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A 2-local characterization of $M(22)$

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Abstract. Let G be a finite group with $O(G) = 1$. We show that if G contains a subgroup A isomorphic to E_{64} such that $C_G(A) = O(N_G(A)) \times A$, $N_G(A)/C_G(A) \cong Sp(6,2)$, and $N_G(A)/O(N_G(A))$ splits over $(O(N_G(A))A)/O(N_G(A))$, then either $G \cong M(22)$ or $G = N_G(A)$.

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§1. Introduction

This paper is concerned with a 2-local characterization of $M(22)$, Fischer's group of order $2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$.

In the study of finite groups having a standard subgroup L isomorphic to a group of Lie type with rank at least 4 over a field of characteristic 2, the case where $L \cong \Omega^+(8,2)$ is exceptional. In fact, when Seitz [18] tried to classify all such groups, the case where $L \cong \Omega^+(8,2)$ remained unsettled, and it was later settled in [7]. In [7], a 2-local characterization of $M(22)$, which is stated in [4], plays a crucial role. However, the proof given in [4] is erroneous. In fact, the 2-structure described in Sections 2 through 4 of [4] does not coincide with the actual 2-structure of $M(22)$. The purpose of this paper is to remedy the argument in [4], and prove the following theorem.

Main Theorem. *Let G be a finite group with $O(G) = 1$, and suppose that G contains a subgroup A isomorphic to E_{64} such that $C_G(A) = O(N_G(A)) \times A$ and $N_G(A)/C_G(A) \cong Sp(6,2)$. Suppose further that $N_G(A)/O(N_G(A))$ splits over $(O(N_G(A))A)/O(N_G(A))$. Then either $G \cong M(22)$ or $G = N_G(A)$.*

In passing, we mention that in Section 5 of [4], it is proved that if $N_G(A)/O(N_G(A))$ does not split over $(O(N_G(A))A)/O(N_G(A))$, then $G = N_G(A)$.

Our notation is standard possibly except for the following:

$E(X)$: the product of the quasisimple subnormal subgroups of X,
X^∞ : the final term of the derived series of X,
$X \wr Y$: the wreath product of X by Y,
$X * Y$: a central product of X and Y,
Z_n : the cyclic group of order n,
E_n : the elementary abelian grouyp of order n,
D_8 : the dihedral group of order 8,
Q_8 : the quaternion group,
Σ_n : the symmetric group of degree n,
Σ'_n : the alternating group of degree n,
$\Gamma L(2,4)$: $Z_3 \times \mathrm{SL}(2,4)$ together with an automotphism of order 2 inverting Z_3 and inducing $\mathrm{Aut}(\mathrm{SL}(2,4))$ on $\mathrm{SL}(2,4)$,
$\Gamma U(4,2)$: $Z_3 \times \mathrm{SU}(4,2)$ together with an automorphism of order 2 inverting Z_3 and inducing $\mathrm{Aut}(\mathrm{SU}(4,2))$ on $\mathrm{SU}(4,2)$,
$GF(q)$: the field with q elements,
$M_n(q)$: the set of $n \times n$ matrices with entries in $GF(q)$.

If we write $X = YZ$, it means that $Y \lhd X$ and $X = \langle Y, Z\rangle$. If $Y \cap Z = 1$ and if an emphasis is to be placed on that fact, then we write $X = Y \cdot Z$.

If X is a 2-group, then by $J(X)$, we denote the usual Thompson subgroup generated by the abelian subgroups of maximum order.

Let G be a group isomorphic to Σ'_5 (resp. Σ_5). Suppose that G acts on a group V isomorphic to E_{16}. If the order of the centralizer in V of an element of order 3 of G is 4, then we refer to this action as a "standard action as $\Omega^-(4,2)$ (resp. $O^-(4,2)$)." If the action of an element of order 3 of G is fixed-point-free, then we refer to this action as a "standard action as $\mathrm{SL}(2,4)$ (resp. $\mathrm{Aut}(\mathrm{SL}(2,4))$)."

We use the "bar" convertion for homomorphic images. Thus if G is a group, N is a normal subgroup and $\bar{G}$ denotes the factor group G/N, then, for any subset X of G, $\bar{X}$ denotes the image of X under the natural projection $G \to \bar{G}$. Similarly we use the "double bar" convention and the "tilde" convertion.

§2. Preliminary Results

In this section, we collect a number of preliminary lemmas to be used in later sections.

The first two lemmas are easy to verify and their proofs are omitted.

Lemma 2.1. *Let z be an involution actiong on a group Y, and let a be an element of Y such that $[a,z]$ is an involution. Then z centralizes $[a,z]$.*

Lemma 2.2. *Let x be an involution acting on an elementary abelian 2-group C. Then the following hold.*

(1) $|C_C(x)|^2 \geq |C|$.

(2) *If A is a subgroup of $C_C(x)$, then* $|C_{C/A}(x)| \geq |[C,x]|$.

(3) *If A is an x-invariant subgroup of C, then* $|C_C(x)| \geq |C_{C/A}(x)|$.

Lemma 2.3. *Let R be a Sylow 2-subgroup of a group G, and let C be an abelian subgroup of R which is weakly closed in R with respect to G. Let $\bar{R} = R/C$, $\Gamma = \{E \mid E$ is a subgroup of R not contained in C such that $E^g \subseteq C$ for some $g \in G.\}$, $\gamma' = \max\{|\bar{E}| \mid E \in \Gamma\}$.*

(1) *If $E \in \Gamma$, then there exists $g \in G$ such that $E^g \subseteq C$ and $N_R(E)^g \subseteq R$.*

(2) *Suppose that x is an involution of R such that $\langle x \rangle \in \Gamma$. Then $|[C,x]| \leq {\gamma'}^2$ and, if C is elementary abelian, then $|[C,x]| \leq \gamma'$.*

Throughout the rest of the statement of this lemma, we assume that C is elementary abelian, and set $\gamma = \max\{|E| \mid E \in \Gamma\}$.

(3) *If $E \in \Gamma$ and $|E| = \gamma$, then $|C_{C/(C\cap E)}(\bar{x})| \leq |\bar{E}|$ for every involution $\bar{x}$ of $\bar{E}$.*

(4) *If $E \in \Gamma$ and $|E| = \gamma$, then $|C/(C \cap E)| \leq |\bar{E}|^2$.*

(5) *If $E \in \Gamma$ and $|E| = \gamma$, then $|[C, \bar{x}]| \leq |\bar{E}|$ for every involution $\bar{x}$ of $\bar{E}$.*

(6) *If $E \in \Gamma$ and $|E| = \gamma$, then $|C/((C \cap E)[C, \bar{x}])| \leq |\bar{E}|$ for every involution $\bar{x}$ of $\bar{E}$.*

Proof. Statement(1) is (9.3) of Goldschmidt [9], and (2) is Corollary 4 (2) of [9]. Statement(3) follows from (1), and an equivalent statement can be found in the proof of Corollary 4 (1) of [9]. Now (4) follows from (3) and Lemma 2.2(1), and is essentially the same as Corollary 4 (1) of [9]. Similarly(5) follows from (3) and Lemma 2.2(2), and (6) follows from (3) and Lemma 2.2(3). □

Lemma 2.4. *Let F be a special 2-group of order 2^{2n+1} with a subgroup B such $Z(F) = \Phi(F) \cong E_{2^n}$, $Z(F) \subseteq B \cong E_{2^{n+1}}$ and $[B,F] = Z(F)$. Set $\widetilde{F} = F/Z(F)$, and $\overline{F} = F/B$. Suppose that an involution x acts on F and B is x-invariant. Then $C_{\bar{F}}(x) = \overline{C_{\widetilde{F}}(x)}$.*

Proof. From the assumption that $[B,F] = Z(F)$, it follows that for each element y of $Z(F)$, there exists an element $t(y)$ of F such that $[B, t(y)] = \langle y \rangle$, and such an element is uniquely determined modulo B. Note that the bijection between $Z(F)$ and $\overline{F}$ which associates $\overline{t(y)}$ with y is an x-isomorphism. Let

z be an involution of $B - Z(F)$. Now suppose that the lemma is false. Then there exists an element y of $C_{Z(F)}(x)$ such that $[\widetilde{t(y)}, x] = \widetilde{z}$. By Lemma 2.1, $\overline{t(y)} \notin [\overline{F}, x]$. Hence $y \notin [Z(F), x]$. By a suitable choice of z, we may assume $[t(y), x] = z$. Then again by Lemma 2.1, $[z, x] = 1$. Calculating in the semi-direct product $F \cdot \langle x \rangle$, we get $1 = [(t(y)x)^2, t(y)x] = [t(y)^2 z, t(y)x] \in y[Z(F), x]$. Since $y \notin [Z(F), x]$, this is a contradiction. □

Lemma 2.5. *Let P be a group. Set $C = O_2(P)$, and suppose that the following three conditions hold.*

(1) $\Phi(\Phi(C)) = 1$.

(2) *$C/\Phi(C)$ is isomorphic to $\Phi(C)$ as a P-module.*

(3) *Either*

(3_1) *there exists a P-orbit $\{a^P\}$ of $\Phi(C)$ such that $C_{\Phi(C)}(C_P(a)) = \langle a \rangle$ and $\Phi(C) = \langle x|$ both x and ax are in $\{a^P\}\rangle$; or*

(3_2) *$\Phi(C) \cong E_{16}$ and $P/C \cong E_9$, and P/C acts faithfully on $C/\Phi(C)$.*

Then C is homocyclic abelian of exponent 4.

Proof. If (3_1) is satisfied, then this is virtually the same as Proposition 1.4 of [6]. Thus assume (3_2) holds, and let K be a complement of C in P. Let $\overline{C} = C/\Phi(C)$. Write $K = L \times M$ with $[\overline{C}, L] \cong [\overline{C}, M] \cong E_4$. Since $\Phi(C) \subseteq Z(C)$ by (2) and since $|[\overline{C}, L]| = 4$, $|[C, L]'| \leq 2$. Since $[C, L]'$ is K-invariant, this means $[C, L]' = 1$. Therefore $[C, L] = C_{[C,L]}(L) \times [C, L, L]$. Since $[C, L, L] = [C, L]$, this implies $C_{[C,L]}(L) = 1$, and hence $|[C, L]| = 16$. Similarly $|[C, K]| = 16$. Consequently $[C, L] \cap [C, K] = 1$. Since $[C, L]$ and $[C, K]$ are both normal in C, the desired conclusion immediately follows from this. □

Lemma 2.6. *Let A be a vector space of even dimension over $GF(2)$ with a quadratic form.*

(1) *Assume either that the dimension is greater than or equal to 4 or that the quadratic form is of minus type. Then $A = \langle x \mid x$ is non-singular $\rangle$.*

(2) *Assume either that the dimension is greater than or equal to 4 or that the quadratic form is of plus type. Then $A = \langle x \mid x$ is singular $\rangle$.*

Proof. This is easy to verify. □

Lemma 2.7. *Let $G = Z_2 \wr \Sigma'_n$, $n \geq 5$. Let $P = (G/Z(G))'$ (thus $P \cong E_{2^{n-1}} \cdot \Sigma'_n$ or $E_{2^{n-2}} \cdot \Sigma'_n$ according as n is odd or even). Then there are one or two classes of complements of $O_2(P)$ in P according as n is odd or even (note that in the* *case where n is even, this implies that if $I \cong E_{2^{n-1}}$, $I \supseteq I_0 \cong E_{2^{n-2}}$, Σ'_n acts on I, I_0 is Σ'_n-invariant, and the action of Σ'_n on I_0 is the same as that of $P/O_2(P)$ on $O_2(P)$, then there are two possibilities for the action of Σ'_n on I, one of which is decomposable and the other one is indecomposable).*

Proof. See Lemma 11.3 of Aschbacher [3]. $\square$

The following lemmas is verified by straightforward calculations.

Lemma 2.8. *Let $C = E_{16}$ and $M = Sp(4,2)$, and let M act on C in a standard way. Then the following hold for every elementary abelian subgroup E of order 8 of M.*

(1) $|C_C(E)| \leq 4$.

(2) $[C,x] \subseteq C_C(E)$ *for every involution x of E such that* $|[C,x]| = 2$.

Arguing as in [1] and [2], we obtains the following two lemmas.

Lemma 2.9. *Let $M = \mathrm{Aut}(\mathrm{SU}(4,2))$. Let a be an involution of $M - M'$ such that $C_{M'}(a) \cong Sp(4,2)$. Set $D = C_{M'}(a)$. Let $C = \langle e_i, f_i \mid 3 \leq i \leq 6\rangle \cong E_{256}$. Suppose that M act on C, $[f_i, a] = e_i$ for each i, $\langle e_i \mid 3 \leq i \leq 6\rangle$ and $\langle f_i \mid 3 \leq i \leq 6\rangle$ are D-invariant, and D acts on $\langle e_i \mid 3 \leq i \leq 6\rangle$ as $Sp(4,2)$ so that it fixes the alternating form θ defined by $\theta(e_i, e_j) = \delta_{i,9-j}$, where $\delta_{i,9-j}$ is Kronecker's delta. Set $I = C_D(\langle e_3, e_4\rangle)$. Let J be an elementary abelian subgroup of order 16 of M' such that $J \supseteq I$, and set $N = N_M(J)$. Then the following hold.*

(1) *M has four classes of involutions, and $|[C,x]| \geq 4$ for every involution x of M.*

(2) *For every noncentral involution x of M, $|[C,x]| = 16$; for every central involution x of M, $|[C,x]| = 4$.*

(3) *If E is an elementary abelian subgroup of order 4 of M, then E contains a noncentral involution, and hence $|C_C(E)| \leq 16$ by (2).*

(4) *$O_2(N) = J$, and $N/O_2(N) \cong \Sigma_5$.*

(5) *$[C, O_2(N)] = C_C(O_2(N)) = \langle e_3, f_3, e_4, f_4\rangle$, and $N/O_2(N)$ acts as $\mathrm{Aut}(\mathrm{SL}(2,4))$ in a standard way on both $\langle e_3, f_3, e_4, f_4\rangle$ and $C/\langle e_3, f_3, e_4, f_4\rangle$.*

(6) *$N/O_2(N)$ acts on $O_2(N)$ as $O^-(4,2)$ in a standard way. Hence by Lemma 2.6, $O_2(N) = \langle x \in O_2(N) \mid x$ is noncentral $\rangle$ and $O_2(N) = \langle x \in O_2(N) \mid x$ is central $\rangle$.*

(7) *$O_2(N)$ is the unique elementary abelian subgroup of order* 16 *of* N'.

(8) *If E is an elementary abelian subgroup of order* 16 *of M containing a, then $E \subseteq \langle a\rangle \times D$ and $C_C(E) \subseteq C_C(a) = \langle e_i \mid 3 \le i \le 6\rangle$, and hence by Lemma 2.8, $|C_C(E)| \le 4$, and $|\langle C_C(E), [C,x]\rangle| \le 8$ for every central involution x contained in E.*

(9) *If E is an elementary abelian subgroup of order* 16 *of M not contained in M', then E contains an involution conjugate to a, and hence by* (8), *$|C_C(E)| \le 4$, and $|\langle C_C(E), [C,x]\rangle| \le 8$ for every central involution x contained in E.*

(10) *For each element v of $C - \langle e_3, f_3, e_4, f_4\rangle$, $|[v, O_2(N)]| = 8$.*

(11) *There exist elements e_5', f_5', e_6', f_6' of C with $e_i'\langle e_3, f_3, e_4, f_4\rangle = e_i\langle e_3, f_3, e_4, f_4\rangle$ and $f_i'\langle e_3, f_3, e_4, f_4\rangle = f_i\langle e_3, f_3, e_4, f_4\rangle$ for each $i = 5, 6$ such that if we regard C as a vector space of dimension* 4 *over $GF(4)$ such that an element $\alpha(\ne 0,1)$ of $GF(4)$ acts so that ${e_i'}^{\alpha} = f_i$ for each i, and define a hermitian form φ by $\varphi(e_i', e_j') = \delta_{i,9-j}$ (here for $i = 3, 4$, we let $e_i' = e_i$ and $f_i' = f_i$), then M acts on C in a standard way.*

Lemma 2.10. *Let $M = \mathrm{Aut}(\mathrm{GL}(4,2))$. Let a be an involution of $M - M'$ such that $C_{M'}(a) \cong Sp(4,2)$. Set $D = C_{M'}(a)$. Let $C = \langle e_i, f_i \mid 3 \le i \le 6\rangle \cong E_{256}$. Suppose that M act on C, $[f_i, a] = e_i$ for each i, $\langle e_i \mid 3 \le i \le 6\rangle$ and $\langle f_i \mid 3 \le i \le 6\rangle$ are D-invariant, and D acts on $\langle e_i \mid 3 \le i \le 6\rangle$ as $Sp(4,2)$ so that it fixes the alternating form θ defined by $\theta(e_i, e_j) = \delta_{i,9-j}$, where $\delta_{i,9-j}$ is Kronecker's delta. Set $I = C_D(\langle e_3, e_4\rangle)$. Let J be an elementary abelian subgroup of order* 16 *of M' such that $J \supseteq I$, and set $N = N_M(J)$. Then the following hold.*

(1) *M has four classes of involutions, and $|[C,x]| \ge 4$ for every involution x of M.*

(2) *For every noncentral involution x of M, $|[C,x]| = 16$; for every central involution x of M, $|[C,x]| = 4$.*

(3) *If E is an elementary abelian subgroup of order* 4 *of M, then $|C_C(E)| \le 32$; if E is an elementary abelian subgroup of order* 8 *of M, then $|C_C(E)| \le 16$.*

(4) *$O_2(N) = J$, and $N/O_2(N) \cong \mathrm{GL}(2,2) \wr Z_2 \cong O^+(4,2)$.*

(5) *$[C, O_2(N)] = C_C(O_2(N)) = \langle e_3, f_3, e_4, f_4\rangle$, and $N/O_2(N)$ acts on $\langle e_3, e_4, f_3, f_4\rangle$, $C/\langle e_3, e_4, f_3, f_4\rangle$ and $O_2(N)$ in a standard way (note that the standard action of $\mathrm{GL}(2,2) \wr Z_2$ is the same as that of $O^+(4,2)$). Hence by Lemma 2.6, $O_2(N) = \langle x \in O_2(N) \mid x$ is noncentral $\rangle$ and $O_2(N) = \langle x \in O_2(N) \mid x$ is central $\rangle$.*

(6) *If* E *is an elementary abelian subgroup of order* 16 *of* N *with* $E \neq O_2(N)$, *then* $|C_C(E)| \leq 8$.

Lemma 2.11. *Let* N *be a group with* $N/O(N) \cong \mathrm{PSL}(3,4)$. *Suppose that* N *acts faithfully and irreducibly on an elementary abelian* 2*-group* J *with* $2^8 \leq |J| \leq 2^{10}$. *Let* s,t *be elements of* N *such that* $sO(N)$ *has order* 5 *in* $N/O(N)$ *and* $tO(N)$ *has order* 3 *in* $N/O(N)$, *and suppose that* $|[J,s]| = 2^8$ *and* $|[J,t]| = 2^6$. *Then* $|J| = 512$ *and* $N \cong \mathrm{PSL}(3,4)$.

Proof. Let
$$O(N) = P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots \supseteq P_k = 1$$
be a characteristic composition series of $O(N)$ (namely, for each i, P_{i-1} is a minimal characteristic subgroup of $O(N)$ properly containing P_i). In general, if p is an odd prime, then the 2-rank of $\mathrm{GL}(n,p)$ is n and every elementary abelian 2-subgroup of order 2^n contains the involution of the center of $\mathrm{GL}(n,p)$. On the other hand, since N is isomorphic to a subgroup of $\mathrm{GL}(10,2)$, the rank of P_{i-1}/P_i is less than or equal to 4 for each i, and hence N^∞ acts on P_{i-1}/P_i trivially. By the three-subgroup lemma, $[O(N),N^\infty] = [O(N),N^\infty,\ldots,N^\infty,N^\infty] = 1$. Thus $N^\infty \cong \mathrm{PSL}(3,4)$ or $\mathrm{SL}(3,4)$. Now we obtain the desired conclusion by arguing as in [1] and [2]. □

The following lemma also follows from [1] and [2].

Lemma 2.12. *Let* N,J *be as in Lemma 2.11, and let* $R \in \mathrm{Syl}_2(N)$. *Then the following hold.*

(1) R *contains precisely two elementary abelian subgroups* A, B *of order* 16.

(2) N *acts on* J *irreducibly.*

(3) *The order of the centralizer in* J *of one of* A *or* B *is* 16, *while that of the other is* 2. *We choose our notation so that* $|C_J(A)| = 16$ *and* $|C_J(B)| = 2$

(4) $|[J,A]| = 256$, $[J,A,A] = C_J(A)$, $|[J,B]| = 32$, $[J,B,B] = C_J(B)$.

(5) N *has only one class of involutions, and* $|[J,x]| = 16$ *for every involution* x *of* N.

Lemma 2.13. *Let* N *be a group such that* $N/O(N) \cong M_{22}$. *Suppose* N *acts faithfully on an elementary abelian group* J *of order* 1024. *Then* $N \cong M_{22}$, *and the lengths of the* N*-orbits of involutions of* J *are either*

(1) 22,231 *and* 770*; or*

(2) 77,330 *and* 616.

Proof. Arguing as in Lemma 2.12, we obtain $N \cong M_{22}$ or $3M_{22}$. Hence the desired conclusion follows from Hunt [15] and Smith [19]. $\square$

Lemma 2.14. *Let $N = M_{22}$, and let N act faithfully on an elementary abelian group J of order* 1024. *Suppose the action is the same as the one described in Lemma 2.13*(1). *Let $R \in \mathrm{Syl}_2(N)$. Then*

(1) *R contains precisely two elementary abelian subgroups A, B of order* 16. *We choose our notation so that $N_N(A)/A \cong \Sigma_6'$ and $N_N(B)/B \cong \Sigma_5$. N has only one class of involutions.*

(2) *Let a, b and c be elements of J such that $|\{a^N\}| = 22$, $|\{b^N\}| = 231$ and $|\{c^N\}| = 770$, respectively. Then $C_N(a) \cong \mathrm{PSL}(3,4)$, $C_N(b) \cong E_{16}\cdot\Sigma_5$ and $C_N(c) \cong E_{16} \cdot (E_9 \cdot Z_4)$. We choose our notation so that $O_2(C_N(b)) = B$ and $O_2(C_N(c)) = A$.*

(3) *$|C_J(x)| = 64$ for every involution x of N.*

(4) *$|C_J(A)| = 32$, $|[C_J(A), N_N(A)]| = 16$, and $N_N(A)$ acts on $C_J(A)$ indecomposably. $|[C_J(A), x]| = 4$ for every involution x of $N_N(A) - A$.*

(5) *All involutions of $[C_J(A), N_N(A)]$ are conjugate to b under the action of N. Six of the involutions of $C_J(A) - [C_J(A), N_N(A)]$ are conjugate to a. The remaining ten involutions of $C_J(A) - [C_J(A), N_N(A)]$ are conjugate to c.*

(6) *Let X be the set of pairs (x, y) of involutions of $C_J(A)$ such that $x \in \{b^N\}$ and $y \in \{c^N\}$. Then X splits into two $N_N(A)$-orbits.*

(7) *$|[J/C_J(A), A]| = 16$, $[J/C_J(A), A] = C_{J/C_J(A)}(A)$, and $[J/C_J(A), A]$ and A are isomorphic as an $N_N(A)/A$-module.*

(8) *Under the action of $C_N(a)$, the involutions of $J/\langle a\rangle$ split into three orbits.*

Proof. Statements(1) and (2) are well-known, and (3) follows from (2) by a counting argument. By (3), $|C_J(A)| \le 64$. By (2), $C_N(x)$ contains some conjugate of A for every involution x of J (see Lemma 2.12(1)), and hence $C_J(A)$ contains some involution of each of the three orbits. In view of the action of $N_N(A)/A$ on $C_J(A)$, this implies $|C_J(A)| \ge 32$. If $|C_J(A)| = 64$, then by Lemma 2.7, at least one involution of $C_J(A)$ is centralized by $N_N(A)$, which contradicts (2) (note that arguing as in [1], we see that if $\Sigma_6' \cong Sp(4,2)'$ acts faithfully and irreducibly on an elementary abelian 2-group I_0 of order at most 64, then $|I_0| = 16$, and the action is the standard action of $Sp(4,2)'$, which is the same as the action described in Lemma 2.7). Thus $|C_J(A)| = 32$. The other assertions in (4) also follow from the fact that no involution of $C_J(A)$ is centralized by $N_N(A)$. Since all involutions of $[C_J(A), N_N(A)]$ are centralized by a Sylow 2-subgroup of $N_N(A)$, they belong to $\{b^N\}$. Since $C_J(A)$ splits into three orbits of involutions under the action of $N_N(A)$, different orbits under this action must correspond to different N-orbits. Let y be an involution of $C_J(A) - [C_J(A), N_N(A)]$ such that $|\{y^{N_N(A)}\}| = 6$. Then since $5 || C_{N_N(A)}(y)|$ and since y does not belong to $\{b^N\}$, y must belong to $\{a^N\}$. This also shows that an involution z of $C_J(A) - [C_J(A), N_N(A)]$ such that $|\{z^{N_N(A)}\}| = 10$ belongs to $\{c^N\}$. Thus (5) is proved. Since $[C_J(A), N_N(A)]$ splits into two orbits of involutions with lengths 6 and 9 under the action of $C_{N_N(A)}(c)$ ($= C_N(c)$), (6) follows. If we choose the element a so that $C_N(a)$ contains both A and B and consider the action of $C_N(a)$ on $J/\langle a\rangle$, then by Lemma 2.12(3), A and B of this lemma correspond to A and B of Lemma 2.12, respectively. Hence $|[J/\langle a\rangle, A]| = 256$. Therefore $|[J/C_J(A), A]| = 16$ and, if we let $dC_J(A)$ denote a fixed element of $J/C_J(A) - [J/C_J(A), A]$, then the bijection from A to $[J/C_J(A), A]$ which associates $[dC_J(A), x] \in [J/C_J(A), A]$ with $x \in A$ gives an $(N_N(A)/A)$-isomorphism. Since the action of $N_N(A)$ on A is irreducible, this also implies $[J/C_J(A), A] = C_{J/C_J(A)}(A)$. Finally (8) follows from (2), (3), (5) and Lemma 2.12. □

§3. Notation and Initial Reduction

Throughout the rest of this paper, we let G, A be as in the Main Theorem.

Let Y be a complement of A in $N_G(A)$, i.e., $N_G(A) = A\cdot Y$. Then $C_Y(A) = O(Y) = O(N_G(A))$ and $N_G(A)/O(N_G(A)) = (O(N_G(A))A/O(N_G(A)))\cdot(Y/O(Y)) \cong E_{64}\cdot Sp(6,2)$. Write $A = \langle e_i \mid 2 \le i \le 7\rangle$ (we here let i range from 2 to 7 so that the notation will be consistent with that in [7]). By a result of Dempwolff [5], the action of $Y/O(Y)$ on A is uniquely determined. In particular, we may assume that Y leaves invariant the alternating form θ on A defined by $\theta(e_i, e_j) = \delta_{i,9-j}$, where $\delta_{i,9-j}$ is Kronecker's delta.

We fix the following notation.

Notation 1. Let f_2 be an involution of Y such that

$$[e_7, f_2] = e_2,\ [e_k, f_2] = 1 \text{ for } k \neq 7.$$

For $3 \le i \le 6$, let f_i be an involution of Y such that

$$[e_7, f_i] = e_i,\ [e_{9-i}, f_i] = e_2,$$
$$[e_k, f_i] = 1 \text{ for } k \neq 7, 9-i.$$

Let g_3 be an involution of Y such that

$$[e_6, g_3] = e_3,\ [e_k, g_3] = 1 \text{ for } k \neq 6.$$

For $4 \le i \le 5$, let g_i be an involution of Y such that

$$[e_6, g_i] = e_i,\ [e_{9-i}, g_i] = e_3,$$
$$[e_k, g_i] = 1 \text{ for } k \ne 6, 9-i.$$

Let h be an involution of Y such that

$$[e_5, h] = e_4,\ [e_k, h] = 1 \text{ for } k \ne 5.$$

We choose these involutions so that $S = \langle f_i, g_j, h \mid 2 \le i \le 6, 3 \le j \le 5\rangle$ is a Sylow 2-Subgroup of Y.

Lemma 3.1. (1) *Each involution of the coset f_2A is conjugate to either f_2 or f_2e_3 in $N_G(A)$.*

(2) *Each involution of the coset f_3A is conjugate to either f_3 or f_3e_4 in $N_G(A)$.*

(3) *Each involution of the coset f_2g_3A is conjugate to either f_2g_3 or $f_2g_3e_4$ in $N_G(A)$.*

(4) *Each involution of the coset f_2g_3hA is conjugate to f_2g_3h in $N_G(A)$.*

Proof. This immediately follows from Notation 1. □

Let $C_1 = \langle e_i, f_j \mid 2 \le i \le 6, 3 \le j \le 6\rangle$ and $C = \langle C_1, f_2\rangle$. Then $C_1 = \langle e_3, f_6\rangle * \langle e_4, f_5\rangle * \langle e_5, f_4\rangle * \langle e_6, f_3\rangle$, $C = \langle f_2\rangle \times C_1$, and $\langle e_i, f_{9-i}\rangle \cong D_8$ for each $3 \le i \le 6$. In the rest of this section, we determine possible structures for $N_G(C)/C$ in the following sequence of lemmas.

Lemma 3.2. *The group generated by all the involutions of $\langle C, e_7\rangle - C$ is A.*

Proof. Let e_7x be an involution of $\langle C, e_7\rangle - C$. Since $C/\langle e_2\rangle$ is elementary abelian, $x\langle e_2\rangle \in C_{C/\langle e_2\rangle}(e_7) = \langle e_i, f_2 \mid 2 \le i \le 6\rangle/\langle e_2\rangle$, and hence $x \in \langle e_i, f_2 \mid 2 \le i \le 6\rangle$. But $\langle e_i, f_2 \mid 2 \le i \le 6\rangle$ is also elementary abelian, and hence $x \in C_{\langle e_i, f_2 \mid 2 \le i \le 6\rangle}(e_7) = \langle e_i \mid 2 \le i \le 6\rangle$. Therefore $x \in A$. On the other hand, $A = \langle e_7x \mid x \in \langle e_i \mid 2 \le i \le 6\rangle\rangle$. Thus the lemma is proved. □

Set $F = C_Y(\langle e_2, e_7\rangle) \cap N_Y(\langle f_i \mid 2 \le i \le 6\rangle)$. Then $O(Y)F/O(Y)$ $= C_Y(\langle e_2, e_7\rangle)/O(Y) \cong Sp(4,2) \cong \Sigma_6$. Set $M = N_G(C)$ and $D = M \cap N_G(A)$. Then $O(D) = O(F)$, and $[C, O(D)] = 1$ and $D = C\cdot(\langle e_7\rangle \times F)$. Set $\bar{M} = M/C$, then $\bar{D}/O(\bar{D}) \cong Z_2 \times \Sigma_6$.

Lemma 3.3. $C_{\overline{M}}(\overline{e_7}) = \overline{D}$

Proof. This follows from Lemma 3.2. □

Let $\overline{H} = C_{\overline{M}}(C/Z(C))$.

Lemma 3.4. $|\overline{H}|$ *is odd.*

Proof. This is because $|C_{\overline{H}}(\overline{e_7})| = |O(\overline{D})|$ is odd by Lemma 3.3. $\square$

Let $\overline{\overline{\mathrm{Aut(C)}}} = \mathrm{Aut(C)}/\mathrm{C}_{\mathrm{Aut(C)}}(\mathrm{C}/\mathrm{Z(C)})$. Then $\overline{\overline{\mathrm{Aut(C)}}} \cong O^+(8,2)$. We also let $\overline{\overline{M}} = \overline{M}/\overline{H}$, and regard $\overline{\overline{M}}$ as a subgroup of $\overline{\overline{\mathrm{Aut(C)}}}$.

Lemma 3.5. $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong Z_2 \times \Sigma_6$, Σ_8 *or* $\mathrm{Aut(SU}(4,2))$.

Proof. Since $C_{\overline{\overline{M}}}(\overline{\overline{e_7}}) = \overline{\overline{C_{\overline{M}}(\overline{e_7})}} = \overline{\overline{D}} = \langle\overline{\overline{e_7}}\rangle \times \overline{\overline{F}}$ by Lemmas 3.4 and 3.3, $C_{\overline{\overline{M}}}(\overline{\overline{e_7}}) \cong Z_2 \times \Sigma_6$. By Harris and Solomon [14], $E(\overline{\overline{M}}/O(\overline{\overline{M}}))$ is isomorphic to one of the following groups:

(1) Σ_6' or $\Sigma_6' \times \Sigma_6'$;

(2) Σ_8', $\mathrm{SU}(4,2)$, $\mathrm{SL}(5,2)$, $\mathrm{SU}(5,2)$ or $Sp(4,4)$;

(3) $\mathrm{PSU}(4,3)$.

By considering the orders of these groups and $O^+(8,2)$, we can eliminate $\mathrm{SL}(5,2)$, $\mathrm{SU}(5,2)$, $Sp(4,4)$ and $\mathrm{PSU}(4,3)$. In $O^+(8,2)$, no element of order 5 is centralized by a subgroup isomorphic to Σ_6' (see Frame [8]). Hence we can eliminate $\Sigma_6' \times \Sigma_6'$. Thus the lemma follows. $\square$

We examine the three cases of Lemma 3.5 separately in subsequent sections.

§4. Conjugacy Classes of Involutions

In this section and the next section, we assume that $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong \mathrm{Aut(SU}(4,2))$ and prove that $G \cong M(22)$. The principal aim of this section is to show that G has three classes of involutions.

Lemma 4.1. $\overline{\overline{M}}$ *is isomorphic to* $\mathrm{Aut(SU}(4,2))$ *or* $\Gamma U(4,2)$ $(\cong (Z_3 \times \mathrm{SU}(4,2)) \cdot Z_2)$.

Proof. Since 5 divides $|\mathrm{SU}(4,2)|$ and $|O(\overline{\overline{M}})|$ divides $3\cdot5\cdot7$, $(O(\overline{\overline{M}})C_{\overline{\overline{M}}}(O(\overline{\overline{M}})))/O(\overline{\overline{M}}) \supseteq E(\overline{\overline{M}}/O(\overline{\overline{M}}))$. Now the lemma follows from the class list of $O^+(8,2)$ ([8]). $\square$

Lemma 4.2. *Every involution of* $\overline{\overline{M}}$ *is conjugate to some involution of* $\langle\overline{\overline{e_7}}\rangle\times\overline{\overline{F}}$ *in* $\overline{\overline{M}}$.

Proof. $\overline{\overline{M}}$ has 4 classes of involutions, and their representatives are $\overline{\overline{g_3}}$(inner central), $\overline{\overline{g_4}}$(inner non-central), $\overline{\overline{e_7}}$(field) and $\overline{\overline{e_7g_3}}$((field) × (inner)), which are all contained in $\langle\overline{\overline{e_7}}\rangle \times \overline{\overline{F}}$. $\square$

Having Lemma 2.9 in mind, we fix the following notation.

Notation 2. Let x_1 be an element of M' such that $\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle \cong E_{16}$. Then $N_{\overline{\overline{M}}}(\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle)/\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle$ is isomorphic to either $\mathrm{Aut}(\mathrm{SL}(2,4))$ or $\Gamma L(2,4)$ ($\cong (Z_3 \times \mathrm{SL}(2,4)) \cdot Z_2$). This factor group acts both on $(Z(C)\langle e_3, f_3, e_4, f_4\rangle)/Z(C)$ and on $C/(Z(C)\langle e_3, f_3, e_4, f_4\rangle)$ in a standard way as $\mathrm{Aut}(\mathrm{SL}(2,4))$ or $\Gamma L(2,4)$, whereas the action on $\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle$ is the same as that of $O^-(4,2)$ on a standard module. Thus we can choose x_1 so that $\overline{\overline{x_1}}$, $\overline{\overline{g_3 x_1}}$ and $\overline{\overline{h x_1}}$ are noncentral involutions. Then $\overline{x_1}$ is conjugate to $\overline{g_4}$. Since g_4 is an involution, we can choose x_1 as an involution. Let x_2 be an element of M' such that

$$\begin{aligned} [e_4, x_2] \in f_3 Z(C), \quad [f_4, x_2] \in e_3 f_3 Z(C), \\ [\langle e_3, f_3\rangle, x_2] \subseteq Z(C) \quad . \end{aligned}$$

We choose x_2 so that $\langle\overline{\overline{g_5}}, \overline{\overline{x_2}}\rangle \cong E_4$ and x_2 is an involution. Moreover, we choose x_1 and x_2 so that $\langle AS, x_1, x_2\rangle$ is a Sylow 2-subgroup of M. Set $R = \langle AS, x_1, x_2\rangle$.

We prove $R \in \mathrm{Syl}_2(G)$ in the following sequence of lemmas.

Lemma 4.3. *If x is an element of M such that $x^2 \in \langle e_2\rangle$ and $\overline{\overline{x}}$ is a noncentral involution of $E(\overline{\overline{M}})$, then $C_C(x)$ contains an abelian subgroup of order* 64.

Proof. By taking a suitable conjugate of x, we may assume $\overline{x} = \overline{g_4}$. Since $x^2 \in \langle e_2\rangle$, $x = g_4 y$, where $y\langle e_2\rangle \in C_{C/\langle e_2\rangle}(g_4) = \langle e_i, f_i \mid 2 \le i \le 4\rangle/\langle e_2\rangle$. Consequently $C_C(x)$ contains $\langle e_i, f_i \mid 2 \le i \le 4\rangle$, as desired. □

Lemma 4.4. *C is weakly closed in R with respect to $C_G(e_2)$.*

Proof. By way of contradiction, let C_2 be a subgroup of R such that $C_2 \ne C$, $C_2 \cong C$ and $C_2' = \langle e_2\rangle$. Then $C_2/\langle e_2\rangle \cong E_{512}$. If $|\overline{C_2}| = 2$, then, by Lemma 2.9(1), $|C_{C/\langle e_2\rangle}(\overline{C_2})| \le 2 \cdot 64$. This means that $|C_2/\langle e_2\rangle| \le |\overline{C_2}| \cdot |C_{C/\langle e_2\rangle}(\overline{C_2})| \le 256$, which is a contradiction. If $4 \le |\overline{C_2}| \le 8$, then by Lemma 2.9(3), $|C_{C/\langle e_2\rangle}(\overline{C_2})| \le 2 \cdot 16$, which leads to a similar contradiction. Thus $|\overline{C_2}| = 16$. If $\overline{C_2} \ne \overline{\langle g_3, g_4, h, x_1\rangle}$, then by (7) and (9) of Lemma 2.9, $|C_{C/\langle e_2\rangle}(\overline{C_2})| \le 2 \cdot 4$, which again leads to the same kind of contradiction. Consequently $\overline{C_2} = \overline{\langle g_3, g_4, h, x_1\rangle}$. We also have $|C_2 \cap C| = 1024/16 = 64$. Since $|C_{C/\langle e_2\rangle}(g_4)| \le 2 \cdot 16 = 32$, this means that $C_2 \ni g_4$ and $C_2/\langle e_2\rangle \supseteq C_{C/\langle e_2\rangle}(g_4)$. On the other hand, by Lemma 4.3, the group generated by g_4 and the inverse image of $C_{C/\langle e_2\rangle}(g_4)$ contains an abelian subgroup of order 128. Since $C_2 \cong C \cong Z_2 \times (D_8 * D_8 * D_8 * D_8)$, this is a contradiction. □

Lemma 4.5. $R \in \mathrm{Syl}_2(G)$.

Proof. Since $Z(R) = \langle e_2 \rangle$, this immediately follows from Lemma 4.4. $\square$

We next determine $J(R)$ and $N_G(J(R))$. Set $J = \langle e_i, f_i, g_3, g_4, h, x_1 \mid 2 \leq i \leq 4 \rangle$. We prove $J = J(R)$.

Lemma 4.6. $J \cong E_{1024}$

Proof. Let $\widetilde{M} = M/Z(C)$. Also set $J_0 = \langle e_i, f_i \mid 2 \leq i \leq 4 \rangle$, $J_1 = \langle J_0, g_3, g_4, h \rangle$ and $J_2 = \langle J_0, g_3, h, x_1 \rangle$. Calculating in $N_G(A)$, we see that $J_1 \cong E_{512}$. We show that $J_2 \cong E_{512}$. By Lemma 2.9(6), there exists $\overline{y} \in N_{\overline{M}}(J_0)$ such that $\overline{g_3}^{\overline{y}} = \overline{g_3}$, $\overline{h}^{\overline{y}} = \overline{h}$ and $\overline{g_4}^{\overline{y}} = \overline{x_1}$. Note that $[\widetilde{C}, \overline{g_3 h}] = [\widetilde{C}, \overline{x_1}] = \widetilde{J_0}$. Since both $\widetilde{g_4}^{\widetilde{y}}$ and $\widetilde{x_1}$ are involutions, we get $g_4^y \in x_1 J_0$. Similarly $(g_3 h)^y \in g_3 h J_0$. Since $[\widetilde{g_3 h}^{\widetilde{y}}, \widetilde{g_3}^{\widetilde{y}}] = [\widetilde{g_3 h}^{\widetilde{y}}, \widetilde{h}^{\widetilde{y}}] = 1$, we also obtain $g_3^y \in g_3 J_0$ and $h^y \in h J_0$. Consequently $J_1^y = J_2$, and hence $J_2 \cong J_1 \cong E_{512}$. Note that this implies $[J_0, g_4] = [J_0, x_1] = 1$.

Now in order to prove $J \cong E_{1024}$, it suffices to show that $[g_4, x_1] = 1$. Note that $\overline{g_3 g_4}$, $\overline{g_3 h}$ and $\overline{g_4 h}$ are noncentral involutions of $\overline{M}$, and $\overline{g_4}$, $\overline{x_1}$ and $\overline{g_4 x_1}$ are also noncentral involutions of $\overline{M}$. Hence by Lemma 2.9(6), there exists $\overline{z} \in N_{\overline{M}}(J_0)$ such that $\overline{g_3 g_4}^{\overline{z}} = \overline{g_4}$ and $\overline{g_3 h}^{\overline{z}} = \overline{x_1}$. As before, we have $(g_3 g_4)^z \in g_4 J_0$ and $(g_3 h)^z \in x_1 J_0$. On the other hand, $[(g_3 g_4)^z, (g_3 h)^z] = [g_3 g_4, g_3 h]^z = 1$. Since $[J_0, g_4] = [J_0, x_1] = 1$, we now obtain $[g_4, x_1] = [(g_3 g_4)^z, (g_3 h)^z] = 1$, as desired. $\square$

Lemma 4.7. *Let $\widetilde{M} = M/Z(C)$. Then $\widetilde{I} \subseteq \widetilde{J}$ for every abelian subgroup $\widetilde{I}$ of $\widetilde{R}$ such that $\overline{I} = \overline{J}$.*

Proof. Let $\widetilde{I}$ be an abelian subgroup of $\widetilde{R}$ such that $\overline{I} = \overline{J}$. First note that $\widetilde{I} \cap \widetilde{C} \subseteq C_{\widetilde{C}}(\langle g_3, g_4, h, x_1 \rangle) = \widetilde{C} \cap \widetilde{J}$. With each element x of $\langle g_3, g_4, h, x_1 \rangle$, we associate an element $\varphi(x)$ of C such that $\widetilde{x\varphi(x)} \in \widetilde{I}$. Then for any elements x, y of $\langle g_3, g_4, h, x_1 \rangle$, we have $[\widetilde{\varphi(x)}, \widetilde{y}] = [\widetilde{\varphi(y)}, \widetilde{x}]$ because $[\widetilde{x\varphi(x)}, \widetilde{y\varphi(y)}] = 1$. Now let a be an element of $\langle g_3, g_4, h, x_1 \rangle$ such that $\overline{\overline{a}}$ is a central involution of $E(\overline{\overline{M}})$. Suppose that $\widetilde{\varphi(a)} \notin \widetilde{C} \cap \widetilde{J}$. Then, by Lemma 2.9(10), $|[\widetilde{\varphi(a)}, \widetilde{\langle g_3, g_4, h, x_1 \rangle}]| = 8$. Since $[\widetilde{\varphi(y)}, \widetilde{a}] = [\widetilde{\varphi(a)}, \widetilde{y}]$ must hold for every $y \in \langle g_3, g_4, h, x_1 \rangle$, $[\widetilde{C}, \widetilde{a}] \supseteq [\widetilde{\varphi(a)}, \widetilde{\langle g_3, g_4, h, x_1 \rangle}]$. But since $\overline{\overline{a}}$ is central, $|[\widetilde{C}, \widetilde{a}]| = 4$ by Lemma 2.9(2), a contradiction. Thus $\widetilde{\varphi(a)} \in \widetilde{J}$. Since a is arbitrary, the desired conclusion follows from Lemma 2.9(6). $\square$

Lemma 4.8. $J = J(R)$

Proof. Let I be an abelian subgroup of order at least 1024 of R. We have only to show that $I \subseteq J$. Arguing as in Lemma 4.4, we can easily show that $\overline{I} = \overline{J}$. Therefore $I \subseteq J$ by Lemma 4.7. $\square$

Lemma 4.9. *$N_G(J)/C_G(J) \cong M_{22}$, and the action is the same as the one studied in Lemma* 2.14*.*

Proof. Let $\widetilde{N_G(J)} = N_G(J)/C_G(J)$. Since $N_M(J)$ contains a Sylow 2-subgroup of G, $N_M(J)$ contains a Sylow 2-subgroup of $N_G(J)$. Note that $O_2(\widetilde{N_M(J)}) = \widetilde{\langle e_5, e_6, f_5, f_6\rangle}$, and $\widetilde{N_M(J)}/O_2(\widetilde{N_M(J)})$ is isomorphic to either $\mathrm{Aut}(\mathrm{SL}(2,4))$ or $\Gamma L(2,4)$, where the action on $O_2(\widetilde{N_M(J)})$ is the same as that on a standard module. Hence a Sylow 2-subgroup of $\widetilde{N_G(J)}$ is isomorphic to a Sylow 2-subgroup of M_{22}. Note also that $e_2 \sim e_3$ in G. Thus in view of Lemmas 4.8, 2.11 and 2.13, it follows from Gorenstein and Harada [11] that $\widetilde{N_G(J)} \cong M_{22}$. By Lemma 2.14(1), this implies $\widetilde{N_M(J)}/O_2(\widetilde{N_M(J)}) \cong \mathrm{Aut}(\mathrm{SL}(2,4)) \cong \Sigma_5$ (so $\overline{\overline{M}} \cong \mathrm{Aut}(\mathrm{SU}(4,2))$). Suppose that the action of $\widetilde{N_G(J)}$ on J is the same as the one described in Lemma 2.13(2). Then since $C_{\widetilde{N_G(J)}}(e_2) \supseteq \widetilde{N_M(J)}$ and $|\widetilde{N_G(J)} : \widetilde{N_M(J)}| = |M_{22}|/2^4|\mathrm{Aut}(\mathrm{SL}(2,4))| = 231$, we have $|\widetilde{N_G(J)} : C_{\widetilde{N_G(J)}}(e_2)| = 77$, i.e., $|C_{\widetilde{N_M(J)}}(e_2) : \widetilde{N_M(J)}| = 3$. This implies $\widetilde{N_M(J)}' \lhd C_{\widetilde{N_G(J)}}(e_2)$. Consequently $\widetilde{\langle e_5, e_6, f_5, f_6\rangle} = O_2(\widetilde{N_M(J)}') \lhd C_{\widetilde{N_G(J)}}(e_2)$, which contradicts the structure of M_{22} described in Lemma 2.14(1). Thus the desired conclusion follows from Lemma 2.13. $\square$

Lemma 4.10. *G has precisely three classes of involutions.*

Proof. In view of Lemmas 4.8 and 4.9, it suffices to show that each involution of G is conjugate to an involution of C. Let x be an involution. Since R is a Sylow 2-subgroup of G, we may assume $x \in R$. Then $x \in M$. By Lemma 4.2, we may assume $x \in N_G(A)$. If $x \in A$, x is conjugate to e_2. Thus we may assume $x \notin A$. Note that $N_G(A)/A$ has four classes of involution and their representatives are f_2A, f_3A, f_2g_3A and f_2g_3hA. The desired conclusion now follows from Lemma 3.1. $\square$

We proceed to determine the structures of the centralizers of involutions. We include the proof of the following lemma in this section.

Lemma 4.11. *$C_G(e_2) = O(C_G(e_2))N_G(C)$.*

Proof. Let $\widetilde{C_G(e_2)} = C_G(e_2)/\langle e_2\rangle$. We show $\widetilde{C}$ is strongly closed in $\widetilde{R}$ with respect to $C_G(e_2)$. By Lemma 4.4, $\widetilde{C}$ is weakly closed. Let Γ, γ be as in Lemma 2.3. Let $\widetilde{E}$ be an element of Γ such that $|\widetilde{E}| = \gamma$. First suppose that $|\overline{E}| = 2$, and let $\overline{x}$ be the involution of $\overline{E}$. Then by Lemma 2.9(1), $|[\widetilde{C},\overline{x}]| \geq 4$, which contradicts Lemma 2.3(5). Next suppose that $8 \geq |\overline{E}| \geq 4$. Then by Lemma 2.9(3), there exists an involution $\overline{x}$ of $\overline{E}$ such that $|[\widetilde{C},\overline{x}]| = 16$. This again contradicts Lemma 2.3(5). Thus $|\overline{E}| = 16$. Suppose that $\overline{E} \not\subseteq \overline{M'}$. Then by Lemma 2.9(9), there exists an involution $\overline{x}$ of $\overline{E}$ such that $|\langle C_{\widetilde{C}}(\overline{E}), [\widetilde{C},\overline{x}]\rangle| \leq 2\cdot 8 = 16$, which contradicts Lemma 2.3(6). Thus $\overline{E} \subseteq \overline{M'}$. By Lemma 2.9(7) and Notation 2, $\overline{E} = \overline{\langle g_3, g_4, h, x_1\rangle}$. If $|\widetilde{C} \cap \widetilde{E}| \leq 2$, then by Lemma 2.9(2), $|\langle(\widetilde{C} \cap \widetilde{E}), [\widetilde{C},\overline{x}]\rangle| \leq 16$ for every central involution $\overline{x}$ contained in $\overline{E}$, which contradicts Lemma 2.3(6). Thus $|\widetilde{C} \cap \widetilde{E}| \geq 4$, which implies $|E| \geq 128$, where E denotes the full inverse image of $\widetilde{E}$. But by Lemma 4.7, $E \subseteq J$, and hence E is elementary abelian, which is a contradiction because C does not contain an elementary abelian subgroup of order 128. Consequently $\widetilde{C}$ is strongly closed in $\widetilde{R}$ with respect to $\widetilde{C_G(e_2)}$. Since $N_{\widetilde{G}}(\widetilde{C})$ controls the fusion of $\widetilde{C}$, we obtain $(O(C_G(e_2))\langle e_2\rangle)\langle f_2\rangle/(O(C_G(e_2))\langle e_2\rangle) \lhd C_G(e_2)/(O(C_G(e_2))\langle e_2\rangle)$ by Glauberman's Z^*-Theorem.

Now let $\widetilde{C_G(e_2)} = C_G(e_2)/(O(C_G(e_2))\langle e_2, f_2\rangle)$. Considering the action of $\overline{\overline{F}}$ on $\widetilde{C}$, we see that each involution of $\widetilde{C}$ is conjugate to $\widetilde{e_3}$, $\widetilde{f_3}$, $\widetilde{e_3f_3}$, $\widetilde{e_3f_4}$ or $\widetilde{e_3f_6}$. In view of the action of $N_{\overline{\overline{M}}}(\overline{\overline{J}})/\overline{\overline{J}}$, it follows that each involution of $\widetilde{C}$ is conjugate to $\widetilde{e_3}$ or $\widetilde{e_3f_6}$. Therefore by the main theorem of Goldschmidt [9], $\widetilde{C} \lhd \widetilde{C_G(e_2)}$, as desired. □

§5. Centralizers of Involutions

We continue with the notation of the preceding section, and complete the proof for the case where $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong \mathrm{Aut}(\mathrm{SU}(4,2))$.

Our aim is to show that $O(C_G(e_2)) = 1$. For this purpose, we need to determine the structure of $C_G(e_3f_2)$ and $C_G(f_2)$. Before this is done, we make some more preparations. Let $I = \langle e_2, f_2, e_3, f_3, g_3\rangle$.

Lemma 5.1. *Let $\widetilde{N_G(J)} = N_G(J)/C_G(J)$. Then the following hold:*

(1) *$\widetilde{R}$ contains exactly two elementary abelian subgroups $\widetilde{\langle e_5, f_5, e_6, f_6\rangle}$, $\widetilde{\langle e_5, f_5, g_5, x_2\rangle}$ of order 16.*

(2) *$N_{\widetilde{N_G(J)}}(\widetilde{\langle e_5, f_5, e_6, f_6\rangle}) \cong E_{16}\cdot \mathrm{Aut}(\mathrm{SL}(2,4))$, $N_{\widetilde{N_G(J)}}(\widetilde{\langle e_5, f_5, g_5, x_2\rangle}) \cong E_{16}\cdot Sp(4,2)'$, where the actions are standard.*

(3) *$C_J(\widetilde{\langle e_5, f_5, g_5, x_2\rangle}) = I$.*

(4) *e_2,f_2 and e_3f_2 are the representatives of the three conjugacy classes of involutions of G.*

(5) *If x, y are elements of I such that $x \in \widetilde{\{e_2^G\}}$ and $y \in \{(e_3f_2)^G\}$, then there exists an element $\widetilde{g} \in N_{\widetilde{N_G(J)}}(\langle \widetilde{e_5, f_5, g_5, x_2} \rangle)$ such that $x^{\widetilde{g}} = e_2$ and $y^{\widetilde{g}} = e_3f_2$ or such that $x^{\widetilde{g}} = e_2$ and $y^{\widetilde{g}} = e_2g_3$.*

(6) $O_2(C_{\widetilde{N_G(J)}}(e_3f_2)) = \langle \widetilde{e_5, f_5, g_5, x_2} \rangle$.

(7) $I \lhd C_{N_G(J)}(e_3f_2)$.

Proof. Statement(1) follows from Lemma 2.14(1) and Notation 2. As noted in the proof of Lemma 4.9, $\widetilde{N_M(J)}/\langle \widetilde{e_5, f_5, e_6, f_6} \rangle \cong \Sigma_5$. Hence (2) follows from Lemma 2.14(1). We now prove (3). By Lemma 2.9(5), $[C/Z(C), \overline{J}] = C_{C/Z(C)}(\overline{J}) = \langle e_i, f_i \mid 2 \le i \le 4 \rangle / Z(C)$. Hence it follows from Lemma 2.9(2) that if $\overline{x}$ is an involution of $\overline{J}$ which centralizes $Z(C)\langle e_5, f_5 \rangle / Z(C)$, then $\overline{x}$ is central. In view of Lemma 2.9(6), it follows form the choice of x_1 that every involution in $\overline{\langle g_4, h, x_1 \rangle} - \overline{\langle h \rangle}$ is noncentral. Since $\overline{h}$ does not centralize $Z(C)\langle e_5, f_5 \rangle / Z(C)$ and $\overline{g_3}$ centralizes $Z(C)\langle e_5, f_5 \rangle / Z(C)$, we get $C_{\overline{J}}(Z(C)\langle e_5, f_5 \rangle / Z(C)) = \overline{\langle g_3 \rangle}$. Consequently, calculating in M, we obtain $C_J(\langle e_5, f_5 \rangle) = \langle e_2, f_2, e_3, f_3, g_3 \rangle$. On the other hand, by Lemma 2.14(4), $|C_J(\langle \widetilde{e_5, f_5, g_5, x_2,} \rangle)| = 32$. Therefore (3) holds. We proceed to the proof of (4). In view of Lemmas 4.8 and 4.10, it suffices to show that $e_2 \not\sim f_2 \not\sim e_3f_2 \not\sim e_2$ in $N_G(J)$. Set $\widetilde{X} = N_{\widetilde{N_G(J)}}(\langle \widetilde{e_5, f_5, g_5, x_2} \rangle)$. We first prove $[I, \widetilde{X}] = \langle e_2, f_3, f_2f_3, f_2g_3 \rangle$. Clearly $e_2 = [f_2, e_7] \in [I, \widetilde{R}] \subseteq [I, \widetilde{X}]$. Since $e_2 \sim e_3$ in $N_G(A)$, $e_2 \sim e_3$ in $N_G(J)$ by Lemma 4.8. Hence $e_3 \in [I, \widetilde{X}]$ by Lemma 2.14(5). Suppose that $f_2f_3 \notin [I, \widetilde{X}]$. Then since $f_2f_3 \sim f_2g_3$ in Y (recall that Y is a complement of A in $N_G(A)$), it follows from Lemmas 4.8 and 2.14(5) that $f_2g_3 \notin [I, \widetilde{X}]$. Since $|I : [I, \widetilde{X}]| = 2$ by Lemma 2.14(4), this implies $f_3g_3 = (f_2f_3)(f_2g_3) \in [I, \widetilde{X}]$. But since $f_3g_3 \sim f_2f_3$, this is a contradiction. Thus $f_2f_3 \in [I, \widetilde{X}]$, and hence $f_2g_3 \in [I, \widetilde{X}]$. Therefore $[I, \widetilde{X}] = \langle e_2, e_3, f_2f_3, f_2g_3 \rangle$. By Lemma 2.14(5), this implies $e_2 \not\sim f_2$ and $e_2 \not\sim e_3f_2$ in $N_G(J)$. We have $f_2 \sim e_2f_2 \sim g_3 \sim e_3g_3 \sim f_2f_3g_3 \sim e_2e_3f_2f_3g_3$, $e_3f_2 \sim e_2e_3f_2 \sim e_2g_3 \sim e_2e_3g_3 \sim e_2f_2f_3g_2 \sim e_3f_2f_3g_3$ and $f_3 \sim e_2f_3 \sim e_3f_3 \sim e_2e_3f_3$ in $N_G(A)$. In view of Lemma 2.14(5), this implies $f_2 \not\sim e_3f_2$. Note that $C_{\widetilde{N_G(J)}}(f_2)$ contains $\widetilde{N_M(J)}'$. By (2) and (5) of Lemma 2.14, this means that f_2 corresponds to the element a in Lemma 2.14 and e_3f_2 corresponds to c (so $f_2 \not\sim f_3$ and $e_3f_2 \sim f_3$). Recall that $e_3f_2 \sim e_2g_3$. On the other hand, it follows from Lemma 4.11 that e_3f_2 and e_2g_3 are not conjugate in $C_G(e_2)$. Consequently (5) follows from Lemma 2.14(6). Now (6) follows from Lemma 2.14(2), and (7) follows from (3) and (6). □

Lemma 5.2. $O(C_G(\langle e_2, e_3f_2 \rangle))I \lhd C_G(\langle e_2, e_3f_2 \rangle)$

Proof. Let $\widetilde{C_G(e_2)} = C_G(e_2)/(O(C_G(e_2))Z(C))$. Clearly $\widetilde{C_G(\langle e_2, e_3f_2\rangle)} \subseteq C_{\widetilde{C_G(e_2)}}(\widetilde{e_3})$. In the sense of Lemma 2.9(11), $\widetilde{\langle e_3, f_3\rangle}$ is the "1-dimensional subspace of $\widetilde{C}$ over $GF(4)$ spanned by $\widetilde{e_3}$. " Hence $\widetilde{\langle e_3, f_3\rangle} \lhd C_{\widetilde{C_G(e_2)}}(\widetilde{e_3})$. Since $\widetilde{g_3}$ is the "transvection with respect to $\widetilde{\langle e_3, f_3\rangle}$," $\widetilde{C}\langle\widetilde{g_3}\rangle \lhd C_{\widetilde{C_G(e_2)}}(\widetilde{e_3})$. By way of contradiction, suppose that $\widetilde{I} \ntriangleleft \widetilde{C_G(\langle e_2, e_3f_2\rangle)}$. Then there exists an element x of $C_G(\langle e_2, e_3f_2\rangle)$ such that $g_3^x = g_3a$ where $a \in C$, and $\widetilde{a} \in C_{\widetilde{C}}(\widetilde{g_3}) - [\widetilde{C}, \widetilde{g_3}]$. Since the full inverse image of $C_{\widetilde{C}}(\widetilde{g_3}) = \langle \widetilde{e_i}, \widetilde{f_i} \mid 2 \le i \le 5\rangle$ in $C_G(e_2)/O(C_{E_G}(e_2))$ is centralized by $g_3O(C_{E(G)}(e_2))$ and since g_3 and g_3a are both involutions, $aO(C_G(e_2))$ is an involution. Hence there exists an element g of $C_Y(\langle e_2, e_3, f_2, g_3\rangle)$ $(\subseteq C_Y(\langle e_2, e_3f_2\rangle) \cap C_Y(g_3))$ such that $a^g \in \langle e_i, f_i \mid 2 \le i \le 4\rangle$. Thus this fusion must occur in $N_{C_G(\langle e_2, e_3f_2\rangle)}(J)$. But this contradicts Lemma 5.1(7). $\square$

The proof of Lemma 5.3 is similar to and easier than that of Lemma 5.2, and so it is omitted.

Lemma 5.3. *$O(C_G(\langle e_2, e_2g_3\rangle))I \lhd C_G(\langle e_2, e_2g_3\rangle)$.*

Lemma 5.4. *If x, y are elements of I such that $x \in \{e_2^G\}$ and $y \in \{(e_3f_2)^G\}$, then*

$$O(C_G(\langle x, y\rangle))I \lhd C_G(\langle x, y\rangle).$$

Proof. This follows from (3) and (5) of Lemma 5.1 and Lemmas 5.2 and 5.3. $\square$

We now determine the structure of $C_G(e_3f_2)$.

Lemma 5.5. *$O(C_G(e_3f_2))I \lhd C_G(e_3f_2)$.*

Proof. By Lemma 5.1(6), $J\langle e_5, f_5, g_5, x_2\rangle \lhd C_R(e_3f_2)$. Since $e_6, e_7f_6 \in C_R(e_3f_2)$, this implies $|C_R(e_3f_2)| = 2^{16}$. Since $Z(R) = \langle e_2\rangle$, it follows from Lemma 5.1(4) that $C_R(e_3f_2) \in \mathrm{Syl}_2(C_G(e_3f_2))$. Note also that $(e_7f_6)^2 = e_6$, $C_{J\langle e_5, f_5, g_5, x_2\rangle/J}(e_6) = J\langle e_5, f_5\rangle/J$ and $C_I(e_6) = \langle e_2, f_2, e_3\rangle$.

We show that I is strongly closed in $C_R(e_3f_2)$ with respect to $C_G(e_3f_2)$. By way of contradiction, let x be an element of $C_R(e_3f_2)$ such that $x \notin I$ and $x^g \in I$ for some $g \in C_G(e_3f_2)$. Since $C_{N_G(J)}(e_3f_2) = N_{C_G(e_3f_2)}(J)$ controls the fusion of J in $C_G(e_3f_2)$, $x \notin J$ (see Lemma 5.1(7)). Hence by Lemma 2.3(2), we may assume $C_{C_R(e_3f_2)}(x)^g \subseteq C_R(e_3f_2)$. First suppose that $x \in J\langle e_5, f_5, g_5, x_2\rangle$. By Lemma 2.14(5), I contains a subgroup I_1 of order 16 all of whose involutions are conjugate to e_2 in G. Let y be an involution of I_1. Suppose that $y^g \in J$. Then since $C_{N_G(J)}(e_3f_2)$ controls the fusion, there exists an element g_1 of $C_{N_G(J)}(e_3f_2)$ such that $(y^g)^{g_1} = y$. But then by Lemma 5.1(7), $x^{gg_1} \in I$, which contradicts Lemma 5.4. Thus $y^g \notin J$. Since y is arbitrary, $I_1^g \cap J = 1$. Since $I_1 \subseteq C_{C_R(e_3f_2)}(x)$, $I_1^g \subseteq C_R(e_3f_2)$. Since $(e_7f_6)^2 = e_6$ and $C_{J\langle e_5,f_5,g_5,x_2\rangle/J}(e_6) = J\langle e_5, f_5\rangle/J$, $J\langle e_5, f_5, g_5, x_2\rangle/J$ is the only elementary abelian subgroup of order 16 of $C_R(e_3f_2)/J$. Consequently $JI_1^g = J\langle e_5, f_5, g_5, x_2\rangle$, and hence $C_{C_R(e_3f_2)}(I_1^g) = C_J(I_1^g) \cdot I_1^g = C_J(\langle g_5, x_2, f_5, e_5\rangle) \cdot I_1^g = I \cdot I_1^g$. Since $C_J(x)^g = C_{C_R(e_3f_2)}(x)^g \cap J^g \subseteq C_R(e_3f_2) \cap C_{C_G(e_3f_2)}(I_1)^g = C_{C_R(e_3f_2)}(I_1^g)$, we get $C_J(x)^g \subseteq I \cdot I_1^g$. Since $|C_J(x)| = 64$ by Lemma 2.14(3), it follows that there exists an element z of $C_J(x) - I$ such that $z^g \in I$. But this contradicts the fact that $C_{N_G(J)}(e_3f_2)$ controls the fusion of J. Consequently no element of $J\langle e_5, f_5, g_5, x_2\rangle - I$ is fused into I.

Recall that $(e_7f_6)^2 = e_6$ and $C_I(e_6) = \langle e_2, f_2, e_3\rangle$, and $J\langle e_5, f_5, g_5, x_2\rangle \subseteq C_R(I)$ by Lemma 5.1(3). Thus I is weakly closed in $C_R(e_3f_2)$. Now if we define γ' as in Lemma 2.3, then $\gamma' = 2$, which contradicts Lemma 2.3(2). Therefore I is strongly closed.

Now let $\widetilde{C_G(e_3f_2)} = C_G(e_3f_2)/(O(G)\langle e_3f_2\rangle)$. Suppose that $\widetilde{I} \ntriangleleft \widetilde{C_G(e_3f_2)}$, and set $\widetilde{X} = \langle \widetilde{I}^{\widetilde{C_G(e_3f_2)}}\rangle$. By Lemma 4.8, $N_{\widetilde{C_G(e_3f_2)}}(\widetilde{J})$ controls the fusion of $\widetilde{J}$, and hence it follows from the proof of Lemma 2.14(6) that the involutions of $\widetilde{I}$ split into two classes of sizes 6 and 9. Therefore by the main theorem of [9], $\widetilde{X} \cong \mathrm{PSL}(2,q) \times \mathrm{PSL}(2,q)$ for some q with $q \equiv 3, 5 \pmod 8$. Set $Q = J\langle e_5, f_5, g_5, x_2\rangle$. Then by Lemma 5.1(3), $\widetilde{Q} \subseteq C_{\widetilde{C_G(e_3f_2)}}(\widetilde{I})$. This implies $\widetilde{X}\widetilde{Q} = \widetilde{X} \times C_{\widetilde{Q}}(\widetilde{X})$, and hence $\widetilde{Q} = \widetilde{I} \times C_{\widetilde{Q}}(\widetilde{X})$. But then $\widetilde{I} \cap \widetilde{Q}' = 1$, which contradicts Notation 1. Consequently $\widetilde{I} \triangleleft \widetilde{C_G(e_3f_2)}$, as desired. $\square$

Lemma 5.6. *$C_G(e_3f_2)$ is solvable.*

Proof. Set $B = N_{C_G(e_3f_2)}(I)$ and $\widetilde{B} = B/I$. In veiw of Lemma 5.5, the lemma is equivalent to the assertion that $\widetilde{B}$ is solvable. Note that $J\langle e_5, f_5, g_5, x_2\rangle = C_R(I) \in \mathrm{Syl}_2(C_B(I))$ and $C_R(e_3f_2) = C_R(I)\langle e_7f_6\rangle \in \mathrm{Syl}_2(B)$ (see the first paragraph of the proof of Lemma 5.5). In particular, a Sylow 2-subgroup of $B/C_B(I)$ is a cyclic group of order 4, and hence $B/C_B(I)$ is solvable. Thus it suffices to show that $\widetilde{C_B(I)}$ is solvable. Set $W = N_{N_B(J)}(C_R(I))$ and $Z/J = C_{W/J}(C_R(I)/J)$. By Lemma 2.14(7), $|\widetilde{C_R(I)}/[\widetilde{J}, \widetilde{C_R(I)}]| = 32$. Since $W/Z \cong E_9 \cdot Z_4$, $\widetilde{C_R(I)}/[\widetilde{J}, \widetilde{C_R(I)}] \cong E_{32}$ or $D_8 * D_8$. Since $\langle e_5, f_5, g_5\rangle \cong E_8$, this implies $\widetilde{C_R(I)}/[\widetilde{J}, \widetilde{C_R(I)}] \cong E_{32}$. Hence if we write $[\widetilde{C_R(I)}/[\widetilde{J}, \widetilde{C_R(I)}], O(W/J)] = \widetilde{E}/[\widetilde{J}, \widetilde{C_R(I)}]$, then $|\widetilde{E}/[\widetilde{J}, \widetilde{C_R(I)}]| = 16$, i.e., $|\widetilde{E}| = 256$. Note that $\widetilde{E}/(\widetilde{E} \cap \widetilde{J}) \cong \widetilde{C_R(I)}/\widetilde{J}$ and $\widetilde{E} \cap \widetilde{J} = [\widetilde{J}, \widetilde{C_R(I)}] = C_{\widetilde{J}}(\widetilde{C_R(I)})$, and $\widetilde{C_R(I)}/\widetilde{J}$ and $[\widetilde{J}, \widetilde{C_R(I)}]$ are isomorphic as a (W/Z)-module by Lemma 2.14(7). Consequently $\widetilde{E}$ is abelian by Lemma 2.5. By Lemma 2.3(2), $\widetilde{E}$ is strongly closed in $\widetilde{C_R(I)}$ with respect to $\widetilde{C_B(I)}$.

Suppose that $O(\widetilde{C_B(I)})\widetilde{E} \ntriangleleft \widetilde{C_B(I)}$, and set $X = \langle E^{C_B(I)}\rangle$. In view of the action of W/Z and $\widetilde{J}/(\widetilde{J}\cap\widetilde{E})$ on $\widetilde{E}$, $\widetilde{J}\cap\widetilde{E}$ is the only nontrivial proper $N_{\widetilde{B}}(\widetilde{E})$-invariant subgroup of $\widetilde{E}$. Hence $O_2((O(\widetilde{C_B(I)})\widetilde{X})/O(\widetilde{C_B(I)})) = 1$ by the main theorem of [9]. Considering the action of $O(W/Z)$ on $\widetilde{E}$, it also follows from the main theorem of [9] that $(O(\widetilde{C_B(I)})\widetilde{X})/O(\widetilde{C_B(I)})$ is the direct product of groups isomorphic to $\mathrm{SL}(2,2^n)$ $(n = 2,4,8)$. By Lemma 2.12(4), E' is a subgroup of I with $|E'| \geq 16$. Hence $|X^\infty \cap I| \geq 16$. On the other hand, $I \subseteq Z(E)$ by Lemma 5.1(3), and hence $X^\infty \cap I \subseteq Z(X^\infty)$. Note that Sylow 2-subgroups of $2\,\mathrm{SL}(2,4) \cong \mathrm{SL}(2,5)$ are not abelian. Thus we get a contradiction to the structure of the Schur multiplier of $\mathrm{SL}(2,2^n)$ (see the argument used in the proof of Lemma 2.11). Consequently $O(\widetilde{C_B(I)})\widetilde{E} \lhd \widetilde{C_B(I)}$. Since $|\widetilde{C_R(I)} : \widetilde{E}| = 2$, this implies that $\widetilde{C_B(I)}$ is solvable, as desired. □

Lemma 5.7. *Let $\widetilde{C_G(f_2)} = C_G(f_2)/\langle f_2\rangle$. Then $\widetilde{C_G(f_2)}$ has three classes of involutions with representatives $\widetilde{e_2}$, $\widetilde{e_3}$ and $\widetilde{f_4 g_3}$.*

Proof. By Lemma 2.14(2), $N_{\widetilde{C_G(f_2)}}(\widetilde{J})/C_{\widetilde{C_G(f_2)}}(\widetilde{J}) \cong \mathrm{PSL}(4,3)$ (see the proof of (4) and (5) of Lemma 5.1). Hence by Lemma 2.14(8), the involutions of $\widetilde{J}$ split into three classes under the action of $N_{\widetilde{C_G(f_2)}}(\widetilde{J})$. Let $\widetilde{x}$ be an involution of $N_{\widetilde{C_G(f_2)}}(\widetilde{J}) - \widetilde{J}$. We prove $\widetilde{x}$ is fused into $\widetilde{J}$. By Lemma 2.12(5), we may assume $\widetilde{x}\widetilde{J} = \widetilde{e_5}\widetilde{J}$. By Notation 1, $[g_3, e_5] = 1$. Since $J \subseteq N_G(C)$ and $e_5 \in C$ and $f_2g_3, g_3 \notin C$, we have $f_2g_3, g_3 \notin [J, e_5]$; that is, $\widetilde{g_3} \in C_{\widetilde{J}}(\widetilde{e_5}) - [\widetilde{J}, \widetilde{e_5}]$. Hence by Lemma 2.12(6), $\widetilde{x}$ is conjugate to $\widetilde{e_5}$ or $\widetilde{e_5 g_3}$. Recall that $N_G(A) = A \cdot Y$. Thus there exists $v \in Y$ such that $f_2^v = f_2$, $e_5^v = e_4$ and $g_3^v = g_3$. Hence e_5 and e_5g_3 are conjugate to e_4 and e_4g_3, respectively, in $C_{N_G(A)}(f_2)$. Therefore every involution of $N_{\widetilde{C_G(f_2)}}(\widetilde{J}) - \widetilde{J}$ is fused into $\widetilde{J}$. Since $N_{\widetilde{C_G(f_2)}}(\widetilde{J})$ controls the fusion of $\widetilde{J}$, this means that $\widetilde{C_G(f_2)}$ has precisely three classes of involutions.

We now show that f_4g_3 is conjugate to e_3f_2 in G. In Y, 28 of the involutions of $\langle f_2, f_3, f_4, g_3, g_4, h\rangle$ are conjugate to f_4g_3. For each such involution x, every element of the coset $x\langle e_2, e_3, e_4\rangle$ is conjugate to x in $N_G(A)$ by Lemma 3.1(4). Hence J contains 224 involutions conjugate to f_4g_3 in $N_G(A)$, and none of them is contained in I. By Lemma 2.14(5), 15 of the involutions of I are conjugate to e_2. Since $224 + 15 > 231$, it follows from Lemma 2.14(2) that f_4g_3 cannot be conjugate to e_2 or f_2. Consequently f_4g_3 is conjugate to e_3f_2. Note that f_4g_3 and $f_4g_3f_2$ are conjugate in Y. Thus both elements of the coset $f_4g_3\langle f_2\rangle$ are conjugate to e_3f_2 in G. On the other hand, one element of the coset $e_3\langle f_2\rangle$ is conjugate to e_2 and the other is (conjugate to) e_3f_2 in G, and one element of the coset $e_2\langle f_2\rangle$ is e_2 and the other is conjugate to f_2 in G. Therefore $\widetilde{f_4 g_3} \not\sim \widetilde{e_3} \not\sim \widetilde{e_2} \not\sim f_4g_3$ in $\widetilde{C_G(f_2)}$, and the lemma is proved. □

Lemma 5.8. $C_G(f_2)/(\langle f_2\rangle O(C_G(f_2))) \cong \mathrm{PSU}(6,2)$.

Proof. Let $\widetilde{C_G(f_2)} = C_G(f_2)/(\langle f_2\rangle O(C_G(f_2)))$. By Lemma 5.7, $\widetilde{C_G(f_2)}$ contains three classes of involutions. Let $\widetilde{x}$ be an involution which is conjugate to either $\widetilde{e_3}$ or $\widetilde{f_4 g_3}$. We may assume x is conjugate to $e_3 f_2$ in G (see the second paragraph of the proof of Lemma 5.7). Thus $C_G(x)$ is solvable by Lemma 5.6, and hence $C_{C_G(x)}(f_2)$ is solvable. Since $|C_{\widetilde{C_G(f_2)}}(\widetilde{x}) : \widetilde{C_{C_G(x)}(f_2)}| \leq 2$, this means that $C_{\widetilde{C_G(f_2)}}(\widetilde{x})$ is solvable. Now since $e_2 \not\sim f_2 \sim e_2 f_2$ in G, $C_{\widetilde{C_G(f_2)}}(\widetilde{e_2}) = \widetilde{C_{C_G(f_2)}(e_2)}$. By Lemma 4.11, $\widetilde{C_{C_G(f_2)}(e_2)}/O(\widetilde{C_{C_G(f_2)}(e_2)})$ is an extension of $D_8 * D_8 * D_8 * D_8$ by $\mathrm{SU}(4,2)$ (see the parenthetic remark about $\overline{\overline{M}}$ in the proof of Lemma 4.9). Consequently the centralizer of each involution of $\widetilde{C_{E(G)}(f_2)}$ is 2-constrained. Since $\widetilde{C_G(f_2)}$ is connected in the sense of Gorenstein and Walter [13], $O(C_{\widetilde{C_G(f_2)}}(\widetilde{e_2})) = 1$ by Theorem B of [13]. Therefore $\widetilde{C_{E(G)}(f_2)} \cong \mathrm{PSU}(6,2)$ by a result of Parrot [17]. □

We are now in a position to complete the proof for the case where $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong \mathrm{Aut}(\mathrm{SU}(4,2))$. Note that $\mathrm{PSU}(6,2)$ is 2-generated and 2-balanced. Hence in view of Lemmas 5.1(4), 4.11, 5.6 and 5.8, it follows from Theorem A of [13] that $O(C_G(x)) = 1$ for every involution x of G. Therefore we obtain $E(G) \cong M(22)$ by a result of Hunt [15] or Parrot [17].

§6. Contradiction

In this section, we assume that $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong \Sigma_8$, and derive a contradiction. Arguing as in Section 4, we obtain the following lemmas.

Lemma 6.1. $\overline{\overline{M}} \cong \Sigma_8(\cong \mathrm{Aut}(\mathrm{GL}(4,2)))$.

Lemma 6.2. *Every involution of $\overline{\overline{M}}$ is conjugate to some involution of $\langle\overline{\overline{e_7}}\rangle \times \overline{\overline{F}}$ in $\overline{\overline{M}}$.*

Having Lemma 2.10 in mind, we fix the following notation.

Notation 3. Let x_1 be an element of M' such that $\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle \cong E_{16}$. Then $N_{\overline{\overline{M}}}(\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle)/\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle \cong \mathrm{GL}(2,2) \wr Z_2 \cong O^+(4,2)$, and $N_{\overline{\overline{M}}}(\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle)/\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle$ acts on $(Z(C)\langle e_3, e_4, f_3, f_4\rangle)/Z(C)$, $C/(Z(C)\langle e_3, e_4, f_3, f_4\rangle)$ and $\langle\overline{\overline{g_3}}, \overline{\overline{g_4}}, \overline{\overline{h}}, \overline{\overline{x_1}}\rangle$ in a standard way. Thus we can choose x_1 so that $\overline{\overline{x_1}}$, $\overline{\overline{g_3 x_1}}$ and $\overline{\overline{h x_1}}$ are central involutions. Let x_2 be an element of M' such that

$$[e_4 f_4, x_2] \in e_3 f_3 Z(C), [\langle e_3, f_3, f_4\rangle, x_2] \subseteq Z(C).$$

We choose x_1 and x_2 as involutions. Moreover we choose them so that $\langle \overline{\overline{g_5}}, \overline{\overline{x_2}} \rangle \cong E_4$ and $\langle AS, x_1, x_2 \rangle$ is a Sylow 2-subgroup of M. Set $R = \langle AS, x_1, x_2 \rangle$.

We now argue as in Section 4, using Lemma 2.10 in place of Lemma 2.9. Then we obtain the following lemma.

Lemma 6.3. $R \in \mathrm{Syl}_2(G)$.

Lemma 6.4. *There exists $x_1' \in x_1 C$ such that $\langle e_i, f_i, g_3 h, g_4, x_1' \mid 2 \le i \le 4 \rangle \cong E_{512}$.*

Proof. By Lemma 2.10(5), there exists $\overline{y} \in N_{\overline{M}}(\langle e_i, f_i \mid 2 \le i \le 4 \rangle)$ such that $\overline{g_3 h}^{\overline{y}} = \overline{g_4}$, $\overline{g_4}^{\overline{y}} = \overline{g_3 h}$ and $\overline{h}^{\overline{y}} = \overline{x_1}$. Set $x_1' = h^y$. Then arguing as in the first paragraph of the proof of 4.6, we obtain $\langle e_i, f_i, g_3 h, g_4, x_1' \mid 2 \le i \le 4 \rangle = \langle e_i, f_i, g_3 h, g_4, h \mid 2 \le i \le 4 \rangle^y \cong E_{512}$. □

Let x_1' be as in Lemma 6.4, and set $J = \langle e_i, f_i, g_3, g_4, h, x_1' \mid 2 \le i \le 4 \rangle$.

Lemma 6.5. $J \cong E_{1024}$.

Proof. In view of Lemma 6.4, it suffices to show that $[g_4 h, g_3 h x_1'] = 1$. Note that g_4 and $g_3 h$ are noncentral involutions and $g_3 g_4 h$ is a central involution, and that $g_4 h$ and $g_3 h x_1'$ are noncentral involutions and $g_3 g_4 x_1'$ is a central involution. Hence by Lemma 2.10(5), there exists $\overline{z} \in N_{\overline{M}}(\langle e_i, f_i \mid 2 \le i \le 4 \rangle)$ such that $\overline{g_4}^{\overline{z}} = \overline{g_4 h}$ and $\overline{g_3 h}^{\overline{z}} = \overline{g_3 h x_1'}$. Therefore arguing as in the second paragraph of the proof of Lemma 4.6, we obtain $[g_4 h, g_3 h x_1'] = [g_4^z, (g_3 h)^z] = 1$. □

Arguing as in Section 4, we also obtain the following lemma.

Lemma 6.6. $J = J(R) \cong E_{1024}$.

We are now in a position to derive a contradiction. Let $\widetilde{N_G(J)} = N_G(J)/C_G(J)$. Since $N_M(J)$ contains a Sylow 2-subgroup of G, $N_M(J)$ contains a Sylow 2-subgroup of $N_G(J)$. Note that $O_2(\widetilde{N_M(J)}) = \widetilde{\langle e_5, e_6, f_5, f_6 \rangle}$, $\widetilde{N_M(J)}/O_2(\widetilde{N_M(J)}) \cong \mathrm{GL}(2,2) \wr Z_2$ and $\widetilde{N_M(J)}/O_2(\widetilde{N_M(J)})$ acts on $O_2(\widetilde{N_M(J)})$ in a standard way. Hence a Sylow 2-subgroup of $\widetilde{N_M(J)}$ is isomorphic to a Sylow 2-subgroup of Σ_8. Therefore we see from Gorenstein and Harada [12] that the action of $\widetilde{N_M(J)}$ on J cannot be consistent with the fusion of J, which is a contradiction. This concludes the discussion for the case where $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong \Sigma_8$.

§7. Normal Case

In this section, we assume that $\overline{\overline{M}}/O(\overline{\overline{M}}) \cong Z_2 \times \Sigma_6$, and show that A is normal in G.

Arguing as in Section 4, we obtain the following two lemmas.

Lemma 7.1. $AS \in \mathrm{Syl}_2(G)$

Lemma 7.2. $J(AS) = \langle e_2, e_3, e_4\rangle C_S(\langle e_2, e_3, e_4\rangle) = \langle e_i, f_i, g_3, g_4, h \mid 2 \leq i \leq 4\rangle \cong E_{512}$.

Lemma 7.3. $N_G(J(AS)) = O(N_G(J(AS))N_{N_G(A)}(J(AS)))$.

Proof. Set $W = N_G(J(AS))$, $\widetilde{W} = W/C_W(J(AS))$, $B = \langle e_5, e_6, e_7\rangle$, $W_1 = N_Y(\langle f_i, g_3, g_4, h \mid 2 \leq i \leq 4\rangle)$. Then $\widetilde{N_A(J(AS))} = \widetilde{B} \cdot \widetilde{W_1} \cong E_8 \cdot \mathrm{SL}(3, 2)$ and $\widetilde{AS} \in \mathrm{Syl}_2(\widetilde{BW_1})$. We prove that $\widetilde{B}$ is strongly closed in $\widetilde{AS}$ with respect to $\widetilde{W}$. Define Γ and γ as in Lemma 2.3 (note that we have not yet proved that $\widetilde{B}$ is weakly closed) and, by way of contradiction, suppose that $\Gamma \neq \emptyset$. Let $\widetilde{E}$ be a member of Γ. Then since $[J(AS), \widetilde{y}] = \langle e_2, e_3, e_4\rangle$ for every involution $\widetilde{y}$ of $\widetilde{B}$, $[J(AS), \widetilde{y_1}] = [J(AS), \widetilde{y_2}]$ for any involutions $\widetilde{y_1}$, $\widetilde{y_2}$ of $\widetilde{E}$. But $|[J(AS)/\langle e_2, e_3, e_4\rangle, \widetilde{y}]| = 4$ for each involution $\widetilde{y}$ of $\widetilde{AS} - \widetilde{B}$, and $[J(AS)/\langle e_2, e_3, e_4\rangle, \widetilde{y_1}] \neq [J(AS)/\langle e_2, e_3, e_4\rangle, \widetilde{y_2}]$ if $\widetilde{y_1}\widetilde{B} \neq \widetilde{y_2}\widetilde{B}$. Hence $|\widetilde{E}| = 2$. Since $\widetilde{E}$ is arbitrary, this means $\gamma = 2$. In particular, $\widetilde{B}$ is weakly closed. These contradict Lemma 2.3(6). Thus $\widetilde{B}$ is strongly closed. Consequently $\widetilde{W} = O(\widetilde{W})(\widetilde{BW_1})$ by Goldschmidt [9]. We next prove $O(\widetilde{W}) = 1$. Arguing as in Lemma 2.11, we can easily show that $O(\widetilde{W})$ centralizes $\widetilde{BW_1}$. On the other hand, $\langle e_2, e_3, e_4\rangle$ is the unique minimal $\widetilde{BW_1}$-invariant subgroup of $J(AS)$, and $\langle f_3, f_4, g_4, e_2, e_3, e_4\rangle/\langle e_2, e_3, e_4\rangle$ is the unique minimal $\widetilde{BW_1}$-invariant subgroup of $J(AS)/\langle e_2, e_3, e_4\rangle$. Therefore $O(\widetilde{W})$ centralizes $J(AS)$, and hence $O(\widetilde{W}) = 1$, as desired. □

We can now easily show that A is strongly closed in AS. Note that each involution of AS is conjugate to an involution of J (see the last few sentences of the proof of Lemma 4.10). Since $N_G(J)$ controls the fusion of J by Lemma 7.2. it follow from Lemma 7.3 that no involution of $J - A$ is conjugate to an involution of $J \cap A$. Consequently A is strongly closed. Therefore $A \lhd G$ by Goldschmidt [9], as desired.

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