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The basis number of the cartesian product of certain classes of graphs

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Abstract. In this paper we prove that the basis number of the Cartesian product of paths, cycles and theta graphs with Tenets is exactly 3. However, if we apply Theorem 4.1 of Ali and Marougi [3], which gives a general upper bound of the Cartesian product of disjoint connected graphs, we find that the basis number of these graphs is less than or equal to 4.

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§1. Introduction

Throughout this paper we consider only finite connected simple graphs. For the undefined terminology we refer the reader to [17].

It is well known that any graph G is associated with a q-dimensional vector space over the finite field Z_2, say $(Z_2)^q$, where q is the order of the edge-set, $E(G)$, of the graph G. If $E(G) = \{e_1, e_2, \ldots, e_q\}$, then every subset $S \subset E(G)$ corresponds to a vector $v \in (Z_2)^q$ such that the i-th component is 1 if $e_i \in S$ and 0 if $e_i \notin S$. Since the edges are used to define the vector space, some authors call this space *the edge space.* The subspace of the edge space consisting of all the cycles and the edge-disjoint union of cycles is called *the cycle space of* G and it is denoted by $\mathcal{C}(G)$. The cycle space of a connected graph has dimension given by $\dim\mathcal{C}(G) = q - p + 1$, where p is the order of the vertex-set of G. A basis $\mathcal{B}$ *of* $\mathcal{C}(G)$ in which every edge of G occurs in at most k cycles of $\mathcal{B}$ is called a *k-fold basis.* The minimum nonnegative integer, $\mathrm{b}(G)$, such that $\mathcal{C}(G)$ has a $\mathrm{b}(G)$-*fold* basis is called *the basis number* of G. In 1937 S. MacLane [18] proved that a graph G is planar if and only if $\mathrm{b}(G) \leq 2$. After that, the subject of basis number was left aside until the end of 1981 when Schemeichel [19] determined the basis number of the complete graphs K_n and the complete Bipartite graphs $K_{n,m}$, also he proved the existence of graphs of arbitrary basis numbers. Then, Banks and Schemeichel [11] proved that the basis number of the n-cube is 4. Since 1981, many papers appeared that focus on finding the basis number of certain classes of graphs that obtained from different kinds of products on graphs (like the Cartesian product, the strong product, the semi-strong product, the Lexicographic (or the composition) product, or the semi-composition product), see [1–10] and [12–16].

Definition 1.1. *The Cartesian product of the graphs* $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ *is the graph* $G = G_1 \times G_2$ *with vertex set* $V(G) = V_1 \times V_2$ *and edge set*

$$E(G) = \{(u_1, v_1)(u_2, v_2) : u_1 = u_2 \text{ and } v_1v_2 \in E_2 \text{ or } v_1 = v_2 \text{ and } u_1u_2 \in E_1\}.$$

It is clear that $\dim \mathcal{C}(G_1 \times G_2) = |V(G_1)||E(G_2)| + |V(G_2)||E(G_1)| - |V(G_1)||V(G_2)| + 1$.

The following is Theorem 4.1 in Ali and Marougi [3] in which he found an upper bound of the basis number of the Cartesian product of two disjoint connected graphs.

Theorem 1.1. *If G and H are connected disjoint graphs, then*

$$\mathrm{b}(G \times H) \leq \max\{\mathrm{b}(G) + \triangle(T_H),\ \mathrm{b}(H) + \triangle(T_G)\}$$

where T_H and T_G are spanning trees of H and G, respectively, such that the maximum degrees $\triangle(T_H)$ and $\triangle(T_G)$ are minimum with respect to all spanning trees of H and G. Also, they cited a reference in [4] where they have proved the following result.

Theorem 1.2 *$G \times H$ is nonplanar if G and H are any graphs with $\triangle(G) \geq 2$ and $\triangle(H) \geq 3$.*

The following Lemma of *Jaradat-Alzoubi-Rawashdeh* [15] will be used frequently in our proofs.

Lemma 1.1. *Let S_1, S_2 be sets of cycles of a graph G, and suppose that both $S_1$and S_2 are linearly independent, and $E(S_1) \cap E(S_2)$ induces a forest in G (with the possibility that $E(S_1) \cap E(S_2) = \phi$). Then $S_1 \cup S_2$ is linearly independent.*

The main purpose of this paper is to prove that the basis number of the Cartesian product of paths, cycles and theta graphs with Tenets is exactly 3. However, if we apply the above Theorem of Ali and Marougi, Theorem1.1, we find that the basis number of these graphs is less than or equal to 4.

§2. The Main Results

In this section, $P_n = 12\cdots n$ is a path on n vertices and $n-1$ edges and $C_n = 12\cdots n1$ is a cycle on n vertices and n edges. The theta graph θ_n is a graph obtained from the graph of C_n by adding a new edge that joins two nonadjacent vertices of C_n. We consider the tenet graph, T_{2m+1}; $m \geq 3$, as a graph consisting of a center vertex a and two concentric m-cycles C_u and C_v in addition to m paths of the form au_iv_i for each $i = 1, 2, \ldots, m$; where $C_u = u_1u_2\ldots u_mu_1$ is the inner cycle and $C_v = v_1v_2\ldots v_mv_1$ is the outer cycle. Our object is to prove that the basis number of the graphs $P_n \times T_{2m+1}$, $C_n \times T_{2m+1}$ and $\theta_n \times T_{2m+1}$ is 3.

Since $|V(P_n \times T_{2m+1})| = n(2m+1)$ and $|E(P_n \times T_{2m+1})| = 6nm + n - 2m - 1$, we have $\dim\mathcal{C}(P_n \times T_{2m+1}) = 4nm - 2m$, where $\mathcal{C}(P_n \times T_{2m+1})$ is the cycle space of the graph $P_n \times T_{2m+1}$.

Theorem 2.1 *For each $n \geq 2$ and $m \geq 3$, we have* $\mathrm{b}(P_n \times T_{2m+1}) = 3$.

Proof. It is clear that $\triangle(P_n) \geq 2$, if $n \geq 3$, *and* $\triangle(T_{2m+1}) \geq 3$, if $m \geq 3$, so by Theorem 1.2 the graph of $P_n \times T_{2m+1}$ is nonplanar. For the case when $n = 2$, it is easy to see that the graph $P_2 \times T_{2m+1}$ is nonplanar. Then by MacLane's Theorem we have $\mathrm{b}(P_n \times T_{2m+1}) \geq 3$, for all the graphs with $n \geq 2$ *and* $m \geq 3$.

To prove that $\mathrm{b}(P_n \times T_{2m+1}) \leq 3$, it is sufficient to exhibit a 3-fold basis for the cycle space of the graph $P_n \times T_{2m+1}$, which will be denoted by $\mathcal{B}(P_n \times T_{2m+1})$.

For each $i = 1, 2, \ldots, n$, let $T^{(i)} = \{i\} \times T_{2m+1}$, then it is clear that $T^{(i)}$ is a copy of the planar graph T_{2m+1}. Define $\mathcal{B}^{(i)}$ to be the basis of $\mathcal{C}\left(T^{(i)}\right)$ that consists of all the boundaries of the finite faces of the planar graph $T^{(i)}$, then $\mathcal{B}(T) = \bigcup_{i=1}^{n} \mathcal{B}^{(i)}$ is linearly independent subset of $\mathcal{C}(P_n \times T_{2m+1})$ because $E\left(\mathcal{B}^{(i)}\right) \cap E\left(\mathcal{B}^{(j)}\right) = \emptyset$ for all $i \neq j$ with $1 \leq i, j \leq n$.

If i is odd and $1 \leq i \leq n-1$, then define $\mathcal{B}_{io} = \mathcal{B}_{io}^{v} \cup \mathcal{B}_{io}^{u}$ where

$$\mathcal{B}_{io}^{v} = \{(i, v_j)(i, v_{j+1})(i+1, v_{j+1})(i+1, v_j)(i, v_j) : 1 \leq j \leq m-1\}$$

and

$$\mathcal{B}_{io}^{u} = \{(i, u_j)(i, v_j)(i+1, v_j)(i+1, u_j)(i, u_j) : 1 \leq j \leq m\} \cup \{C_a\},$$

where $C_a = (i, a)(i, u_1)(i+1, u_1)(i+1, a)(i, a)$ is a 4-cycle.

If i is even and $1 \leq i \leq n-1$, then define $\mathcal{B}_{ie} = \mathcal{B}_{ie}^{v} \cup \mathcal{B}_{ie}^{u}$

$$\mathcal{B}_{ie}^{v} = \{(i, v_j)(i, v_{j+1})(i+1, v_{j+1})(i+1, v_j)(i, v_j) : 1 \leq j \leq m-1\} \cup \{C_m\},$$

where $C_m = (i, v_1)(i, v_m)(i+1, v_m)(i+1, v_1)(i, v_1)$ is a 4-cycle, and

$$\begin{aligned}\mathcal{B}_{ie}^{u} = \{&(i, u_j)(i, u_{j+1})(i+1, u_{j+1})(i+1, u_j)(i, u_j) : 1 \leq j \leq m-1\} \\ &\cup \{(i, a)(i, u_m)(i+1, u_m)(i+1, a)(i, a)\}.\end{aligned}$$

For each odd i with $1 \leq i \leq n-1$, the set $\mathcal{B}_{io}^{v}$ is linearly independent because it is the set of all the boundaries of the finite faces of the planar subgraph $i\,(i+1) \times P_v$ which is a basis of the subspace $\mathcal{C}\,(i\,(i+1) \times P_v)$, where P_v is the path $v_1 v_2 \ldots v_m$. The cycles of the set $\mathcal{B}_{io}^{u} \backslash \{C_a\}$ are edge disjoint, so they are linearly independent and C_a is linearly independent with them because it contains the edge $(i+1,a)\,(i,a)$ which doesn't occur in any cycle of $\mathcal{B}_{io}^{u} \backslash \{C_a\}$, so $\mathcal{B}_{io}^{u}$ is linearly independent. Moreover, $E\,(\mathcal{B}_{io}^{v}) \cap E\,(\mathcal{B}_{io}^{u})$ is a forest of paths of order 2, thus by Lemma 1.1 we conclude that $\mathcal{B}_{io}$ is linearly independent set of cycles in $\mathcal{C}\,(P_n \times T_{2m+1})$ for each odd i where $1 \leq i \leq n-1$.

For each even i with $1 \leq i \leq n-1$, it is clear that $\mathcal{B}_{ie}^{v}$ consists of all the boundaries of the finite faces of the planar graph $i\,(i+1) \times P_v$, which form a basis of $\mathcal{C}\,(i\,(i+1) \times P_v)$, in addition to the cycle C_m. The cycle C_m contains the edge $(i, v_1)\,(i, v_m)$ which doesn't occur in any cycle of $\mathcal{B}_{ie}^{v} \backslash \{C_m\}$, so it is linearly independent with the cycles of $\mathcal{B}_{ie}^{v} \backslash \{C_m\}$, then $\mathcal{B}_{ie}^{v}$ is linearly independent. Also, $\mathcal{B}_{ie}^{u}$ is linearly independent set of cycles because its cycles are the boundaries of all the finite faces of the planar graph $i\,(i+1) \times P_{au}$ where P_{au} is the path $u_1 u_2 \ldots u_m a$. It is clear that $E\,(\mathcal{B}_{ie}^{v}) \cap E\,(\mathcal{B}_{ie}^{u}) = \emptyset$, then $\mathcal{B}_{ie} = \mathcal{B}_{ie}^{v} \cup \mathcal{B}_{ie}^{u}$ is linearly independent set of cycles in $\mathcal{C}\,(P_n \times T_{2m+1})$ for each even i where $1 \leq i \leq n-1$.

Define $\mathcal{B}\,(V) = \bigcup\limits_{i=1}^{n-1} \mathcal{B}_i$, where $\mathcal{B}_i = \mathcal{B}_{io}$ if i is odd and $1 \leq i \leq n-1$, and $\mathcal{B}_i = \mathcal{B}_{ie}$ if i is even and $1 \leq i \leq n-1$, then to prove that $\mathcal{B}\,(V)$ is linearly independent we can use induction on n; if $n = 2$, then $\mathcal{B}\,(V) = \mathcal{B}_1$ is linearly independent. Suppose that for $n = k$ the set $\bigcup\limits_{i=1}^{k-1} \mathcal{B}_i$ is linearly independent, then for $n = k+1$the set $\bigcup\limits_{i=1}^{k} \mathcal{B}_i$ is linearly independent because $E\left(\bigcup\limits_{i=1}^{k-1} \mathcal{B}_i\right) \cap E\,(\mathcal{B}_k)$ is a forest, this is obtained by applying Lemma 1.1. Thus $\mathcal{B}\,(V)$ is linearly independent set of cycles in $\mathcal{C}\,(P_n \times T_{2m+1})$.

Now, define $\mathcal{B}\,(P_n \times T_{2m+1}) = \mathcal{B}\,(T) \cup \mathcal{B}\,(V)$. Every linear combination of cycles from $\mathcal{B}\,(V)$ contains an edge either of the form $(i,a)\,(i+1,a)$, $(i, v_j)\,(i+1, v_{j+1})$ or $(i, u_j)\,(i+1, u_j)$ which don't occur in any cycle of $\mathcal{B}\,(T)$, and so it cannot be obtained as a linear combination of cycles from $\mathcal{B}\,(V)$, therefore $\mathcal{B}\,(T) \cup \mathcal{B}\,(V) = \mathcal{B}\,(P_n \times T_{2m+1})$ is linearly independent set of cycles in $\mathcal{C}(P_n \times T_{2m+1})$. Also, note that

$$\begin{aligned}
|\mathcal{B}(P_n \times T_{2m+1})| = |\mathcal{B}(T)| + |\mathcal{B}(V)| &= \left|\bigcup_{i=1}^{n} \mathcal{B}^{(i)}\right| + \left|\bigcup_{i=1}^{n-1} \mathcal{B}_i\right| \\
&= \sum_{n=1}^{n} \left|\mathcal{B}^{(i)}\right| + \sum_{n=1}^{n-1} |\mathcal{B}_i| = n(2m) + (n-1)(2m) \\
&= 4mn - 2m = \dim \mathcal{C}(P_n \times T_{2m+1}),
\end{aligned}$$

therefore $\mathcal{B}(P_n \times T_{2m+1})$ is a basis of the cycle space of the graph of $P_n \times T_{2m+1}$. It is easy to verify that $\mathcal{B}(P_n \times T_{2m+1})$ is a 3-fold basis of $\mathcal{C}(P_n \times T_{2m+1})$. ■

It is clear that $|V(C_n \times T_{2m+1})| = |V(C_n)||V(T_{2m+1})| = n(2m+1) = 2nm+n$ and $|E(C_n \times T_{2m+1})| = |V(C_n)||E(T_{2m+1})| + |E(C_n)||V(T_{2m+1})| = n(4m) + n(2m+1) = 6nm+n$. Then $\dim \mathcal{C}(C_n \times T_{2m+1}) = (6nm+n) - (2nm+n) + 1 = 4nm+1$.

Theorem 2.2 *For each* $n \geq 3$ *and* $m \geq 3$, *we have* $\mathrm{b}(C_n \times T_{2m+1}) = 3$.

Proof. The graph of $C_n \times T_{2m+1}$ is nonplanar because it contains the nonplanar subgraph $P_n \times T_{2m+1}$. Then by MacLane's Theorem we have $\mathrm{b}(C_n \times T_{2m+1}) \geq 3$ for *each* $n \geq 3$ *and* $m \geq 3$. On the other hand, to prove that $\mathrm{b}(C_n \times T_{2m+1}) \leq 3$, we will exhibit a 3-fold basis for $\mathcal{C}(C_n \times T_{2m+1})$. Define $\mathcal{B}(C_n \times T_{2m+1}) = \mathcal{B}(P_n \times T_{2m+1}) \cup \{C^a\} \cup \mathcal{B}^u \cup \mathcal{B}^v \cup \{C^m\}$, where $\mathcal{B}(P_n \times T_{2m+1})$ is the basis of the subspace $\mathcal{C}(P_n \times T_{2m+1})$ which was obtained in Theorem 2.1 where C^a and C^m are n-cycle and they are defined as follows:

$$\begin{aligned}
C^a &= (1,a)(2,a)\cdots(n,a)(1,a), \\
C^m &= (1,v_1)(2,v_1)\cdots(n,v_1)(1,v_1)
\end{aligned}$$

and the sets of cycles $\mathcal{B}^u$ and $\mathcal{B}^v$ are defined as follows:

$$\begin{aligned}
\mathcal{B}^u &= \{(1,u_j)(2,u_j)\cdots(n,u_j)(1,u_j) : j = 1,2,\ldots,m\}, \\
\mathcal{B}^v &= \{(n,v_j)(n,v_{j+1})(1,v_{j+1})(1,v_j)(n,v_j) : j = 1,2,\ldots,m-1\}.
\end{aligned}$$

It is clear that $\mathcal{B}^v$ contains all the boundaries of the finite faces of the planar graph $n1 \times P_v$, where P_v is the path $v_1v_2\ldots v_m$, which form a basis of the subspace $\mathcal{C}(n1 \times P_v)$. Thus $\mathcal{B}^v$ is linearly independent. Since C^m contains the edge $(1,v_1)(2,v_1)$ which doesn't occur in any cycle of $\mathcal{B}^v$, the set $\mathcal{B}^v \cup \{C^m\}$ is linearly independent because C^m cannot be obtained as a linear combination of cycles from $\mathcal{B}^v$. Note that every cycle in $\mathcal{B}^v \cup \{C^m\}$ contains one or two edges of the form $(1,v_j)(n,v_j)$ which don't occur in any cycle of $\mathcal{B}(P_n \times T_{2m+1})$, thus it is linearly independent with all the cycles of $\mathcal{B}(P_n \times T_{2m+1})$, then $\mathcal{B}(P_n \times T_{2m+1}) \cup \mathcal{B}^v \cup \{C^m\}$ is linearly independent set of cycles in $\mathcal{C}(C_n \times T_{2m+1})$. Now, every cycle in $\{C^a\} \cup \mathcal{B}^u$, say C, contains an edge of the form $(n,a)(1,a)$ or $(n,u_j)(1,u_j)$; $1 \leq j \leq m$ which doesn't occur in any other cycle of $\mathcal{B}(C_n \times T_{2m+1}) \setminus \{C\}$. Therefore $\mathcal{B}(P_n \times T_{2m+1}) \cup \mathcal{B}^v \cup \{C^m\} \cup \{C^a\} \cup \mathcal{B}^u = \mathcal{B}(C_n \times T_{2m+1})$ is linearly independent set of cycles in $\mathcal{C}(C_n \times T_{2m+1})$. Since $\mathcal{B}(C_n \times T_{2m+1})$ is linearly independent set and $|\mathcal{B}(C_n \times T_{2m+1})| = 4mn + 1 = \dim \mathcal{C}(C_n \times T_{2m+1})$, we conclude that $\mathcal{B}(C_n \times T_{2m+1})$ is a basis of $\mathcal{C}(C_n \times T_{2m+1})$. It is easy to verify that $\mathcal{B}(C_n \times T_{2m+1})$ is a 3-fold basis by following its construction. ■

In the following Theorem we consider θ_n as a graph obtained from the cycle graph C_n by adding the edge st that joins the nonadjacent vertices s and t of C_n. Without loss of generality we assume that $s < t$. Since $|V(\theta_n)| = n$ and $|E(\theta_n)| = n+1$, it is easy to find that $|V(\theta_n \times T_{2m+1})| = 2nm + n$ and $|E(\theta_n \times T_{2m+1})| = (n+1)(2m+1) + n(4mn) = 6mn + n + 2m + 1$. Thus $\dim\mathcal{C}(\theta_n \times T_{2m+1}) = 4mn + 2m + 2$.

Theorem 2.3 *For each $n \geq 4$ and $m \geq 3$, we have* $\mathrm{b}(\theta_n \times T_{2m+1}) = 3$.

Proof. The graph of $\theta_n \times T_{2m+1}$ is nonplanar because it contains the nonplanar subgraph $P_n \times T_{2m+1}$. Then applying MacLane's Theorem implies that $\mathrm{b}(\theta_n \times T_{2m+1}) \geq 3$ for all $n \geq 4$ *and* $m \geq 3$. To prove that $\mathrm{b}(\theta_n \times T_{2m+1}) \leq 3$ we must exhibit a 3-fold basis for $\mathcal{C}(\theta_n \times T_{2m+1})$. We define $\mathcal{B}(\theta_n \times T_{2m+1}) = \mathcal{B}(P_n \times T_{2m+1}) \cup \mathcal{B}^v \cup \{C^m\} \cup \{C^a\} \cup \mathcal{B}^u \cup \mathcal{B}_{st}$ where

$$\begin{aligned}\mathcal{B}_{st} = &\{(s,u_j)(s+1,u_j)\cdots(t,u_j)(s,u_j) : 1 \leq j \leq m\} \\ &\cup \{(s,v_j)(s+1,v_j)\cdots(t,v_j)(s,v_j) : 1 \leq j \leq m\} \\ &\cup \{(s,a)(s+1,a)\cdots(t,a)(s,a)\},\end{aligned}$$

and $\mathcal{B}(P_n \times T_{2m+1})$ and $\mathcal{B}^v \cup \{C^m\}$ are the same sets that were defined in Theorem 2.2 and the set of cycles $\mathcal{B}^u$ is defined by the set

$$\mathcal{B}^u = \{(1,u_j)(2,u_j)\cdots(s,u_j)(t,u_j)\cdots(n,u_j)(1,u_j) : j = 1,2,\ldots,m\},$$

and the cycle C^a is defined by

$$C^a = (1,a)(2,a)\cdots(s,a)(t,a)\cdots(n,a)(1,a).$$

It is clear that every cycle, say C, in $\mathcal{B}_{st}$ contains an edge of the form $(t,u_j)(s,u_j)$, $(t,v_j)(s,v_j)$ or $(t,a)(s,a)$ which doesn't occur in any cycle of $(\mathcal{B}(P_n \times T_{2m+1}) \cup \mathcal{B}^v \cup \{C^m\} \cup \mathcal{B}_{st}) \setminus \{C\}$, thus $\mathcal{B}(P_n \times T_{2m+1}) \cup \mathcal{B}^v \cup \{C^m\} \cup \mathcal{B}_{st}$ is linearly independent. The cycles of $\{C^a\} \cup \mathcal{B}^u$ are linearly independent with the cyles of $\mathcal{B}(P_n \times T_{2m+1}) \cup \mathcal{B}^v \cup \{C^m\} \cup \mathcal{B}_{st}$ for the same reasons which were mentioned in the proof of Theorem 2.2. Moreover, note that $|\mathcal{B}(\theta_n \times T_{2m+1})| = 4mn + 2m + 2$ $=\dim\mathcal{C}(\theta_n \times T_{2m+1})$. Therefore, $\mathcal{B}(\theta_n \times T_{2m+1})$ is a basis of $\mathcal{C}(\theta_n \times T_{2m+1})$ and it is easy to verify that this basis is a 3-fold basis for $\mathcal{C}(\theta_n \times T_{2m+1})$. ■

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