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On 2-factors in star-free graphs

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Abstract. In this paper we give a sharp minimum degree condition for a 2-connected star-free graph to have a 2-factor containing specified edges. Let G be a 2-connected $K_{1,n}$-free graph with minimum degree $n+d$ and $I \subseteq E(G)$. If one of the followings holds, then G has a 2-factor which contains every edge in I: i) $n = 3$, $d \geq 1$, $|I| \leq 2$ and $|V(G)| \geq 8$ if $|I| = 2$; ii) $n = 4$, $d \geq 1$, $|I| \leq 2$ and $|V(G)| \geq 11$ if $|I| = 2$; iii) $n \geq 5$, $d \geq 1$ and $|I| \leq 1$; iv) $n \geq 5$, $d \geq \lfloor (\sqrt{4n-3}+1)/2 \rfloor$ and $|I| \leq 2$.

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All graphs considered are only finite undirected graphs without loops and multiple edges. A graph is called $K_{1,n}$*-free* if it contains no $K_{1,n}$ as an induced subgraph. We call a spanning r-regular subgraph of a graph *an* r*-factor*.

There have been many results on the existence of 2-factors in star-free graphs. In [4], the following theorem is shown.

Theorem 1 (Ota and Tokuda [4])**.** *Let* $n \geq 3$ *be an integer and* G *be a* $K_{1,n}$*-free graph. If the minimum degree of* G *is at least* $2n-2$*, then* G *has a* 2*-factor.*

Recently in [1], the minimum degree condition in Theorem 1 was improved for 2-connected graphs.

Theorem 2 (Aldred et al. [1])**.** *Let* $n \geq 3$ *be an integer and* G *be a* 2*-connected* $K_{1,n}$*-free graph. If the minimum degree of* G *is at least* n*, then* G *has a* 2*-factor.*

The object of this paper is to prove the following theorem, which considers the existence of a 2-factor containing specified edges.

Theorem 3. *Let G be a 2-connected $K_{1,n}$-free graph with minimum degree $n+d$ and $I \subseteq E(G)$. If one of the following holds, then G has a 2-factor which contains every edge in I.*

i) $n = 3$, $d \geq 1$, $|I| \leq 2$ *and* $|V(G)| \geq 8$ *if* $|I| = 2$;

ii) $n = 4$, $d \geq 1$, $|I| \leq 2$ *and* $|V(G)| \geq 11$ *if* $|I| = 2$;

iii) $n \geq 5$, $d \geq 1$ *and* $|I| \leq 1$;

iv) $n \geq 5$, $d \geq \left\lfloor \dfrac{\sqrt{4n-3}+1}{2} \right\rfloor$ *and* $|I| \leq 2$.

Some examples which show the sharpness of this result are given later.

We must mention at this point that the following result was obtained recently [3], related also to 2-factors with given properties in claw-free graphs.

Theorem 4. *Let G be a 2-connected $K_{1,3}$-free graph with minimum degree at least 4. For every pair of edges e_1, e_2 of G the graph $G^* = G - \{e_1, e_2\}$ has a 2-factor.*

Here we prepare some terminology and notation used in this paper. Let G be a graph. For a vertex v in G, we denote by $N_G(v)$ and $d_G(v)$ the neighborhood and the degree of v, respectively. Let S and T be disjoint subsets of $V(G)$. We denote $\bigcup_{v\in T}(N_G(v) \cap S)$ by $N_S(T)$. The number of edges joining S and T is denoted by $e_G(S,T)$. We define $\mathcal{H}_G(S,T) = \{C \mid C$ is a component of $G-(S\cup T)$, $e_G(V(C),T) \equiv 1 \pmod 2\}$ and $h_G(S,T) = |\mathcal{H}_G(S,T)|$. We often identify a subgraph H of G with its vertex set $V(H)$. For example, $e_G(V(H),T)$ is often denoted by $e_G(H,T)$. Moreover, for a vertex x, we sometimes denote $\{x\}$ by x when there is no fear of confusion. We refer the reader to [2] for basic terminology and notation not defined here.

In our proof of Theorem 3, we use the following theorem, which is a special case of Tutte's f-factor Theorem [5].

Theorem 5 (Tutte [5])**.** *A graph G has a 2-factor if and only if*

$$\delta_G(S,T) = 2|S| + \sum_{x\in T}(d_{G-S}(x) - 2) - h_G(S,T) \geq 0$$

for any disjoint subsets S and T of $V(G)$.

Note that, for any disjoint subsets S and T of $V(G)$, we have

$$\delta_G(S,T) \equiv 0 \pmod 2 \qquad (1)$$

since $\sum_{x\in T} d_{G-S}(x) \equiv h_G(S,T) \pmod 2$.

Let G be a graph which has no 2-factor. If a pair of disjoint subsets (S,T) of $V(G)$ is chosen so that $|S|+|T|$ is minimum among those satisfying $\delta_G(S,T) < 0$, then we call it *a minimal barrier of* G (Note that the existence of a minimal barrier is guaranteed by Theorem 5). We also use the following lemmas in the proof of Theorem 3.

Lemma 1 (Aldred et al. [1])**.** *Let G be a graph which has no 2-factor and let (S,T) be a minimal barrier of G. Then T is independent, and $d_{G-S}(x) = |\{C \in \mathcal{H}_G(S,T) \mid e_G(x,C) > 0\}|$ for every $x \in T$.*

Lemma 2. *Let G be a graph which has no 2-factor and let (S,T) be a minimal barrier of G. Then for every $y \in S$, $d_G(y) > 2$.*

Proof. Suppose that there exists $y \in S$ such that $d_G(y) \leq 2$. Define $S' = S \setminus \{y\}$. Note that

$$\begin{aligned} 2|S'| &= 2|S| - 2, \\ h_G(S',T) &\geq h_G(S,T) - |N_G(y) \setminus (S \cup T)| \text{ and} \\ \sum_{x\in T}(d_{G-S'}(x) - 2) &= \sum_{x\in T}(d_{G-S}(x) - 2) + |N_G(y) \cap T|. \end{aligned}$$

Since $d_G(y) \leq 2$, $|N_G(y) \setminus (S \cup T)| + |N_G(y) \cap T| \leq 2$. Therefore, it follows that

$$\begin{aligned} \delta_G(S',T) &\leq \delta_G(S,T) - 2 + |N_G(y) \setminus (S \cup T)| + |N_G(y) \cap T| \\ &\leq \delta_G(S,T) < 0, \end{aligned}$$

which contradicts that (S,T) is a minimal barrier. □

Proof of Theorem 3. We prove by induction on $|I|$. If $|I| = 0$, then Theorem 2 implies the assertion in every case i) – iv). Hence we consider the case $|I| = p$, where $1 \leq p \leq 2$. By way of contradiction, suppose that G' is the graph which satisfies the assumption of Theorem 3 and none of its 2-factor contains $I \subseteq E(G')$. Let $I = \{e_i \mid 1 \leq i \leq p\}$. For a subset E' of $E(G')$, Let $G'(E')$ be the graph obtained from G' after the subdivision of each edge in E'. (If E' is an emptyset, then $G'(E') = G'$.) Especially, let $G = G'(I)$ and we denote $F = V(G) \setminus V(G') = \{v_i \mid 1 \leq i \leq p\}$, where each v_i corresponds to the original edge e_i in G'. Note that G is 2-connected.

Since there is no 2-factor of G' which contains I, G has no 2-factor. Let (S,T) be a minimal barrier of G and let $U = V(G) \setminus (S \cup T)$. Then by Lemma 1, T is an independent set in G.

Claim 1. *$F \subseteq T$.*

Proof. By Lemma 2, $F \cap S = \emptyset$. Hence it suffices to prove $F \cap U = \emptyset$. Assume the contrary, and let $v_i \in U$ for some i, $1 \le i \le p$. Let $G'' = G'(I \setminus \{e_i\})$ (Note that we can also make G'' by contracting an edge incident to v_i in G, and hence $S, T \subseteq V(G'')$). By the induction hypothesis, there is a 2-factor in G' which contains every edge of $I \setminus \{e_i\}$. Hence G'' has a 2-factor, and it follows from Theorem 5 that $\delta_{G''}(S,T) \ge 0$.

Assume $N_G(v_i) \cap T = \emptyset$, then since $\sum_{x\in T}(d_{G''-S}(x)-2) = \sum_{x\in T}(d_{G-S}(x)-2)$ and $h_{G''}(S,T) = h_G(S,T)$, it holds that $\delta_{G''}(S,T) = \delta_G(S,T) < 0$, a contradiction. Hence there exists $a_i \in N_G(v_i) \cap T$. Let b_i be another neighbor of v_i in G, and let C be the component of $G - (S \cup T)$ which contains v_i.

If $b_i \in S$, then since $C \notin \mathcal{H}_{G''}(S,T)$, it holds that $h_{G''}(S,T) = h_G(S,T) - 1$. Moreover, $\sum_{x\in T}(d_{G''-S}(x) - 2) = \sum_{x\in T}(d_{G-S}(x) - 2) - 1$. Thus it follows that $\delta_{G''}(S,T) = \delta_G(S,T) - 1 + 1 = \delta_G(S,T) < 0$, a contradiction. Next, if $b_i \in T$, then C consists of exactly one vertex v_i. Hence $e_G(C,T) = 2$. This implies that $C \notin \mathcal{H}_G(S,T)$, which contradicts Lemma 1. Finally, if $b_i \in U$, then $h_{G''}(S,T) = h_G(S,T)$ and $\sum_{x\in T}(d_{G''-S}(x) - 2) = \sum_{x\in T}(d_{G-S}(x) - 2)$. Hence $\delta_{G''}(S,T) = \delta_G(S,T) < 0$, a contradiction. □

Following the proof in [1], we now prepare some settings. Let $\mathcal{U} = \mathcal{H}_G(S,T)$, and let

$$\begin{aligned}
\mathcal{U}_1 &= \{C \in \mathcal{U} \mid e_G(T,C) = 1\},\\
\mathcal{U}_{\ge 3} &= \{C \in \mathcal{U} \mid e_G(T,C) \ge 3\},\\
U_1 &= \bigcup_{C\in\mathcal{U}_1} V(C) \text{ and}\\
U_{\ge 3} &= \bigcup_{C\in\mathcal{U}_{\ge 3}} V(C).
\end{aligned}$$

Note that, by the definition of $h_G(S,T)$, $h_G(S,T) = |\mathcal{U}_1| + |\mathcal{U}_{\ge 3}|$.

For every $C \in \mathcal{U}_1$, $N_G(T) \cap C$ consists of precisely one vertex, say w_C. Now we define

$$\begin{aligned}
\mathcal{U}_1^1 &= \{C \in \mathcal{U}_1 \mid N_G(S) \cap C = \{w_C\}\}\\
\mathcal{U}_1^2 &= \mathcal{U}_1 \setminus \mathcal{U}_1^1.
\end{aligned}$$

Then for every $C \in \mathcal{U}_1^2$, it follows that $N_G(S) \cap C \setminus \{w_C\} \ne \emptyset$. Let v_C be a vertex in $N_G(S) \cap C \setminus \{w_C\}$, and let y_C be a vertex in $N_S(v_C)$.

For every $x \in T$, we define the following sets:

$$\begin{aligned}
\mathcal{U}_1^1(x) &= \{C \in \mathcal{U}_1^1 \mid e_G(x, C) = 1\};\\
\mathcal{U}_1^2(x) &= \{C \in \mathcal{U}_1^2 \mid e_G(x, C) = 1\};\\
E_1(x) &= \{w_C y \mid C \in \mathcal{U}_1^1(x),\ y \in N_S(w_C)\};\\
E_2(x) &= \{v_C y_C \mid C \in \mathcal{U}_1^2(x)\};\\
E_3(x) &= \{xy \mid y \in S \cap N_G(x)\};\\
D_1(x) &= E_1(x) \cup E_2(x);\\
D_2(x) &= E_2(x) \cup E_3(x).
\end{aligned}$$

Note that $D_i(x) \cap D_j(x') = \emptyset$ for every $x, x' \in T$ with $x \neq x'$ and $i, j \in \{1, 2\}$.

Let Φ be the set of all mappings from $T \setminus F$ to $\{1, 2\}$, and let

$$\mathcal{D} = \left\{ \bigcup_{x \in T \setminus F} D_{\phi(x)}(x) \;\middle|\; \phi \in \Phi \right\}.$$

Moreover, let

$$D' = \bigcup_{x \in F} E_2(x).$$

Then the following claim holds.

Claim 2. $|D| + |D'| \leq (n-1)|S|$ *for every* $D \in \mathcal{D}$.

Proof. Suppose that $|D| + |D'| > (n-1)|S|$ for some $D \in \mathcal{D}$. Then there exists $y \in S$ which is incident with n edges of $D \cup D'$, say $yz_1, \ldots, yz_n$. Since $E_3(x) \cap (D \cup D') = \emptyset$ for every $x \in F$, it follows that none of z_i is a vertex of F. Therefore, $yz_1, \ldots, yz_n$ are edges in G'. Since G' is $K_{1,n}$-free, $z_i z_j \in E(G')$ for some i and j. Moreover, since $E_1(x) \cap (D \cup D') = \emptyset$ for every $x \in F$, none of z_i is adjacent to a vertex of F in G, and hence we have $z_i z_j \in E(G)$.

By the construction of D and D', $z_i, z_j \in T \cup U_1$. If both of z_i and z_j are in U_1, then they belong to distinct components of $\mathcal{U}_1$ by the definition of E_1 and E_2, and hence they cannot be adjacent in G. Thus $\{z_i, z_j\} \cap T \neq \emptyset$. Without loss of generality, we may assume $z_i \in T$. Then $z_j \in U_1$ because T is independent. Let C be the component that contains z_j, then $C \in \mathcal{U}_1^1(z_i) \cup \mathcal{U}_1^2(z_i)$. If $C \in \mathcal{U}_1^2(z_i)$, then by the construction of D and D', $z_j = v_C$. However, it follows from the definition of v_C that $z_i z_j \notin E(G)$, a contradiction. Consequently $C \in \mathcal{U}_1^1(z_i)$ and $z_j = w_C$. Now $yz_i \in E_3(z_i)$ and $yz_j \in E_1(z_j)$. However, if D contains an edge of $E_3(z_i)$, then D cannot contain any edge of $E_1(z_i)$ by the definition of D_1 and D_2. Moreover, D' cannot contain any edge of $E_1(z_i)$ or $E_3(z_i)$. This contradicts the assumption that $yz_i, yz_j \in D \cup D'$. □

Suppose that there exists $C \in \mathcal{U}_1^1$ such that $|V(C)| \geq 2$. Then for every $z \in V(C) \setminus \{w_C\}$, z is not adjacent to any vertex in T since $C \in \mathcal{U}_1$, and z is not adjacent to any vertex in S by the definition of $\mathcal{U}_1^1$. Hence $G - \{w_C\}$ is disconnected, which contradicts the assumption that G is 2-connected. Therefore, we have $|V(C)| = 1$ for every $C \in \mathcal{U}_1^1$. Since $F \in T$, every vertex in U has degree at least $n + d$ in G. Hence we have

$$e_G(S, C) \geq n + d - 1 \geq n \text{ for every } C \in \mathcal{U}_1^1. \tag{2}$$

For every $x \in T \setminus F$, it follows from Lemma 1 that

$$|\mathcal{U}_1^1(x)| + |\mathcal{U}_1^2(x)| + e_G(x, S) + e_G(x, U_{\geq 3}) = d_G(x) \geq n + d.$$

Hence,

$$|\mathcal{U}_1^1(x)| \geq 1, \text{ or} \tag{3}$$

$$|\mathcal{U}_1^2(x)| + e_G(x, S) + e_G(x, U_{\geq 3}) \geq n + d. \tag{4}$$

Let $T_1 = \{x \in T \setminus F \mid x \text{ satisfies (3)}\}$ and $T_2 = T \setminus (T_1 \cup F)$. If $x \in T_1$, we define $D(x) = D_1(x)$, and if $x \in T_2$, we define $D(x) = D_2(x)$. Then if $x \in T_1$, it follows from (2) that

$$\begin{aligned} |D(x)| + e_G(x, U_{\geq 3}) &= |E_1(x)| + |E_2(x)| + e_G(x, U_{\geq 3}) \\ &\geq n|\mathcal{U}_1^1(x)| + |\mathcal{U}_1^2(x)| + e_G(x, U_{\geq 3}) \quad (5) \\ &\geq n + |\mathcal{U}_1^2(x)| + e_G(x, U_{\geq 3}), \quad (6) \end{aligned}$$

and if $x \in T_2$ it follows from (4) that

$$\begin{aligned} |D(x)| + e_G(x, U_{\geq 3}) &= |E_2(x)| + |E_3(x)| + e_G(x, U_{\geq 3}) \\ &= |\mathcal{U}_1^2(x)| + e_G(x, S) + e_G(x, U_{\geq 3}) \\ &\geq n + d \ \geq \ n + 1. \quad (7) \end{aligned}$$

Moreover, let $T_{\geq 3} = \{x \in T \setminus F \mid e_G(x, U_{\geq 3}) \geq 1\}$. Then, by the above inequalities,

$$|D(x)| + e_G(x, U_{\geq 3}) \geq n + 1 \text{ holds for every } x \in T_{\geq 3}. \tag{8}$$

Let $D = \bigcup_{x \in T \setminus F} D(x)$. It follows from (6), (7) and (8) that

$$
\begin{aligned}
|D| + e_G(T, U_{\geq 3}) &= |D| + e_G(T \setminus F, U_{\geq 3}) + e_G(F, U_{\geq 3}) \\
&= \sum_{x \in T \setminus F} (|D(x)| + e_G(x, U_{\geq 3})) + e_G(F, U_{\geq 3}) \\
&= \sum_{x \in T_{\geq 3}} (|D(x)| + e_G(x, U_{\geq 3})) \\
&\quad + \sum_{x \in (T \setminus F) \setminus T_{\geq 3}} (|D(x)| + e_G(x, U_{\geq 3})) + e_G(F, U_{\geq 3}) \\
&\geq (n+1)|T_{\geq 3}| + n|(T \setminus F) \setminus T_{\geq 3}| + e_G(F, U_{\geq 3}) \\
&= n|T \setminus F| + |T_{\geq 3}| + e_G(F, U_{\geq 3}) \\
&\geq n(|T| - 2) + |T_{\geq 3}| + e_G(F, U_{\geq 3}). \qquad (9)
\end{aligned}
$$

On the other hand, by the definition we have $e_G(T, U_{\geq 3}) \geq 3|\mathcal{U}_{\geq 3}|$ and $e_G(T, U_1) = |\mathcal{U}_1|$. Hence $h_G(S, T) = |\mathcal{U}_1| + |\mathcal{U}_{\geq 3}| \leq e_G(T, U_1) + \frac{1}{3} e_G(T, U_{\geq 3})$. Therefore, it follows from (1) and Lemma 1 that

$$
\begin{aligned}
-2 &\geq \delta_G(S, T) \\
&= 2|S| + \sum_{x \in T} (d_{G-S}(x) - 2) - h_G(S, T) \\
&= 2|S| - 2|T| + \sum_{x \in T} d_{G-S}(x) - h_G(S, T) \\
&= 2|S| - 2|T| + e_G(T, U_1) + e_G(T, U_{\geq 3}) - h_G(S, T) \\
&\geq 2|S| - 2|T| + \frac{2}{3} e_G(T, U_{\geq 3}),
\end{aligned}
$$

which implies

$$
n|T| \geq n|S| + \frac{n}{3} e_G(T, U_{\geq 3}) + n. \qquad (10)
$$

Moreover, Claim 2 yields

$$
(n-1)|S| \geq |D| + |D'|. \qquad (11)
$$

Taking sum of (9), (10) and (11), we have

$$
0 \geq |S| + |T_{\geq 3}| + e_G(F, U_{\geq 3}) + |D'| + \frac{n-3}{3} e_G(T, U_{\geq 3}) - n. \qquad (12)
$$

Claim 3. $|S| \geq 1$.

Proof. Assume $|S| = 0$. Then $\mathcal{U}_1 = \emptyset$, because G is 2-connected. Since T is independent, $e_G(F, \mathcal{U}_{\geq 3}) = 2|F|$, and hence $\mathcal{U}_{\geq 3} \neq \emptyset$. Let $C \in \mathcal{U}_{\geq 3}$. Then, since $e_G(T, C) \geq 3$ and $|F| \leq 2$, it follows from Lemma 1 that there exists a vertex $x \in T \setminus F$.

Considering the neighbor of x, we obtain $|\mathcal{U}_{\geq 3}| \geq n+1$ from Lemma 1. This yields $e_G(T, U_{\geq 3}) \geq 3(n+1)$. Therefore, if $n \geq 4$, $\frac{n-3}{3} e_G(T, U_{\geq 3}) \geq (n-3)(n+1) \geq n+1 > n$, which contradicts (12). On the other hand, if $n = 3$, it holds from Lemma 1 that $e_G(T, U_{\geq 3}) - e_G(F \cup \{x\}, U_{\geq 3}) \geq 3|\mathcal{U}_{\geq 3}| - (4 + |\mathcal{U}_{\geq 3}|) = 2|\mathcal{U}_{\geq 3}| - 4 \geq 2(n+1) - 4 > 0$. Hence there exists $x' \in T \setminus (F \cup \{x\})$. This implies $|T_{\geq 3}| \geq 2$, and hence $|T_{\geq 3}| + e_G(F, U_{\geq 3}) \geq 2+2 > n$, which contradicts (12). □

If $\mathcal{U}_{\geq 3} \neq \emptyset$, then it holds that $\frac{n-3}{3} e_G(T, U_{\geq 3}) \geq n-3$, and Lemma 1 implies that $|T_{\geq 3}| + e_G(F, U_{\geq 3}) \geq 3$. Since $|S| \geq 1$, this contradicts (12). Therefore we have $\mathcal{U}_{\geq 3} = \emptyset$, which implies $|T_{\geq 3}| = e_G(F, U_{\geq 3}) = e_G(T, U_{\geq 3}) = 0$. It follows from (9) and (11) that $(n-1)|S| \geq n|T| - 2n$, which implies $2n - |T| \geq (n-1)(|T| - |S|)$. If $|T| - |S| \geq 2$, then $2n - |T| \leq 2n - (|S| + 2) \leq 2n - 3$ and $(n-1)(|T| - |S|) \geq 2n - 2$, a contradiction. Thus $|T| - |S| \leq 1$. Now $|T| \geq |S| + 1$ holds by (10), and hence $|T| = |S| + 1$.

Assume $|F| = 1$, then it follows from (6) and (7) that $|D| \geq |T \setminus F| \cdot n = (|T| - 1)n = n|S| > (n-1)|S|$, which contradicts Claim 2. This implies $|F| = 2$, and hence we may assume that $|I| = 2$ and G' satisfies i), ii) or iv).

Claim 4. $|S| \geq 2$.

Proof. Assume $|S| = 1$. Since G is 2-connected, $G - S$ is connected. Then, since T is independent and $|T| = 2$, there exists a component C of $G - (S \cup T)$ such that $e_G(C, T) \geq 2$. The fact that $\mathcal{U}_{\geq 3} = \emptyset$ yields $C \notin \mathcal{H}_G(S, T)$, which contradicts Lemma 1. □

Now we divide the rest of the proof.

Case 1. G' satisfies i) or ii).

Note that (12) implies $|S| \leq n$. We consider the following cases with regard to $|S|$ and n.

Case 1a. $|S| = 2$.

Assume that there exists $C \in \mathcal{U}_1^1(x)$ for some $x \in T$. Then (2) implies that $|N_G(z) \cap S| \geq n \geq 3$, where z is the only vertex in C. This contradicts the assumption of this case. Hence we have $\mathcal{U}_1^1 = \emptyset$, and so $T_1 = \emptyset$. Since $\mathcal{U}_{\geq 3} = \emptyset$, it follows from (4) that $|\mathcal{U}_1^2(x)| + e_G(x, S) \geq n+1$ holds for every $x \in T \setminus F$, and hence

$$|D| \geq (n+1)|T \setminus F| = (n+1)(|S| + 1 - |F|) = n+1. \tag{13}$$

Now assume that $e_G(F, U_1) = 0$. Then $N_G(v_i) \subseteq S$ for $i = 1, 2$, because $\mathcal{U}_{\geq 3} = \emptyset$. However, since $|S| = 2$, this contradicts the construction of G from G'. Therefore, $e_G(F, U_1) \geq 1$. Since $\mathcal{U}_1^1 = \emptyset$, there exists $C \in \mathcal{U}_1^2(v_i)$ for some $v_i \in F$, and hence $|D'| \geq 1$. Without loss of generality, we may assume that $C \in \mathcal{U}_1^2(v_1)$.

Case 1a-i). G' satisfies i).

In this case $|D| + |D'| \geq (n+1) + 1 = 5 > (n-1)|S|$, which contradicts Claim 2.

Case 1a-ii). G' satisfies ii).

In this case, by (13), $|D| \geq n + 1 = 5 = (n-1)|S| - 1$. Since $|D'| \geq 1$, Claim 2 yields $|D| = 5$, and hence $|D'| = 1$. Since $\mathcal{U}_1^1 = \mathcal{U}_{\geq 3} = \emptyset$ and $|\mathcal{U}_1^2(v_1)| \leq |D'| = 1$, we obtain $e_G(v_1, S) = 1$. Let $S = \{y_1, y_2\}$, where $v_1 y_1 \in E(G)$. Since G is 2-connected, $G - \{y_1\}$ is connected. By the fact that $N_G(v_1) = \{y_1, w_C\}$ and $N_G(V(C)) \cap T = \{v_1\}$, it follows that y_2 is adjacent to some vertex z in $V(C)$.

Now we have $|D| = 5$ and $|S| = 2$, and hence some vertex $y_i \in S$ is adjacent to at least 3 edges of D, say $y_i z_1, y_i z_2$ and $y_i z_3$. If $i = 1$, then let $z_4 = w_C$, and if $i = 2$, then let $z_4 = z$. Then in either case it follows that $y_i z_j \in E(G')$ for every j, $1 \leq j \leq 4$. Moreover, since $|N_G(v_j) \cap \{z_1, z_2, z_3, z_4\}| \leq 1$ for $j = 1$ and 2, $\{z_1, z_2, z_3, z_4\}$ is an independent set in G'. This contradicts that G' is $K_{1,4}$-free.

Case 1b. $|S| = 3$.

Case 1b-i). G' satisfies i).

In this case, by Claim 2, $|D| \leq 2|S| = 6$. Since $e_G(x, U_{\geq 3}) = 0$ for every $x \in T$, it follows from (6) and (7) that $D(x) \geq 3$ for every $x \in T \setminus F$. Moreover, since $|T \setminus F| = |T| - 2 = |S| + 1 - 2 = 2$, $D(x) = 3$ holds for every $x \in T \setminus F$. Now it follows from (7) that $T \setminus F = T_1$, and hence by (5), $|\mathcal{U}_1^1(x)| = 1$ and $|\mathcal{U}_1^2(x)| = 0$ holds for every $x \in T \setminus F$.

Let x_1 and x_2 be the two vertices of $T \setminus F$ and let $\mathcal{U}_1^1(x_i) = \{C_i\}$ for $i = 1, 2$. We already know that each C_i consists of precisely one vertex, say z_i. Since $|S| = 3$, it follows from (2) that every vertex in S is adjacent to both z_1 and z_2.

Assume that there exists a component C of $G - (S \cup T)$ which is not C_1 or C_2. Since $\mathcal{U}_{\geq 3} = \emptyset$, Lemma 1 implies that $e_G(T, C) \leq 1$. Since G is 2-connected, there exists $y \in S$ and $z \in V(C)$ such that $yz \in E(G)$. Now $yz_1, yz_2, yz \in E(G')$ because $\{z_1, z_2, z\} \cap F = \emptyset$. Moreover, since C_1, $C_2 \in \mathcal{U}_1^1(x_1) \cup \mathcal{U}_1^1(x_2)$, neither z_1 nor z_2 can be adjacent to a vertex in F. Hence $zz_1, zz_2, z_1 z_2 \notin E(G')$, which contradicts that G' is $K_{1,3}$-free.

Therefore C_1 and C_2 are the only components in $G-(S\cup T)$. Now we have $V(G')=S\cup(T\setminus F)\cup\{z_1,z_2\}$, $|S|=3$ and $|T\setminus F|=2$. Hence $|V(G')|=|S|+|T\setminus F|+|\{z_1,z_2\}|=7$, a contradiction.

Case 1b-ii). G' satisfies ii).

Assume that there exists $C\in\mathcal{U}_1^1(x)$ for some $x\in T$. Then (2) implies that $|N_G(z)\cap S|\geq 4$, where z is the only vertex in C. This contradicts the assumption of this case. Therefore, $\mathcal{U}_1^1=\emptyset$, and hence $T_1=\emptyset$. Since $e_G(x,U_{\geq 3})=0$ for every $x\in T$, it follows from (7) that $D(x)\geq 5$ for every $x\in T\setminus F$. Moreover, since $|T\setminus F|=|T|-2=|S|+1-2=2$, $|D|+|D'|\geq 5|T\setminus F|\geq 10>(n-1)|S|$, which contradicts Claim 2.

Case 1c. $|S|=4$.

Note that this case occurs only when G' satisfies ii), because $|S|\leq n$. Now by Claim 2, $|D|\leq 3|S|=12$. Since $e_G(x,U_{\geq 3})=0$ for every $x\in T$, it follows from (6) and (7) that $D(x)\geq 4$ for every $x\in T\setminus F$. Moreover, since $|T\setminus F|=|T|-2=|S|+1-2=3$, $D(x)=4$ holds for every $x\in T\setminus F$. Now it follows from (7) that $T\setminus F=T_1$, and hence by (6), $|\mathcal{U}_1^1(x)|=1$ and $|\mathcal{U}_1^2(x)|=0$ holds for every $x\in T\setminus F$.

Let x_1,x_2 and x_3 be three vertices of $T\setminus F$ and let $\mathcal{U}_1^1(x_i)=\{C_i\}$ for $i=1,2,3$. We already know that each C_i consists of precisely one vertex, say z_i. Since $|S|=4$, it follows from (2) that every vertex in S is adjacent to all of z_1,z_2 and z_3.

Assume that there exists a component C of $G-(S\cup T)$ which is not C_1,C_2 or C_3. Then by Lemma 1 and the fact that $\mathcal{U}_{\geq 3}=\emptyset$, $e_G(T,C)\leq 1$ holds. Since G is 2-connected, there exists $y\in S$ and $z\in V(C)$ such that $yz\in E(G)$. Now $yz_1,yz_2,yz_3,yz\in E(G')$ because $\{z_1,z_2,z_3,z\}\cap F=\emptyset$. Moreover, since C_1,C_2 and $C_3\in\mathcal{U}_1^1$, none of z_1,z_2 and z_3 can be adjacent to a vertex in F. Hence $\{z,z_1,z_2,z_3\}$ is an independent set in G', which contradicts that G' is $K_{1,4}$-free.

Therefore C_1,C_2 and C_3 are the only components in $G-(S\cup T)$. Now we have $V(G')=S\cup(T\setminus F)\cup\{z_1,z_2,z_3\}$, $|S|=4$ and $|T\setminus F|=3$. Hence $|V(G')|=|S|+|T\setminus F|+|\{z_1,z_2,z_3\}|=10$, a contradiction. This completes the proof of Case 1.

Case 2. G' satisfies iv).

Since $|T|=|S|+1\geq 3$, there exists a vertex in $T\setminus F$. Let x be a vertex in $T\setminus F$ and let $N_G(x)\cap U=\{z_1,z_2,\ldots,z_l\}$. Since T is independent, $\{z_1,z_2,\ldots,z_l\}\cap F=\emptyset$, and hence $xz_1,xz_2,\ldots xz_l\in E(G')$. Recall $\mathcal{U}_{\geq 3}=\emptyset$. It follows from Lemma 1 that $z_1,z_2,\ldots,z_l$ belong to distinct components in $\mathcal{U}_1$. Thus $z_iz_j\notin E(G)$ for every i and j. Now it is clear that $\{z_1,z_2,\ldots z_l\}\cap F=\emptyset$, and $\{z_1,z_2,\ldots z_l\}\cap N_G(F)=\emptyset$ since every z_i belongs to a component in $\mathcal{U}_1$.

Therefore, $\{z_1, z_2, \dots z_l\}$ is an independent set in G', and thus $\{x, z_1, z_2, \dots, z_l\}$ induces a star of size $l = |N_G(x) \cap U|$ in G'. Since G' is $K_{1,n}$-free, it follows that $|N_G(x) \cap U| \le n-1$ for every $x \in T \setminus F$. Therefore, we obtain

$$|S| \ge d_G(x) - |N_G(x) \cap U| \ge n + d - (n-1) \ge d+1. \tag{14}$$

Note that $d \ge 2$ holds because $n \ge 5$. Assume that there exists $C \in \mathcal{U}_1^1(x)$ for some $x \in T$. Then (2) implies that $|N_G(z) \cap S| \ge n+d-1 \ge n+1$, where z is the only vertex in C. However (12) yields $|S| \le n$, a contradiction. Thus we have $\mathcal{U}_1^1 = \emptyset$, and so $T_1 = \emptyset$. Therefore, it follows from (7) and (14) that

$$\begin{aligned}
|D| &\ge (n+d)(|T \setminus F|) \\
&= (n+d)(|T| - 2) \\
&= (n+d)(|S| - 1) \\
&= (n-1)(|S|-1) + (d+1)(|S|-1) \\
&= (n-1)|S| - (n-1) + (d+1)(|S|-1) \\
&\ge (n-1)|S| - (n-1) + (d+1)d \\
&\ge (n-1)|S| - (n-1) + \left(\left\lfloor \frac{\sqrt{4n-3}+1}{2} \right\rfloor + 1 \right) \left\lfloor \frac{\sqrt{4n-3}+1}{2} \right\rfloor \\
&= (n-1)|S| - (n-1) + \left(\left\lfloor \frac{\sqrt{4n-3}+1}{2} \right\rfloor + 1 \right) \left(\left\lfloor \frac{\sqrt{4n-3}-1}{2} \right\rfloor + 1 \right) \\
&> (n-1)|S| - (n-1) + \frac{\sqrt{4n-3}+1}{2} \cdot \frac{\sqrt{4n-3}-1}{2} \\
&= (n-1)|S| - (n-1) + \frac{4n-3-1}{4} \\
&= (n-1)|S|,
\end{aligned}$$

which contradicts Claim 2. This completes the proof of Theorem 3. □

Here we give some examples which show that Theorem 3 is in some sense best possible.

Example I. We will show that Theorem 3 doesn't hold for the graphs with connectivity 1. Let H_1, H_2 be complete graphs of order r, where $r \ge n+d+1$. We construct a graph G_1 by adding an edge e_1 which joins H_1 and H_2. Clearly G_1 is a connected $K_{1,n}$-free graph and its minimum degree is at least $n+d$. However, there is no 2-factor of G containing the edge e_1. Moreover, if we add two more edges e_2 and e_3 which join H_1 and H_2, we will construct a graph G_1' which is 2-connected, and it doesn't have a 2-factor containing all of e_1, e_2 and e_3. This example shows that Theorem 3 doesn't hold for $|I| \ge 3$.

Example II. We will show that Theorem 3 doesn't hold for the graphs with minimum degree n. Let $n \geq 3$ and $r \geq n$. Let G_2 be the graph such that

$$\begin{aligned} V(G_2) &= \{x_i, y_i, z_i \mid 1 \leq i \leq r\} \text{ and} \\ E(G_2) &= \{y_i x_j, y_i z_j \mid j - i \in \{0, 1, \dots n-2\} \pmod r\} \\ &\quad \cup \ \{x_i z_i \mid 1 \leq i \leq r\} \cup \{y_1 y_2\}. \end{aligned}$$

Then G_2 is an $(n-1)$-connected $K_{1,n}$-free graph with minimum degree n. We construct the graph G by subdividing $e_1 = y_1 y_2$ as in the proof of Theorem 3, and let $S = \{y_i \mid 1 \leq i \leq r\}$ and $T = \{x_i \mid 1 \leq i \leq r\} \cup \{v_1\}$, then we have $\delta_G(S, T) = -2$. Hence G has no 2-factor, which implies that G_2 has no 2-factor containing e_1.

Example III. We will show that, in case of $n = 3$ or 4 and $|I| = 2$, Theorem 3 doesn't hold for the graphs with $3n - 2$ vertices whose minimum degree is $n + 1$. Let $S' = \{y_1, \dots, y_n\}$, $T' = \{x_1, \dots, x_{n-1}\}$ and $U' = \{z_1, \dots, z_{n-1}\}$. Let G_3 be the graph such that

$$\begin{aligned} &V(G_3) = S' \cup T' \cup U' \text{ and} \\ &E(G_3) = \{uv \mid u, v \in S'\} \cup \{uv \mid u \in S', v \in T' \cup U'\} \cup \{x_i z_i \mid 1 \leq i \leq n-1\}. \end{aligned}$$

Then G_3 is a 2-connected $K_{1,n}$-free graph with minimum degree n. We construct the graph G by subdividing $e_1 = y_1 y_2$ and $e_2 = y_2 y_3$ as in the proof of Theorem 3, and let $S = S'$ and $T = T' \cup \{v_1, v_2\}$, then we have

$$\begin{aligned} \delta_G(S, T) &= 2|S| + \sum_{x \in T} (d_{G-S}(x) - 2) - h_G(S, T) \\ &= 2|S| - 2|T| + \sum_{x \in T} d_{G-S}(x) - h_G(S, T) \\ &= 2n - 2(n - 1 + 2) + (n - 1) - (n - 1) \\ &= -2. \end{aligned}$$

Hence G has no 2-factor, which implies that G_3 has no 2-factor containing e_1 and e_2.

Example IV. Let

$$s_0 = \left\lfloor \frac{\sqrt{4n - 3} + 1}{2} \right\rfloor - 1.$$

We will show that Theorem 3 doesn't hold for the graphs with minimum degree $n + s_0$, in case of $n \geq 7$. Let $\mathcal{C} = \{C_i^j \mid 1 \leq i \leq s_0,\ 1 \leq j \leq n-1\}$ be a set of sufficiently large complete graphs. From each C_i^j, we choose one vertex w_i^j.

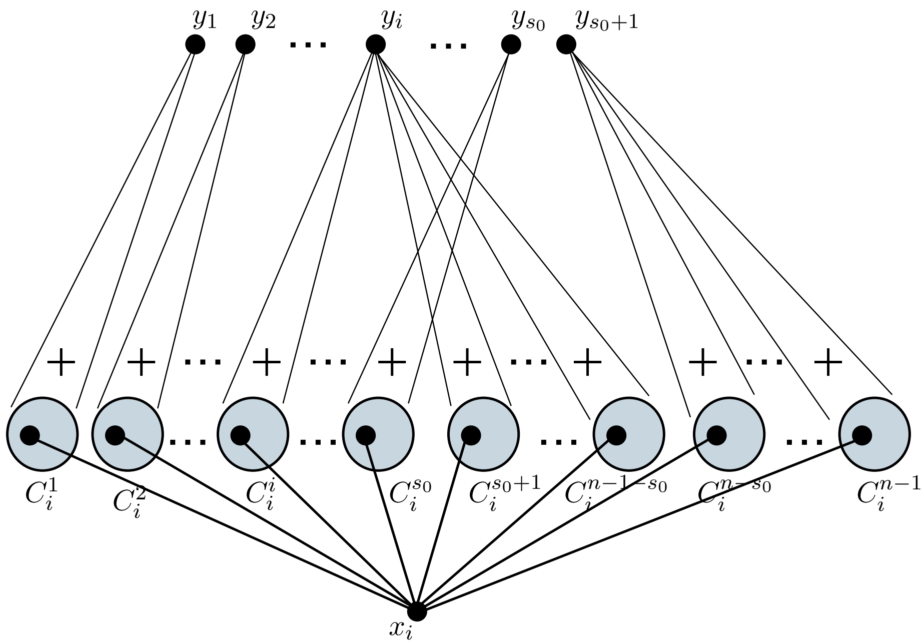

Figure 1: Edges around $C_i^1, \cdots, C_i^{n-1}$.

Let G_4 be the graph such that

$$V(G_4) = \{y_i \mid 1 \le i \le s_0 + 1\} \cup \{x_i \mid 1 \le i \le s_0\} \cup \left(\bigcup_{C \in \mathcal{C}} V(C) \right) \text{ and}$$

$$\begin{aligned} E(G_4) = &\{y_i y_j \mid 1 \le i < j \le s_0 + 1\} \cup \{y_i x_j \mid 1 \le i \le s_0 + 1,\ 1 \le j \le s_0\} \\ &\cup\ \{x_i w_i^j \mid 1 \le i \le s_0,\ 1 \le j \le n-1\} \cup \left(\bigcup_{C \in \mathcal{C}} E(C) \right) \\ &\cup\ \left\{ z_i^j y_j \mid z_i^j \in V(C_i^j),\ 1 \le i, j \le s_0 \right\} \\ &\cup\ \left\{ z_i^j y_i \mid z_i^j \in V(C_i^j),\ 1 \le i \le s_0,\ s_0 + 1 \le j \le n - 1 - s_0 \right\} \\ &\cup\ \left\{ z_i^j y_{s_0+1} \mid z_i \in V(C_i^j),\ 1 \le i \le s_0,\ n - s_0 \le j \le n - 1 \right\}. \end{aligned}$$

(See Figure 1.) Note that $n \ge 2s_0 + 2$ holds when $n \ge 7$. This implies that $s_0 + 1 \le n - 1 - s_0$.

Let $S = \{y_i \mid 1 \le i \le s_0 + 1\}$ and $T = \{x_i \mid 1 \le i \le s_0\}$. Then, the neighborhood of y_i ($1 \le i \le s_0$) is contained in $S \cup T$ and $n - s_0 - 1$

components which are elements of C. Since $xy \in E(G_4)$ for every $x \in T$ and $y \in S$, the maximum number of the size of an independent set in $N_{G_4}(y_i)$ is $n - s_0 - 1 + |T| = n - 1$. Therefore, there is no induced $K_{1,n}$ with center y_i for $1 \le i \le s_0$.

The neighborhood of y_{s_0+1} is contained in $S \cup T$ and ${s_0}^2$ components which are elements of C. Hence the maximum number of the size of an independent set in $N_{G_4}(y_{s_0+1})$ is

$$\begin{aligned} {s_0}^2 + |T| = s_0(s_0+1) = \left(\left\lfloor \frac{\sqrt{4n-3}+1}{2} \right\rfloor - 1 \right) \left\lfloor \frac{\sqrt{4n-3}+1}{2} \right\rfloor \\ \le \frac{\sqrt{4n-3}-1}{2} \cdot \frac{\sqrt{4n-3}+1}{2} = n-1, \end{aligned}$$

and so there is no induced $K_{1,n}$ with center y_{s_0+1}.

For any i with $1 \le i \le s_0$, $N_{G_4}(x_i) = S \cup \{w_i^j \mid 1 \le j \le n-1\}$. Since every $y \in S$ has a neighbor in $\{w_i^j \mid 1 \le j \le n-1\}$, the maximum number of the size of an independent set in $N_{G_4}(x_i)$ is $|\{w_i^j \mid 1 \le j \le n-1\}| = n-1$. Therefore, there is no induced $K_{1,n}$ with center x_i for $1 \le i \le s_0$.

By the above observation, it follows that G_4 is a 2-connected $K_{1,n}$-free graph with minimum degree $n + s_0$. We construct the graph G by subdividing $e_1 = y_1y_2$ and $e_2 = y_2y_3$ as in the proof of Theorem 3 and let $S' = S$ and $T' = T \cup \{v_1, v_2\}$ (Note that $s_0 \ge 2$ holds in case of $n \ge 7$, and hence $|S| \ge 3$). Then we have $\delta_G(S', T') = -2$. Hence G has no 2-factor, which implies that G_4 has no 2-factor containing e_1 and e_2.

Example V. If $n = 5$ or 6, then

$$\left\lfloor \frac{\sqrt{4n-3}+1}{2} \right\rfloor = 2.$$

We will show that, in case of $n = 5$ or 6, Theorem 3 doesn't hold for the graphs with minimum degree $n+1$. Let $\mathcal{C} = \{C_i^j \mid 1 \le i \le 2,\ 1 \le j \le 4\}$ be a set of sufficiently large complete graphs. From each C_i^j, we choose one vertex w_i^j. Let G_5 be the graph such that

$$\begin{aligned} V(G_4) &= \{y_1, y_2, y_3, x_1, x_2\} \cup \left(\bigcup_{C \in \mathcal{C}} V(C) \right) \text{ and} \\ E(G_4) &= \{y_iy_j \mid 1 \le i < j \le 3\} \cup \{y_ix_j \mid 1 \le i \le 3,\ 1 \le j \le 2\} \\ &\quad \cup \ \{x_iw_i^j \mid 1 \le i \le 2,\ 1 \le j \le 4\} \cup \left(\bigcup_{C \in \mathcal{C}} E(C) \right) \\ &\quad \cup \ \left\{ z_i^j y_i \mid z_i^j \in V(C_i^j),\ 1 \le i \le 2,\ 1 \le j \le 3 \right\} \\ &\quad \cup \ \left\{ z_i^4 y_3 \mid z_i^j \in V(C_i^j),\ 1 \le i \le 2 \right\}. \end{aligned}$$

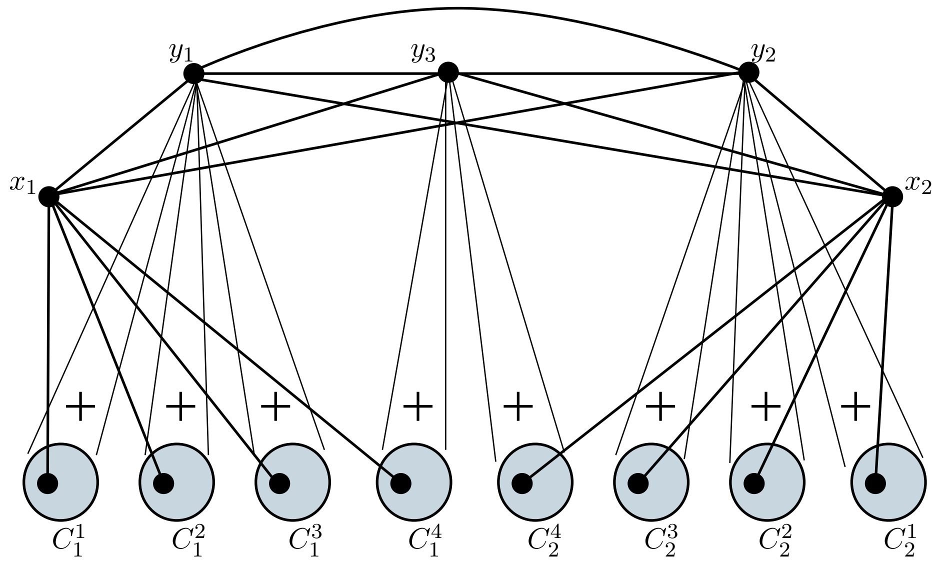

Figure 2: Example V.

(See Figure 2.) Then G_5 is a $K_{1,6}$-free graph with minimum degree 7. Let $S' = \{y_1, y_2, y_3\}$ and $T' = \{x_1, x_2\}$. We construct the graph G by subdividing $e_1 = y_1y_2$ and $e_2 = y_2y_3$ as in the proof of Theorem 3 and let $S = S'$ and $T = T' \cup \{v_1, v_2\}$, then we have $\delta_G(S, T) = -2$. Hence G has no 2-factor, which implies that G_5 has no 2-factor containing e_1 and e_2. By removing the components C_1^1 and C_2^1 from G_5, we obtain a $K_{1,5}$-free graph with minimum degree 6 which has no 2-factor containing e_1 and e_2.

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