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A generalized Yoneda algebra of an algebra associated with a cyclic quiver

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Abstract. Let $A = K\Gamma/(X^k)$, where $K\Gamma$ is the path algebra of a cyclic quiver Γ over a field K, X is the sum of all arrows of Γ and k is a positive integer. In this paper, we describe the ring structure of the generalized Yoneda algebra $\bigoplus_{i\geq 0}\operatorname{Ext}^i_A(A/J^l, A/J^l)$ of A with multiplication given by the Yoneda product, where J denotes the Jacobson radical of A and l is a positive integer with $l \leq k$.

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§1. Introduction

Let $K\Gamma$ be the path algebra over a field K of the cyclic quiver Γ with s vertices $e_1, \dots, e_s$ and s arrows $a_1, \dots, a_s$, where s is a positive integer. We set $X = a_1 + \cdots + a_s$, $A = K\Gamma/(X^k)$ with a positive integer k and J the Jacobson radical of A, that is, the ideal of A generated by X. Let l be a positive integer with $l \leq k$. Then we call the algebra $\mathcal{E}(A/J^l) = \bigoplus_{i\geq 0}\operatorname{Ext}^i_A(A/J^l, A/J^l)$ with multiplication given by the Yoneda product the *generalized Yoneda algebra* of A, because the algebra $\mathcal{E}(A/J)$ is the usual Yoneda algebra of A.

A. I. Generalov [4] has determined the ring structure of the usual Yoneda algebra $\mathcal{E}(A/J)$ of A by using the diagrammatic method which is presented by D. J. Benson and J. F. Carlson in [1] (cf. Remark in Section 3.1). Our purpose of this paper is to describe the ring structure of the generalized Yoneda algebra $\mathcal{E}(A/J^l)$ of A by basic calculations. By the way, a basic self-injective Nakayama algebra over K is of the form $A = K\Gamma/(X^k)$ with $k \geq 2$ and K. Erdmann and T. Holm [2] determined the ring structure of the Hochschild cohomology ring $\mathrm{HH}^*(A) = \bigoplus_{i\geq 0}\operatorname{Ext}^i_{A^e}(A, A)$ of A. Here, A^e denotes the enveloping algebra $A \otimes_K A^\circ$ of A, where A° is the opposite ring of A.

This paper is organized as follows: In Section 2, we construct an A-projective resolution of A/J^l (Proposition 2.1) and calculate the group $\mathrm{Ext}^i_A(A/J^l, A/J^l)$ for $i \geq 0$ (Propositions 2.2 and 2.4). In Section 3, we calculate the Yoneda product in $\mathcal{E}(A/J^l)$ (Propositions 3.1 and 3.5) and describe the ring structure of $\mathcal{E}(A/J^l)$ (Theorems 3.4 and 3.8) by referring to [3].

§2. Calculation of the group $\mathrm{Ext}^i_A(A/J^l, A/J^l)$

Let s be a positive integer, Γ the cyclic quiver with s vertices $e_1, e_2, \ldots, e_s$ and s arrows $a_1, a_2, \ldots, a_s$ such that each a_i starts at e_i and ends at e_{i+1}, where we regard the subscripts i of e_i modulo s. Let K be a field and $K\Gamma$ the path algebra of Γ over K. In $K\Gamma$, $a_i = e_{i+1}a_ie_i$ holds for each $1 \leq i \leq s$. Let X be the sum of all arrows: $X = a_1 + a_2 + \cdots + a_s$. Note that X is a non-zero divisor in $K\Gamma$.

We fix a positive integer k, and we denote $K\Gamma/(X^k)$ by A. Then A is a finite dimensional algebra, since $A = \bigoplus_{p=0}^{k-1} \bigoplus_{q=1}^{s} KX^pe_q$ and $\dim_K A = ks$. Let $J = AX = XA = (X)/(X^k)$, then J is the radical of A because J is a nilpotent ideal and $A/J \simeq K\Gamma/(X) \simeq \prod_{i=1}^{s} Ke_i$ is semi-simple.

Let l be a fixed positive integer with $l \leq k$. In this section, we calculate the group $\mathrm{Ext}^i_A(A/J^l, A/J^l)$ for $i \geq 0$ in order to consider the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i\geq 0} \mathrm{Ext}^i_A(A/J^l, A/J^l)$ of A. First, we give an A-projective resolution of A/J^l for the calculation.

Proposition 2.1. *Let $A = K\Gamma/(X^k)$, $J = XA$ the radical of A, l a positive integer with $l \leq k$. Then there exists the following periodic right A-projective resolution of A/J^l:*

$$(2.1) \quad \cdots \xrightarrow{\kappa} A \xrightarrow{d} A \xrightarrow{\kappa} A \xrightarrow{d} A \xrightarrow{\pi} A/J^l \longrightarrow 0,$$

where $\pi : A \to A/J^l$ is the natural right A-epimorphism, $d : A \to A$ and $\kappa : A \to A$ are the right A-homomorphisms defined by

$$d(x) = X^l x, \qquad \kappa(x) = X^{k-l}x$$

for all $x \in A$.

Proof. Since $\mathrm{Ker}\ \pi = J^l = X^lA = \mathrm{Im}\ d$, $d\kappa = 0$ and $\kappa d = 0$, it suffices to show that $\mathrm{Ker}\ d \subseteq \mathrm{Im}\ \kappa$ and $\mathrm{Ker}\ \kappa \subseteq \mathrm{Im}\ d$.

Let $a \in \mathrm{Ker}\ d$, where $a = u + (X^k)$ for some $u \in K\Gamma$. Then we have $0 = d(a) = X^lu + (X^k)$ in A, hence there exists an element $v \in K\Gamma$ such that $X^lu = X^kv$ in $K\Gamma$. Since X is a non-zero divisor in $K\Gamma$, we have $u = X^{k-l}v$. Hence $a = X^{k-l}v + (X^k) = \kappa(v + (X^k)) \in \mathrm{Im}\ \kappa$, so we have $\mathrm{Ker}\ d \subseteq \mathrm{Im}\ \kappa$. Similarly, we also have $\mathrm{Ker}\ \kappa \subseteq \mathrm{Im}\ d$. □

In the rest of this section, we calculate the group $\mathrm{Ext}^i_A(A/J^l, A/J^l)$. We denote the functor $\mathrm{Hom}_A(-, A/J^l)$ by $(-)^*$. By applying the functor to the projective resolution (2.1) of A/J^l, we have the following commutative diagram of left A/J^l-modules:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & A^* & \xrightarrow{d^*} & A^* & \xrightarrow{\kappa^*} & A^* & \xrightarrow{d^*} & A^* & \xrightarrow{\kappa^*} & \cdots \\ & & \mu \wr \downarrow & & \mu \wr \downarrow & & \mu \wr \downarrow & & \mu \wr \downarrow & & \\ 0 & \longrightarrow & A/J^l & \xrightarrow{d^\#} & A/J^l & \xrightarrow{\kappa^\#} & A/J^l & \xrightarrow{d^\#} & A/J^l & \xrightarrow{\kappa^\#} & \cdots, \end{array} \tag{2.2}$$

where we set

$$\mu : A^* = \mathrm{Hom}_A(A, A/J^l) \xrightarrow{\sim} A/J^l; \quad \phi \longmapsto \phi(1_A),$$

$d^\# = \mu d^* \mu^{-1}$ and $\kappa^\# = \mu \kappa^* \mu^{-1}$. Note that the inverse μ^{-1} of μ is given by $\mu^{-1}(a + J^l)(x) = ax + J^l$ for all $x \in A$ and $a + J^l \in A/J^l$. Since the left A/J^l-module A/J^l is generated by $1_A + J^l$, the left A/J^l-module $\mathrm{Hom}_A(A, A/J^l)$ is generated by $\mu^{-1}(1_A + J^l) = \pi$, that is,

$$\mathrm{Hom}_A(A, A/J^l) = (A/J^l)\pi.$$

By the left module action of A/J^l on $\mathrm{Hom}_A(A, A/J^l)$, for $a + J^l \in A/J^l$, we have

$$((a + J^l)\pi)(x) = (a + J^l)\pi(x) = (a + J^l)(x + J^l) = ax + J^l$$

for all $x \in A$. Moreover, for the left A/J^l-homomorphisms d^* and κ^*, we have

$$d^* = 0, \qquad \kappa^*(\pi) = (X^{k-l} + J^l)\pi,$$

since $d^*(\pi)(x) = (\pi d)(x) = X^l x + J^l = 0$ and $\kappa^*(\pi)(x) = (\pi\kappa)(x) = X^{k-l}x + J^l$ for all $x \in A$. Hence the left A/J^l-homomorphisms $d^\#$ and $\kappa^\#$ satisfy that $d^\# = 0$ and

$$\kappa^\#(1_A + J^l) = (\mu\kappa^*)(\pi) = \mu((X^{k-l} + J^l)\pi) = X^{k-l} + J^l. \tag{2.3}$$

If $k \geq 2l$ then $\kappa^* = 0$, and hence we easily obtain the following proposition.

Proposition 2.2. *In the case $k \geq 2l$, we have the following isomorphisms of left A/J^l-modules*:

$$\mathrm{Ext}^i_A(A/J^l, A/J^l) = A^* \xrightarrow{\sim} A/J^l; \quad \phi \longmapsto \phi(1_A),$$

for $i \geq 0$, where $A^* = \mathrm{Hom}_A(A, A/J^l) = (A/J^l)\pi$ with the natural right A-epimorphism $\pi : A \to A/J^l$.

Next we consider the case $k < 2l$. We prepare the following lemma in order to compute the group $\mathrm{Ext}_A^i(A/J^l, A/J^l)$ for $i \geq 0$.

Lemma 2.3. *In the case $k < 2l$, we have the following equations*:

$$\mathrm{Im}\ \kappa^{\#} = J^{k-l}/J^l, \qquad \mathrm{Ker}\ \kappa^{\#} = J^{2l-k}/J^l,$$

where $\kappa^{\#}$ is the left A/J^l-homomorphism as above and J^0 denotes A.

Proof. By the equation (2.3), we have $\mathrm{Im}\ \kappa^{\#} = (AX^{k-l} + J^l)/J^l = J^{k-l}/J^l$ and $\kappa^{\#}(J^{2l-k}/J^l) = (J^{2l-k}X^{k-l})/J^l = 0$. Hence it suffices to show that $\mathrm{Ker}\ \kappa^{\#} \subset J^{2l-k}/J^l$.

Let $a + J^l \in \mathrm{Ker}\ \kappa^{\#}$, where $a = u + (X^k)$ for some $u \in K\Gamma$. Then we have $0 = \kappa^{\#}(a + J^l) = aX^{k-l} + J^l$, hence there exists an element $v \in K\Gamma$ such that $aX^{k-l} = (v + (X^k))X^l$. It follows that $uX^{k-l} + (X^k) = vX^l + (X^k)$, so there exists an element $w \in K\Gamma$ such that $uX^{k-l} - vX^l = wX^k$. Since X is a non-zero divisor in $K\Gamma$, we have $u = vX^{2l-k} + wX^l = (v + wX^{k-l})X^{2l-k}$. Let $a' = v + wX^{k-l} + (X^k) \in A$, then $a = a'X^{2l-k} \in J^{2l-k}$ holds. Therefore we have $a + J^l \in J^{2l-k}/J^l$. □

So we have the following theorem by Lemma 2.3 and the commutative diagram (2.2).

Proposition 2.4. *In the case $k < 2l$, we have the following isomorphisms of left A/J^l-modules*:

$$\mathrm{Ext}_A^i(A/J^l, A/J^l)$$
$$= \begin{cases} A^* \xrightarrow{\sim} A/J^l; & \phi \longmapsto \phi(1_A) \quad \text{if } i = 0, \\ \mathrm{Ker}\ \kappa^* \xrightarrow{\sim} J^{2l-k}/J^l; & \phi \longmapsto \phi(1_A) \quad \text{if } i \text{ is odd}, \\ A^*/\mathrm{Im}\ \kappa^* \xrightarrow{\sim} A/J^{k-l}; & [\phi] \longmapsto a + J^{k-l} \quad \text{if } i \text{ is even}, \end{cases}$$

where $[\phi]$ is the element represented by $\phi \in A^*$ and $\phi(1_A) = a + J^l$ for some $a \in A$.

Proof. For the proof, we use the commutative diagram (2.2) of left A/J^l-modules and Lemma 2.3.

If $i = 0$, then the left A/J^l-isomorphism

$$\mu : \mathrm{Ext}_A^0(A/J^l, A/J^l) = A^* \xrightarrow{\sim} A/J^l; \quad \phi \longmapsto \phi(1_A),$$

is the desired isomorphism.

If i is odd, then the left A/J^l-isomorphism

$$\mathrm{Ext}_A^i(A/J^l, A/J^l) = \mathrm{Ker}\ \kappa^* \xrightarrow{\sim} \mathrm{Ker}\ \kappa^{\#} = J^{2l-k}/J^l; \quad \phi \longmapsto \phi(1_A),$$

which is induced by μ is the desired isomorphism.

If i is even, then the left A/J^l-isomorphism

$$\mathrm{Ext}_A^i(A/J^l, A/J^l) = A^*/\mathrm{Im}\ \kappa^* \simeq (A/J^l)/\mathrm{Im}\ \kappa^{\#}$$

is induced by μ. Since $\mathrm{Im}\ \kappa^{\#} = J^{k-l}/J^l$, the composition of left A/J^l-isomorphisms

$$\begin{array}{ccccc} A^*/\mathrm{Im}\ \kappa^* & \xrightarrow{\sim} & (A/J^l)/(J^{k-l}/J^l) & \xrightarrow{\sim} & A/J^{k-l}; \\ [\phi] & \longmapsto & \phi(1_A) + J^{k-l}/J^l & \longmapsto & a + J^{k-l}, \end{array}$$

where $\phi(1_A) = a + J^l$ for some $a \in A$ is the desired isomorphism. $\square$

§3. Calculation of the Yoneda product

In this section, we calculate the Yoneda product in the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i \geq 0} \mathrm{Ext}_A^i(A/J^l, A/J^l)$ of A by means of the resolution (2.1). Then we determine the ring structure of $\mathcal{E}(A/J^l)$.

We recall the definition of the Yoneda product $\times$ in $\mathcal{E}(A/J^l)$. Denote the right A-projective resolution (2.1) by

$$\cdots \xrightarrow{d_4} A_3 \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{\pi} A/J^l \longrightarrow 0,$$

where we set $A_i = A$, $d_{2i+1} = d$ and $d_{2i+2} = \kappa$ for $i \geq 0$. Let $[\phi] \in \mathrm{Ext}_A^i(A/J^l, A/J^l)$ and $[\psi] \in \mathrm{Ext}_A^j(A/J^l, A/J^l)$ be the elements which are represented by $\phi \in \mathrm{Ker}\ d_{i+1}^*$ and $\psi \in \mathrm{Ker}\ d_{j+1}^*$, respectively. There exists the following commutative diagram of right A-modules:

$$\begin{tikzcd} \cdots \arrow[r, "d_{i+j+1}"] & A_{i+j} \arrow[r, "d_{i+j}"] \arrow[d, "\sigma_i"'] & \cdots \arrow[r, "d_{j+2}"] & A_{j+1} \arrow[r, "d_{j+1}"] \arrow[d, "\sigma_1"'] & A_j \arrow[d, "\sigma_0"'] \arrow[dr, "\psi"] & & \\ \cdots \arrow[r, "d_{i+1}"] & A_i \arrow[r, "d_i"] & \cdots \arrow[r, "d_2"] & A_1 \arrow[r, "d_1"] & A_0 \arrow[r, "\pi"] & A/J^l \arrow[r] & 0, \end{tikzcd}$$

where $\sigma_\nu\,(0 \leq \nu \leq i)$ are liftings of ψ. Then the Yoneda product $[\phi] \times [\psi]$ is given by $[\phi\sigma_i] \in \mathrm{Ext}_A^{i+j}(A/J^l, A/J^l)$.

Define the ring automorphism $\beta : A \to A$ by

$$(3.1) \qquad \beta(e_i) = e_{i-1}, \qquad \beta(a_i) = a_{i-1}$$

for $1 \leq i \leq s$. Then it is easily verified that $\beta(X) = X$ and $aX^t = X^t\beta^t(a)$ for all $a \in A$ and $t \geq 0$, where β^0 denotes the identity map on A. We use these equations in the following calculations.

3.1. The case $k \geq 2l$

In this subsection, we consider the case $k \geq 2l$. In order to clearly describe the degree of the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i\geq 0} \operatorname{Ext}_A^i(A/J^l, A/J^l)$, by Proposition 2.2, we write $\operatorname{Ext}_A^i(A/J^l, A/J^l) = (A/J^l)\pi_i$ for $i \geq 0$, where π_i denotes the natural right A-epimorphism $\pi : A \to A/J^l$. Note that if $\phi \in \operatorname{Ext}_A^i(A/J^l, A/J^l)$ then there exists some $a \in A$ such that $\phi = (a + J^l)\pi_i$, and hence $\phi(x) = ax + J^l$ for all $x \in A$.

Proposition 3.1. *In the case $k \geq 2l$, for $(a + J^l)\pi_i \in \operatorname{Ext}_A^i(A/J^l, A/J^l)$ and $(b + J^l)\pi_j \in \operatorname{Ext}_A^j(A/J^l, A/J^l)$ with a, $b \in A$, the Yoneda product $(a + J^l) \times (b + J^l) \in \operatorname{Ext}_A^{i+j}(A/J^l, A/J^l)$ is given as follows:*

$$\begin{aligned}&(a + J^l)\pi_i \times (b + J^l)\pi_j \\ &\quad = \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\pi_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even},\\ (a\beta^{\frac{i-1}{2}k+l}(b) + J^l)\pi_{i+j} & \text{if } i \text{ is odd}, \ j = 0 \text{ or } j \text{ is even},\\ (aX^{k-2l}\beta^{\frac{i+1}{2}k-l}(b) + J^l)\pi_{i+j} & \text{if } i \text{ is odd}, \ j \text{ is odd},\end{cases}\end{aligned}$$

where β is the ring automorphism of A as in (3.1)*. In particular, π_0 is the identity element of the generalized Yoneda algebra $\mathcal{E}(A/J^l)$.*

Proof. Let $\phi = (a + J^l)\pi_i$ and $\psi = (b + J^l)\pi_j$, then we have $\phi(x) = ax + J^l$ and $\psi(x) = bx + J^l$ for all $x \in A$.

First, we consider the case $j = 0$ or j is even. Define the right A-homomorphism $\sigma_i : A_{i+j} \to A_i$ by

$$\sigma_i(x) = \begin{cases} \beta^{\frac{i}{2}k}(b)x & \text{if } i = 0 \text{ or } i \text{ is even},\\ \beta^{\frac{i-1}{2}k+l}(b)x & \text{if } i \text{ is odd},\end{cases} \tag{3.2}$$

for $x \in A_{i+j}$. Then there exists the following commutative diagram of right A-modules:

$$\begin{tikzcd} \cdots \arrow[r, "d_{i+j+1}"] & A_{i+j} \arrow[r, "d_{i+j}"] \arrow[d, "\sigma_i"'] & \cdots \arrow[r, "\kappa"] & A_{j+1} \arrow[r, "d"] \arrow[d, "\sigma_1"'] & A_j \arrow[d, "\sigma_0"'] \arrow[dr, "\psi"] & & \\ \cdots \arrow[r, "d_{i+1}"] & A_i \arrow[r, "d_i"] & \cdots \arrow[r, "\kappa"] & A_1 \arrow[r, "d"] & A_0 \arrow[r, "\pi"] & A/J^l \arrow[r] & 0. \end{tikzcd}$$

Indeed, we check this as follows. Since $\sigma_0(x) = bx$ for $x \in A_{i+j}$, it follows that $\pi\sigma_0 = \psi$. If $i = 0$ or i is even, then we have

$$(\sigma_i d)(x) = \beta^{\frac{i}{2}k}(b)X^l x = X^l\beta^{\frac{i}{2}k+l}(b)x = (d\sigma_{i+1})(x)$$

for $x \in A_{i+j+1}$ If i is odd, then we have

$$(\sigma_i\kappa)(x) = \beta^{\frac{i-1}{2}k+l}(b)X^{k-l}x = X^{k-l}\beta^{\frac{i+1}{2}k}(b)x = (\kappa\sigma_{i+1})(x)$$

for $x \in A_{i+j+1}$. Therefore σ_i is a lifting of ψ, and hence we have

$$(a+J^l)\pi_i \times (b+J^l)\pi_j = \phi\sigma_i = \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\pi_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (a\beta^{\frac{i-1}{2}k+l}(b) + J^l)\pi_{i+j} & \text{if } i \text{ is odd.} \end{cases}$$

Next, we consider the case j is odd. Define the right A-homomorphism $\sigma_i : A_{i+j} \to A_i$ by

$$\sigma_i(x) = \begin{cases} \beta^{\frac{i}{2}k}(b)x & \text{if } i = 0 \text{ or } i \text{ is even,} \\ X^{k-2l}\beta^{\frac{i+1}{2}k-l}(b)x & \text{if } i \text{ is odd,} \end{cases}$$

for $x \in A_{i+j}$. Then there exists the following commutative diagram of right A-modules:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+j+1}"] & A_{i+j} \arrow[r, "d_{i+j}"] \arrow[d, "\sigma_i"'] & \cdots \arrow[r, "d"] & A_{j+1} \arrow[r, "\kappa"] \arrow[d, "\sigma_1"'] & A_j \arrow[d, "\sigma_0"'] \arrow[dr, "\psi"] & & \\
\cdots \arrow[r, "d_{i+1}"] & A_i \arrow[r, "d_i"] & \cdots \arrow[r, "\kappa"] & A_1 \arrow[r, "d"] & A_0 \arrow[r, "\pi"] & A/J^l \arrow[r] & 0.
\end{tikzcd}$$

Indeed, we check this as follows. It is clear that $\pi\sigma_0 = \psi$. If $i = 0$ or i is even, then we have

$$(\sigma_i\kappa)(x) = \beta^{\frac{i}{2}k}(b)X^{k-l}x = X^l X^{k-2l}\beta^{\frac{i+2}{2}k-l}(b)x = (d\sigma_{i+1})(x)$$

for $x \in A_{i+j+1}$. If i is odd, then we have

$$(\sigma_i d)(x) = X^{k-2l}\beta^{\frac{i+1}{2}k-l}(b)X^l x = X^{k-l}\beta^{\frac{i+1}{2}k}(b)x = (\kappa\sigma_{i+1})(x)$$

for $x \in A_{i+j+1}$. Therefore σ_i is a lifting of ψ, and hence we have

$$\begin{aligned}(a+J^l)\pi_i &\times (b+J^l)\pi_j \\ &= \phi\sigma_i = \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\pi_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (aX^{k-2l}\beta^{\frac{i+1}{2}k-l}(b) + J^l)\pi_{i+j} & \text{if } i \text{ is odd.} \end{cases}\end{aligned}$$

This completes the proof of the proposition. □

Then we have the following lemma.

Lemma 3.2. *In the case $k \geq 2l$, we have the following equations*:

$$\pi_i = \begin{cases} {\pi_2}^{\frac{i}{2}} & \text{if } i = 0 \text{ or } i \text{ is even}, \\ \pi_1 \times {\pi_2}^{\frac{i-1}{2}} & \text{if } i \text{ is odd}, \end{cases}$$

where we set ${\pi_2}^0 = \pi_0$.

Proof. We shall show the statement by induction on i. For $i = 0, 1, 2$, the equation is true, since we set ${\pi_2}^0 = \pi_0$ and π_0 is the identity element by Proposition 3.1. Suppose as the induction hypothesis that the equation is true for $i \geq 1$. If i is odd, then we have

$$\pi_{i+2} = \pi_i \times \pi_2 = \pi_1 \times {\pi_2}^{\frac{i-1}{2}} \times \pi_2 = \pi_1 \times {\pi_2}^{\frac{i+1}{2}}$$

by Proposition 3.1 and the induction hypothesis. If i is even, then we also have

$$\pi_{i+2} = \pi_i \times \pi_2 = {\pi_2}^{\frac{i}{2}} \times \pi_2 = {\pi_2}^{\frac{i+2}{2}}.$$

Therefore the equation is true for $i + 2$ and hence the statement follows. $\square$

By Proposition 3.1, we have

$$(a + J^l)\pi_i = (a + J^l)\pi_0 \times \pi_i$$

for $a + J^l \in A/J^l$ and $i \geq 0$. Hence we have the following lemma by Lemma 3.2.

Lemma 3.3. *In the case $k \geq 2l$, the set $\{(a + J^l)\pi_0, \pi_1, \pi_2 \,|\, a \in A\}$ is a set of generators of the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i \geq 0}(A/J^l)\pi_i$. Moreover, for $(a + J^l)\pi_0$, $(b + J^l)\pi_0 \in (A/J^l)\pi_0$, π_1 and π_2, we have the following equations*:

$$\begin{aligned} (a + J^l)\pi_0 \times (b + J^l)\pi_0 &= (ab + J^l)\pi_0, \\ \pi_1 \times (b + J^l)\pi_0 &= (\beta^l(b) + J^l)\pi_0 \times \pi_1, \\ \pi_2 \times (b + J^l)\pi_0 &= (\beta^k(b) + J^l)\pi_0 \times \pi_2, \\ \pi_1 \times \pi_1 &= (X^{k-2l} + J^l)\pi_2, \\ \pi_2 \times \pi_1 &= \pi_1 \times \pi_2. \end{aligned}$$

The following theorem immediately follows by Lemma 3.3.

Theorem 3.4. *In the case $k \geq 2l$, the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_A(A/J^l, A/J^l)$ is isomorphic to the ring*

$$(A/J^l)[\zeta, \eta] \Big/ \Big(\zeta\eta - \eta\zeta,\, \zeta^2 - (X^{k-2l} + J^l)\eta\Big),$$

where $\deg \zeta = 1$, $\deg \eta = 2$, $(A/J^l)[\zeta, \eta]$ *is the non-commutative polynomial ring over* A/J^l *with the commutative laws*

$$\zeta(b + J^l) = (\beta^l(b) + J^l)\zeta, \qquad \eta(b + J^l) = (\beta^k(b) + J^l)\eta$$

for $b + J^l \in A/J^l$, *and* β *is the ring automorphism of* A *as in* (3.1).

In particular, if $k \geq 3l$ *then the relation* $\zeta^2 = 0$ *holds in the above, and if* $k = 2l$ *then* $\mathcal{E}(A/J^l)$ *is isomorphic to the ring* $(A/J^l)[\zeta]$, *where* $\deg \zeta = 1$ *and* $\zeta(b + J^l) = (\beta^l(b) + J^l)\zeta$ *for* $b + J^l \in A/J^l$.

Remark. Let $l = 1$ in the above theorem, then we have the result for the usual Yoneda algebra $\mathcal{E}(A/J) = \bigoplus_{i \geq 0} \operatorname{Ext}_A^i(A/J, A/J)$ as follows. If $k = 2$, then $\mathcal{E}(A/J)$ is isomorphic to the ring $(A/J^l)[\zeta]$, where $\deg \zeta = 1$ and $\zeta(b + J) = (\beta(b) + J)\zeta$ for $b + J \in A/J$. If $k \geq 3$, then $\mathcal{E}(A/J)$ is isomorphic to the ring

$$(A/J)[\zeta, \eta]/(\zeta\eta - \eta\zeta,\, \zeta^2),$$

where $\deg \zeta = 1$, $\deg \eta = 2$, $(A/J)[\zeta, \eta]$ is the non-commutative polynomial ring over A/J with the commutative laws

$$\zeta(b + J) = (\beta(b) + J)\zeta, \qquad \eta(b + J) = (\beta^k(b) + J)\eta,$$

for $b + J \in A/J$. This result is equal to that obtained by A. I. Generalov in [4].

3.2. The case $k < 2l$

In this subsection, we consider the case $k < 2l$. In order to clearly describe the degree of the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i \geq 0} \operatorname{Ext}_A^i(A/J^l, A/J^l)$, by Proposition 2.4, we write

$$\operatorname{Ext}_A^i(A/J^l, A/J^l) = \begin{cases} A^* = (A/J^l)\pi_0 & \text{if } i = 0, \\ \operatorname{Ker} \kappa^* = (J^{2l-k}/J^l)\pi_i & \text{if } i \text{ is odd}, \\ A^*/\operatorname{Im} \kappa^* = (A/J^l)\pi_i/(J^{k-l}/J^l)\pi_i & \text{if } i \text{ is even}, \end{cases}$$

where π_i denotes the natural right A-epimorphism $\pi : A \to A/J^l$. Furthermore, let

$$\varepsilon_i = \begin{cases} \pi_0 & \text{if } i = 0, \\ (X^{2l-k} + J^l)\pi_i & \text{if } i \text{ is odd}, \\ [\pi_i] = \pi_i + \operatorname{Im} \kappa^* & \text{if } i \text{ is even}, \end{cases}$$

then the group $\operatorname{Ext}_A^i(A/J^l, A/J^l)$ is the left A/J^l-module generated by ε_i, that is,

$$\operatorname{Ext}_A^i(A/J^l, A/J^l) = (A/J^l)\varepsilon_i \qquad \text{for } i \geq 0.$$

Proposition 3.5. *In the case $k < 2l$, for $(a + J^l)\varepsilon_i \in \operatorname{Ext}_A^i(A/J^l, A/J^l)$ and $(b + J^l)\varepsilon_j \in \operatorname{Ext}_A^j(A/J^l, A/J^l)$ with $a, b \in A$, the Yoneda product $(a + J^l)\varepsilon_i \times (b + J^l)\varepsilon_j \in \operatorname{Ext}_A^{i+j}(A/J^l, A/J^l)$ is given as follows*:

$$\begin{aligned}
&(a + J^l)\varepsilon_i \times (b + J^l)\varepsilon_j \\
&= \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\varepsilon_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (a\beta^{\frac{i+1}{2}k-l}(b) + J^l)\varepsilon_{i+j} & \text{if } i \text{ is odd, } j = 0 \text{ or } j \text{ is even,} \\ (aX^{2l-k}\beta^{\frac{i-1}{2}k+l}(b) + J^l)\varepsilon_{i+j} & \text{if } i \text{ is odd, } j \text{ is odd.} \end{cases}
\end{aligned}$$

In particular, ε_0 is the identity element of the generalized Yoneda algebra $\mathcal{E}(A/J^l)$.

Proof. Let

$$\phi_i = \begin{cases} (a + J^l)\pi_i & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (aX^{2l-k} + J^l)\pi_i & \text{if } i \text{ is odd,} \end{cases}$$

$$\psi_j = \begin{cases} (b + J^l)\pi_j & \text{if } j = 0 \text{ or } j \text{ is even,} \\ (bX^{2l-k} + J^l)\pi_j & \text{if } j \text{ is odd,} \end{cases}$$

then $(a + J^l)\varepsilon_i$ and $(b + J^l)\varepsilon_j$ are represented by ϕ_i and ψ_j, respectively. Therefore, we have $(a + J^l)\varepsilon_i = [\phi_i]$ and $(b + J^l)\varepsilon_j = [\psi_j]$.

First, we consider the case $j = 0$ or j is even. In this case, we can use the same lifting σ_i of $\psi_j = (b + J^l)\pi_j$ as in (3.2). Since

$$\phi_i\sigma_i = \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\pi_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (a\beta^{\frac{i+1}{2}k-l}(b)X^{2l-k} + J^l)\pi_{i+j} & \text{if } i \text{ is odd,} \end{cases}$$

holds and the Yoneda product is given by $(a + J^l)\varepsilon_i \times (b + J^l)\varepsilon_j = [\phi_i\sigma_i]$, we have

$$(a + J^l)\varepsilon_i \times (b + J^l)\varepsilon_j = \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\varepsilon_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (a\beta^{\frac{i+1}{2}k-l}(b) + J^l)\varepsilon_{i+j} & \text{if } i \text{ is odd.} \end{cases}$$

Next, we consider the case j is odd. Define the right A-homomorphism $\sigma_i : A_{i+j} \to A_i$ by

$$\sigma_i(x) = \begin{cases} \beta^{\frac{i}{2}k}(b)X^{2l-k}x & \text{if } i = 0 \text{ or } i \text{ is even,} \\ \beta^{\frac{i-1}{2}k+l}(b)x & \text{if } i \text{ is odd,} \end{cases}$$

for $x \in A_{i+j}$. Then there exists the following commutative diagram of right A-modules:

$$\begin{tikzcd}
\cdots \arrow[r, "d_{i+j+1}"] & A_{i+j} \arrow[r, "d_{i+j}"] \arrow[d, "\sigma_i"'] & \cdots \arrow[r, "d"] & A_{j+1} \arrow[r, "\kappa"] \arrow[d, "\sigma_1"'] & A_j \arrow[d, "\sigma_0"'] \arrow[dr, "\psi_j"] & & \\
\cdots \arrow[r, "d_{i+1}"] & A_i \arrow[r, "d_i"] & \cdots \arrow[r, "\kappa"] & A_1 \arrow[r, "d"] & A_0 \arrow[r, "\pi"] & A/J^l \arrow[r] & 0.
\end{tikzcd}$$

Indeed, we check this as follows. It is clear that $\pi\sigma_0 = \psi_j$. If $i = 0$ or i is even, then we have

$$(\sigma_i\kappa)(x) = \beta^{\frac{i}{2}k}(b)X^{2l-k}X^{k-l}x = X^l\beta^{\frac{i}{2}k+l}(b)x = (d\sigma_{i+1})(x)$$

for $x \in A_{i+j+1}$. If i is odd, then we have

$$(\sigma_i d)(x) = \beta^{\frac{i-1}{2}k+l}(b)X^l x = X^{k-l}\beta^{\frac{i+1}{2}k}(b)X^{2l-k}x = (\kappa\sigma_{i+1})(x)$$

for $x \in A_{i+j+1}$. Therefore σ_i is a lifting of ψ_j. Since

$$\phi_i\sigma_i = \begin{cases} (a\beta^{\frac{i}{2}k}(b)X^{2l-k} + J^l)\pi_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (aX^{2l-k}\beta^{\frac{i-1}{2}k+l}(b) + J^l)\pi_{i+j} & \text{if } i \text{ is odd,} \end{cases}$$

holds and the Yoneda product is given by $(a + J^l)\varepsilon_i \times (b + J^l)\varepsilon_j = [\phi_i\sigma_i]$, we have

$$(a + J^l)\varepsilon_i \times (b + J^l)\varepsilon_j = \begin{cases} (a\beta^{\frac{i}{2}k}(b) + J^l)\varepsilon_{i+j} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ (aX^{2l-k}\beta^{\frac{i-1}{2}k+l}(b) + J^l)\varepsilon_{i+j} & \text{if } i \text{ is odd.} \end{cases}$$

This completes the proof of the proposition. $\square$

Then we have the following lemma by the similar proof to Lemma 3.2.

Lemma 3.6. *In the case $k < 2l$, we have the following equations*:

$$\varepsilon_i = \begin{cases} {\varepsilon_2}^{\frac{i}{2}} & \text{if } i = 0 \text{ or } i \text{ is even,} \\ \varepsilon_1 \times {\varepsilon_2}^{\frac{i-1}{2}} & \text{if } i \text{ is odd,} \end{cases}$$

where we set ${\varepsilon_2}^0 = \varepsilon_0$.

By Proposition 3.5, we have

$$(a + J^l)\varepsilon_i = (a + J^l)\varepsilon_0 \times \varepsilon_i$$

for $a + J^l \in A/J^l$ and $i \geq 0$. Hence we have the following lemma by Lemma 3.6.

Lemma 3.7. *In the case $k < 2l$, the set $\{(a + J^l)\varepsilon_0, \varepsilon_1, \varepsilon_2 \,|\, a \in A\}$ is a set of generators of the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i\geq 0}(A/J^l)\varepsilon_i$. Moreover, for $(a+J^l)\varepsilon_0$, $(b+J^l)\varepsilon_0 \in (A/J^l)\varepsilon_0$, ε_1 and ε_2, we have the following equations*:

$$\begin{aligned} (a + J^l)\varepsilon_0 \times (b + J^l)\varepsilon_0 &= (ab + J^l)\varepsilon_0, \\ \varepsilon_1 \times (b + J^l)\varepsilon_0 &= (\beta^{k-l}(b) + J^l)\varepsilon_0 \times \varepsilon_1, \\ \varepsilon_2 \times (b + J^l)\varepsilon_0 &= (\beta^k(b) + J^l)\varepsilon_0 \times \varepsilon_2, \\ \varepsilon_1 \times \varepsilon_1 &= (X^{2l-k} + J^l)\varepsilon_2, \\ \varepsilon_2 \times \varepsilon_1 &= \varepsilon_1 \times \varepsilon_2. \end{aligned}$$

The following theorem immediately follows by Lemma 3.7.

Theorem 3.8. *In the case $k < 2l$, the generalized Yoneda algebra $\mathcal{E}(A/J^l) = \bigoplus_{i\geq 0} \operatorname{Ext}_A^i(A/J^l, A/J^l)$ is isomorphic to the ring*

$$(A/J^l)[\zeta,\eta]\Big/\Big(\zeta\eta-\eta\zeta,\, \zeta^2-(X^{2l-k}+J^l)\eta\Big),$$

where $\deg\zeta = 1$, $\deg\eta = 2$, *$(A/J^l)[\zeta,\eta]$ is the non-commutative polynomial ring over A/J^l with the commutative laws*

$$\zeta(b+J^l) = (\beta^{k-l}(b)+J^l)\zeta, \qquad \eta(b+J^l) = (\beta^k(b)+J^l)\eta$$

for $b+J^l \in A/J^l$, and β is the ring automorphism of A as in (3.1).

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