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Local orbit types of s-representations for exceptional semisimple symmetric spaces

Kurando Baba

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Abstract. We investigate the local orbit types of hyperbolic or elliptic orbits of the linear isotropy representations of exceptional semisimple (pseudo-Riemannian) symmetric spaces in terms of the restricted root systems with respect to maximal split abelian subspace.

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Introduction

In 1999, H. Tamaru [7] investigated the local orbit types of the isotropy actions of compact semisimple Riemannian symmetric spaces. According to his results, the set of all local orbit types can be determined in terms of the restricted roots system. By Tamaru's results, K. Kondo [3] completed the lists of all local orbit types of linear isotropy representations of symmetric R-spaces of low rank.

In this paper, we investigate the linear isotropy representations of semisimple symmetric spaces in terms of the restricted root systems with respect to maximal split abelian subspaces. Let G be a connected semisimple Lie group. Let (G, H) be a semisimple symmetric pair, $(\mathfrak{g}, \mathfrak{h})$ be its infinitesimal pair (which is also called a semisimple symmetric pair) and σ be an involution of $\mathfrak{g}$ such that the set of all fixed points of σ coincides with $\mathfrak{h}$. If we put $\mathfrak{q} := \{X \in \mathfrak{g} \mid \sigma(X) = -X\}$, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ with respect to the Killing form of $\mathfrak{g}$. Then the quotient space G/H is a semisimple symmetric space, and its tangent space at eH is identified with the subspace $\mathfrak{q}$. By this identification the linear isotropy representation coincides with the adjoint representation Ad of H on $\mathfrak{q}$, which is called an *s-representation*. An element $X \in \mathfrak{q}$ is said to be *semisimple* if the complexification $\mathrm{ad}(X)^{\mathbf{C}}$ of the endomorphism $\mathrm{ad}(X)$ of $\mathfrak{g}$ is diagonalizable, where ad is the adjoint representation of $\mathfrak{g}$. A semisimple element $X \in \mathfrak{q}$ is said to be *hyperbolic* (resp. *elliptic*) if any eigenvalue of $\mathrm{ad}(X)^{\mathbf{C}}$ is real (resp. pure imaginary). We call the orbit of the s-representation through an element $X \in \mathfrak{q}$ a *hyperbolic orbit* (resp. an *elliptic orbit*) if X is hyperbolic (resp. elliptic). Let $\mathfrak{a}$ be a maximal split abelian subspace of $\mathfrak{q}$ (i.e., a maximal abelian subspace of $\mathfrak{q}$ which consists of only hyperbolic elements or only elliptic elements). In this paper, we say that $\mathfrak{a}$ is *vector-type* (resp. *troidal-type*) if $\mathfrak{a}$ consists of only hyperbolic elements (resp. elliptic elements). It is shown that $\mathfrak{a}$ is vector-type (resp. troidal-type) if and only if there exists a Cartan involution θ of $\mathfrak{g}$ satisfying the following two conditions: (i) $\theta \circ \sigma = \sigma \circ \theta$, (ii) $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p} \cap \mathfrak{q}$ (resp. $\mathfrak{k} \cap \mathfrak{q}$), where $\mathfrak{p} := \mathrm{Ker}(\theta + \mathrm{id})$ and $\mathfrak{k} := \mathrm{Ker}(\theta - \mathrm{id})$. Note that any vector-type maximal split abelian subspace is split in the sense of [5]. Then the isotropy subalgebra of the $\mathrm{Ad}(H)$-orbit through an element of $\mathfrak{a}$ can be expressed by means of the restricted root system Δ of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$. The theory of restricted root systems for semisimple symmetric spaces is developed by W. Rossmann [6], T. Oshima and J. Sekiguchi [5]. In this paper, we say that the isotropy subalgebra $\mathfrak{h}_X$ at a hyperbolic element (resp. an elliptic element) X is *hyperbolic principal* (resp. *elliptic principal*) if the local orbit type $[\mathfrak{h}_X]$ is the smallest one in the set of all local orbit types of hyperbolic orbits (resp. elliptic orbits) of the s-representation. A subset Δ' of Δ is called a *closed subsystem* if Δ' satisfies the following two conditions: (i) if $\lambda, \mu \in \Delta'$ and $\lambda + \mu \in \Delta$ then $\lambda + \mu \in \Delta'$, (ii) $\Delta' = -\Delta'$.

Theorem. *Let Δ' be a closed subsystem of Δ. Then there exists a semisimple symmetric pair $(\mathfrak{g}', \mathfrak{h}')$ which satisfies the following three conditions*:

(i) $\mathfrak{g}'$ *and* $\mathfrak{h}'$ *are subalgebras of* $\mathfrak{g}$ *and* $\mathfrak{h}$, *respectively,*

(ii) *the restricted root system of* $(\mathfrak{g}', \mathfrak{h}')$ *is isomorphic to* Δ',

(iii) *the hyperbolic principal isotropy subalgebra of* $(\mathfrak{g}', \mathfrak{h}')$ *is an ideal of the centralizer* $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ *of* $\mathfrak{a}$ *in* $\mathfrak{h}$.

Remark 1. *We use this theorem to determine local orbit types of the s-representations of exceptional semisimple symmetric pairs in Section* 3.

For any closed subsystem, we explicitly construct a semisimple symmetric pair $(\mathfrak{g}', \mathfrak{h}')$ as in Theorem in terms of the root system of the complexification $\mathfrak{g}^{\mathbf{C}}$ of $\mathfrak{g}$ and the restricted root system Δ. Let Δ' be a closed subsystem of Δ and $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ be a semisimple symmetric pair. In this paper, we call $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ a *subsymmetric pair of* $(\mathfrak{g}, \mathfrak{h})$ *associated with* Δ' if $(\tilde{\mathfrak{g}}, \tilde{\mathfrak{h}})$ satisfies the conditions (i)–(iii) as in Theorem.

The organization of this paper is as follows. In Section 1, we give preliminaries for the restricted root systems with respect to maximal split abelian subspaces for semisimple symmetric pairs. In Section 2, we give a brief review of properties of the isotropy subalgebras of hyperbolic orbits of s-representations. In Subsection 2.1, we prove Theorem. In Section 3, we shall investigate the local orbit types of orbits of the s-representations for the following nine exceptional semisimple symmetric pairs (Subsection 3.1–Subsection 3.9):$(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$, $(\mathfrak{e}_{6(2)}, \mathfrak{sp}(3,1))$, $(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3,1))$, $(\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$, $(\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6)+\mathfrak{su}(2))$, $(\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$, $(\mathfrak{e}_{6(6)} + \mathfrak{e}_{6(6)}, \mathfrak{e}_{6(6)})$, $(\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{sp}(6, \mathbf{C}))$ and $(\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{e}_{6(6)})$.

§1. Preliminaries

Let G be a connected semisimple Lie group and σ be an involution of G. We denote by G_σ the set of all fixed points of σ and by $(G_\sigma)_0$ its identity component. For a closed subgroup H of G with $(G_\sigma)_0 \subset H \subset G_\sigma$, the quotient space G/H is a semisimple symmetric space. Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and H, respectively. We call both the pairs (G, H) and $(\mathfrak{g}, \mathfrak{h})$ the *semisimple symmetric pairs*. We denote by Ad_G (resp. $\mathrm{ad}_{\mathfrak{g}}$) the adjoint representation of G (resp. $\mathfrak{g}$). For simplicity, we omit the subscripts G and $\mathfrak{g}$ in the sequel. The involution σ of G induces an involution of $\mathfrak{g}$, which is also denoted by the same symbol σ. Then we have $\mathfrak{h} = \{X \in \mathfrak{g} \,|\, \sigma(X) = X\}$ and an orthogonal decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ with respect to the Killing form B of $\mathfrak{g}$, where $\mathfrak{q} := \{X \in \mathfrak{g} \,|\, \sigma(X) = -X\}$. Then B restricted to $\mathfrak{q} \times \mathfrak{q}$ is nondegenerate and $\mathrm{Ad}(H)$-invariant. Since $\mathfrak{q}$ is identified with the tangent space of G/H at eH, the bilinear form on $\mathfrak{q} \times \mathfrak{q}$ determines a G-invariant nondegenerate metric on G/H, where e is the identity element of G. Thus any semisimple symmetric pair gives rise to a semisimple pseudo-Riemannian symmetric space. It follows from Lemma 10.2 of [1] that there exists a Cartan involution θ of $\mathfrak{g}$ commuting with σ. Any such Cartan involution is $\mathrm{Ad}(H_0)$-conjugate to θ (Theorem 2.1, Chapter IV of [4]). Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition corresponding to θ, where $\mathfrak{k} = \{X \in \mathfrak{g} \,|\, \theta(X) = X\}$ and $\mathfrak{p} = \{X \in \mathfrak{g} \,|\, \theta(X) = -X\}$. Then we have the simultaneous eigenspace decomposition $\mathfrak{g} = \mathfrak{k}\cap\mathfrak{h}+\mathfrak{p}\cap\mathfrak{h}+\mathfrak{k}\cap\mathfrak{q}+\mathfrak{p}\cap\mathfrak{q}$ of σ and θ. Since $\theta\circ\sigma$ is an involution of $\mathfrak{g}$, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{h}^a + \mathfrak{q}^a$, where $\mathfrak{h}^a := \{X \in \mathfrak{g} \,|\, \theta\circ\sigma(X) = X\}$ and $\mathfrak{q}^a := \{X \in \mathfrak{g} \,|\, \theta\circ\sigma(X) = -X\}$. Set $\mathfrak{g}^d := \mathfrak{k} \cap \mathfrak{h} + \sqrt{-1}(\mathfrak{p} \cap \mathfrak{h}) + \sqrt{-1}(\mathfrak{k} \cap \mathfrak{q}) + \mathfrak{p} \cap \mathfrak{q} (\subset \mathfrak{g}^{\mathbf{C}})$, where $\mathfrak{g}^{\mathbf{C}}$ denotes the complexification of $\mathfrak{g}$. With the bracket operation inherited from $\mathfrak{g}^{\mathbf{C}}$, $\mathfrak{g}^d$ is another real form of $\mathfrak{g}^{\mathbf{C}}$. The restrictions of the complexifications of σ and θ to $\mathfrak{g}^d$ are involutions of $\mathfrak{g}^d$, which are also denoted by the same symbols σ and θ, respectively. Note that σ is a Cartan involution of $\mathfrak{g}^d$. We put $\mathfrak{h}^d := \{X \in \mathfrak{g}^d \,|\, \theta(X) = X\}$ and $\mathfrak{q}^d := \{X \in \mathfrak{g}^d \,|\, \theta(X) = -X\}$. Then we have an orthogonal decomposition $\mathfrak{g}^d = \mathfrak{h}^d + \mathfrak{q}^d$. The pair $(\mathfrak{g}, \mathfrak{h}^a)$ (resp. $(\mathfrak{g}^d, \mathfrak{h}^d)$)

is called the *associated* (resp. *dual*) *symmetric pair* of $(\mathfrak{g},\mathfrak{h})$. For simplicity, we write $(\mathfrak{g},\mathfrak{h})^a$ and $(\mathfrak{g},\mathfrak{h})^d$ instead of $(\mathfrak{g},\mathfrak{h}^a)$ and $(\mathfrak{g}^d,\mathfrak{h}^d)$, respectively.

Next, we recall the notion of the restricted root systems with respect to maximal split abelian subspaces for semisimple symmetric spaces. An element $X\in\mathfrak{q}$ is said to be *semisimple* if the endomorphism $\mathrm{ad}(X)^{\mathbf{C}}$ is diagonalizable. A semisimple element $X\in\mathfrak{q}$ is said to be *hyperbolic* (resp. *elliptic*) if any eigenvalue of $\mathrm{ad}(X)^{\mathbf{C}}$ is real (resp. pure imaginary). Let $\mathfrak{a}$ be a maximal split abelian subspace of $\mathfrak{q}$ (i.e., a maximal abelian subspace of $\mathfrak{q}$ which consists of only hyperbolic elements or only elliptic elements). We say that $\mathfrak{a}$ is *vector-type* (resp. *troidal-type*) if all elements of $\mathfrak{a}$ are hyperbolic (resp. elliptic). It is known that $\mathfrak{a}$ is vector-type (resp. troidal-type) if and only if there exists a Cartan involution of $\mathfrak{g}$ whose (-1)-eigenspace (resp. $(+1)$-eigenspace) contains $\mathfrak{a}$. In the sequel, we assume that $\mathfrak{a}$ is contained in $\mathfrak{p}$. For each element λ of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, we put $\mathfrak{g}_\lambda:=\{X\in\mathfrak{g}\,|\,[A,X]=\lambda(A)X,\ \forall A\in\mathfrak{a}\}$. Then $\Delta:=\{\lambda\in\mathfrak{a}^*\setminus\{0\}\,|\,\mathfrak{g}_\lambda\neq\{0\}\}$ is called the *restricted root system* with respect to $\mathfrak{a}$. It follows from Theorem 5 of [6] that Δ is a root system. All vector-type maximal split abelian subspaces have the same dimension, so the *split rank* of a semisimple symmetric pair is defined as the dimension of any vector-type maximal split abelian subspace. Since all $\mathrm{ad}(A)$'s ($A\in\mathfrak{a}$) are simultaneously diagonalizable, we have a direct sum decomposition

$$\mathfrak{g}=\mathfrak{g}_0+\sum_{\lambda\in\Delta}\mathfrak{g}_\lambda,$$

which is called the *restricted root space decomposition* with respect to $\mathfrak{a}$.

Remark 2. (1) *In general, a maximal abelian subspace of* $\mathfrak{q}$ *may contain an element which is not semisimple. However, it is known that any maximal abelian subspace of* $\mathfrak{q}$ *containing a maximal split abelian subspace consists of only semisimple elements* (*see Lemma* 2.2 *of* [5]).

(2) *The restricted root system of* $(\mathfrak{g},\mathfrak{h})$ *with respect to a troidal-type maximal split abelian subspace coincides with that of* $(\mathfrak{g}^{ad},\mathfrak{h})$ *with respect to a vector-type maximal split abelian subspace.*

For any $\lambda,\mu\in\Delta\cup\{0\}$, we have $[\mathfrak{g}_\lambda,\mathfrak{g}_\mu]\subset\mathfrak{g}_{\lambda+\mu}$, $\sigma(\mathfrak{g}_\lambda)=\mathfrak{g}_{-\lambda}$ and $\theta(\mathfrak{g}_\lambda)=\mathfrak{g}_{-\lambda}$. We denote by $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ (resp. $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$) the centralizer of $\mathfrak{a}$ in $\mathfrak{h}$ (resp. $\mathfrak{q}$). Note that $\mathfrak{a}$ coincides with $\mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$ if and only if $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$ (for example, in the case where G/H is a Riemannian symmetric space). We put $\mathfrak{h}_\lambda:=(\mathfrak{g}_\lambda+\mathfrak{g}_{-\lambda})\cap\mathfrak{h}$ and $\mathfrak{q}_\lambda:=(\mathfrak{g}_\lambda+\mathfrak{g}_{-\lambda})\cap\mathfrak{q}$ for any $\lambda\in\Delta$. In this paper, we call the dimension of $\mathfrak{q}_\lambda$ and the pair $(\dim(\mathfrak{p}\cap\mathfrak{q}_\lambda),\dim(\mathfrak{k}\cap\mathfrak{q}_\lambda))$ the *multiplicity* and the *signature* of λ, respectively. We denote by $(m^+(\lambda),m^-(\lambda))$ the signature of $\lambda\in\Delta$.

Lemma 1. *Let* Δ_+ *be the positive root system of* Δ *with respect to some lexicographic ordering of* $\mathfrak{a}^*$. *Then* $\mathfrak{h}$ *and* $\mathfrak{q}$ *are orthogonally decomposed as*

$$\mathfrak{h} = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}) + \sum_{\lambda \in \Delta_+} \mathfrak{h}_\lambda,$$

and

$$\mathfrak{q} = \mathfrak{z}_{\mathfrak{q}}(\mathfrak{a}) + \sum_{\lambda \in \Delta_+} \mathfrak{q}_\lambda,$$

respectively.

Proof. Since $\mathfrak{g}_0$ is σ-invariant, we have $\mathfrak{g}_0 = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}) + \mathfrak{z}_{\mathfrak{q}}(\mathfrak{a})$. Similarly, $\mathfrak{g}_\lambda + \mathfrak{g}_{-\lambda} = \mathfrak{h}_\lambda + \mathfrak{q}_\lambda$ holds for any $\lambda \in \Delta_+$. Hence we obtain the orthogonal decompositions of $\mathfrak{h}$ and $\mathfrak{q}$ as in the statement. □

Finally, we explain how to construct two kinds of Satake diagrams of $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{a}_{\mathfrak{p}}$ (resp. $\mathfrak{a}_{\mathfrak{q}}$) be a maximal abelian subspace of $\mathfrak{p}$ (resp. $\mathfrak{q}$) containing $\mathfrak{a}$. By Lemma 2.4 of [5], $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{a}_{\mathfrak{q}}$ is an abelian subalgebra of $\mathfrak{g}$. Let $\tilde{\mathfrak{a}}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_{\mathfrak{p}} + \mathfrak{a}_{\mathfrak{q}}$. It follows from Lemma 3.2, Chapter VI of [2] that $\tilde{\mathfrak{a}}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$. We denote by R the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\tilde{\mathfrak{a}}^{\mathbf{C}}$ and by A_α ($\alpha \in R$) the vector of $\tilde{\mathfrak{a}}^{\mathbf{C}}$ defined by $B^{\mathbf{C}}(A, A_\alpha) = \alpha(A)$ for all $A \in \tilde{\mathfrak{a}}^{\mathbf{C}}$, where $B^{\mathbf{C}}$ denotes the Killing form of $\mathfrak{g}^{\mathbf{C}}$. We put $(A_1, \ldots, A_r)$ is a basis of $\tilde{\mathfrak{a}}_{\mathbf{R}} (:= \mathrm{Span}_{\mathbf{R}}\{A_\alpha | \alpha \in R\})$ such that $(A_1, \ldots, A_m)$ is a basis of $\mathfrak{a}$ and $(A_{m+1}, \ldots, A_n)$ is a basis of $\sqrt{-1}(\mathfrak{k} \cap \mathfrak{a}_{\mathfrak{q}})$, where r (resp. n) is the rank of $\mathfrak{g}^{\mathbf{C}}$ (resp. $(\mathfrak{g}, \mathfrak{h})$) and m is the split rank of $(\mathfrak{g}, \mathfrak{h})$. Let $\Psi(R)$ be a simple root system of R for the lexicographic ordering of $(\tilde{\mathfrak{a}}_{\mathbf{R}})^*$ with respect to the above basis. We put $\Psi(R)_0 := \{\alpha \in \Psi(R) \mid \bar{\alpha} = 0\}$, where $\bar{\cdot}$ denotes the restriction to $\mathfrak{a}$. Then we construct the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ as follows. In Dynkin diagram of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\Psi(R)$, every root of $\Psi(R)_0$ is denoted by a black circle $\bullet$ and every root of $\Psi(R) \setminus \Psi(R)_0$ by a white circle $\circ$. If $\alpha, \beta \in \Psi(R) \setminus \Psi(R)_0$ satisfy $\bar{\alpha} = \bar{\beta}$, α and β are joined by a curved arrow. Similarly, we construct the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$ from the Dynkin diagram of $\mathfrak{g}^{\mathbf{C}}$.

§2. Isotropy subalgebras of orbits of s-representations

We shall use the notations of previous sections. The linear isotropy representation of a semisimple pseudo-Riemannian symmetric space is called an *s-representation*. By the identification $T_{eH}G/H \cong \mathfrak{q}$, the linear isotropy representation of G/H coincides with the adjoint representation Ad of H on $\mathfrak{q}$. The $\mathrm{Ad}(H)$-orbit through an element $X \in \mathfrak{q}$ is said to be *hyperbolic* (resp. *elliptic*) if X is hyperbolic (resp. elliptic). We denote by H_X the isotropy subgroup of H at $X \in \mathfrak{q}$. We denote by $[\mathfrak{h}_X]$ the local orbit type of $\mathrm{Ad}(H)$-orbit through an element X, that is, $[\mathfrak{h}_X] := \{\mathrm{Ad}(h)\mathfrak{h}_X \mid h \in H\}$. In this paper, we say that the isotropy subalgebra $\mathfrak{h}_X$ at a hyperbolic point X is *hyperbolic principal* (abbreviated to *h-principal*) if $[\mathfrak{h}_X]$ is the smallest local orbit type in the set of all local orbit types of $\mathrm{Ad}(H)$-orbits through hyperbolic elements of $\mathfrak{q}$. We also say that the isotropy subalgebra $\mathfrak{h}_X$ at an elliptic element X is *elliptic principal* (abbreviated to *e-principal*) if $[\mathfrak{h}_X]$ is the smallest local orbit type in the set of all local orbit types of $\mathrm{Ad}(H)$-orbits through elliptic elements of $\mathfrak{q}$. We note that the e-principal isotropy subalgebra of $(\mathfrak{g}, \mathfrak{h})$ coincides with the h-principal isotropy subalgebra of $(\mathfrak{g}^{ad}, \mathfrak{h})$. We call that two orbits $\mathrm{Ad}(H)X_1$ and $\mathrm{Ad}(H)X_2$ are of the *same orbit type* if the isotropy subgroups H_{X_1} and H_{X_2} are conjugate, and of the *same local orbit type* if the isotropy subalgebras $\mathfrak{h}_{X_1}$ and $\mathfrak{h}_{X_2}$ are conjugate.

Proposition 2. *If we put* $\Delta_A := \{\lambda \in \Delta \mid \lambda(A) = 0\}$ *for any* $A \in \mathfrak{a}$,

$$\mathfrak{h}_A = \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}) + \sum_{\lambda \in \Delta_A \cap \Delta_+} \mathfrak{h}_\lambda.$$

Proof. It is clear that $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ is contained in $\mathfrak{h}_A$. Let Z_λ be any element of $\mathfrak{h}_\lambda$ with $\lambda(A) = 0$. Then there exists $X_\lambda \in \mathfrak{g}_\lambda$ which satisfies the relation $Z_\lambda = X_\lambda + \sigma(X_\lambda)$. Then we have $[Z_\lambda, A] = -\lambda(A)(X_\lambda - \sigma(X_\lambda)) = 0$. Hence the subspace $\mathfrak{h}_\lambda$ is contained in $\mathfrak{h}_A$. Conversely, let Z be any element of $\mathfrak{h}_A$. From Lemma 1 we have $Z = Z_0 + \sum_{\lambda \in \Delta_+} Z_\lambda$ for some $Z_0 \in \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}), Z_\lambda \in \mathfrak{h}_\lambda$. Since $[Z, A] = 0$ holds, we have $Z_\lambda = 0$ for all $\lambda \in \Delta_+$ with $\lambda(A) \neq 0$. Hence the desired relation follows. □

It follows from Proposition 2 that $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ is equal to the h-principal isotropy subalgebra. Since any hyperbolic orbit meets the subspace $\mathfrak{a}$, the set of all local orbit types of $\mathrm{Ad}(H)$-orbits through hyperbolic elements of $\mathfrak{q}$ coincides with $\{[\mathfrak{h}_A] \mid A \in \mathfrak{a}\}$. A subset Δ' of Δ is called a *closed subsystem* if the following two properties hold:

(i) if $\lambda, \mu \in \Delta'$ and $\lambda + \mu \in \Delta$ then $\lambda + \mu \in \Delta'$,

(ii) $\Delta' = -\Delta'$.

We define the multiplicity and the signature of $\lambda \in \Delta'$ by those of $\lambda \in \Delta$, respectively. It is clear that Δ_A is a closed subsystem for all $A \in \mathfrak{a}$.

2.1. Proof of Theorem

In this subsection, we prove Theorem stated in Introduction.

Proof of Theorem. Let $\mathfrak{a}_{\mathfrak{q}}$ and $\mathfrak{a}_{\mathfrak{p}}$ be maximal abelian subspaces of $\mathfrak{q}$ and $\mathfrak{p}$ containing $\mathfrak{a}$, respectively. Let $\tilde{\mathfrak{a}}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_{\mathfrak{q}} + \mathfrak{a}_{\mathfrak{p}}$. By imitating the proof of Lemma 2.2 of [5] we can show that $\tilde{\mathfrak{a}}$ is invariant under σ and θ. Moreover, we have

$$\begin{aligned}\tilde{\mathfrak{a}} &= \tilde{\mathfrak{a}} \cap \mathfrak{k} \cap \mathfrak{h} + \tilde{\mathfrak{a}} \cap \mathfrak{p} \cap \mathfrak{h} + \tilde{\mathfrak{a}} \cap \mathfrak{k} \cap \mathfrak{q} + \tilde{\mathfrak{a}} \cap \mathfrak{p} \cap \mathfrak{q} \\ &= \tilde{\mathfrak{a}} \cap \mathfrak{k} \cap \mathfrak{h} + \mathfrak{a}_{\mathfrak{p}} \cap \mathfrak{h} + \mathfrak{a}_{\mathfrak{q}} \cap \mathfrak{k} + \mathfrak{a}.\end{aligned}$$

Let R be the root system of $\mathfrak{g}^{\mathbf{C}}$ with respect to $\tilde{\mathfrak{a}}^{\mathbf{C}}$, $\mathfrak{g}^{\mathbf{C}}_{\alpha}$ be the root space for $\alpha \in R$, and A_α the vector of $\tilde{\mathfrak{a}}^{\mathbf{C}}$ defined by $B^{\mathbf{C}}(A, A_\alpha) = \alpha(A)$ for all $A \in \tilde{\mathfrak{a}}^{\mathbf{C}}$, where $\mathfrak{g}^{\mathbf{C}}$ and $B^{\mathbf{C}}$ are the complexifications of $\mathfrak{g}$ and its Killing form, respectively. We extend σ and θ to $\mathfrak{g}^{\mathbf{C}}$ as $\mathbf{C}$-linear involutions, which are also denoted by the same symbols σ and θ, respectively. We put $\tilde{\mathfrak{a}}_{\mathbf{R}} := \mathrm{Span}_{\mathbf{R}}\{A_\alpha \,|\, \alpha \in R\}$. Then we have $\tilde{\mathfrak{a}}_{\mathbf{R}} = \sqrt{-1}(\tilde{\mathfrak{a}} \cap \mathfrak{k} \cap \mathfrak{h}) + \mathfrak{a}_{\mathfrak{p}} \cap \mathfrak{h} + \sqrt{-1}(\mathfrak{a}_{\mathfrak{q}} \cap \mathfrak{k}) + \mathfrak{a}$.

We put
$$R' := \big\{\alpha \in R \,|\, \bar{\alpha} \in \Delta' \cup \{0\}\big\},$$

where $\bar{\cdot}$ denotes the restriction to $\mathfrak{a}$. Since Δ' is a closed subsystem of Δ, R' is that of R. We put $\tilde{\mathfrak{a}}'_{\mathbf{R}} := \mathrm{Span}_{\mathbf{R}}\{A_\alpha \,|\, \alpha \in R'\}$. Then $\tilde{\mathfrak{a}}'_{\mathbf{R}}$ is invariant under σ and θ. We put

$$\begin{aligned}&\tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{h}} := (\sqrt{-1}\tilde{\mathfrak{a}}'_{\mathbf{R}}) \cap \mathfrak{k} \cap \mathfrak{h}, \quad \tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{h}} := \tilde{\mathfrak{a}}'_{\mathbf{R}} \cap \mathfrak{p} \cap \mathfrak{h},\\ &\tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{q}} := (\sqrt{-1}\tilde{\mathfrak{a}}'_{\mathbf{R}}) \cap \mathfrak{k} \cap \mathfrak{q}, \quad \tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{q}} := \tilde{\mathfrak{a}}'_{\mathbf{R}} \cap \mathfrak{p} \cap \mathfrak{q},\end{aligned}$$

and $\tilde{\mathfrak{a}}' := \tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{h}} + \tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{h}} + \tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{q}} + \tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{q}}$, which are subspaces of $\tilde{\mathfrak{a}}$. Note that $\tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{q}}$ coincides with $\mathrm{Span}_{\mathbf{R}}\{A_\lambda \,|\, \lambda \in \Delta'\}$, where the vector A_λ of $\mathfrak{a}$ is defined by $B(A_\lambda, A) = \lambda(A)$ for all $A \in \mathfrak{a}$. We put

$$\mathfrak{g}' := \tilde{\mathfrak{a}}' + \Big(\sum_{\alpha \in R \text{ such that } \bar{\alpha}=0} \mathfrak{g}^{\mathbf{C}}_{\alpha}\Big) \cap \mathfrak{g} + \sum_{\lambda \in \Delta'} \mathfrak{g}_\lambda.$$

It is a subalgebra of $\mathfrak{g}$, and invariant under σ and θ. Therefore σ restricted to $\mathfrak{g}'$ is an involution of $\mathfrak{g}'$. We put $\mathfrak{h}' := \{X \in \mathfrak{g}' \,|\, \sigma(X) = X\}$ and $\mathfrak{q}' := \{X \in \mathfrak{g}' \,|\, \sigma(X) = -X\}$. To show that $\mathfrak{g}'$ is semisimple it suffices to show that it has no center and is reductive in $\mathfrak{g}$. We denote by $\mathfrak{z}'$ the center of $\mathfrak{g}'$. Then $\mathfrak{z}'$ is contained in $\tilde{\mathfrak{a}}'$. For any $\alpha \in R'$, we choose a nonzero vector $X_\alpha \in \mathfrak{g}^{\mathbf{C}}_{\alpha}$. Then for any $Z \in \mathfrak{z}'$, $0 = [Z, X_\alpha] = \alpha(Z)X_\alpha$ holds. The symmetric bilinear form B_θ on $\mathfrak{g} \times \mathfrak{g}$ given by $B_\theta(X, Y) := -B(X, \theta(Y))$ for any $X, Y \in \mathfrak{g}$ is positive definite. Then we have $B_\theta(Z, A) = 0$ for all $A \in \tilde{\mathfrak{a}}'$, i.e., $Z = 0$. From Proposition 1.1.5.1 and Corollary 1.1.5.4 Chapter 1 of [8] we conclude that $\mathfrak{g}'$ is semisimple.

Finally, we show that the pair $(\mathfrak{g}', \mathfrak{h}')$ satisfies (ii) and (iii). It is clear that the restricted root system of $(\mathfrak{g}', \mathfrak{h}')$ with respect to $\tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{q}}$ is isomorphic to Δ'. It follows from the definition of $(\mathfrak{g}', \mathfrak{h}')$ that its h-principal isotropy subalgebra coincides with $\tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{h}}+\tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{h}}+(\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}$. Note that the h-principal isotropy subalgebra of $(\mathfrak{g},\mathfrak{h})$ coincides with $\tilde{\mathfrak{a}}\cap\mathfrak{k}\cap\mathfrak{h}+\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h}+(\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}$. Then we can show that

$$\begin{aligned}
[\tilde{\mathfrak{a}}\cap\mathfrak{k}\cap\mathfrak{h},(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}]&\subset(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h},\\
[\mathfrak{a}_{\mathfrak{p}}\cap\mathfrak{h},(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}]&\subset(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h},\\
[\tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{h}},(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}]&\subset(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h},\\
[\tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{h}},(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}]&\subset(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h},
\end{aligned}$$

and

$$[(\textstyle\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h},(\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}]\subset\tilde{\mathfrak{a}}'_{\mathfrak{k}\cap\mathfrak{h}}+\tilde{\mathfrak{a}}'_{\mathfrak{p}\cap\mathfrak{h}}+(\sum_{\bar{\alpha}=0}\mathfrak{g}_\alpha^{\mathbf{C}})\cap\mathfrak{h}$$

hold. Hence $(\mathfrak{g}',\mathfrak{h}')$ satisfies (iii). □

Let Δ' be a closed subsystem of Δ and $(\mathfrak{g}',\mathfrak{h}')$ be a semisimple symmetric pair. We call $(\mathfrak{g}',\mathfrak{h}')$ a *subsymmetric pair of* $(\mathfrak{g},\mathfrak{h})$ *associated with* Δ' if $(\mathfrak{g}',\mathfrak{h}')$ satisfies the conditions (i)–(iii) as in Theorem.

§3. Determination of local orbit types of s-representations

Let (G,H) be an exceptional semisimple symmetric pair. We shall use the notations of previous sections. Let Δ be the restricted root system of $(\mathfrak{g},\mathfrak{h})$ with respect to a vector-type maximal split abelian subspace $\mathfrak{a}$. We take a simple root system $\{\lambda_1,\ldots,\lambda_r\}(=:\Psi)$ of Δ and denote by $(A_{\lambda_1},\ldots,A_{\lambda_r})$ its dual basis. Denote by $\mathcal{L}_h(\mathfrak{g},\mathfrak{h},\Psi)$ the set of all local orbit types of the $\mathrm{Ad}(H)$-orbits through $A=\sum_{i=1}^r a_iA_{\lambda_i}$ with all $a_i\geq 0$. Note that $\mathcal{L}_h(\mathfrak{g},\mathfrak{h},\Psi)$ depends on the choice of Ψ. We consider the case where Ψ satisfies the condition that there exists a λ_{i_0} such that $m^+(\lambda_j)\geq m^-(\lambda_j)$ holds for all $j\neq i_0$ (see Section 6 of [5]). In this case, we determine the set $\mathcal{L}_h(\mathfrak{g},\mathfrak{h},\Psi)$. We denote by Δ_Θ the intersection of $\sum_{\lambda\in\Theta}\mathbf{R}\lambda$ and Δ for a subset Θ of Ψ. If we put $\Theta_A:=\{\lambda_i\mid a_i=0\}$, then Δ_A coincides with Δ_{Θ_A} and Θ_A is a simple root system of Δ_A. From Theorem stated in Introduction there exists a subsymmetric pair $(\mathfrak{g}(\Delta_{\Theta_A}),\mathfrak{h}(\Delta_{\Theta_A}))$ of $(\mathfrak{g},\mathfrak{h})$ associated with Δ_{Θ_A}. We denote by $\mathfrak{h}_0(\Delta_{\Theta_A})$ the h-principal isotropy subalgebra of $(\mathfrak{g}(\Delta_{\Theta_A}),\mathfrak{h}(\Delta_{\Theta_A}))$. Then we have

$$\begin{aligned}
\mathfrak{h}_A&=\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})+\sum_{\lambda\in\Delta_A\cap\Delta_+}\mathfrak{h}_\lambda\\
&=\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})/\mathfrak{h}_0(\Delta_{\Theta_A})+\mathfrak{h}_0(\Delta_{\Theta_A})+\sum_{\lambda\in\Delta_{\Theta_A}\cap\Delta_+}\mathfrak{h}_\lambda\\
&=\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})/\mathfrak{h}_0(\Delta_{\Theta_A})+\mathfrak{h}(\Delta_{\Theta_A}).
\end{aligned}\tag{3.1}$$

For any subset Θ of Ψ, we call

$$\mathfrak{h}_\Theta := \mathfrak{z}_{\mathfrak{h}}(\mathfrak{a}) + \sum_{\lambda \in \Delta_\Theta \cap \Delta_+} \mathfrak{h}_\lambda$$

the *corresponding subalgebra* of Θ. The isotropy subalgebra $\mathfrak{h}_A$ coincides with the corresponding subalgebra of Θ_A. By the above argument, we have the following proposition which is useful to determine $\mathcal{L}_h(\mathfrak{g}, \mathfrak{h}, \Psi)$.

Proposition 3. $\mathcal{L}_h(\mathfrak{g}, \mathfrak{h}, \Psi) = \{[\mathfrak{h}_\Theta] \mid \Theta \subset \Psi\}$.

Now, we give a recipe to determine $\mathcal{L}_h(\mathfrak{g}, \mathfrak{h}, \Psi)$ as follows.

(Step 1) By investigating the split rank of $(\mathfrak{g}, \mathfrak{h})$ and the Satake diagram of $(\mathfrak{g}^d, \mathfrak{k}^d)$, we determine the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$, where $\mathfrak{k}^d$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}^d$. We determine the h-principal isotropy subalgebra $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ of $(\mathfrak{g}, \mathfrak{h})$ in terms of the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$, $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a}_{\mathfrak{q}})$ and $(\mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ is a maximal compactly imbedded subalgebra of $\mathfrak{g}$.

(Step 2) For each $\Theta(\subset \Psi)$ we find a subsymmetric pair $(\mathfrak{g}', \mathfrak{h}')$ associated with Δ_Θ in terms of the Dynkin diagram of Δ with respect to Ψ. From $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$, $(\mathfrak{g}', \mathfrak{h}')$ and the h-principal isotropy subalgebra of $(\mathfrak{g}', \mathfrak{h}')$ we determine $\mathfrak{h}_\Theta$ and hence $\mathcal{L}_h(\mathfrak{g}, \mathfrak{h}, \Psi)$.

By using this recipe, we investigate the local orbit types of hyperbolic orbits for the nine exceptional semisimple symmetric pairs stated in Introduction. For any Lie algebra $\mathfrak{l}$, $\mathfrak{l}^k$ denotes the direct sum of k copies of $\mathfrak{l}$. For convenience, we use the following diagram.

$$\begin{array}{ccccc}
(\mathfrak{g}, \mathfrak{h}) & \xleftrightarrow{\textit{associated}} & (\mathfrak{g}, \mathfrak{h}^a) & \xleftrightarrow{\textit{dual}} & (\mathfrak{g}^{ad}, \mathfrak{h}^d) \\
\updownarrow \textit{dual} & & & & \updownarrow \textit{associated} \\
(\mathfrak{g}^d, \mathfrak{h}^d) & \xleftrightarrow{\textit{associated}} & (\mathfrak{g}^d, \mathfrak{h}^a) & \xleftrightarrow{\textit{dual}} & (\mathfrak{g}^{ad}, \mathfrak{h})
\end{array}$$

In the case of $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$, we have the following diagram (cf. (1.16) of [5]).

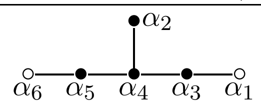
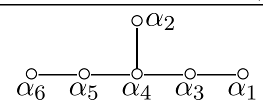
$$\begin{array}{ccccc}
(\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)}) & \xleftrightarrow{\textit{associated}} & (\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6) + \mathfrak{su}(2)) & \xleftrightarrow{\textit{dual}} & (\mathfrak{e}_{6(2)}, \mathfrak{sp}(3,1)) \\
\updownarrow \textit{dual} & & & & \updownarrow \textit{associated} \\
(\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3,1)) & \xleftrightarrow{\textit{associated}} & (\mathfrak{e}_{6(-26)}, \mathfrak{su}^*(6) + \mathfrak{su}(2)) & \xleftrightarrow{\textit{dual}} & (\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})
\end{array}$$

3.1. $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$-case

(Step 1) Let $\mathfrak{a}$ be a vector-type maximal split abelian subspace of $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{a}_\mathfrak{q}$ (resp. $\mathfrak{a}_\mathfrak{p}$) be a maximal abelian subspace of $\mathfrak{q}$ (resp. $\mathfrak{p}$) containing $\mathfrak{a}$. Let $\tilde{\mathfrak{a}}$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_\mathfrak{q} + \mathfrak{a}_\mathfrak{p}$. We denote by R the root system of $\mathfrak{g}^\mathbf{C}$ with respect to $\tilde{\mathfrak{a}}^\mathbf{C}$ and by $R_0 := \{\alpha \in R \mid \alpha(A) = 0, \forall A \in \mathfrak{a}\}$. We have the root space decomposition

$$\mathfrak{g}^\mathbf{C} = \tilde{\mathfrak{a}}^\mathbf{C} + \sum_{\alpha \in R} \mathfrak{g}_\alpha^\mathbf{C}$$

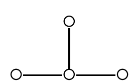
with respect to $\tilde{\mathfrak{a}}^\mathbf{C}$. Note that a maximal compactly imbedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is $\mathfrak{sp}(4)$. In this case, we have $\mathfrak{a} = \mathfrak{a}_\mathfrak{q}$. Hence the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ coincides with that of $(\mathfrak{g}^d, \mathfrak{k}^d)$. Since $(\mathfrak{g}^d, \mathfrak{k}^d) = (\mathfrak{e}_{6(-26)}, \mathfrak{f}_4)$ holds, the Satake diagrams of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{k})$ are given as follows (cf. Table VI, Chapter X of [2]).

the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$	the Satake diagram of $(\mathfrak{g}, \mathfrak{k})$
	

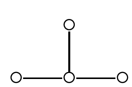
It follows from the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ that $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ is a simple root system of R_0. Since $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$, we have

$$\mathfrak{z}_\mathfrak{h}(\mathfrak{a}) = \tilde{\mathfrak{a}} \cap \mathfrak{h} + \Big(\sum_{\alpha \in R_0} \mathfrak{g}_\alpha^\mathbf{C} \Big) \cap \mathfrak{g}.$$

Since both the dimension of $\tilde{\mathfrak{a}} \cap \mathfrak{h}$ and the rank of R_0 are equal to 4, $\mathfrak{z}_\mathfrak{h}(\mathfrak{a})$ is semisimple and the Dynkin diagram of the complexification $(\mathfrak{z}_\mathfrak{h}(\mathfrak{a}))^\mathbf{C}$ of $\mathfrak{z}_\mathfrak{h}(\mathfrak{a})$ is given as follows:

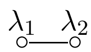


It follows from the Satake diagram of $(\mathfrak{g}, \mathfrak{k})$ that, for any root $\alpha_i (i = 2, 3, 4, 5)$ of $(\mathfrak{z}_\mathfrak{h}(\mathfrak{a}))^\mathbf{C}$, the restriction of α_i to $\mathfrak{a}_\mathfrak{p} \cap \mathfrak{h}$ is not equal to neither zero nor the restriction of α_j $(j \neq i)$ to $\mathfrak{a}_\mathfrak{p} \cap \mathfrak{h}$. Hence the Satake diagram of $(\mathfrak{z}_\mathfrak{h}(\mathfrak{a}), \mathfrak{k} \cap \mathfrak{z}_\mathfrak{h}(\mathfrak{a}))$ is given as follows:



Hence $\mathfrak{z}_\mathfrak{h}(\mathfrak{a})$ is isomorphic to $\mathfrak{so}(4, 4)$.

(Step 2) The Dynkin diagram of the restricted root system Δ of $(\mathfrak{g}, \mathfrak{h})$ with respect to $\mathfrak{a}$ is given as follows (cf. Table V of [5]).



$$\begin{pmatrix} m^+(\lambda_i) & m^+(2\lambda_i) \\ m^-(\lambda_i) & m^-(2\lambda_i) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 4 & 0 \end{pmatrix} (i = 1, 2)$$

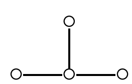
We denote by $\Psi(\mathfrak{a}) = \{\lambda_1, \lambda_2\}$. In the case of $\Psi(\mathfrak{a})$ and $\emptyset$, these corresponding subalgebra coincide with $\mathfrak{h}$ and $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$, respectively. We put $\Theta_i = \{\lambda_i\}$ $(i = 1, 2)$. By Theorem stated in Introduction, there exists a subsymmetric pair of $(\mathfrak{g}, \mathfrak{h})$ associated with Δ_{Θ_i}. In this case, it is shown that a subsymmetric pair of $(\mathfrak{g}, \mathfrak{h})$ associated with Δ_{Θ_i} is isomorphic to $(\mathfrak{so}(5,5), \mathfrak{so}(4,5))$ and its h-principal isotropy subalgebra is isomorphic to $\mathfrak{so}(4,4)$. It follows from (3.1) that the corresponding subalgebra of Θ_i is isomorphic to $\mathfrak{so}(4,5)$. Thus, by Proposition 3 we have

$$\mathcal{L}_h(\mathfrak{g}, \mathfrak{h}, \Psi(\mathfrak{a})) = \Big\{[\mathfrak{f}_{4(4)}], [\mathfrak{so}(4,5)], [\mathfrak{so}(4,4)]\Big\},$$

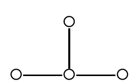
where $\mathfrak{so}(4,5)$ (resp. $\mathfrak{so}(4,4)$) is a subalgebra of $\mathfrak{f}_{4(4)}$ isomorphic to $\mathfrak{so}(4,5)$ (resp. $\mathfrak{so}(4,4)$).

3.2. $(\mathfrak{g}^d, \mathfrak{h}^d) = (\mathfrak{e}_{6(-26)}, \mathfrak{sp}(3,1))$-case

(Step 1) In this case, the vector-type maximal split abelian subspace $\mathfrak{a}$ of $(\mathfrak{g}^d, \mathfrak{h}^d)$ coincides with that of $(\mathfrak{g}, \mathfrak{h})$, and is a maximal abelian subspace of $\mathfrak{p}^d$. Note that a maximal compactly imbedded subalgebra $\mathfrak{k}^d$ of $\mathfrak{g}^d$ is $\mathfrak{f}_4$. Since $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a})$ coincides with $\mathfrak{z}_{\mathfrak{k}^d}(\mathfrak{a})$, $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a})$ is compact. Let $\mathfrak{a}_{\mathfrak{q}^d}$ be a maximal abelian subspace of $\mathfrak{q}^d$ containing $\mathfrak{a}$. Then the Satake diagrams of $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a})$ and $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a}_{\mathfrak{q}^d})$ coincide with those of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{k})$, respectively. It is shown that $\mathfrak{a}$ coincides with the center of $\mathfrak{z}_{\mathfrak{g}^d}(\mathfrak{a})$. We denote by $\hat{\mathfrak{z}}_{\mathfrak{g}^d}(\mathfrak{a})$ the semisimple part of $\mathfrak{z}_{\mathfrak{g}^d}(\mathfrak{a})$. It follows from the Satake diagram of $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a})$ that the Dynkin diagram of $(\hat{\mathfrak{z}}_{\mathfrak{g}^d}(\mathfrak{a}))^{\mathbf{C}}$ is given as follows:



It follows from the Satake diagram of $(\mathfrak{g}, \mathfrak{k})$ that the Satake diagram of $(\hat{\mathfrak{z}}_{\mathfrak{g}^d}(\mathfrak{a}), \hat{\mathfrak{z}}_{\mathfrak{g}^d}(\mathfrak{a}) \cap \mathfrak{h}^d)$ is given as follows:



Since $\hat{\mathfrak{z}}_{\mathfrak{g}^d}(\mathfrak{a}) \cap \mathfrak{h}^d$ coincides with $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a})$, $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a})$ is isomorphic to $\mathfrak{so}(4) + \mathfrak{so}(4) (\cong \mathfrak{sp}(1)^4)$.

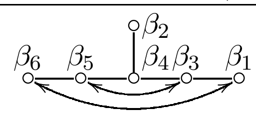
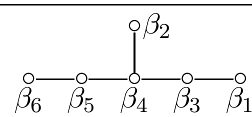
(Step 2) The restricted root system of $(\mathfrak{g}^d, \mathfrak{h}^d)$ with respect to $\mathfrak{a}$ coincides with the restricted root system Δ (including their multiplicities and signatures) of $(\mathfrak{g}, \mathfrak{h})$. We put $\Theta_i = \{\lambda_i\}$ $(i = 1, 2)$. By Theorem stated in Introduction, there exists a subsymmetric pair of $(\mathfrak{g}^d, \mathfrak{h}^d)$ associated with Δ_{Θ_i}. In this case, it is shown that a subsymmetric pair of $(\mathfrak{g}^d, \mathfrak{h}^d)$ associated with Δ_{Θ_i} is isomorphic to $(\mathfrak{so}(1,9), \mathfrak{so}(1,4) + \mathfrak{so}(5))$ and its h-principal isotropy subalgebra is isomorphic to $\mathfrak{so}(4) + \mathfrak{so}(4) (\cong \mathfrak{sp}(1)^4)$. Note that $\mathfrak{so}(1,4) + \mathfrak{so}(5)$ is isomorphic to $\mathfrak{sp}(1,1) + \mathfrak{sp}(2) (\subset \mathfrak{sp}(3,1))$. It follows from (3.1) that the corresponding subalgebra of Θ_i is isomorphic to $\mathfrak{sp}(1,1) + \mathfrak{sp}(2)$. Thus, by Proposition 3 we have

$$\mathcal{L}_h(\mathfrak{g}^d, \mathfrak{h}^d, \Psi(\mathfrak{a})) = \left\{ [\mathfrak{sp}(3,1)], [\mathfrak{sp}(1,1) + \mathfrak{sp}(2)], [\mathfrak{sp}(1)^4] \right\},$$

where $\mathfrak{sp}(1,1) + \mathfrak{sp}(2)$ (resp. $\mathfrak{sp}(1)^4$) is a subalgebra of $\mathfrak{sp}(3,1)$ isomorphic to $\mathfrak{sp}(1,1) + \mathfrak{sp}(2)$ (resp. $\mathfrak{sp}(1)^4$).

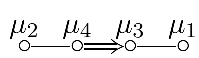
3.3. $(\mathfrak{g}, \mathfrak{h}^a) = (\mathfrak{e}_{6(6)}, \mathfrak{su}^*(6) + \mathfrak{su}(2))$-case

(Step 1) Let $\mathfrak{a}^a$ be a vector-type maximal split abelian subspace of $(\mathfrak{g}, \mathfrak{h}^a)$. In this case, $\mathfrak{a}^a$ is a maximal abelian subspace of $\mathfrak{q}^a$. Let $\mathfrak{a}^a_{\mathfrak{p}}$ be a maximal abelian subspace of $\mathfrak{p}$ containing $\mathfrak{a}$ and $\tilde{\mathfrak{a}}^a$ be a maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}^a_{\mathfrak{p}}$. Then the Satake diagram of $(\mathfrak{g}, \mathfrak{h}^a, \mathfrak{a}^a)$ coincides with that of the Riemannian symmetric pair $(\mathfrak{e}_{6(2)}, \mathfrak{su}(6) + \mathfrak{su}(2))$, which is given as follows (cf. Table VI, Chapter X of [2]).

the Satake diagram of $(\mathfrak{g}, \mathfrak{h}^a, \mathfrak{a}^a)$	the Satake diagram of $(\mathfrak{g}, \mathfrak{k})$
	

Since there exists no black circle in the Satake diagram of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$, $\mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}^a)$ has no semisimple part. Since the dimension of $\tilde{\mathfrak{a}}^a \cap \mathfrak{h}^a$ is equal to two, $\mathfrak{z}_{\mathfrak{h}^a}(\mathfrak{a}^a)$ is isomorphic to $\mathbf{R}^2$.

(Step 2) The Dynkin diagram of the restricted root system Δ^a of $(\mathfrak{g}, \mathfrak{h}^a)$ with respect to $\mathfrak{a}^a$ is given as follows (cf. Table V of [5]).



$$\begin{pmatrix} m^+(\mu_i) & m^+(2\mu_i) \\ m^-(\mu_i) & m^-(2\mu_i) \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} (i = 2, 4), \\ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} (i = 1, 3) \end{cases}$$

We denote by $\Psi(\mathfrak{a}^a) = \{\mu_i \mid i = 1, \ldots, 4\}$. We put $\Theta = \{\mu_i \mid i = 2, 3, 4\}$. By Theorem stated in Introduction, there exists a subsymmetric pair of $(\mathfrak{g}, \mathfrak{h}^a)$ associated with Δ_Θ. In this case, it is shown that a subsymmetric pair of

$(\mathfrak{g},\mathfrak{h}^a)$ associated with Δ_Θ is isomorphic to $(\mathfrak{so}(4,4),\mathfrak{so}(1,4)+\mathfrak{so}(3))$ and its h-principal isotropy subalgebra is isomorphic to $\mathbf{R}$. It follows from (3.1) that the corresponding subalgebra of Θ is isomorphic to $\mathbf{R}+\mathfrak{so}(1,4)+\mathfrak{so}(3)$. By the same argument, we can determine the other isotropy subalgebras as in Table 1. Thus, we can determine $\mathcal{L}_h(\mathfrak{g},\mathfrak{h}^a,\Psi(\mathfrak{a}^a))$.

$\Theta(\subset\Psi(\mathfrak{a}^a))$	$\mathfrak{h}^a_\Theta$
$\Psi(\mathfrak{a}^a)$	$\mathfrak{su}^*(6)+\mathfrak{su}(2)$
$\{\mu_i \mid i=2,3,4\}$	$\mathbf{R}+\mathfrak{so}(1,4)+\mathfrak{so}(3)$
$\{\mu_i \mid i=1,2,4\}$	$\mathbf{R}+\mathfrak{so}(3)+\mathfrak{sl}(2,\mathbf{R})$
$\{\mu_i \mid i=1,2,3\}$	$\mathfrak{so}(2)+\mathfrak{sl}(3,\mathbf{R})$
$\{\mu_i \mid i=1,3,4\}$	$\mathfrak{sl}(3,\mathbf{C})+\mathfrak{so}(2)$
$\{\mu_2,\mu_4\}$	$\mathbf{R}^2+\mathfrak{so}(3)$
$\{\mu_3,\mu_4\}$	$\mathbf{R}+\mathfrak{so}(2)+\mathfrak{sl}(2,\mathbf{C})$
$\{\mu_1,\mu_2\},\{\mu_1,\mu_4\},\{\mu_2,\mu_3\}$	$\mathbf{R}+\mathfrak{so}(2)+\mathfrak{sl}(2,\mathbf{R})$
$\{\mu_1,\mu_3\}$	$\mathfrak{sl}(3,\mathbf{R})$
$\{\mu_2\},\{\mu_4\}$	$\mathbf{R}^2+\mathfrak{so}(2)$
$\{\mu_1\},\{\mu_3\}$	$\mathbf{R}+\mathfrak{sl}(2,\mathbf{R})$
$\emptyset$	$\mathbf{R}^2$

Table 1: the corresponding subalgebras in the case of $(\mathfrak{g},\mathfrak{h}^a)=(\mathfrak{e}_{6(6)},\mathfrak{su}^*(6)+\mathfrak{su}(2))$

3.4. $(\mathfrak{g}^{ad},\mathfrak{h}^d)=(\mathfrak{e}_{6(2)},\mathfrak{sp}(3,1))$-case

(Step 1) In this case, the vector-type maximal split abelian subspace $\mathfrak{a}^a$ of $(\mathfrak{g}^{ad},\mathfrak{h}^d)$ coincides with that of $(\mathfrak{g},\mathfrak{h}^a)$ and is a maximal abelian subspace of $\mathfrak{p}^{ad}$. Note that a maximal compactly imbedded subalgebra $\mathfrak{k}^{ad}$ of $\mathfrak{g}^{ad}$ is $\mathfrak{su}(6)+\mathfrak{su}(2)$. Since $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a}^a)$ coincides with $\mathfrak{z}_{\mathfrak{k}^{ad}}(\mathfrak{a})$, $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a}^a)$ is compact. Let $\mathfrak{a}^a_{\mathfrak{q}^{ad}}$ be a maximal abelian subspace of $\mathfrak{q}^{ad}$ containing $\mathfrak{a}^a$. Then the Satake diagram of $(\mathfrak{g}^{ad},\mathfrak{h}^d,\mathfrak{a}^a)$ and $(\mathfrak{g}^{ad},\mathfrak{h}^d,\mathfrak{a}^a_{\mathfrak{q}^{ad}})$ coincide with those of $(\mathfrak{g},\mathfrak{h}^a,\mathfrak{a}^a)$ and $(\mathfrak{g},\mathfrak{k})$, respectively. Therefore $\mathfrak{z}_{\mathfrak{h}^d}(\mathfrak{a}^a)$ is $\{0\}$.

(Step 2) The restricted root system of $(\mathfrak{g}^{ad},\mathfrak{h}^d)$ with respect to $\mathfrak{a}^a$ coincides with the restricted root system Δ^a (including their multiplicities and signatures) of $(\mathfrak{g},\mathfrak{h}^a)$. We put $\Theta=\{\mu_i \mid i=2,3,4\}$. By Theorem stated in Introduction, there exists a subsymmetric pair of $(\mathfrak{g}^{ad},\mathfrak{h}^d)$ associated with Δ_Θ. In this case, it is shown that a subsymmetric pair of $(\mathfrak{g}^{ad},\mathfrak{h}^d)$ associated with Δ_Θ is isomorphic to $(\mathfrak{so}(5,3),\mathfrak{so}(1,3)+\mathfrak{so}(4))$ and its h-principal isotropy

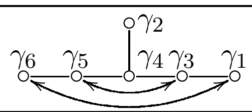
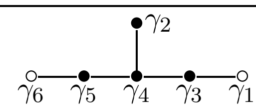
$\Theta(\subset \Psi(\mathfrak{a}^a))$	$\mathfrak{h}^d_\Theta$
$\Psi(\mathfrak{a}^a)$	$\mathfrak{sp}(3,1)$
$\{\mu_i \mid i=2,3,4\}$	$\mathfrak{so}(1,3)+\mathfrak{so}(4)$
$\{\mu_i \mid i=1,2,4\}$	$\mathfrak{so}(3)+\mathfrak{so}(2,\mathbf{C})$
$\{\mu_i \mid i=1,2,3\}$	$\mathfrak{so}(2)+\mathfrak{so}(3,\mathbf{C})$
$\{\mu_i \mid i=1,3,4\}$	$\mathfrak{so}^*(6)$
$\{\mu_2,\mu_4\}$	$\mathfrak{so}(3)$
$\{\mu_3,\mu_4\}$	$\mathfrak{so}^*(4)$
$\{\mu_1,\mu_2\},\{\mu_1,\mu_4\},$ $\{\mu_2,\mu_3\}$	$\mathfrak{so}(2)+\mathfrak{so}(2,\mathbf{C})$
$\{\mu_1,\mu_3\}$	$\mathfrak{so}(3,\mathbf{C})$
$\{\mu_2\},\{\mu_4\}$	$\mathfrak{so}(2)$
$\{\mu_1\},\{\mu_3\}$	$\mathfrak{so}(2,\mathbf{C})$
$\emptyset$	$\{0\}$

Table 2: the corresponding subalgebras in the case of $(\mathfrak{g}^{ad},\mathfrak{h}^d) = (\mathfrak{e}_{6(2)},\mathfrak{sp}(3,1))$

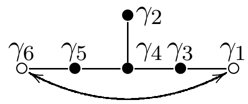
subalgebra is $\{0\}$. It follows from (3.1) that the corresponding subalgebra of Θ is isomorphic to $\mathfrak{so}(1,3)+\mathfrak{so}(4)$. By the same argument, we can determine the other isotropy subalgebras as in Table 2. Thus, we can determine $\mathcal{L}_h(\mathfrak{g}^{ad},\mathfrak{h}^d,\Psi(\mathfrak{a}^a))$.

3.5. $(\mathfrak{g}^d,\mathfrak{h}^a)=(\mathfrak{e}_{6(-26)},\mathfrak{su}^*(6)+\mathfrak{su}(2))$-case

(Step 1) Let $\mathfrak{a}^{da}$ be a vector-type maximal split abelian subspace of $(\mathfrak{g}^d,\mathfrak{h}^a)$ and $\mathfrak{a}^{da}_{\mathfrak{q}^{da}}$ (resp. $\mathfrak{a}^{da}_{\mathfrak{p}^{da}}$) be a maximal abelian subspace of $\mathfrak{q}^{da}$ (resp. $\mathfrak{p}^{da}$) containing $\mathfrak{a}^{da}$. In this case, $\mathfrak{a}^{da}$ is not equal to neither $\mathfrak{a}^{da}_{\mathfrak{q}^{da}}$ nor $\mathfrak{a}^{da}_{\mathfrak{p}^{da}}$. Note that a maximal compactly imbedded subalgebra $\mathfrak{k}^{da}$ of $\mathfrak{g}^{da}$ coincides with $\mathfrak{k}^d$ by (1.4.1) of [5]. First, we determine the Satake diagram of $(\mathfrak{g}^d,\mathfrak{h}^a,\mathfrak{a}^{da})$. The Satake diagram of $(\mathfrak{g}^d,\mathfrak{h}^a,\mathfrak{a}^{da}_{\mathfrak{q}^{da}})$ coincides with that of the Riemannian symmetric pair $(\mathfrak{e}_{6(2)},\mathfrak{su}(6)+\mathfrak{su}(2))$. Therefore the Satake diagrams of $(\mathfrak{g}^d,\mathfrak{h}^a,\mathfrak{a}^{da}_{\mathfrak{q}^{da}})$ and $(\mathfrak{g}^d,\mathfrak{k}^d)$ are given as follows (cf. Table VI, Chapter X of [2]).

the Satake diagram of $(\mathfrak{g}^d,\mathfrak{h}^a,\mathfrak{a}^{da}_{\mathfrak{q}^{da}})$	the Satake diagram of $(\mathfrak{g}^d,\mathfrak{k}^d)$
	

Hence the Satake diagram of $(\mathfrak{g}^d,\mathfrak{h}^a,\mathfrak{a}^{da})$ is given as follows:



From above Satake diagrams it is shown that the h-principal isotropy subalgebra is isomorphic to $\mathbf{R} + \mathfrak{sp}(2) + \mathfrak{su}(2)$.

(Step 2) Let $\Psi(\mathfrak{a}^{da})$ be a simple root system of the restricted root system Δ^{da} of $(\mathfrak{g}^d, \mathfrak{h}^a)$ with respect to $\mathfrak{a}^{da}$, whose rank is equal to 1. Thus, we have

$$\mathcal{L}_h(\mathfrak{g}^d, \mathfrak{h}^a, \Psi(\mathfrak{a}^{da})) = \Big\{[\mathfrak{su}^*(6) + \mathfrak{su}(2)], [\mathbf{R} + \mathfrak{sp}(2) + \mathfrak{su}(2)]\Big\},$$

where $\mathbf{R} + \mathfrak{sp}(2) + \mathfrak{su}(2)$ is a subalgebra of $\mathfrak{su}^*(6) + \mathfrak{su}(2)$ isomorphic to $\mathbf{R} + \mathfrak{sp}(2) + \mathfrak{su}(2)$.

3.6. $(\mathfrak{g}^{ad}, \mathfrak{h}) = (\mathfrak{e}_{6(2)}, \mathfrak{f}_{4(4)})$-case

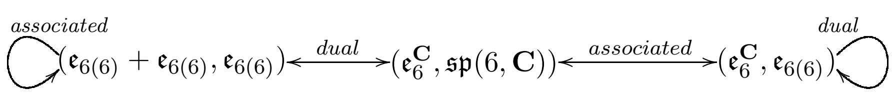
(Step 1) In this case, the vector-type maximal split abelian subspace $\mathfrak{a}^{da}$ of $(\mathfrak{g}^{ad}, \mathfrak{h})$ coincides with that of $(\mathfrak{g}^d, \mathfrak{h}^a)$. Let $\mathfrak{a}^{da}_{\mathfrak{q}^{dad}}$ (resp. $\mathfrak{a}^{da}_{\mathfrak{p}^{dad}}$) be a maximal abelian subspace of $\mathfrak{q}^{dad}$ (resp. $\mathfrak{p}^{dad}$) containing $\mathfrak{a}^{da}$. Then the Satake diagrams of $(\mathfrak{g}^{ad}, \mathfrak{h}, \mathfrak{a}^{da})$ and $(\mathfrak{g}^{da}, \mathfrak{h}, \mathfrak{a}^{da}_{\mathfrak{q}^{dad}})$ coincide with those of $(\mathfrak{g}^d, \mathfrak{h}^a, \mathfrak{a}^{da})$ and $(\mathfrak{g}^d, \mathfrak{k}^d)$, respectively. Hence it is shown that the h-principal isotropy subalgebra is isomorphic to $\mathfrak{so}(5, 3)$.

(Step 2) The restricted root system of $(\mathfrak{g}^{ad}, \mathfrak{h})$ with respect to $\mathfrak{a}^{da}$ coincides with the restricted root system Δ^{da} (including their multiplicities and signatures) of $(\mathfrak{g}^d, \mathfrak{h}^a)$. Thus, we have

$$\mathcal{L}_h(\mathfrak{g}^{ad}, \mathfrak{h}, \Psi(\mathfrak{a}^{da})) = \Big\{[\mathfrak{f}_{4(4)}], [\mathfrak{so}(5, 3)]\Big\},$$

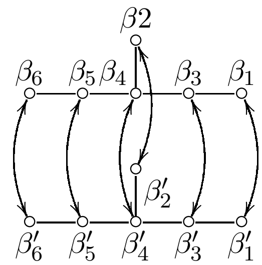
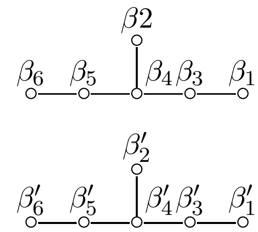
where $\mathfrak{so}(5, 3)$ is a subalgebra of $\mathfrak{f}_{4(4)}$ isomorphic to $\mathfrak{so}(5, 3)$.

In the case of $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{e}_{6(6)} + \mathfrak{e}_{6(6)}, \mathfrak{e}_{6(6)})$, we have the following diagram by Lemma 1.13.1 of [5].



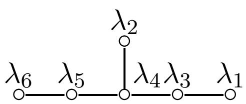
3.7. $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{e}_{6(6)} + \mathfrak{e}_{6(6)}, \mathfrak{e}_{6(6)})$-case

(Step 1) Let $\mathfrak{a}$ be a vector-type maximal split abelian subspace of $(\mathfrak{g}, \mathfrak{h})$. In this case, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{q}$, and a maximal compactly imbedded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is $\mathfrak{sp}(4)+\mathfrak{sp}(4)$. Therefore the Satake diagram of $(\mathfrak{g},\mathfrak{h},\mathfrak{a})$ coincides with that of the Riemannian symmetric pair $(\mathfrak{e}_6^{\mathbf{C}},\mathfrak{e}_6)$. Then the Satake diagrams of $(\mathfrak{g},\mathfrak{h},\mathfrak{a})$ and $(\mathfrak{g},\mathfrak{k})$ are given as follows.

the Satake diagram of $(\mathfrak{g},\mathfrak{h},\mathfrak{a})$	the Satake diagram of $(\mathfrak{g},\mathfrak{k})$
	

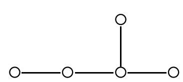
By these Satake diagrams it is shown that $\mathfrak{z}_{\mathfrak{h}}(\mathfrak{a})$ is isomorphic to $\mathbf{R}^6$.

(Step 2) The Dynkin diagram of the restricted root system Δ of $(\mathfrak{g},\mathfrak{h})$ with respect to $\mathfrak{a}$ is given as follows.



$$\begin{pmatrix} m^+(\mu_i) & m^+(2\mu_i) \\ m^-(\mu_i) & m^-(2\mu_i) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} (i=1,\dots,6)$$

Note that, in this case, all restricted roots have the same multiplicity and signature. We put $\Psi(\mathfrak{a}) := \{\lambda_i \mid i=1,\dots,6\}$. Since all simple roots have the same multiplicity and signature, the corresponding subalgebra of $\Theta(\subset \Psi(\mathfrak{a}))$ only depends on the form of the Dynkin diagram of Δ_Θ. If a subdiagram of the Dynkin diagram of Δ has the form



then it is shown that a subsymmetric pair of $(\mathfrak{g},\mathfrak{h})$ associated with the restricted root system whose Dynkin diagram coincides with the above subdiagram is isomorphic to $(\mathfrak{so}(5,5)+\mathfrak{so}(5,5),\mathfrak{so}(5,5))$, and its h-principal isotropy subalgebra is isomorphic to $\mathbf{R}^5$. It follows from (3.1) that the corresponding subalgebra is isomorphic to $\mathbf{R}+\mathfrak{so}(5,5)$. By the same argument, we can determine the other isotropy subalgebras as in Table 3. Thus, by Proposition 3 we can determine $\mathcal{L}_h(\mathfrak{g},\mathfrak{h},\Psi(\mathfrak{a}))$.

3.8. $(\mathfrak{g}^d,\mathfrak{h}^d)=(\mathfrak{e}_6^{\mathbf{C}},\mathfrak{sp}(6,\mathbf{C}))$-case

(Step 1) In this case, the vector-type maximal split abelian subspace $\mathfrak{a}$ of $(\mathfrak{g}^d,\mathfrak{h}^d)$ coincides with that of $(\mathfrak{g},\mathfrak{h})$. Note that a maximal compactly imbedded subalgebra $\mathfrak{k}^d$ of $\mathfrak{g}^d$ is $\mathfrak{e}_6$. Let $\mathfrak{a}_{\mathfrak{q}^d}$ be a maximal abelian subspace of $\mathfrak{q}^d$

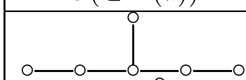
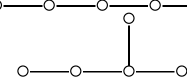
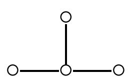
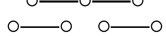
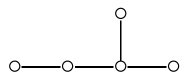
$\Theta(\subset \Psi(\mathfrak{a}))$	$\mathfrak{h}_\Theta$
	$\mathfrak{e}_{6(6)}$
	$\mathbf{R} + \mathfrak{so}(5,5)$
	$\mathbf{R} + \mathfrak{sl}(6,\mathbf{R})$
	$\mathbf{R} + \mathfrak{sl}(5,\mathbf{R}) + \mathfrak{sl}(2,\mathbf{R})$
	$\mathbf{R} + \mathfrak{sl}(3,\mathbf{R})^2 + \mathfrak{sl}(2,\mathbf{R})$
	$\mathbf{R}^2 + \mathfrak{sl}(5,\mathbf{R})$
	$\mathbf{R}^2 + \mathfrak{sl}(3,\mathbf{R}) + \mathfrak{sl}(2,\mathbf{R})^2$
	$\mathbf{R}^2 + \mathfrak{sl}(4,\mathbf{R}) + \mathfrak{sl}(2,\mathbf{R})$
	$\mathbf{R}^2 + \mathfrak{so}(4,4)$
	$\mathbf{R}^2 + \mathfrak{sl}(3,\mathbf{R})^2$
	$\mathbf{R}^3 + \mathfrak{sl}(4,\mathbf{R})$
	$\mathbf{R}^3 + \mathfrak{sl}(3,\mathbf{R}) + \mathfrak{sl}(2,\mathbf{R})$
	$\mathbf{R}^3 + \mathfrak{sl}(2,\mathbf{R})^3$
	$\mathbf{R}^4 + \mathfrak{sl}(3,\mathbf{R})$
	$\mathbf{R}^4 + \mathfrak{sl}(2,\mathbf{R})^2$
	$\mathbf{R}^5 + \mathfrak{sl}(2,\mathbf{R})$
$\emptyset$	$\mathbf{R}^6$

Table 3: the corresponding subalgebras in the case of $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{e}_{6(6)} + \mathfrak{e}_{6(6)}, \mathfrak{e}_{6(6)})$

containing $\mathfrak{a}$. The Satake diagrams of $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a})$ and $(\mathfrak{g}^d, \mathfrak{h}^d, \mathfrak{a}_{\mathfrak{q}^d})$ coincide with those of $(\mathfrak{g}, \mathfrak{h}, \mathfrak{a})$ and $(\mathfrak{g}, \mathfrak{k})$, respectively. Hence it is shown that the h-principal isotropy subalgebra is $\{0\}$.

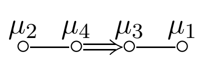
(Step 2) The restricted root system of $(\mathfrak{g}^d, \mathfrak{h}^d)$ with respect to $\mathfrak{a}$ coincides with the restricted root system Δ (including their multiplicities and signatures) of $(\mathfrak{g}, \mathfrak{h})$. If a subdiagram of the Dynkin diagram of Δ has the form



then it is shown that a subsymmetric pair of $(\mathfrak{g}^d, \mathfrak{h}^d)$ associated with the restricted root system whose Dynkin diagram coincides with the above subdiagram is isomorphic to $(\mathfrak{so}(10, \mathbf{C}), \mathfrak{so}(5, \mathbf{C}) + \mathfrak{so}(5, \mathbf{C}))$, and its h-principal isotropy subalgebra is $\{0\}$. Note that $\mathfrak{so}(5, \mathbf{C}) + \mathfrak{so}(5, \mathbf{C})$ is isomorphic to $\mathfrak{sp}(2, \mathbf{C}) + \mathfrak{sp}(2, \mathbf{C}) (\subset \mathfrak{sp}(6, \mathbf{C}))$. It follows from (3.1) that the corresponding subalgebra is isomorphic to $\mathfrak{sp}(2, \mathbf{C}) + \mathfrak{sp}(2, \mathbf{C})$. By the same argument we can determine the other isotropy subalgebras as in Table 4. Thus, by Proposition 3 we can determine $\mathcal{L}_h(\mathfrak{g}^d, \mathfrak{h}^d, \Psi(\mathfrak{a}))$.

3.9. $(\mathfrak{g}^d, \mathfrak{h}^a) = (\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{e}_{6(6)})$-case

(Step 1) Let $\mathfrak{a}^{da}$ be a vector-type maximal split abelian subspace of $(\mathfrak{g}^d, \mathfrak{h}^a)$ and $\mathfrak{a}^{da}_{\mathfrak{q}^{da}}$ (resp. $\mathfrak{a}^{da}_{\mathfrak{p}^{da}}$) be a maximal abelian subspace of $\mathfrak{q}^{da}$ (resp. $\mathfrak{p}^{da}$) containing $\mathfrak{a}^{da}$. Note that $\mathfrak{a}^{da}$ is not equal to neither $\mathfrak{a}^{da}_{\mathfrak{q}^{da}}$ nor $\mathfrak{a}^{da}_{\mathfrak{p}^{da}}$. Then the Satake diagram of $(\mathfrak{g}^d, \mathfrak{h}^a, \mathfrak{a}^{da}_{\mathfrak{q}^{da}})$ coincides with that of $(\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{e}_6)$. The Dynkin diagram of $(\mathfrak{g}^d, \mathfrak{h}^a)$ with respect to $\mathfrak{a}^{da}$ is given as follows (cf. Table V of [5]).



$$\begin{pmatrix} m^+(\mu_i) & m^+(2\mu_i) \\ m^-(\mu_i) & m^-(2\mu_i) \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} (i = 2), \\ \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} (i = 4) \\ \begin{pmatrix} 2 & 0 \\ 2 & 0 \end{pmatrix} (i = 1, 3) \end{cases}$$

Therefore it is shown that the Satake diagram of $(\mathfrak{g}^d, \mathfrak{h}^a, \mathfrak{a}^{da})$ is given as follows.

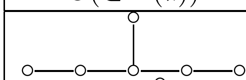
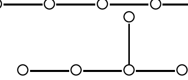
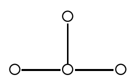
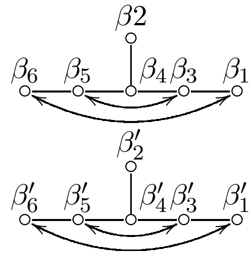
$\Theta(\subset \Psi(\mathfrak{a}))$	$\mathfrak{h}^d_\Theta$
	$\mathfrak{sp}(6, \mathbf{C})$
	$\mathfrak{sp}(2, \mathbf{C})^2$
	$\mathfrak{sl}(4, \mathbf{C})$
	$\mathfrak{sp}(2, \mathbf{C}) + \mathbf{C}$
	$\mathfrak{sp}(1, \mathbf{C})^2 + \mathbf{C}$
	$\mathfrak{sp}(2, \mathbf{C})$
	$\mathfrak{sp}(1, \mathbf{C}) + \mathbf{C}^2$
	$\mathfrak{sp}(1, \mathbf{C})^2 + \mathbf{C}$
	$\mathfrak{sp}(1, \mathbf{C})^4$
	$\mathfrak{sp}(1, \mathbf{C})^2$
	$\mathfrak{sp}(1, \mathbf{C})^2$
	$\mathfrak{sp}(1, \mathbf{C}) + \mathbf{C}$
	$\mathbf{C}^3$
	$\mathfrak{sp}(1, \mathbf{C})$
	$\mathbf{C}^2$
	$\mathbf{C}$
$\emptyset$	$\{0\}$

Table 4: the corresponding subalgebras in the case of $(\mathfrak{g}^d, \mathfrak{h}^d) = (\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{sp}(6, \mathbf{C}))$



(Remark. β_i and β_i' are joined by a curved arrow for each i)

Hence the h-principal isotropy subalgebra is isomorphic to $\mathbf{R}^2 + \mathfrak{so}(2)^4$.

(Step 2) We put $\Psi(\mathfrak{a}^{da}) := \{\mu_i \mid i = 1, \ldots, 4\}$ and $\Theta := \{\mu_i \mid i = 2, 3, 4\}$. By Theorem stated in Introduction, there exists a subsymmetric pair of $(\mathfrak{g}^d, \mathfrak{h}^a)$ associated with Δ_Θ. In this case, it is shown that a subsymmetric pair of $(\mathfrak{g}^d, \mathfrak{h}^a)$ associated with Δ_Θ is isomorphic to $(\mathfrak{so}(8, \mathbf{C}), \mathfrak{so}(3, 5))$ and its h-principal isotropy subalgebra is isomorphic to $\mathfrak{so}(2)^4$. It follows from (3.1) that the corresponding subalgebra of Θ is isomorphic to $\mathbf{R}^2 + \mathfrak{so}(3, 5)$. By the same argument we can determine the other corresponding subalgebras as in Table 5. Thus, by Proposition 3 we can determine $\mathcal{L}_h(\mathfrak{g}^d, \mathfrak{h}^a, \Psi(\mathfrak{a}^{da}))$.

$\Theta(\subset \Psi(\mathfrak{a}^{da}))$	$\mathfrak{h}^{da}_\Theta$
$\Psi(\mathfrak{a}^a)$	$\mathfrak{e}_{6(6)}$
$\{\mu_i \mid i = 2, 3, 4\}$	$\mathbf{R}^2 + \mathfrak{so}(3, 5)$
$\{\mu_i \mid i = 1, 3, 4\}$	$\mathfrak{so}(2) + \mathfrak{su}^*(6)$
$\{\mu_i \mid i = 1, 2, 4\}$	$\mathbf{R}^2 + \mathfrak{su}(1, 2) + \mathfrak{so}^*(4)$
$\{\mu_i \mid i = 1, 2, 3\}$	$\mathbf{R}^2 + \mathfrak{su}(1, 1) + \mathfrak{so}^*(6)$
$\{\mu_3, \mu_4\}$	$\mathbf{R} + \mathfrak{so}(2)^2 + \mathfrak{su}^*(4)$
$\{\mu_2, \mu_4\}$	$\mathbf{R}^2 + \mathfrak{so}(2)^2 + \mathfrak{su}(1, 2)$
$\{\mu_2, \mu_3\}$	$\mathbf{R}^2 + \mathfrak{so}(2) + \mathfrak{su}(1, 1) + \mathfrak{so}^*(4)$
$\{\mu_1, \mu_4\}$	$\mathbf{R}^2 + \mathfrak{so}(2) + \mathfrak{su}(2) + \mathfrak{so}^*(4)$
$\{\mu_1, \mu_3\}$	$\mathbf{R}^2 + \mathfrak{so}(2) + \mathfrak{so}^*(6)$
$\{\mu_1, \mu_2\}$	$\mathbf{R}^2 + \mathfrak{so}(2) + \mathfrak{su}(1, 1) + \mathfrak{so}^*(4)$
$\{\mu_4\}$	$\mathbf{R}^2 + \mathfrak{so}(2)^3 + \mathfrak{su}(2)$
$\{\mu_3\}, \{\mu_1\}$	$\mathbf{R}^2 + \mathfrak{so}(2)^2 + \mathfrak{so}^*(4)$
$\{\mu_2\}$	$\mathbf{R}^2 + \mathfrak{so}(2)^3 + \mathfrak{su}(1, 1)$
$\emptyset$	$\mathbf{R}^2 + \mathfrak{so}(2)^4$

Table 5: the corresponding subalgebras in the case of $(\mathfrak{g}^d, \mathfrak{h}^a) = (\mathfrak{e}_6^{\mathbf{C}}, \mathfrak{e}_{6(6)})$

Remark 3. (1) *If* $\mathfrak{b}$ *is a troidal-type maximal split abelian subspace of* $\mathfrak{q}$*, the subspace* $\sqrt{-1}\mathfrak{b}$ *is a vector-type maximal split abelian subspace of* $\mathfrak{q}^{dad}$*.*

$\Theta(\subset \Phi(\mathfrak{b}))$	$\mathfrak{h}_\Theta$
$\Phi(\mathfrak{b})$	$\mathfrak{f}_{4(4)}$
$\emptyset$	$\mathfrak{so}(5,3)$

Table 6: Local orbit types of elliptic orbits of $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{e}_{6(6)}, \mathfrak{f}_{4(4)})$

Therefore the restricted root system Σ of $(\mathfrak{g},\mathfrak{h})$ with respect to $\mathfrak{b}$ coincides with that of $(\mathfrak{g}^{ad},\mathfrak{h})$ with respect to $\sqrt{-1}\mathfrak{b}$. Let $\{\nu_1,\dots,\nu_l\}(=: \Phi(\mathfrak{b}))$ be a simple root system of Σ. Then $\mathcal{L}_h(\mathfrak{g}^{ad},\mathfrak{h},\Phi(\mathfrak{b}))$ coincides with the set $\mathcal{L}_e(\mathfrak{g},\mathfrak{h},\Phi(\mathfrak{b}))$ of all local orbit types of the Ad(H)*-orbits through $B = \sum_{i=1}^{l} b_i B_{\nu_i}$ with all $b_i \geq 0$, where $(B_{\nu_1},\dots,B_{\nu_l})$ is the dual basis of $(\nu_1,\dots,\nu_l)$. For example, in the case of $(\mathfrak{g},\mathfrak{h}) = (\mathfrak{e}_{6(6)},\mathfrak{f}_{4(4)})$, we can determine $\mathcal{L}_e(\mathfrak{g},\mathfrak{h},\Phi(\mathfrak{b})(=\Psi(\mathfrak{a}^{da})))$ by Table 6.*

(2) *The set $\mathcal{L}_h(\mathfrak{g},\mathfrak{h},\Psi)$ is not necessarily equal to the set of all local orbit types of the hyperbolic* Ad(H)*-orbits. For example, if $\mathfrak{h}^a$ is semisimple and, for any $\lambda \in \Delta$, $\mathfrak{g}_\lambda \cap \mathfrak{h}^a \neq \{0\}$ holds, then $\mathcal{L}_h(\mathfrak{g},\mathfrak{h},\Psi)$ coincides with the set of all local orbit types of hyperbolic* Ad(H)*-orbits, where Δ denotes the restricted root system of $(\mathfrak{g},\mathfrak{h})$ with respect to a vector-type maximal split abelian subspace.*

Research plan in the future. We plan to determine the set of all local orbit types of hyperbolic orbits (or elliptic orbits) of s-representations, completely.

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Kurando Baba
Department of Mathematics, Science University of Tokyo
Wakamiya-cho 26, Shinjuku-ku, Tokyo 162-0827, Japan
E-mail: `baba@math.kagu.tus.ac.jp`