SUT Journal of Mathematics
Vol. 44, No. 1 (2008), 23–31

Magic covering of chain of an arbitrary 2-connected simple graph

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(Received August 19, 2007; Revised November 30, 2007)

Abstract. A simple graph $G = (V, E)$ admits an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. We say that G is H-magic if there is a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ such that for each subgraph $H' = (V', E')$ of G isomorphic to H, $\sum_{v\in V'} f(v) + \sum_{e\in E'} f(e)$ is constant. When $f(V) = \{1, 2, \ldots, |V|\}$, then G is said to be H-supermagic. In this paper we show that a chain of any 2-connected simple graph H is H-supermagic.

AMS 2000 *Mathematics Subject Classification.* 20J06.

Key words and phrases. Chain of graph, Magic covering and H-supermagic.

§1. Introduction

The concept of H-magic graphs was introduced in [2]. An edge-covering of a graph G is a family of different subgraphs $H_1, H_2, \ldots, H_k$ such that each edge of E belongs to at least one of the subgraphs H_i, $1 \le i \le k$. Then, it is said that G admits an $(H_1, H_2, \ldots, H_k)$-edge covering. If every H_i is isomorphic to a given graph H, then we say that G admits an H-covering.

Suppose that $G = (V, E)$ admits an H-covering. We say that a bijective function $f : V \cup E \to \{1, 2, 3, \ldots, |V| + |E|\}$ is an H-magic labeling of G if there is a positive integer $m(f)$, which we call magic sum, such that for each subgraph $H' = (V', E')$ of G isomorphic to H, we have, $f(H') = \sum_{v\in V'} f(v) + \sum_{e\in E'} f(e) = m(f)$. In this case we say that the graph G is H-magic. When $f(V) = \{1, 2, \ldots, |V|\}$, we say that G is H-supermagic and we denote its supermagic-sum by $s(f)$.

We use the following notations. For any two integers $n < m$, we denote by $[n, m]$, the set of all consecutive integers from n to m. For any set $\mathbb{I} \subset \mathbb{N}$ we write $\sum \mathbb{I} = \sum_{x\in\mathbb{I}} x$ and for any integers k, $\mathbb{I} + k = \{x + k : x \in \mathbb{I}\}$. Thus

$k + [n, m]$ is the set of consecutive integers from $k + n$ to $k + m$. It can be easily verified that $\sum(\mathbb{I} + k) = \sum \mathbb{I} + k|\mathbb{I}|$. Finally, given a graph $G = (V, E)$ and a total labeling f on it we denote by $f(G) = \sum f(V) + \sum f(E)$.

In [2], A. Gutierrez, and A. Llado studied the families of complete and complete bipartite graphs with respect to the star-magic and star-supermagic properties and proved the following results.

- The star $K_{1,n}$ is $K_{1,h}$-supermagic for any $1 \leq h \leq n$.
- Let G be a d-regular graph. Then G is not $K_{1,h}$-magic for any $1 < h < d$.
- (a) The complete graph K_n is not $K_{1,h}$-magic for any $1 < h < n - 1$.
  (b) The complete bipartite graph $K_{n,n}$ is not $K_{1,h}$-magic for any $1 < h < n$.
- The complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for $n \geq 1$.
- The complete bipartite graph $K_{n,n}$ is not $K_{1,n}$-supermagic for any integer $n > 1$.
- For any pair of integers $1 < r < s$, the complete bipartite graph $K_{r,s}$ is $K_{1,h}$-supermagic if and only if $h = s$.

The following results regarding path-magic and path-supermagic coverings are also proved in [2].

- The path P_n is P_h-supermagic for any integer $2 \leq h \leq n$.
- Let G be a P_h-magic graph, $h > 2$. Then G is C_h-free.
- The complete graph K_n is not P_h-magic for any $2 < h \leq n$.
- The cycle C_n is P_h-supermagic for any integer $2 \leq h < n$ such that $\gcd(n, h(h-1)) = 1$.

Also in [2], the authors constructed some families of H-magic graphs for a given graph H by proving the following results.

- Let H be any graph with $|V(H)| + |E(H)|$ even. Then the disjoint union $G = kH$ of k copies of H is H-magic.

Let G and H be two graphs and $e \in E(H)$ a distinguished edge in H. We denote by $G * eH$ the graph obtained from G by gluing a copy of H to each edge of G by the distinguished edge $e \in E(H)$.

- Let H be a 2-connected graph and G an H-free supermagic graph. Let k be the size of G and $h = |V(H)| + |E(H)|$. Assume that h and k are not both even. Then, for each edge $e \in E(H)$, the graph $G * eH$ is H-magic.

In [3], P. Selvagopal and P. Jeyanthi proved that for any positive integer n, k- polygonal snake of length n is C_k-supermagic.

In this paper we construct a chain graph Hn of 2-connected graph H of length n, and prove that a chain graph Hn is H-supermagic.

§2. Preliminary Results

Let $P = \{X_1, X_2, \ldots, X_k\}$ be partition set of a set X of integers. When all sets have the same cardinality we say then P is a k-equipartition of X. We denote the set of subsets sums of the parts of P by $\sum P = \{\sum X_1, \sum X_2, \ldots, \sum X_k\}$. The following lemmas are proved in [2].

Lemma 1. *Let h and k be two positive integers and let $n = hk$. For each integer $0 \leq t \leq \left\lfloor \frac{h}{2} \right\rfloor$ there is a k-equipartition P of $[1, n]$ such that $\sum P$ is an arithmetic progression of difference $d = h - 2t$.*

Lemma 2. *Let h and k be two positive integers and let $n = hk$. In the two following cases there exists a k-equipartition P of a set X such that $\sum P$ is a set of consecutive integers.*

(i) *h or k are not both even and $X = [1, hk]$*

(ii) *$h = 2$ and k is even and $X = [1, hk + 1] - \left\{\frac{k}{2} + 1\right\}$.*

We have the following four results from the above two lemmas.

(a) If h is odd, then there exists a k-equipartition $P = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum P$ is a set of consecutive integers and $\sum P = \frac{(h-1)(hk+k+1)}{2} + [1, k]$.

(b) If h is even, then there exists a k-equipartition $P = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that subsets sum are equal and is equal to $\frac{h(hk+1)}{2}$.

(c) If h is even and k is odd, then there exists a k-equipartition $P = \{X_1, X_2, \ldots, X_k\}$ of $X = [1, hk]$ such that $\sum P$ is a set of consecutive integers and $\sum P = \frac{h(hk+1)}{2} + \left[-\frac{k-1}{2}, \frac{k-1}{2}\right]$.

(d) If $h = 2$ and k is even, and $X = [1, 2k + 1] - \left\{\frac{k}{2} + 1\right\}$ then there exists a k-equipartition $P = \{X_1, X_2, \ldots, X_k\}$ of X such that $\sum P$ is a set of consecutive integers and $\sum P = \left[\frac{3k}{2} + 3, \frac{5k}{2} + 2\right]$.

We generalise the second part of Lemma 2.

Corollary 1. *Let h and k be two even positive integers and $h \geq 4$. If $X = [1, hk + 1] - \left\{\frac{k}{2} + 1\right\}$, there exists a k-equipartition P of X such that $\sum P$ is a set of consecutive integers.*

Proof. Let $Y = [1, 2k+1] - \left\{\frac{k}{2}+1\right\}$ and $Z = (2k+1) + [1, (h-2)k]$. Then $X = Y \cup Z$. By (d), there exists a k-equipartition $P_1 = \{Y_1, Y_2, \ldots, Y_k\}$ of Y such that

$$\sum P_1 = \left[\frac{3k}{2}+3, \frac{5k}{2}+2\right].$$

As $h-2$ is even, by (b) there exists a k-equipartition $P_2' = \{Z_1', Z_2', \ldots, Z_k'\}$ of $[1, (h-2)k]$ such that

$$\sum P_2' = \left\{\frac{(h-2)(hk-2k+1)}{2}\right\}.$$

Hence, there exists a k-equipartition $P_2 = \{Z_1, Z_2, \ldots, Z_k\}$ of Z such that

$$\sum P_2 = \left\{(h-2)(2k+1) + \frac{(h-2)(hk-2k+1)}{2}\right\}.$$

Let $X_i = Y_i \cup Z_i$ for $1 \le i \le k$. Then $P = \{X_1, X_2, \ldots, X_k\}$ is a k-equipartition of X such that $\sum P$ is a set of consecutive integers and

$$\sum P = (h-2)(2k+1) + \frac{(h-2)(hk-2k+1)}{2} + \left[\frac{3k}{2}+3, \frac{5k}{2}+2\right].$$

□

§3. Chain of an arbitrary simple connected graph

Let $H_1, H_2, \ldots, H_n$ be copies of a graph H. Let u_i and v_i be two distinct vertices of H_i for $i = 1, 2, \ldots, n$. We construct a chain graph Hn of H of length n by identifying two vertices u_i and v_{i+1} for $i = 1, 2, \ldots, n-1$. See Figures 1 and 2.

§4. Main Result

Theorem 1. *Let H be a 2-connected (p, q) simple graph. Then Hn is H-supermagic if any one of the following conditions is satisfied.*

(i) *$p + q$ is even*

(ii) *$p + q + n$ is even*

Proof. Let $G = (V, E)$ be a chain of n copies of H. Let us denote the i^{th} copy of H in Hn by $H_i = (V_i, E_i)$. Note that $|V| = np - n + 1$ and $|E| = nq$. Moreover, we remark that by H is a 2-connected graph, Hn does not contain a subgraph H other than H_i.

Let v_i be the vertex in common with H_i and H_{i+1} for $1 \leq i \leq n-1$. Let v_0 and v_n be any two vertices in H_1 and H_n respectively so that $v_0 \neq v_1$ and $v_n \neq v_{n-1}$. Let $V_i' = V_i - \{v_{i-1}, v_i\}$ for $1 \leq i \leq n$.

Case (i): $p+q$ is even

Suppose p and q are odd. As $p-2$ is odd, by (a) there exists an n-equipartition $P_1' = \{X_1', X_2', \ldots, X_n'\}$ of $[1, n(p-2)]$ such that

$$\sum P_1' = \frac{(p-3)(np-n+1)}{2} + [1, n]\,.$$

Adding $n+1$ to $[1, n(p-2)]$, we get an n-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that

$$\sum P_1 = (p-2)(n+1) + \frac{(p-3)(np-n+1)}{2} + [1, n]$$

Similarly, since q is odd there exists an n-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1) + [1, nq]$ such that

$$\sum P_2 = q(np-n+1) + \frac{(q-1)(nq+n+1)}{2} + [1, n]$$

Define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, np+nq-n+1\}$ as follows:

(i) $f(v_i) = i+1$ for $0 \leq i \leq n$.

(ii) $f(V_i') = X_{n-i+1}$ for $1 \leq i \leq n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(H_i) &= f(v_{i-1}) + f(v_i) + \sum f(V_i') + \sum f(E_i) \\ &= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \\ &= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2} \end{aligned}$$

As $H_i \cong H$ for $1 \leq i \leq n$, Hn is H-supermagic.

Suppose both p and q are even. As p is even, by Lemma 1, there exists an n-equipartition $P_1' = \{X_1', X_2', \ldots, X_n'\}$ of $[1, n(p-2)]$ such that $\sum P_1'$ is arithmetic progression of difference 2 and

$$\sum P_1' = \left\{ \frac{n\left[(p-2)^2 - 2\right] + p - 4}{2} + 2r : 1 \leq r \leq n \right\}.$$

Adding $n+1$ to $[1, n(p-2)]$, we get an n-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that

$$\sum P_1 = \left\{(p-2)(n+1) + \frac{n\left[(p-2)^2-2\right]+p-4}{2} + 2i : 1 \leq i \leq n\right\}$$

As q is even, by (b), there exists an n-equipartition $P_2' = \{Y_1', Y_2', \ldots, Y_n'\}$ of $[1, nq]$ such that $\sum P_2' = \left\{\frac{q(nq+1)}{2}\right\}$.

Adding $np-n+1$ to $[1, nq]$ there exists an n-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1)+[1, nq]$ such that

$$\sum P_2 = \left\{q(np-n+1) + \frac{q(nq+1)}{2}\right\}$$

Define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, np+nq-n+1\}$ as follows:

(i) $f(v_i) = i+1$ for $0 \leq i \leq n$.

(ii) $f(V_i') = X_{n-i+1}$ for $1 \leq i \leq n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \leq i \leq n$.

Then for $1 \leq i \leq n$,

$$\begin{aligned} f(H_i) &= f(v_{i-1}) + f(v_i) + \sum f(V_i') + \sum f(E_i) \\ &= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \\ &= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2} \end{aligned}$$

As $H_i \cong H$ for $1 \leq i \leq n$, Hn is H-supermagic.

Case (ii): $p+q+n$ is even: Suppose p is odd, q is even and n is odd. Since p is odd as in proof of Case (i), there exists an n-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that

$$\sum P_1 = (p-2)(n+1) + \frac{(p-3)(np-n+1)}{2} + [1, n]$$

Since q is even and n is odd, by (c) there exists an n-equipartition $P_2' = \{Y_1', Y_2', \ldots, Y_n'\}$ of $[1, nq]$ such that

$$\sum P_2' = \frac{q(nq+1)}{2} + \left[-\frac{n-1}{2}, \frac{n-1}{2}\right].$$

Adding $np-n+1$ to $[1, nq]$ there exists an n-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1)+[1, nq]$ such that

$$\sum P_2 = q(np-n+1) + \frac{q(nq+1)}{2} + \left[-\frac{n-1}{2}, \frac{n-1}{2}\right]$$

Define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, np+nq-n+1\}$ as follows:

(i) $f(v_i) = i + 1$ for $0 \le i \le n$.

(ii) $f(V_i') = X_{n-i+1}$ for $1 \le i \le n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \le i \le n$.

Then for $1 \le i \le n$,

$$\begin{aligned} f(H_i) &= f(v_{i-1}) + f(v_i) + \sum f(V_i') + \sum f(E_i) \\ &= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \\ &= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2} \end{aligned}$$

As $H_i \cong H$ for $1 \le i \le n$, Hn is H-supermagic.
Suppose p is even, q is odd and n is odd. Since $p-2$ is even and n is odd, by (c) there exists an n-equipartition $P_1' = \{X_1', X_2', \ldots, X_n'\}$ of $[1, n(p-2)]$ such that

$$\sum P_1' = \frac{(p-2)\,[n(p-2)+1]}{2} + \left[-\frac{n-1}{2}, \frac{n-1}{2}\right].$$

Adding $n{+}1$ to $[1, n(p-2)]$, we get an n-equipartition $P_1 = \{X_1, X_2, \ldots, X_n\}$ of $[n+2, np-n+1]$ such that such that

$$\sum P_1 = (p-2)(n+1) + \frac{(p-2)\,[n(p-2)+1]}{2} + \left[-\frac{n-1}{2}, \frac{n-1}{2}\right]$$

Since q is odd, as in Case (i) there exists an n-equipartition $P_2 = \{Y_1, Y_2, \ldots, Y_n\}$ of $(np-n+1) + [1, nq]$ such that

$$\sum P_2 = q(np-n+1) + \frac{(q-1)(nq+n+1)}{2} + [1, n]$$

Define a total labeling $f : V \cup E \to \{1, 2, 3, \ldots, np + nq - n + 1\}$ as follows:

(i) $f(v_i) = i + 1$ for $0 \le i \le n$.

(ii) $f(V_i') = X_{n-i+1}$ for $1 \le i \le n$.

(iii) $f(E_i) = Y_{n-i+1}$ for $1 \le i \le n$.

Then for $1 \le i \le n$,

$$\begin{aligned} f(H_i) &= f(v_{i-1}) + f(v_i) + \sum f(V_i') + \sum f(E_i) \\ &= f(v_{i-1}) + f(v_i) + \sum X_{n-i+1} + \sum Y_{n-i+1} \\ &= \frac{n(p+q)^2 + 3(p+q) - 2n(p+q) + 2n - 2}{2} \end{aligned}$$

As $H_i \cong H$ for $1 \le i \le n$, Hn is H-supermagic. $\square$

§5. Illustrations

A chain of a 2-connected $(5,7)$ simple graph H of length 5 is shown in Figure 1 and a chain of a 2-connected $(6,9)$ simple graph H of length 3 is shown in Figure 2.

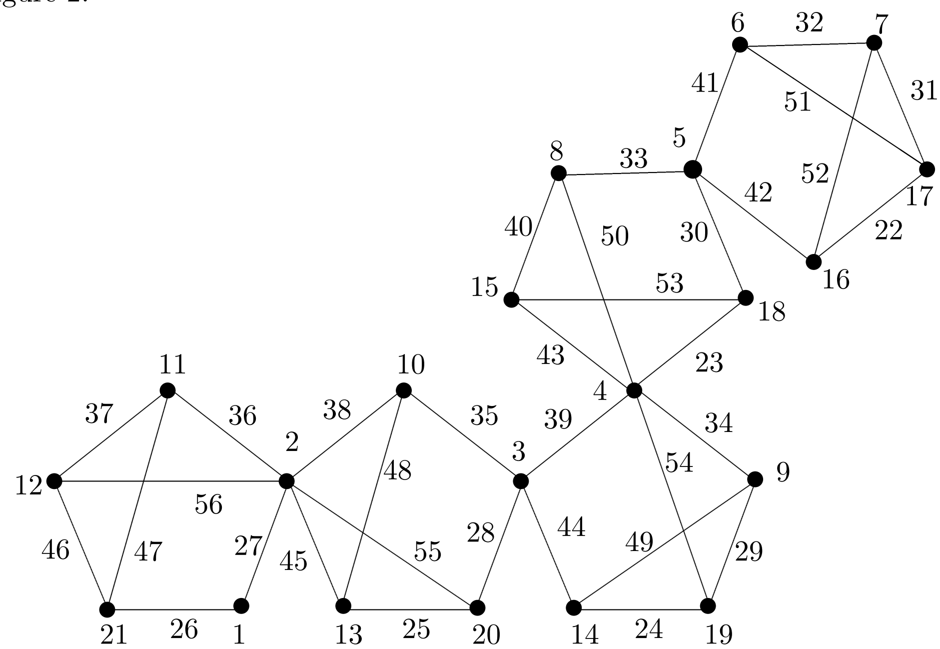

Figure 1. $p = 5$, $q = 7$, $s(f) = 322$.

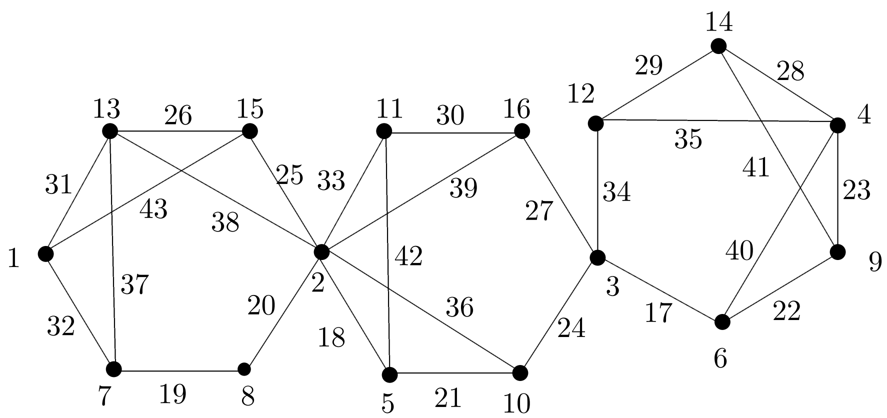

Figure 2. $p = 6$, $q = 9$, $s(f) = 317$.

References

[1] J.A. Gallian, *A Dynamic Survey of Graph labeling*, The Electronic Journal of Combinatorics. **5** (2005).

[2] A. Gutierrez and A. Llado, *Magic coverings*, J. Combin. Math. Combin. Comput. **55** (2005), 43–56.

[3] P. Selvagopal, P. Jeyanthi, *On C_k-supermagic graphs*, International Journal of Mathematics and Computer Science, **3.1** (2008), 25–30.

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