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On conformal equivalence of Berwald manifolds all of whose indicatrices have positive curvature

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Abstract. The problem given by M. Matsumoto in his paper [10] is that whether there exist conformally equivalent Berwald, or locally Minkowski manifolds. In this paper we are interested in case of positive definite Berwald manifolds of dimension $n \geq 3$ solving the problem under a further condition: we shall suppose that one, and therefore all indicatrices have positive curvature. Then the conformal change must be homothetic unless the Berwald manifolds are Riemannian.

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§1. Preliminaries

1.1. Throughout the paper we use the terminology and conventions described in [13]. Now we briefly summarize the basic notations.

(i) M is an n (> 1)-dimensional, C^∞, connected, paracompact manifold; $C^\infty(M)$ is the ring of real-valued smooth functions on M.

(ii) $\pi\colon TM \to M$ is the tangent bundle of M, $\pi_0\colon \mathcal{T}M \to M$ is the bundle of nonzero tangent vectors.

(iii) $\mathfrak{X}(M)$ denotes the $C^\infty(M)$-module of vector fields on M.

(iv) $\Omega^k(M)$ is the module of scalar k-forms on M; $\Omega^0(M) := C^\infty(M)$.

(v) $\psi^k(M)$ is the module of vector k-forms on M; $\psi^0(M) := \mathfrak{X}(M)$.

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(vi) ι_X, $\mathcal{L}_X$ are the *insertion operator* and the *Lie-derivative* with respect to the vector field $X \in \mathfrak{X}(M)$, respectively. The *exterior derivative* is denoted by d as usual. It is well-known that

$$\mathcal{L}_X = \iota_X \circ d + d \circ \iota_X, \ \mathcal{L}_X \circ d = d \circ \mathcal{L}_X.$$

1.2. *Vertical apparatus.* ([13]; see also [9] and [19]) Consider the tangent bundle $\pi\colon TM \to M$. $\mathfrak{X}^v(TM)$ denotes the $C^\infty(M)$-module of vertical vector fields on TM. $C \in \mathfrak{X}^v(TM)$ and $J \in \psi^1(TM)$ are the Liouville vector field and the vertical endomorphism, respectively. We have:

$$Im\ J = Ker\ J = \mathfrak{X}^v(TM), \ J^2 = 0, \tag{1}$$

$$d_J\varphi = d\varphi \circ J, \tag{2}$$

where $\varphi \in C^\infty(TM)$ and d_J is the derivation induced by J. The vertical and complete lifts of a vector field $X \in \mathfrak{X}(M)$ are denoted by X^v and X^c, respectively. As it is well-known

$$[J, X^v] = 0 \Rightarrow d_J \circ \mathcal{L}_{X^v} = \mathcal{L}_{X^v} \circ d_J \tag{3}$$

and, furthermore, the collection $(X_1^v, \dots, X_n^v, X_1^c, \dots, X_n^c)$ is a local basis for $\mathfrak{X}(TM)$ provided that $(X_1, \dots, X_n)$ is a local basis of $\mathfrak{X}(M)$.

1.3. *Horizontal endomorphisms.* ([3], [4]; see also [13]) A vector 1-form $h \in \psi^1(TM)$ is said to be a *horizontal endomorphism* on M if the following conditions are satisfied:

(HE 1) h is smooth on $\mathcal{T}M$,

(HE 2) h is a projector, i.e. $h^2 = h$,

(HE 3) $Ker\ h = \mathfrak{X}^v(TM)$.

J and h are obviously related as follows:

$$h \circ J = 0, \ J \circ h = J \tag{4}$$

and, furthermore, any horizontal endomorphism h determines an almost complex structure $F \in \psi^1(TM)$ ($F^2 = -1$, F is smooth on $\mathcal{T}M$) such that

$$F \circ J = h, \ F \circ h = -J \text{ and } J \circ F = \nu, \tag{5}$$

where $\nu := 1 - h$ is the so-called vertical projector. The *horizontal lift* of a vector field $X \in \mathfrak{X}(M)$ is given by the formula

$$X^h = FX^v; \tag{6}$$

as it is well-known the collection $(X_1^v, \dots, X_n^v, X_1^h, \dots, X_n^h)$ is a local basis for $\mathfrak{X}(TM)$ provided that $(X_1, \dots, X_n)$ is a local basis of $\mathfrak{X}(M)$.

Let a Riemannian metric g on the vertical subbundle be given. The mapping

$$\begin{aligned} &g_h\colon \mathfrak{X}(TM) \times \mathfrak{X}(TM) \to C^\infty(\mathcal{T}M), \\ &g_h(X,Y) := g(JX, JY) + g(\nu X, \nu Y) \end{aligned} \tag{7}$$

is said to be the *prolongation* of g *along* h. (Note that g_h is generally smooth only over $\mathcal{T}M$!)

1.4. *Finsler manifolds.* (for the details see [13]) Let a function $E\colon TM \to \mathbb{R}$ be given. The pair (M, E) is said to be a *Finsler manifold* if the following conditions are satisfied:

(FM 1) $\forall v \in \mathcal{T}M : E(v) > 0;\ E(0) = 0,$

(FM 2) E is of class C^1 on TM and smooth over $\mathcal{T}M$,

(FM 3) $CE = 2E$, i.e. E is homogeneous of degree 2,

(FM 4) the fundamental form $\omega := dd_J E \in \Omega^2(\mathcal{T}M)$ is symplectic.

Under these conditions the mapping

$$g\colon \mathfrak{X}^v(TM) \times \mathfrak{X}^v(TM) \to C^\infty(\mathcal{T}M),\ g(JX, JY) := \omega(JX, Y) \tag{8}$$

is a well-defined, nondegenerate symmetric bilinear form which is said to be the *Riemann-Finsler metric* of (M, E). The Finsler manifold is called *positive definite* if g is positive definite.

Let h be the canonical horizontal endomorphism (the so-called Barthel endomorphism) associated with the *canonical spray* S, i.e.

$$\iota_S \omega = -dE.$$

The tensor field $\mathcal{C}$ satisfying the condition

$$\omega(\mathcal{C}(X,Y), Z) = \frac{1}{2}(\mathcal{L}_{JX} J^* g_h)(Y, Z) \tag{9}$$

is called the *first Cartan tensor* of the Finsler manifold. $\tilde{C}$ denotes its semibasic trace:

$$\tilde{C}(X) := trace(F \circ \iota_X \mathcal{C}); \tag{10}$$

for a general definition see [5]. It is easy to check that the first Cartan tensor is semibasic and its lowered tensor $\mathcal{C}_\flat$ is totally symmetric. Moreover, for any vector field $X, Y, Z \in \mathfrak{X}(M)$

$$\mathcal{C}_\flat(X^h, Y^h, Z^h) = \frac{1}{2} X^v g(Y^v, Z^v) \text{ and } \mathcal{C}^o := \iota_S \mathcal{C} = 0, \tag{11}$$

where S is an arbitrary semispray on M, i.e. $JS = C$. The *second Cartan tensor* $\mathcal{C}'$ is defined by the formula

$$\omega(\mathcal{C}'(X, Y), Z) = \frac{1}{2}(\mathcal{L}_{hX} g_h)(JY, JZ); \tag{12}$$

the vanishing of the second Cartan tensor characterizes the so-called *Landsberg manifolds.*

1.5. *Further formulas (a practical summary).* Let (M, E) be a Finsler manifold. The covariant derivatives with respect to the *Cartan connection* can be explicitly calculated by the following formulas:

(C1) $D_{JX} JY = J[JX, Y] + \mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}\, JY + \mathcal{C}(X, Y)$,

(C2) $D_{hX} JY = \nu[hX, JY] + \mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}\, JY + \mathcal{C}'(X, Y)$,

(C3) $D_{JX} hY = h[JX, Y] + F\mathcal{C}(X, Y) = \overset{\circ}{D}_{JX}\, hY + F\mathcal{C}(X, Y)$,

(C3) $D_{hX} hY = hF[hX, JY] + F\mathcal{C}'(X, Y) = \overset{\circ}{D}_{hX}\, hY + F\mathcal{C}'(X, Y)$,

where $\overset{\circ}{D}$ denotes the *Berwald connection* on the Finsler manifold. The vertical covariant differential of the first Cartan tensor is totally symmetric:

$$(D_{JX}\mathcal{C})(Y, Z) = (D_{JY}\mathcal{C})(X, Z); \tag{13}$$

for a proof see [4]. The v-curvature tensor $\mathbb{Q}$ of the Cartan connection can be calculated by the formula

$$\mathbb{Q}(X, Y)Z = \mathcal{C}(F\mathcal{C}(X, Z), Y) - \mathcal{C}(X, F\mathcal{C}(Y, Z)). \tag{14}$$

It is well-known that the vanishing of the hv-curvature tensor $\overset{\circ}{\mathbb{P}}$ characterizes the so-called *Berwald manifolds* and, consequently, the Barthel endomorphism is just the horizontal lift of a linear connection on the underlying manifold M.

Let a smooth function $\varphi\colon TM \to R$ (or $\varphi\colon \mathcal{T}M \to R$) be given. Since the fundamental form ω is symplectic, there exists a unique vector field $\operatorname{grad}\varphi \in \mathfrak{X}(\mathcal{T}M)$ such that

$$\iota_{\operatorname{grad}\varphi}\omega = d\varphi \Rightarrow \iota_{J\operatorname{grad}\varphi}\omega = -d_J\varphi; \tag{15}$$

this vector field is called the *gradient* of φ.

Lemma 1. *Consider the vertical lift $\alpha^v := \alpha \circ \pi$ of a function $\alpha \in C^\infty(M)$; then $grad\,\alpha^v$ is a vertical vector field with the following properties:*

(i) $[C, grad\,\alpha^v] = -\,grad\,\alpha^v$,

(ii) $grad\,\alpha^v(E) = \alpha^c$*, where* $\alpha^c := S\alpha^v$ *is the complete lift of* α,

(iii) $\iota_{F\,grad\,\alpha^v}\mathcal{C} = -\frac{1}{2}[J, grad\,\alpha^v]$.

If $grad\,\alpha^v = \mu C$, where $\mu \in C^\infty(\mathcal{T}M)$, then $\mu = 0$ and, consequently, the function α is constant.

For the proof see [12] and [17].

Lemma 2. *Let (M, E) be a positive definite Berwald manifold of dimension $n \geq 3$. Then the following assertions are equivalent:*

(i) *The indicatrix hypersurface*

$$S_p := \{v \in T_pM \mid L(v) = 1, \textit{where } E = \tfrac{1}{2}L^2\} \subset T_pM$$

has positive curvature with respect to the Riemann-Finsler metric restricted on the punctured vector space $T_pM \setminus \{0\}$;

(ii) *for any $q \in M$ the indicatrix hypersurface $S_q \subset T_qM$ has positive curvature.*

Proof. Since (M, E) is a Berwald manifold we have a unique linear connection ∇ on the underlying manifold M such that the canonical Barthel endomorphism h coincides the horizontal structure induced by ∇. The Barthel endomorphism is conservative, i.e. the h-covariant derivatives of the energy function E vanish. This means that the linear isomorphisms induced by the parallel transport between the different tangent spaces preserve the Finslerian norm $L(v)$ of any tangent vector $v \in TM$. Therefore the indicatrices are invariant under these isomorphisms. On the other hand, as an easy calculation shows,

$$\tau^* g \mid_{\mathcal{T}_qM} = g \mid_{\mathcal{T}_pM},$$

where $\mathcal{T}_pM := T_pM \setminus \{0\}$ and $\tau\colon T_pM \to T_qM$ is the corresponding linear isomorphism induced by the parallel transport with respect to ∇ along a curve joining p and q. Taking into account the fact that M is connected, the non-trivial implication $(i) \Rightarrow (ii)$ follows immediately. □

Remark 1. Note that this argumentation holds without any modification in case of Finsler manifolds which have a linear connection on the underlying manifold M such that the induced horizontal endomorphism is conservative: they are the so-called *generalized Berwald manifolds*, especially the *Wagner manifolds*; see e.g. [8],[15] and [17].

Definition 1. A positive definite generalized Berwald manifold (M, E) of dimension $n \geq 3$ is called *almost spherical* if one, and therefore all of its indicatrices have positive curvature.

§2. Conformal equivalence of Riemann-Finsler metrics

Definition 2. Consider the Finsler manifolds (M, E) and $(M, \tilde{E})$ with Riemann-Finsler metrics g and $\tilde{g}$, respectively; g and $\tilde{g}$ are said to be *conformally equivalent* if there exists a positive smooth function $\varphi\colon \mathcal{T}M \to \mathbb{R}$ such that $\tilde{g} = \varphi g$. The function φ is called the *scale function* or the *proportionality function.* If the scale function is constant, then we say that the conformal change is *homothetic*

Remark 2. If $\tilde{g} = \varphi g$ then

$$\tilde{E} = \frac{1}{2}\tilde{g}(C, C) = \frac{1}{2}\varphi g(C, C) = \varphi E. \tag{16}$$

It is also well-known due to M.S. Knebelman, that the scale function between conformally equivalent Finsler manifolds is a vertical lift, i.e. φ can always be written in the form

$$\varphi = exp \circ \alpha^v := exp \circ \alpha \circ \pi. \tag{17}$$

Moreover, if a Finsler manifold (M, E) with Riemann-Finsler metric g and a function $\alpha \in C^\infty(M)$ are given, then

$$g_\alpha := \varphi g \quad (\varphi = exp \circ \alpha^v) \tag{18}$$

is the Riemann-Finsler metric of the Finsler manifold (M, E_α), where the energy function E_α is defined by the formula $E_\alpha := \varphi E$. According to these elementary facts we also speak of a *conformal change* $g_\alpha = \varphi g$ of the metric g; for the details see [11], [12] and [17].

In what follows, we summarize some of the basic transformation formulas; for the proof and notations we can refer to Hashiguchi's fundamental work [7] and [17], [18]. Let us define first of all the tensor fields $\mathbb{B}_i^1$ $(1 \leq i \leq 4), \mathbb{V}$ and $\mathbb{H}$ in the following way:

$$
\begin{aligned}
\mathbb{B}_1^1(X) &= d_J E \otimes C(X) - EJX, \\
\mathbb{B}_2^1(X,Y) &= E\mathcal{C}(X,Y) + \frac{1}{2}\Big(d_J E \wedge J(X,Y) + g(JX,JY)C\Big), \\
\mathbb{B}_3^1(X,Y,Z) &= E\Big((D_{JX}\mathcal{C})(Y,Z) - \mathcal{C}(F\mathcal{C}(X,Y),Z) - \mathbb{Q}(X,Y)Z\Big) + \\
&+ \frac{1}{2}\Big(g(JX,JY)JZ + g(JX,JZ)JY - g(JY,JZ)JX\Big) + \\
&+ d_J E \otimes \iota_X \mathcal{C}(Y,Z) + d_J E \otimes \iota_X \mathcal{C}(Z,Y), \\
\mathbb{V}(X,Y,Z) &= \frac{1}{2}\Big(d_J E \otimes \mathcal{C}(X,Y,Z) + d_J E \otimes \mathcal{C}(Z,X,Y) + \\
&+ d_J E \otimes \mathcal{C}(Y,Z,X) + \mathcal{C}_\flat(X,Y,Z)C\Big) + \\
&+ E(D_{JX}\mathcal{C})(Y,Z), \\
\mathbb{B}_4^1(X,Y,Z,W) &= (D_{JW}\mathbb{B}_3^1)(X,Y,Z) - \mathbb{B}_3^1(F\mathcal{C}(X,W),Y,Z) + \\
&+ \mathbb{B}_3^1(X,F\mathcal{C}(Y,W),Z) + \mathbb{B}_3^1(X,Y,F\mathcal{C}(Z,W)) - \\
&- \mathcal{C}(F\mathcal{B}_3^1(X,Y,Z),W), \\
\mathbb{H}(X,Y,Z,W) &= \mathbb{B}_4^1(X,Y,Z,W) + \mathcal{C}(F\mathbb{B}_3^1(X,Y,Z),W).
\end{aligned}
$$

Lemma 3. *Let (M,E) and (M,E_α) be conformally equivalent Finsler manifolds; then*

$$S_\alpha = S - \iota_{F\,grad\,\alpha^v}\mathbb{B}_1^1, \tag{19}$$

$$h_\alpha = h - \iota_{F\,grad\,\alpha^v}\mathbb{B}_2^1, \tag{20}$$

$$\mathcal{C}'_\alpha = \mathcal{C}' - \iota_{F\,grad\,\alpha^v}\mathbb{V}, \tag{21}$$

$$\mathring{\mathbb{P}}_\alpha = \mathring{\mathbb{P}} - \iota_{F\,grad\,\alpha^v}\mathbb{B}_4^1. \tag{22}$$

Definition 3. Let (M,E) be a Finsler manifold; the change

$$g_\alpha = \varphi g$$

is called a *Landsberg-, Berwald-, or locally Minkowski-type* conformal change of the metric g if the resulting Finsler manifold (M, E_α) is a Landsberg, Berwald, or a locally Minkowski manifold. The manifold (M, E) is also said to be a *conformally Landsberg*, a *conformally Berwald* manifold (in an equivalent terminology: a Wagner manifold), or *conformally flat* Finsler manifold, respectively. We set

$$\begin{aligned}\mathbb{L} &:= \{\alpha \in C^\infty(M) \mid g_\alpha = \varphi g \text{ is Landsberg-type}\},\\ \mathbb{B} &:= \{\alpha \in C^\infty(M) \mid g_\alpha = \varphi g \text{ is Berwald-type}\},\\ \mathbb{M} &:= \{\alpha \in C^\infty(M) \mid g_\alpha = \varphi g \text{ is locally Minkowski-type}\}\end{aligned}$$

and, for any $p \in M$

$$\mathbb{L}_p := \{d_p\alpha \mid \alpha \in \mathbb{L}\},\ \mathbb{B}_p := \{d_p\alpha \mid \alpha \in \mathbb{B}\},\ \mathbb{M}_p := \{d_p\alpha \mid \alpha \in \mathbb{M}\}.$$

Lemma 4. *For any $p \in M$ the sets $\mathbb{L}_p$ and $\mathbb{B}_p$ are affine subspaces of the dual vector space T_p^*M; they are linear subspaces provided that (M, E) is a Landsberg, or a Berwald manifold, respectively.*

For a proof see [18].

Definition 4. We set

$$l(p) := \dim\ \mathbb{L}_p,\ b(p) := \dim\ \mathbb{B}_p,\ m(p) := \dim\ Aff(\mathbb{M}_p),$$

where $Aff(\mathbb{M}_p)$ denotes the affine hull of the set $\mathbb{M}_p$.

§3. An observation on the existence of nontrivial conformal changes preserving the (hv)-curvature tensor of the Berwald connection

Lemma 5. *Let (M, E) and (M, E_α) be conformally equivalent Finsler manifolds, i.e.*

$$g_\alpha = \varphi g \quad (\varphi = exp \circ \alpha^v)$$

and $X := F\, grad\, \alpha^v$. Suppose that the second Cartan tensor is invariant under this conformal change; then

$$\begin{aligned} &-\frac{1}{3} g\Big(\mathbb{B}_4^1(X, F\mathcal{C}(X,X), X, X), JX\Big) = \\ (23) \qquad &= \frac{1}{2}\Big(\|\mathcal{C}(X,X)\|^2 (\|JX\|^2 - \frac{(\alpha^c)^2}{2E}) - g^2(\mathcal{C}(X,X), JX)\Big) + \\ &+ Eg(\mathbb{Q}(X, F\mathcal{C}(X,X))F\mathcal{C}(X,X), JX). \end{aligned}$$

Proof. Since $\mathcal{C}'_\alpha = \mathcal{C}'$, it follows by (21) that $\iota_X\mathbb{V}$ vanishes and, consequently,

$$\begin{aligned} E(D_{JX}\mathcal{C})(Y,Z) = -\frac{1}{2}\Big(&\alpha^c\mathcal{C}(Y,Z) + d_JE\otimes\iota_X\mathcal{C}(Y,Z) + \\ &+ d_JE\otimes\iota_X\mathcal{C}(Z,Y) + \mathcal{C}_\flat(X,Y,Z)C\Big). \end{aligned} \tag{24}$$

On the other hand, for any vector field $W \in \mathfrak{X}(M)$ we have that

$$\begin{aligned} &g(\mathbb{B}^1_4(X,W^c,X,X),JX) = \\ &= g((D_{JX}\mathbb{B}^1_3)(X,W^c,X),JX) - g(\mathbb{B}^1_3(F\mathcal{C}(X,X),W^c,X),JX) + \\ &+ g(\mathbb{B}^1_3(X,F\mathcal{C}(W^c,X),X),JX) + g(\mathbb{B}^1_3(X,W^c,F\mathcal{C}(X,X)),JX) - \\ &- g(\mathbb{B}^1_3(X,W^c,X),\mathcal{C}(X,X)), \end{aligned} \tag{25}$$

where, according to (C1), $D_{JX}W^v = \mathcal{C}(W^c,X)$ and, by Lemma 1. (iii)

$$D_{W^v}JX = -\mathcal{C}(W^c,X) \Rightarrow D_{JX}JX = -\mathcal{C}(X,X). \tag{26}$$

Using the metrical property of the classical Cartan connection, (25) reduces to the following simple form

$$\begin{aligned} &g(\mathbb{B}^1_4(X,W^c,X,X),JX) = \\ &= JXg(\mathbb{B}^1_3(X,W^c,X),JX) + 2g(\mathbb{B}^1_3(X,W^c,F\mathcal{C}(X,X)),JX). \end{aligned} \tag{27}$$

Since the v-covariant differential $D_{JX}\mathcal{C}$ can be expressed in a specil way, we have from the definition of $\mathbb{B}^1_3$ the relations

$$\begin{aligned} &\mathbb{B}^1_3(X,W^c,X) = \\ &\quad = \frac{1}{2}\Big(W^vE\ \mathcal{C}(X,X) + \|JX\|^2\ W^v - g(\mathcal{C}(X,X),W^v)\ C\Big) - \\ &\quad -E\Big(\mathcal{C}(F\mathcal{C}(X,W^c),X) + \mathbb{Q}(X,W^c)X\Big), \\ &\mathbb{B}^1_3(X,W^c,F\mathcal{C}(X,X)) = \\ &\quad = \frac{1}{2}\Big(W^vE\ \mathcal{C}(X,F\mathcal{C}(X,X)) + g(\mathcal{C}(X,X),JX)W^v + g(JX,W^v)\mathcal{C}(X,X) - \\ &\quad -\alpha^c\mathcal{C}(W^c,F\mathcal{C}(X,X)) - g(\mathcal{C}(X,X),W^v)JX - g(\mathcal{C}(X,X),\mathcal{C}(X,W^c))\ C\Big) - \\ &\quad -E\Big(\mathcal{C}(F\mathcal{C}(X,X),F\mathcal{C}(X,W^c)) + \mathbb{Q}(X,W^c)F\mathcal{C}(X,X)\Big) \end{aligned}$$

and, consequently,

$$\begin{aligned}
&JXg(\mathbb{B}^1_3(X,W^c,X),JX)=\\
&=-\frac{1}{2}\Big(g(JX,W^v)g(\mathcal{C}(X,X),JX)+\|JX\|^2g(\mathcal{C}(X,X),W^v)+\\
&+3W^vE\ \|\mathcal{C}(X,X)\|^2+\alpha^cg(\mathcal{C}(X,X),\mathcal{C}(X,W^c))\Big)+\\
&+E\Big(g(\mathcal{C}(X,F\mathcal{C}(X,W^c)),\mathcal{C}(X,X))+g(\mathcal{C}(F\mathcal{C}(X,X),W^c),\mathcal{C}(X,X))\Big)+\\
&+\frac{1}{2}\Big(W^vE\ g((D_{JX}\mathcal{C})(X,X),JX)-\alpha^cg((D_{JX}\mathcal{C})(X,X),W^v)\Big)-\\
&-E\Big(g((D_{JX}\mathcal{C})(X,X),\mathcal{C}(X,W^c))+g((D_{JX}\mathcal{C})(X,W^c),\mathcal{C}(X,X))\Big).
\end{aligned}$$

By the help of (24) we can set this formula free from the v-covariant differential of the first Cartan tensor $\mathcal{C}$. Together with our previous result (27) this process gives the following expression:

$$\begin{aligned}
&-\frac{1}{3}g(\mathbb{B}^1_4(X,W^c,X,X),JX)=\\
&=\frac{1}{2}\Big(\|JX\|^2g(\mathcal{C}(X,X),W^v)-g(JX,W^v)g(\mathcal{C}(X,X),JX)\Big)+\\
&+\frac{\alpha^c}{4E}\Big(W^vEg(\mathcal{C}(X,X),JX)-\alpha^cg(\mathcal{C}(X,X),W^v)\Big)+\\
&+Eg(\mathbb{Q}(X,W^c)F\mathcal{C}(X,X),JX).
\end{aligned}\tag{28}$$

Since it has a tensorial character in the second argument, we get the desired relation by the substitution of the vector field $F\mathcal{C}(X,X)$ into (28). □

Definition 5. Let (M,E_R) be a Riemannian manifold, $\alpha\in C^\infty(M)$ such that

(i) $d_p\alpha\neq 0$ and $\alpha(p)=0$; this means that α is regular on a connected open neighbourhood U of the point $p\in M$.

(ii) The gradient of α with respect to the Riemannian structure has a constant unit length on the neighbourhood U, i.e.

$$L_R(\operatorname{grad}_R\alpha)\ |_U\equiv 1,$$

where the fundamental function L_R is defined by the conditions

$$E_R=\frac{1}{2}L_R^2\ \text{ and }\ L_R\geq 0$$

as usual. Consider a smooth function

$$K\colon M \to \mathbb{R} \;\text{ such that }\; -4 < K(q) < 4 \quad (q \in U)$$

and let $\tilde{v} \in T_qM$ be an arbitrary tangent vector. Then, of course,

$$\tilde{v} = v + t \operatorname{grad}_R \alpha(q),$$

where $v \in T_qM$ is tangential to the level hypersurface $N_r := \alpha^{-1}(r) \cap U$ containing the point $q \in U$; $r := \alpha(q)$. The energy function

$$E(\tilde{v}) := (E_R(\tilde{v}) + K(q)L_R(v)\frac{t}{4}) \tag{29}$$
$$\exp \frac{2K(q)}{\sqrt{16 - K^2(q)}}(\arctan \frac{4t + K(q)L_R(v)}{L_R(v)\sqrt{16 - K^2(q)}} - \arctan \frac{K(q)}{\sqrt{16 - K^2(q)}})$$

constructed on the neighbourhood U is called an *Asanov-type Finslerian metric function*; for the terminology see [2]. Furthermore,

$$\begin{aligned} E(0) &:= 0, \\ E(\operatorname{grad}_R \alpha) &:= \frac{1}{2} exp\left\{\frac{2K}{\sqrt{16 - K^2}}\left(\frac{\pi}{2} - \arctan \frac{K}{\sqrt{16 - K^2}}\right)\right\}, \\ E(-\operatorname{grad}_R \alpha) &:= \frac{1}{2} exp\left\{\frac{-2K}{\sqrt{16 - K^2}}\left(\frac{\pi}{2} + \arctan \frac{K}{\sqrt{16 - K^2}}\right)\right\}. \end{aligned} \tag{30}$$

Remark 3. As a special case of our definition a similar, but not exactly the same construction can be found in Asanov's paper [2], see also [1]; now we briefly summarize the basic ideas. Finslerian metric functions proposed by G.S. Asanov to study are given first of all on the product manifold $M := N \times \mathbb{R}$; for brevity let us set

$$\alpha\colon N \times \mathbb{R} \to \mathbb{R},\; \alpha(p, r) := r \Rightarrow N \cong \alpha^{-1}(0).$$

The Riemannian structures on the different level hypersurfaces with respect to the function α are induced by the help of a Riemannian energy function E_R on the manifold N. This means that they are isometric to each other under the natural identification

$$p \in N \longrightarrow (p, r) \in N_r,\; \text{where } N_r := \alpha^{-1}(r).$$

Moreover, *the function K does not depend on the value of* r, i.e. for any scalars $r, s \in \mathbb{R}$

$$K(q, r) = K(q, s) \Rightarrow K(q) := K(q, r);$$

the function E is given by the formula

$$E(\tilde{v}) := (E_R(\tilde{v}) + K(q)L_R(v)\frac{\mid t \mid}{4}) \tag{31}$$
$$\exp\frac{2K(q)}{\sqrt{16-K^2(q)}}(\arctan\frac{\sqrt{16-K^2(q)} \mid t \mid}{K(q) \mid t \mid + 4L_R(v)} - \arctan\frac{\sqrt{16-K^2(q)}}{K(q)}),$$

where

$$\tilde{v} = v + t\frac{\partial}{\partial r} \in T_{(q,r)}M \text{ and } v \in T_{(q,r)}N_r \cong T_qN.$$

As we can see, *Asanov's energy function (31) is reversible* and, consequently, it has lots of singularities along the equatorial section defined by the equation $t = 0$ unless the metric is Riemannian, i.e. $K \equiv 0$: *... At the points of the equatorial section, the generatrix of the indicatrix has a corner whose angle is* $\beta^* = 180° - 2arctg\frac{K}{2}$*... this angle may be regarded as "the parameter of non-Riemannianity"...* (cf. Theorem 7; [1]).

Suppose that $\dim M \geq 3$; Asanov proved that the indicatrices of the Finsler manifold $(N \times \mathbb{R}, E)$ have constant curvature $1 - \frac{K^2}{16}$ with respect to the Riemann-Finsler metric restricted on the tangent spaces (except, of course, at the origin). *Now we are going to show that the metric (29) in Definition 5 also has this property*; the argumentation is based on the fact that the sectional curvature of the indicatrices with respect to their own restricted Riemann-Finsler metric is invariant under any conformal change. Indeed, due to Knebelman's observation, the conformal change works as a scalar multiplication for the tangent spaces as Riemannian manifolds; notations as above.

Proposition 1. *Suppose that* $\dim M \geq 3$*; for any* $q \in U$ *the indicatrix hypersurface* S_q *of an Asanov-type Finslerian metric function has constant curvature.*

Proof. It is enough to prove our statement at the point $p \in M$; the proof is similar in case of any other point. Let $N := \alpha^{-1}(0) \cap U$ be the level hypersurface cointaining p. First of all we investigate the upper half indicatrix

$$S_p^+ := S_p \cap \{\tilde{v} = v + t \operatorname{grad}_R \alpha(p) \in T_pM \mid t > 0\}$$

of the metric (29) by the help of the function

$$\Theta_+(t) := \arctan\frac{\sqrt{16-K^2(p)} \mid t \mid}{K(p) \mid t \mid + 4L_R(v)} - \arctan\frac{4t + K(p)L_R(v)}{L_R(v)\sqrt{16-K^2(p)}},$$

where $v \in T_pN \setminus \{0\}$ is an arbitrarily fixed tangent vector. Differentiating with respect to t, an easy calculation shows that ${\Theta_+}'(t) = 0$ for any positive real number $t \in \mathbb{R}$. Therefore, for example

$$\Theta_+(t) = \lim_{t \to 0^+} \Theta_+(t) = -\arctan \frac{K(p)}{\sqrt{16 - K^2(p)}},$$

provided that $K(p) > 0$; if $K(p) < 0$ then the domain of parameters $t \in \mathbb{R}^+$ must be divided into connected parts! This means that the upper half indicatrix of an *Asanov-type* Finslerian metric function (29) consists of such parts which are homothetic to the upper half indicatrix of the metric (31) at the point p. For the lower half indicatrix S_p^- let us form the auxiliary metric (31) by the help of $-K$ instead of K, i.e.

$$E(\tilde{v}) := (E_R(\tilde{v}) - K(p)L_R(v)\frac{\mid t \mid}{4}) \tag{32}$$
$$\exp \frac{-2K(p)}{\sqrt{16 - K^2(p)}}(\arctan \frac{\sqrt{16 - K^2(p)} \mid t \mid}{-K(p) \mid t \mid + 4L_R(v)} - \arctan \frac{\sqrt{16 - K^2(p)}}{-K(p)}).$$

Differentiating the function

$$\Theta_-(t) := \arctan \frac{\sqrt{16 - K^2(p)} \mid t \mid}{-K(p) \mid t \mid + 4L_R(v)} + \arctan \frac{4t + K(p)L_R(v)}{L_R(v)\sqrt{16 - K^2(p)}}$$

with respect to t, an easy calculation shows that ${\Theta_-}'(t) = 0$ for any real number $t < 0$ and, consequently, the lower half indicatrix of an *Asanov-type* Finslerian metric function (29) consists of such parts which are homothetic to the lower half indicatrix of the metric (32) at the point p. Since the indicatrix hypersurfaces of (31) and (32) have the same constant sectional curvature

$$1 - \frac{K^2(p)}{16} = 1 - \frac{(-K)^2(p)}{16},$$

this means that S_p also has constant sectional curvature as was to be stated; of course, it is just $1 - \frac{K^2(p)}{16}$. $\square$

Proposition 2. *Let (M, E) be a positive definite Finsler manifold of dimension $n \geq 3$ with an almost spherical indicatrix hypersurface at a point $p \in M$, i.e. suppose that it has positive curvature. If there exists a conformal change*

$$g_\alpha = \varphi g \quad (\varphi = exp \circ \alpha^v)$$

of the metric such that

(i) *the scale funtion is regular at the point p,*

(ii) *the (hv)-curvature tensor of the classical Berwald connection is invariant,*

then E is conformal equivalent to an Asanov-type Finslerian metric function on a connected open neighbourhood U of the point p.

The proof consists of more steps presented below; conditions of the theorem are used without any further comment. Keeping in mind that our result has a local character, consider a connected open neighbourhood U of the point p such that $d_q\alpha \neq 0$, where $q \in U$ and, for the sake of brevity, let us set $X := F \operatorname{grad} \alpha^v \Rightarrow JX = \operatorname{grad} \alpha^v$ as above (cf. Lemma 5).

Lemma 6. *The vector fields $\mathcal{C}(X,X)$ and $JX - \frac{\alpha^c}{2E}C$ are linearly dependent at the points $v \in T_pM \setminus \{0\}$, i.e.*

$$G_g\big(\mathcal{C}(X,X), JX - \frac{\alpha^c}{2E}C\big)(v) = 0, \tag{33}$$

where G_g forms the Gram-determinant of its arguments with respect to the Riemann-Finsler metric g.

Proof. It is well-known (see e.g. [13], p. 44) that for any vector field $Y, Z, W \in \mathfrak{X}(TM)$:

$$\mathcal{C}'_\flat(Y,Z,W) = -\frac{1}{2}g(\overset{\circ}{\mathbb{P}}(Y,Z)W, C) \tag{34}$$

which implies the second Cartan tensor to be also invariant under the conformal change $g_\alpha = \varphi g$. Using Lemma 5 we have from the vanishing of the tensor field $\iota_{F \operatorname{grad} \alpha^v}\mathbb{B}_4^1$ that for any $v \in T_pM \setminus \{0\}$:

$$\begin{aligned} 0 = \frac{1}{r^2}\bigg(\|\mathcal{C}(X,X)\|^2\big(\|JX\|^2 - \frac{(\alpha^c)^2}{r^2}\big) - g^2(\mathcal{C}(X,X), JX)\bigg)(v) + \\ + g(\mathbb{Q}(X, F\mathcal{C}(X,X))F\mathcal{C}(X,X), JX)(v), \end{aligned} \tag{35}$$

where $r := L(v)$. If the plane determined by the vertical tangent vectors $\mathcal{C}(X,X)(v)$ and $JX_v - \frac{\alpha^c}{2E}(v)C_v$ exists, then (35) shows the vanishing of the corresponding sectional curvature for the hypersurface $rS_p \subset T_pM$. Since (M,E) is almost spherical at the point p, this is a contradiction. □

Lemma 7. *$E_R := E\|JX\|^2$, where the norm is taken with respect to the Riemann-Finsler metric g is a Riemannian energy function on $\pi^{-1}(U)$. For any vector fields $Y, Z \in \mathfrak{X}(U)$ the associated Riemannian metric γ and g are related as follows:*

$$\begin{aligned} g_R(Y^v, Z^v) = \|JX\|^2 g(Y^v, Z^v) - Y^v(E)g(\mathcal{C}(X,X), Z^v) - \\ - Z^v(E)g(\mathcal{C}(X,X), Y^v) + 2\alpha^c g(\mathcal{C}(X,Y^c), Z^v) + \\ + 2Eg(\mathcal{C}(X,Y^c), \mathcal{C}(X,Z^c)) + 2Eg(\mathbb{Q}(X,Y^c)Z^c, JX), \end{aligned} \tag{36}$$

where $g_R(Y^v, Z^v) := \gamma(Y, Z) \circ \pi$. *We have:*

$$
\begin{aligned}
grad\,\alpha^v = \|JX\|^2 grad_R^v\alpha - grad_R^v\alpha(E)\mathcal{C}(X,X)- \\
- g(\mathcal{C}(X,X), grad_R^v\alpha)C + 2\alpha^c\mathcal{C}(X, grad_R^c\alpha) + \\
+ 2E\mathcal{C}(X, F\mathcal{C}(X, grad_R^c\alpha)) - \mathbb{Q}(X, grad_R^c\alpha)X,
\end{aligned}
\tag{37}
$$

where $grad_R\,\alpha \in \mathfrak{X}(U)$ *is the Riemannian gradient of the function* α*,* $grad_R^v\alpha$ *and* $grad_R^c\alpha$ *are its vertical and complete lifts, respectively.*

Proof. Since the $hv-$curvature tensor of the Berwald connection is invariant, we have that for any vector field $Y, Z, W \in \mathfrak{X}(U)$:

$$
\begin{aligned}
0 &= \overset{\circ}{\mathbb{P}}_\alpha (Y^c, Z^c, W^c) - \overset{\circ}{\mathbb{P}} (Y^c, Z^c, W^c) = \\
&= [[Y^{h_\alpha}, Z^v], W^v] - [[Y^h, Z^v], W^v] = [[Y^{h_\alpha}, Z^v] - [Y^h, Z^v], W^v],
\end{aligned}
$$

which means that the vector field $[Y^{h_\alpha}, Z^v] - [Y^h, Z^v]$ is a vertical lift (see e.g. [13], p. 37). Therefore, as an easy local calculation shows, the components of the difference tensor $h_\alpha - h$ are linear on the tangent spaces and, consequently the difference of the associated semisprays is a quadratic vector field. From the transformation formula (19) it follows at the same time that

$$S_\alpha - S = -\alpha^c C + E grad\alpha^v;$$

applying both sides to the function α^c:

$$E\|JX\| := E\|grad\alpha^v\|^2 = (S_\alpha - S)\alpha^c + (\alpha^c)^2,$$

where the function on the right hand side is quadratic. We have:

$$
\begin{aligned}
g_R(Y^v, Z^v) = Y^v(Z^v E_R) = Y^v\Big((Z^v E)\|JX\|^2 + E Z^v\|JX\|^2\Big) = \\
= \|JX\|^2 g(Y^v, Z^v) + (Z^v E) Y^v\|JX\|^2 + (Y^v E) Z^v\|JX\|^2 + \\
+ E Y^v\big(Z^v\|JX\|^2\big).
\end{aligned}
\tag{38}
$$

Here

$$
\begin{aligned}
Z^v\|JX\|^2 &= 2g(D_{Z^v} JX, JX) \overset{(26)}{=} -2g(\mathcal{C}(X, Z^c), JX) = \\
&= -2g(\mathcal{C}(X,X), Z^v) \Rightarrow Y^v\|JX\|^2 = -2g(\mathcal{C}(X,X), Y^v), \\
Y^v\big(Z^v\|JX\|^2\big) &= -2Y^v g(\mathcal{C}(X,X), Z^v) = \\
&= -2g(D_{Y^v}\mathcal{C}(X,X), Z^v) - 2g(\mathcal{C}(X,X), \mathcal{C}(Y^c, Z^c)) \overset{(26)}{=} \\
&= -2g((D_{Y^v}\mathcal{C})(X,X), Z^v) + 2g(\mathcal{C}(F\mathcal{C}(X, Y^c), X), Z^v) - \\
&\quad -2g(\mathcal{C}(X,X), \mathcal{C}(Y^c, Z^c)) \overset{(13)}{=} -2g((D_{JX}\mathcal{C})(X, Y^c), Z^v) + \\
&\quad +2g(\mathcal{C}(F\mathcal{C}(X, Y^c), X), Z^v) - 2g(\mathcal{C}(X,X), \mathcal{C}(Y^c, Z^c)) \overset{(14)}{=} \\
&= -2g((D_{JX}\mathcal{C})(X, Y^c), Z^v) + 2g(\mathbb{Q}(X, Y^c), Z^c), JX)
\end{aligned}
$$

taking into account the fact that the lowered first Cartan tensor is totally symmetric. Since the vertical covariant differential $D_{JX}\mathcal{C}$ has a special form (24), by the substitution of these expressions into (38) we get immediately the relation between the metrics; (37) is a direct consequence of the previous formula (36). □

Lemma 8. *$\| grad_R\, \alpha\|^2 := \gamma(grad_R\, \alpha, grad_R\, \alpha) \leq 1$ and for any $q \in U$ the following assertions are equivalent:*

(i) *$\| grad_R\, \alpha\|^2(q) = 1$,*

(ii) *$G_g(grad\, \alpha^v, C)(v) = 0$, where $v = \pm\, grad_R\, \alpha(q)$, i.e. the Liouville vector field and $grad\, \alpha^v$ are linearly dependent at the points*

$$v := \pm\, grad_R\, \alpha(q).$$

Proof. Using the Cauchy-Schwarz inequality (with respect to the Riemann-Finsler metric g) we have that

$$-\| \operatorname{grad} \alpha^v\| \leq \frac{C\alpha^c}{\sqrt{g(C,C)}} \leq \| \operatorname{grad} \alpha^v\|;$$

here, as it is well-known, $C\alpha^c = \alpha^c$ and $g(C,C) = 2E$. Therefore

$$\frac{(\alpha^c)^2}{2E} \leq \| \operatorname{grad} \alpha^v\|^2 \Rightarrow (\alpha^c)^2 \leq 2E_R. \tag{39}$$

Evaluating both sides along one of the vector fields $\pm \operatorname{grad}_R \alpha$ it follows that

$$\begin{gathered} \| \operatorname{grad}_R \alpha\|^4 \leq \| \operatorname{grad}_R \alpha\|^2 \text{ and, consequently,} \\ 0 \leq \| \operatorname{grad}_R \alpha\|^2(1 - \| \operatorname{grad}_R \alpha\|^2) \Rightarrow \| \operatorname{grad}_R \alpha\|^2 \leq 1; \end{gathered} \tag{40}$$

the norm in the last formula (40) is, of course, taken with respect to the Riemannian metric γ and equality holds if and only if the condition (ii) is satisfied. □

Lemma 9. *For any tangent vector $v \in T_pM \setminus \{0\}$*

$$G_g\big(\, grad\, \alpha^v, grad^v_R \alpha, C\big)(v) = 0, \tag{41}$$

i.e. the system of vector fields $(grad\, \alpha^v, grad^v_R\alpha, C)$ are linearly dependent at the points of the punctured tangent space $T_pM \setminus \{0\}$.

Proof. Let $v \in T_pM \setminus \{0\}$ be an arbitrary tangent vector; we can obviously suppose that the Liouville vector field C and $\operatorname{grad} \alpha^v$ are linearly *independent* at the point v. In this case, according to Lemma 6 we have that

$$\mathcal{C}(X,X)_v = \theta_v(JX - \frac{\alpha^c}{2E}C)_v, \text{ where } \theta_v := \frac{g(\mathcal{C}(X,X), JX)}{\|JX\|^2 - \frac{(\alpha^c)^2}{2E}}(v) \tag{42}$$

is the Fourier coefficient of the tangent vector $\mathcal{C}(X,X)_v$ with respect to $(JX - \frac{\alpha^c}{2E}C)_v$. It is clear that the formula (42) also holds on a connected open neighbourhood $\mathcal{W} \subset T_pM$ of the point v. In what follows we restrict our investigations to the neighbourhood $\mathcal{W}$ without any further comment; the sign of the restriction will be omitted. Now we are going to calculate again the relation betwen the Riemann-Finsler metric g and γ. For the sake of brevity let us introduce the functions

$$\zeta := g(\mathcal{C}(X,X), JX) \text{ and } \eta := \|JX\|^2 - \frac{(\alpha^c)^2}{2E};$$

then, of course, $\theta = \frac{\zeta}{\eta}$. For any vector fields $Y, Z \in \mathfrak{X}(U)$ we have:

$$\begin{aligned}
Z^v\|JX\|^2 &= 2g(D_{Z^v}JX, JX) \overset{(26)}{=} -2g(\mathcal{C}(X,Z^c), JX) = \\
&\quad -2g(\mathcal{C}(X,X), Z^v) \overset{(42)}{=} -2\theta\Big((Z\alpha)\circ\pi - \frac{\alpha^c}{2E}Z^vE\Big), \\
Y^v\|JX\|^2 &= -2\theta\Big((Y\alpha)\circ\pi - \frac{\alpha^c}{2E}Y^vE\Big), \\
Y^v(Z^v\|JX\|^2) &= -2(Y^v\theta)\Big((Z\alpha)\circ\pi - \frac{\alpha^c}{2E}Z^vE\Big) + \\
&\quad + \frac{\theta}{E}\Big((Y\alpha)\circ\pi Z^vE + \alpha^c g(Y^v, Z^v) - \frac{\alpha^c}{E}(Y^vE)(Z^vE)\Big),
\end{aligned}$$

where

$$Y^v\theta = (Y^v\zeta)\frac{1}{\eta} - \frac{\zeta}{\eta^2}\Big(Y^v\|JX\|^2 - \frac{\alpha^c}{E}(Y\alpha)\circ\pi + \frac{(\alpha^c)^2}{2E^2}Y^vE\Big).$$

Since the lowered first Cartan tensor is totally symmetric, (26) shows that

$$
\begin{aligned}
&Y^v g(\mathcal{C}(X,X),JX) = g(D_{Y^v}\mathcal{C}(X,X),JX) - g(\mathcal{C}(X,X),\mathcal{C}(X,Y^c)) = \\
&= \; g((D_{Y^v}\mathcal{C})(X,X),JX) - 3g(\mathcal{C}(X,X),\mathcal{C}(X,Y^c)) \overset{(13)}{=} \\
&= \; g((D_{JX}\mathcal{C})(X,Y^c),JX) - 3g(\mathcal{C}(X,X),\mathcal{C}(X,Y^c)) \overset{(24)}{=} \\
&= \; -\frac{1}{2E}\Big((Y^vE)g(\mathcal{C}(X,X),JX) + 3\alpha^c g(\mathcal{C}(X,X),Y^v)\Big) + \\
&\quad -3g(\mathcal{C}(X,X),\mathcal{C}(X,Y^c)) \overset{(42)}{=} \\
&= \; -\frac{1}{2E}\Big((Y^vE)g(\mathcal{C}(X,X),JX) + 3\theta\alpha^c(Y\alpha)\circ\pi - 3\theta\frac{(\alpha^c)^2}{2E}Y^vE\Big) - \\
&\quad -3\theta^2\Big((Y\alpha)\circ\pi - \frac{\alpha^c}{2E}Y^vE\Big).
\end{aligned}
$$

Substituting these new formulas into (38) the relation between the metrics reduces to the following simple form:

$$
\begin{aligned}
&g_R(Y^v,Z^v) = Ag(Y^v,Z^v) + P(Y^vE)(Z^vE) + \\
&+ Q\Big((Y\alpha)\circ\pi Z^vE + (Z\alpha)\circ\pi Y^vE\Big) + R(Y\alpha)\circ\pi(Z\alpha)\circ\pi,
\end{aligned}
\tag{43}
$$

where, after a very long calculation, the coeffitients can be given in the following explicite way:

$$
\begin{aligned}
P \; &:= \; \alpha^c\frac{\theta}{2E}\Big(1 + \frac{\alpha^c}{\eta}(\frac{\alpha^c}{2E} + \theta)\Big), \; Q := -\Big(\theta + \alpha^c\frac{\theta}{\eta}(\frac{\alpha^c}{2E} + \theta)\Big), \\
R \; &:= \; 2E\frac{\theta}{\eta}(\frac{\alpha^c}{2E} + \theta)
\end{aligned}
$$

and the "main coefficient" $A := \|JX\|^2 + \theta\alpha^c$ must be positive on the neighbourhood $\mathcal{W}$ because the dimension of the tangent space T_pM is no less than 3. As a direct consequence of (43) we get the relation between the gradient vector fields $\operatorname{grad}\alpha^v$ and $\operatorname{grad}^v_R\alpha$:

$$
\begin{aligned}
A\operatorname{grad}^v_R\alpha \; &= \; \Big(1 - Q\operatorname{grad}^v_R\alpha(E) - R\|\operatorname{grad}_R\alpha\|^2\circ\pi\Big)grad\alpha^v - \\
&\quad -\Big(Q\|\operatorname{grad}_R\alpha\|^2\circ\pi + P\operatorname{grad}^v_R\alpha(E)\Big)C
\end{aligned}
$$

as was to be stated. □

Lemma 10. $\| \mathit{grad}_R\,\alpha\|^2(p) = 1$.

Proof. Suppose that $\|\operatorname{grad}_R \alpha\|^2(p) < 1$; then, by Lemma 8, it follows that the Liouville vector field C and $\operatorname{grad}\alpha^v$ is linearly independent at the point $v := \operatorname{grad}_R \alpha(p)$. On the other hand, since $\operatorname{grad}_R^v \alpha(v) = C_v$, the relation (37) reduces to the following simple form:

$$\operatorname{grad}\alpha^v(v) = \|JX\|^2(v)C_v - 2E(v)\mathcal{C}(X,X)_v \overset{(42)}{=} A(v)C_v - 2E(v)\theta(v)JX_v,$$

where, of course, $JX = \operatorname{grad}\alpha^v$. This means that the "main coefficient" A vanishes at the point v which is a contradiction. □

Remark 4. Without loss of generality we can suppose that $\alpha(p) = 0$; consider now the submanifold $N := \alpha^{-1}(0) \cap U$ together with the induced energy function $E\mid_{TN}$.

Lemma 11. *The functions*

$$\|\,grad\,\alpha^v\|^2 \;\; and \;\; Lg(\mathcal{C}(F\,grad\,\alpha^v, F\,grad\,\alpha^v), grad\,\alpha^v),$$

where L is the fundamental function of the Finsler manifold (M, E), are constant on the tangent space T_pN.

Proof. First of all we are going to prove that the Finsler manifolds $(N, E\mid_{TN})$ and $(N, E_R\mid_{TN})$ are conformally equivalent at the point p; more precisely, for any tangent vector $v \in T_pN \setminus \{0\}$,

$$g_R(Y^v, Z^v)(v) = \|grad\alpha^v\|^2(v)g(Y^v, Z^v)(v), \tag{44}$$

where the vector fields $Y, Z \in \mathfrak{X}(U)$ are, of course, tangential to the submanifold N at the point p. (Note that g_R is just the vertical lift of the Riemannian metric γ!) The following relations are trivial:

$$\begin{aligned} &v \in T_pN \Longleftrightarrow \alpha^c(v) = 0,\\ &\operatorname{grad}\alpha^v(v) \perp C_v \text{ with respect to the metric g},\\ &\mathcal{C}(X,X)_v = \tfrac{\zeta}{\|\operatorname{grad}\alpha\|^2}(v)\operatorname{grad}\alpha^v(v); \end{aligned} \tag{45}$$

here, as above, $\zeta := g(\mathcal{C}(X,X), JX)$ and $X := F\operatorname{grad}\alpha^v \Rightarrow JX = \operatorname{grad}\alpha^v$. Since the Liouville vector field C and $\operatorname{grad}\alpha^v$ are perpendicular at any point $v \in T_pN \setminus \{0\}$, they are linearly independent at the same time. The formulas in the proof of Lemma 9 shows that

$$Z^v\|JX\|^2\mid_{T_pN} = 0,\; Y^v\|JX\|^2\mid_{T_pN} = 0 \text{ and } Y^v(Z^v\|JX\|^2)\mid_{T_pN} = 0$$

which imply the relation (44). Using Knebelman's observation at the point $p \in N$, it follows that the "scale function"

$$\|JX\|^2 = \|\operatorname{grad}\alpha^v\|^2$$

is constant on the tangent space T_pN. On the other hand, for any point $v \in T_pN$,

$$(Y^v)_v\Big(Lg(\mathcal{C}(X,X),JX)\Big) = (Y^vL)_v g(\mathcal{C}(X,X),JX)_v + \\ +L(v)(Y^v)_v g(\mathcal{C}(X,X),JX) = (Y^vL)_v g(\mathcal{C}(X,X),JX)_v - \\ -\frac{L}{2E}(v)(Y^vE)_v g(\mathcal{C}(X,X),JX)_v = 0$$

using the formulas in the proof of Lemma 9 again. □

Lemma 12. *Let $v \in T_pN \setminus \{0\}$ be an arbitrarily fixed tangent vector and consider the integral curve*

$$c\colon \mathbb{R} \to T_pN,\ c(t) := v + t\, grad_R\, \alpha(p)$$

of the vector field $grad_R^v \alpha$. The function

$$y(t) := E \circ c(t)$$

satisfies the following differential equation:

$$2E_R(v)y(t)y''(t) + 2ty(t)y'(t) - (E_R(v) + \frac{t^2}{2})(y')^2(t) - 2y^2(t) = 0. \tag{46}$$

Proof. The differential equation (46) can be deduced from the relation (41), which implies that

$$G_g\big(\operatorname{grad}\alpha^v, \operatorname{grad}_R^v\alpha, C\big) \circ c(t) = 0.$$

Taking into account the following simple facts

$$g(\operatorname{grad}\alpha^v, \operatorname{grad}_R^v\alpha) \circ c = \|\operatorname{grad}_R\alpha\|^2(p) = 1 \ \ \text{(see Lemma 10)}, \\ g(\operatorname{grad}_R^v\alpha, \operatorname{grad}_R^v\alpha) \circ c = \operatorname{grad}_R^v\alpha\big(\operatorname{grad}_R^v\alpha(E)\big) \circ c = y'', \\ g(\operatorname{grad}_R^v\alpha, C) = \operatorname{grad}_R^v\alpha(E) \ \text{ and } \ \operatorname{grad}_R^v\alpha(E) \circ c = y', \\ g(C,C) = 2E \ \text{ and } \ \alpha^c \circ c(t) = t^2, \\ E\|\operatorname{grad}\alpha^v\|^2 = E_R \ \text{ and } \ E_R \circ c(t) = E_R(v) + \frac{1}{2}t^2,$$

the proof is a straightforward calculation. □

Now we are going to solve this differential equation to complete the proof of Proposition 2. As it can be easily seen, if $z := \frac{y'}{y}$ then z satisfies the following first order Ricatti-type differential equation:

$$2E_R(v)z'(t) + 2tz(t) + \frac{2E_R(v) - t^2}{2}z^2(t) - 2 = 0. \tag{47}$$

Since

$$\begin{aligned} y'(0) &= (\mathrm{grad}^v_R\alpha)_v E = (\mathrm{grad}^v_R\alpha)_v(\frac{E_R}{\|JX\|^2}) = \\ &= \frac{\alpha^c}{\|JX\|^2}(v) - \frac{E_R}{\|JX\|^4}(v)(\mathrm{grad}^v_R\alpha)_v\|JX\|^2 = \\ &= -\frac{E_R}{\|JX\|^4}(v)(\mathrm{grad}^v_R\alpha)_v\|JX\|^2 = \\ &= -2\frac{E_R}{\|JX\|^4}(v)g(D_{\mathrm{grad}^v_R\alpha}JX, JX)(v) \stackrel{(26)}{=} \\ &= 2\frac{E_R}{\|JX\|^4}(v)g(\mathcal{C}(X,X), \mathrm{grad}^v_R\alpha)(v) \stackrel{(45)}{=} 2\frac{E_R}{\|JX\|^6}(v)\zeta(v) = \\ &= \frac{L_R^2}{L\|JX\|^6}(v)(L\zeta)(v) = \frac{L_R}{\|JX\|^5}(L\zeta)(v), \\ y(0) &= E(v) = \frac{E_R}{\|JX\|^2}(v) = \frac{L_R^2}{2\|JX\|^2}(v), \end{aligned}$$

we have, by Lemma 11, the initial condition

$$z(0) = 2\frac{L\zeta}{\|JX\|^3}(v)\frac{1}{L_R(v)} = \frac{K(p)}{L_R(v)}, \tag{48}$$

where $K(p) \in \mathbb{R}$ is a constant. As it can be easily seen, the function

$$z\colon \mathcal{I} \to \mathbb{R},\ z(t) := 2\frac{2t + K(p)L_R(v)}{2t^2 + tK(p)L_R(v) + 4E_R(v)} \tag{49}$$

is the uniquely solution of the Cauchy-problem. Therefore

$$\frac{(E\circ c)'}{E\circ c}\,|_{\mathcal{I}} = z;$$

since the left hand side is well-defined on the *whole* set of real numbers it follows that $-4 < K(p) < 4$ and, consequently,

$$\frac{(E\circ c)'}{E\circ c}(t) = 2\frac{2t + K(p)L_R(v)}{2t^2 + tK(p)L_R(v) + 4E_R(v)} \tag{50}$$

for any real number $t \in \mathbb{R}$. Integrating (50) with respect to t, we have that

$$y(t) = 4K^*(p)\left(E_R\circ c(t) + K(p)L_R(v)\frac{t}{4}\right) \tag{51}$$

$$\exp\frac{2K(p)}{\sqrt{16-K^2(p)}}(\arctan\frac{4t + K(p)L_R(v)}{L_R(v)\sqrt{16-K^2(p)}} - \arctan\frac{K(p)}{\sqrt{16-K^2(p)}}),$$

where

$$K^*(p) := \frac{1}{4\|grad\alpha^v\|^2(v)};$$

the right hand side is depend only on the "position" as we have proved in Lemma 11. In view of Remark 3 this result shows that S_p has *constant* positive curvature. Therefore, we can suppose that for any $q \in U$ the indicatrix hypersurface S_q also has positive curvature and the argumentation is similar as above.

§4. On conformal equivalence of almost spherical Berwald manifolds

Proposition 3. *Keeping our previous notations let E be a non-Riemannian Asanov-type Finslerian metric function at the point $p \in M$, i.e. suppose that $K(p) \neq 0$. If*

$$T_pM = W \oplus \{tw \mid \ t \in \mathbb{R}\} =: W \oplus \mathcal{L}(w)$$

is a direct composition such that for any $v \in W$ and $t \in \mathbb{R}$ the symmetry property

$$E(v + tw) = E(-v + tw) \tag{52}$$

is satisfied, then

$$W = Ker(\alpha^c \mid_{T_pM}) = T_pN \quad and \quad w \in \mathcal{L}(grad_R\, \alpha(p)).$$

Proof. Let $v \in W$ be an arbitrary tangent vector such that

$$v = v_0 + t_0 \operatorname{grad}_R \alpha(p), \quad \text{where} \quad v_0 \in T_pN;$$

first of all we suppose that $v_0 \neq 0$. Using the symmetry property (52) it follows that $E(v) = E(-v)$ and, consequently,

$$\left(E_R(v) + K(p)L_R(v_0)\frac{t_0}{4}\right) f(t_0) = \left(E_R(v) - K(p)L_R(v_0)\frac{t_0}{4}\right) f(-t_0), \tag{53}$$

where the function f is defined by the formula

$$f(t) := \exp \frac{2K(q)}{\sqrt{16 - K^2(q)}} \arctan \frac{4t + K(q)L_R(v_0)}{L_R(v_0)\sqrt{16 - K^2(q)}}.$$

It can be easily seen that

$$f \text{ is strictly increasing} \Leftrightarrow K(p) > 0,$$
$$f \text{ is strictly decreasing} \Leftrightarrow K(p) < 0.$$

Since f is positive, these observations are also true for the function f^2; therefore, for any $t \in \mathbb{R}$

$$tK(p)\big(f^2(t) - f^2(-t)\big) \geq 0.$$

On the other hand, according to the relation (53)

$$E_R(v)\big(f(t_0) - f(-t_0)\big) + K(p)L_R(v_0)\frac{t_0}{4}\big(f(t_0) + f(-t_0)\big) = 0$$

and both members on the left hand side have the same sign because their product is no less than 0 as we have seen above. This means that $t_0 = 0$, i.e. $v \in T_pN$. If $v_0 = 0$, then $v = t_0 \operatorname{grad}_R \alpha(p)$ and the symmetry property (52) gives that

$$t_0^2\left(\exp\frac{K(p)}{\sqrt{16-K^2(p)}}\pi - \frac{1}{\exp\frac{K(p)}{\sqrt{16-K^2(p)}}\pi}\right) = 0$$

and, consequently, $t_0 = 0$.

Consider now the subspace $\mathcal{L}(w) \subset T_pM$; we put

$$w = w_0 + t_0 \operatorname{grad}_R \alpha(p),$$

where $w_0 \in T_pN$ and $t_0 := w(\alpha)$. If $v := \frac{1}{t_0}w_0$ and $t := \frac{1}{t_0}$ then the symmetry property (52) gives that

$$E(\operatorname{grad}_R \alpha(p)) = E(\frac{2}{t_0}w_0 + \operatorname{grad}_R \alpha(p)).$$

Let us define a function

$$j\colon [0,2] \to \mathbb{R},\ j(t) := E(\frac{t}{t_0}w_0 + \operatorname{grad}_R \alpha(p)) \Rightarrow j(0) = j(2);$$

since j is continuous and it is differentiable at any inner point, there exists a real number $0 < t < 2$ such that $j'(t) = 0$. On the other hand, according to (29) an easy calculation shows that for *any* inner point t:

$$j'(t) = 2tE_R(\frac{1}{t_0}w_0)$$
$$\exp\frac{2K(p)}{\sqrt{16-K^2(p)}}(\arctan\frac{4 + K(p)L_R(\frac{t}{t_0}w_0)}{L_R(\frac{t}{t_0}w_0)\sqrt{16-K^2(p)}} - \arctan\frac{K(p)}{\sqrt{16-K^2(p)}}),$$

and, consequently, $E(\frac{1}{t_0}w_0) = 0 \Rightarrow w_0 = 0$, which implies our statement. □

Theorem 1. *Let* (M, E) *be an almost spherical Berwald manifold of dimension* $n \geq 3$*; then the Berwald-type conformal changes of its Riemann-Finsler metric must be homothetic unless the manifold is Riemannian, i.e. one of the following cases is satisfied:*

(i) $b \equiv 0$*;*

(ii) $b \equiv n$ *and, consequently,* (M, E) *is a Riemannian manifold.*

Proof. Suppose that there exists a nontrivial Berwald-type conformal change

$$g_\alpha = \varphi g \quad (\varphi = \exp \circ \alpha^v)$$

of the metric g, i.e. (M, E_α) is a Berwald manifold and $d_p\alpha \neq 0$ $(p \in M)$. Proposition 2 implies the energy function E to be conformal to an Asanov-type Finslerian metric function on a connected open neighbourhood U of the point p. This means that for any $q \in U$ the indicatrix hypersurface S_q has constant sectional curvature and the symmetry property

$$E(v + t \operatorname{grad}_R \alpha(p)) = E(v - t \operatorname{grad}_R \alpha(p)),$$

where $v \in T_qM$ is tangential to the level hypersurface $N_r := \alpha^{-1}(r) \cap U$ containing q is satisfied; notations as in the proof of Proposition 2. Since the canonical Barthel endomorphism h arises from a linear connection ∇ on the underlying manifold M, it follows that the punctured tangent spaces as Riemannian manifolds are isometric to each other (cf. the proof of Lemma 2). Therefore, the indicatrices have the *same* constant curvature which means that the function K is constant on the neighbourhood U. We can obviously suppose that this Asanov-type Finslerian metric function is non-Riemannian, i.e. $K \neq 0$. Taking into account the fact that the parallel transport with respect to ∇ preserves the Finslerian norm, Proposition 3 implies that the tangent spaces of the level hypersurfaces N_r are also invariant under the parallel transport with respect to ∇. In other words, these hypersurfaces are totally geodesic submanifolds of the Berwald manifold (M,E), i.e. for example

$$S\alpha^c \mid_{TN} = 0,$$

where $N := \alpha^{-1}(0) \cap U$ - without loss of generality we can suppose that $\alpha(p) = 0$. Starting out from the Berwald manifold (M, E_α) and the Berwald-type conformal change $g = \frac{1}{\varphi} g_\alpha$ of the metric g_α, we also have that

$$S_\alpha \alpha^c \mid_{TN} = 0 \Rightarrow E \| \operatorname{grad} \alpha^v \|^2 \mid_{TN} = 0$$

using the transformation formula (19), see also the proof of Lemma 7. This is obviously contradicts to the regularity property $d_p\alpha \neq 0$. Therefore, the exterior derivative of the function α vanishes and the conformal change is homothetic. If $K = 0$ then the manifold is locally Riemannian and, consequently, it is a Riemannian manifold; the proof can be easily realized by the help of the parallel transport with respect to ∇, see e.g. Proposition 3 in [18]. □

§5. An application: on the uniqueness of Wagner stuctures for Finsler manifolds

Corollary 1. *Suppose that (M, E) is a positive definite almost spherical Wagner manifold of dimension $n \geq 3$; then the Wagner structure or, in an equivalent way, the linear Wagner connection on the underlying manifold M is uniquely determined unless the manifold is Riemannian.*

Proof. As it is well-known (see e.g. [17], [15] and [8]), if there exists a linear Wagner connection on a Finsler manifold (M, E), then it is conformal to a Berwald manifold and vica-verse. Explicitly, if

$$g_\alpha = \varphi g \quad (\varphi = \exp \circ \alpha^v)$$

is a Berwald-type conformal change of the metric g, then the Wagner connection induced by $-\frac{1}{2}\alpha$ is linear. According to Theorem 1, for any positive definite almost spherical Wagner manifold $b \equiv 0$, i.e. the Berwald-type conformal changes can be written in the form

$$\varphi_\lambda := \exp \circ \big(\alpha^v + \lambda\big),$$

where λ is an arbitrary constant. Since the exterior derivative of a constant function vanishes, the Wagner connections induced by the functions $-\frac{1}{2}\alpha$ and $-\frac{1}{2}(\alpha + \lambda)$ coincide as was to be stated; for the details see [17], [16] and [6]. □

Remark 5. For a detailed discussion of the two-dimensional case, see [10].

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