ \text{s.t. } j\equiv 2 \text{ or } 3 \,(\mathrm{mod}\, 4)}} z_j u^{j-2}\right) w = 0$$

in degree 2. Therefore we get the desired isomorphisms. □

We remark that the argument of Remark 5.5 holds in the case $s = 1$.

§6. Applications

In this section, we will give some applications of the results of Section 5. Let Δ be a separable R-algebra as usual.

Let s be an integer with $s \geq 2$ and $\alpha_1, \alpha_2, \cdots, \alpha_s$ be nonzero elements of $Z(\Delta)$ such that α_i is not a zero divisor in Δ for each $1 \leq i \leq s$. Let E_{ij} be the matrix unit in the $s \times s$ matrix ring $M_s(\Delta)$ for $1 \leq i, j \leq s$ and

$$C := \begin{bmatrix} 0 & \cdots & \cdots & 0 & \alpha_s \\ \alpha_1 & 0 & & & 0 \\ 0 & \alpha_2 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & \alpha_{s-1} & 0 \end{bmatrix}.$$

Define the R-subalgebra B of $M_s(\Delta)$ as follows:

$$B = \Delta[E_{11}, E_{22}, \ldots, E_{ss}, C].$$

Note that, in particular, if $\alpha_1 = \alpha_2 = \cdots = \alpha_{s-1} = 1$ then the algebra has the form

$$\begin{bmatrix} \Delta & \alpha_s\Delta & \cdots & \alpha_s\Delta \\ \vdots & \Delta & \ddots & \vdots \\ \vdots & & \ddots & \alpha_s\Delta \\ \Delta & \cdots & \cdots & \Delta \end{bmatrix}_{s\times s}$$

which is similar to a basic hereditary order (cf. [SS]). We calculate the Hochschild cohomology ring of B. The following lemma shows that B is isomorphic to $\Delta\Gamma/(f(X^s))$ for some $f(x) \in Z(\Delta)[x]$, where we note that Δ needs not to be R-separable.

Lemma 6.1. *Let B be the R-algebra as above. Then B is isomorphic to $\Delta\Gamma/(X^s - \alpha)$ as R-algebras, where we set $\alpha = \alpha_1\alpha_2\cdots\alpha_s$.*

Proof. We have

$$aC = Ca \text{ for all } a \in \Delta \text{ and } C^s = \alpha E,$$

where E denotes the identity matrix. We also have

$$C^j E_{ii} = E_{i+j,i+j}C^j \text{ for } 1 \le i \le s \text{ and } 0 \le j \le s-1,$$

where we regard the subscripts of matrix units modulo s. Since α_i is not a zero divisor in Δ for each $1 \le i \le s$, the set $\{C^j E_{ii} \,|\, 1 \le i \le s,\, 0 \le j \le s-1\}$ gives a Δ-basis of B. Therefore there exists the following isomorphism of Δ-modules:

$$\Delta\Gamma/(X^s - \alpha) \xrightarrow{\sim} B; \quad X^j e_i \longmapsto C^j E_{ii}.$$

Moreover, it is clear that the isomorphism is an isomorphism of R-algebras. This completes the proof of the lemma. □

Proposition 6.2. *Let Δ be a separable R-algebra and B the R-algebra as above. Then there exists the following isomorphism of $Z(\Delta)$-algebras*:

$$\mathrm{HH}^*(B) \simeq Z(\Delta)[w]/(\alpha w),$$

where $\deg w = 2$ *and* $\alpha = \alpha_1\alpha_2\cdots\alpha_s$.

Proof. By Lemma 6.1 and Theorem 4.4, we have

$$\mathrm{HH}^t(B) \simeq \mathrm{Ann}_{Z(\Delta)[x]/(x-\alpha)}(x) \simeq \mathrm{Ann}_{Z(\Delta)}(\alpha) = 0$$

for t odd, since α is not a zero divisor in Δ. Hence $\mathrm{HH}^*(B) \simeq \mathrm{HH}^{\mathrm{ev}}(B)$ holds. Moreover, by Proposition 5.1, we have

$$\mathrm{HH}^{\mathrm{ev}}(B) \simeq Z(\Delta)[u,w]/(u-\alpha, uw) \simeq Z(\Delta)[w]/(\alpha w),$$

where $\deg u = 0$ and $\deg w = 2$. $\square$

We remark that if $\Delta = R$ then the result of Proposition 6.2 coincides with [KSS, Theorem 1.1].

Next, we calculate the Hochschild cohomology ring of the truncated polynomial R-algebra $A_n := \Delta[x]/(x^n)$ with $n \geq 2$.

Proposition 6.3. *Let Δ be a separable R-algebra, $Z(\Delta)$ a unique factorization domain with* $\mathrm{char}\, Z(\Delta) = p \geq 0$*, and A_n the truncated polynomial R-algebra as above. Then there exists the following isomorphism of $Z(\Delta)$-algebras*:

$$\mathrm{HH}^*(A_n) \simeq \begin{cases} Z(\Delta)[u,v,w]/(u^n, u^{n-1}v, v^2, nu^{n-1}w) & \text{if } p \nmid n, \\ Z(\Delta)[u,v,w]/(u^n, v^2) & \text{if } 2 \neq p \mid n \text{ or} \\ & \text{if } 2 = p \mid n \text{ and } 4 \mid n, \\ Z(\Delta)[u,v,w]/(u^n, v^2 - u^{n-2}w) & \text{if } 2 = p \mid n \text{ and } 4 \nmid n, \end{cases}$$

where $\deg u = 0$, $\deg v = 1$ *and* $\deg w = 2$.

Proof. Let $s = 1$ and $f(x) = x^n$ for $n \geq 2$, then $\Lambda = \Delta[x]/(x^n) = A_n$, $z_n = 1$ and $z_j = 0$ for $0 \leq j \leq n-1$ in our previous notation.

First, we consider the case $p \nmid n$. Then, since $f'(x) \neq 0$, we can apply Theorem 5.13 to A_n. If $p = 2$, then we have

$$\mathrm{HH}^*(A_n) \simeq Z(\Delta)[u,v,w]/(u^n, u^{n-1}v, v^2, nu^{n-1}w)$$

where $\deg u = 0$, $\deg v = 1$ and $\deg w = 2$, since $\sum_{\substack{2 \leq j \leq n \\ \text{s.t. } j \equiv 2 \text{ or } 3 \,(\mathrm{mod}\, 4)}} z_j u^{j-2}$ is equal to u^{n-2} or 0. If $p \neq 2$, then we also have the same isomorphism.

Second, we consider the case $p \mid n$. Then, since $f'(x) = 0$, we can apply Theorem 5.11 to A_n. If $p \neq 2$, then $\mathrm{HH}^*(A_n) \simeq Z(\Delta)[u,v,w]/(u^n, v^2)$. If $p = 2$, then we have the desired isomorphisms, since the sum $\sum_{\substack{1 \leq j \leq n/2 \\ \text{s.t. } j \text{ is odd}}} z_{2j} u^{2j-2}$ is equal to u^{n-2} if $n/2$ is odd and 0 if $n/2$ is even. $\square$

We remark that if $\Delta = R$ then the result of Proposition 6.3 coincides with [H, Theorem 7.1].

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References

[EH] K. Erdmann and T. Holm, Twisted bimodules and Hochschild cohomology for self-injective algebras of class A_n, Forum Math. **11** (1999), 177–201.

[FS] T. Furuya and K. Sanada, Hochschild cohomology of an algebra associated with a circular quiver, Comm. Algebra **34** (2006), 2019–2037.

[H] T. Holm, Hochschild cohomology rings of algebras $k[X]/(f)$, Contributions to Algebra and Geometry **41** (2000), 291–301.

[KSS] S. Koenig, K. Sanada and N. Snashall, On Hochschild cohomology of orders, Arch. Math. **81** (2003), 627–635.

[P] R. Pierce, Associative algebras, GTM **88** (1982), Springer-Verlag.

[SS] M. Suda and K. Sanada, Periodic projective resolutions and Hochschild cohomology for basic hereditary orders, J. Algebra **305** (2006), 48–67.

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