

A note on a theorem of A. Saeki and R. Ikehata

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Abstract. We extend the theorem of A. Saeki and R. Ikehata for the wave equation with linear dissipation.

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1. Introduction

In this note we consider the wave equation of the form

$$(1.1) \quad w_{tt} - \Delta w + b(x)w_t = 0 \quad \text{in } (0, +\infty) \times \Omega,$$

$$(1.2) \quad w(0, x) = w_0(x), \quad w_t(0, x) = w_1(x) \quad \text{in } \Omega,$$

$$(1.3) \quad w(t, x) = 0 \quad \text{on } (0, +\infty) \times \partial\Omega,$$

where $N \geq 3$, $\Omega(\subset \mathbb{R}^N)$ is an unbounded domain with smooth boundary $\partial\Omega$ and $b(\cdot) \in C^1(\Omega)$ is a positive and bounded function: $0 < b_0 \leq b(x) \leq b_1$ in Ω for some constants b_0 and b_1 . In the following we denote $L^2 = L^2(\Omega)$, $H_0^1 = H_0^1(\Omega)$ e.t.c.. If we assume $\{w_0, w_1\} \in H_0^1 \times L^2$ then we know the following estimate holds for the solutions $w(t, \cdot) \in C^0([0, \infty); H_0^1) \cap C^1([0, \infty); L^2)$ of (1.1) – (1.3):

$$\begin{aligned} (1+t)\|w(t)\|_E^2 + \|w(t)\|_{L^2}^2 + \int_0^t \{(1+\tau)\|w_t(\tau)\|_{L^2}^2 + \|\nabla w(\tau)\|_{L^2}^2\} d\tau \\ \leq C_1\{\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2\} \end{aligned}$$

for some positive constant C_1 (cf. Hirosawa–Nakazawa[1]), where

$$\|w(t)\|_E^2 = \frac{1}{2} (\|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2)$$

is the energy at time $t(\geq 0)$.

Recently, A. Saeki and R. Ikehata showed the following

Theorem 1 ([3]). (1) Assume $\{w_0, w_1\} \in H_0^1 \cap L^{2,1} \times L^{2,1}$, where

$$L^{2,1} = \left\{ f \mid \|f\|_{L^{2,1}}^2 = \int_{\Omega} (1 + |x|)^2 |f(x)|^2 dx < \infty \right\}.$$

Then the solutions $w(t, \cdot)$ of (1.1) – (1.3) satisfy the following inequalities:

$$(1.4) \quad (1+t)^2 \|w(t)\|_E^2 \leq C_2 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$(1.5) \quad (1+t) \|w(t)\|_{L^2}^2 \leq C_3 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2$$

for some positive constants C_2 and C_3 , where

$$\|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2 \equiv \|w_0\|_{H^1}^2 + \|w_0\|_{L^{2,1}}^2 + \|w_1\|_{L^{2,1}}^2.$$

(2) Moreover assume that $w_1 + b(x)w_0 = 0$. Then the following inequalities hold:

$$(1.6) \quad (1+t)^3 \|w(t)\|_E^2 \leq C_4 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$(1.7) \quad (1+t)^2 \|w(t)\|_{L^2}^2 \leq C_5 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2$$

for some positive constants C_4 and C_5 .

In this theorem, essential lemma is the following:

Lemma 2 ([3]). Under the same assumption as in Theorem 1 (1),

$$(1.8) \quad \|w(t)\|_{L^2}^2 + \int_0^t \|w(\tau)\|_{L^2}^2 d\tau \leq C_6 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2$$

for some positive constant C_6 .

We shall give the simple proof of Theorem 1 and its extension:

Theorem 3. (1) Under the same condition as in Theorem 1 (1),

$$(1.9) \quad (1+t)^2 \|w(t)\|_E^2 + \int_0^t \{ (1+\tau)^2 \|w_t(\tau)\|_{L^2}^2 + (1+\tau) \|\nabla w(t)\|_{L^2}^2 \} d\tau \leq C_7 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$(1.10) \quad \lim_{t \rightarrow +\infty} (1+t)^2 \|w(t)\|_E^2 = 0,$$

$$(1.11) \quad \lim_{t \rightarrow +\infty} (1+t) \|w(t)\|_{L^2}^2 = 0,$$

hold for some positive constant C_7 .

(2) Under the same assumption as in Theorem 1 (2),

$$(1.12) \quad (1+t)^4 \|w(t)\|_E^2 + \int_0^t \{(1+\tau)^4 \|w_t(\tau)\|_{L^2}^2 + (1+\tau)^3 \|\nabla w(\tau)\|_{L^2}^2\} d\tau \\ \leq C_8 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$(1.13) \quad (1+t)^2 \|w(t)\|_{L^2}^2 + \int_0^t (1+\tau)^2 \|w(\tau)\|_{L^2}^2 d\tau \\ \leq C_9 \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$(1.14) \quad \lim_{t \rightarrow +\infty} (1+t)^4 \|w(t)\|_E^2 = 0,$$

$$(1.15) \quad (1+t)^3 \|w(t)\|_{L^2}^2 \leq C_{10} \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2,$$

$$(1.16) \quad \lim_{t \rightarrow +\infty} (1+t)^3 \|w(t)\|_{L^2}^2 = 0,$$

hold for some positive constants C_8 , C_9 and C_{10} .

Remark. Obviously, we can treat (1.1) – (1.3) with $N = 2$ and the Cauchy problem in $\mathbb{R}^N$ with $N \geq 3$ ([3]).

2. Proof of Theorem 3 (1)

We assume that $\varphi(t)$ and $\psi(t)$ are the smooth, non-decreasing and non-negative functions of t . Multiplying $\{\varphi w_t + \psi w\}$ the both sides of (1.1), and integrating on Ω , we obtain (c.f., Mochizuki–Nakazawa [2])

$$(2.1) \quad \frac{d}{dt} \int_{\Omega} X(t, x) dx + \int_{\Omega} Z(t, x) dx = 0,$$

where

$$(2.2) \quad X(t, x) = \frac{\varphi(t)}{2} \{w_t^2(t, x) + |\nabla w(t, x)|^2\} + \psi(t) w_t(t, x) w(t, x) \\ + \frac{b(x)\psi(t) - \psi_t(t)}{2} w(t, x)^2,$$

$$(2.3) \quad Z(t, x) = \left\{ b(x)\varphi(t) - \frac{\varphi_t(t)}{2} - \psi(t) \right\} w_t(t, x)^2 \\ + \left\{ \psi(t) - \frac{\varphi_t(t)}{2} \right\} |\nabla w(t, x)|^2 + \frac{\psi_{tt}(t) - b(x)\psi_t(t)}{2} w(t, x)^2.$$

Firstly we shall show (1.9). Put $\varphi(t) = 2(\frac{3}{b_0} + t)^2$ and $\psi(t) = 3(\frac{3}{b_0} + t)$. Then easy computations give

$$\begin{aligned} b(x)\varphi(t) - \frac{\varphi_t(t)}{2} - \psi(t) &\geq C_{11}(1+t)^2, \\ \psi(t) - \frac{\varphi_t(t)}{2} &\geq C_{12}(1+t), \\ \frac{\psi_{tt}(t) - b(x)\psi_t(t)}{2} &\geq -\frac{b(x)\psi_t(t)}{2} \geq -C_{13} \end{aligned}$$

for some positive constants C_{11} , C_{12} and C_{13} . Thus we have

$$(2.4) \quad \int_{\Omega} Z(t, x) dx \geq C_{14} \{ (1+t)^2 \|w_t(t)\|_{L^2}^2 + (1+t) \|\nabla w(t)\|_{L^2}^2 - \|w(t)\|_{L^2}^2 \}$$

for some positive constant C_{14} . Next choosing η as $0 < \eta < 1$, we find

$$\begin{aligned} X(t, x) &\geq \frac{(1-\eta)}{2} \varphi(t) \{ w_t(t, x)^2 + |\nabla w(t, x)|^2 \} \\ &\quad + \left\{ \frac{b(x)\psi(t) - \psi_t(t)}{2} - \frac{\psi(t)^2}{2\eta\varphi(t)} \right\} w(t, x)^2. \end{aligned}$$

As is easily seen, there exists a positive constant C_{15} such that

$$\frac{b(x)\psi(t) - \psi_t(t)}{2} - \frac{\psi(t)^2}{2\eta\varphi(t)} \geq -\frac{\psi(t)^2}{2\eta\varphi(t)} \geq -C_{15}$$

holds. From this, we have

$$(2.5) \quad \int_{\Omega} X(t, x) dx \geq C_{16} \{ (1+t)^2 \|w(t)\|_E^2 - \|w(t)\|_{L^2}^2 \}.$$

On the other hand,

$$(2.6) \quad \int_{\Omega} X(0, x) dx \leq C_{17} (\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2)$$

Integrating (2.1) over $[0, t]$ and using (2.4), (2.5), (2.6) and Lemma 2, we obtain (1.9).

Next we shall show (1.10). By (1.9), we find

$$\liminf_{t \rightarrow +\infty} (1+t)^2 \|w(t)\|_E^2 = 0.$$

Integration the both sides of

$$\frac{d}{dt} \{ (1+t)^2 \|w(t)\|_E^2 \} = 2(1+t) \|w(t)\|_E^2 - (1+t)^2 \int_{\Omega} b(x) w_t(t, x)^2 dx,$$

where we have used

$$\frac{d}{dt} \|w(t)\|_E^2 = - \int_{\Omega} b(x) w_t(t, x)^2 dx,$$

on $[t_1, t_2]$ ($0 \leq t_1 \leq t_2 < +\infty$) gives

$$(2.7) \quad \begin{aligned} & |(1+t_2)^2 \|w(t_2)\|_E^2 - (1+t_1)^2 \|w(t_1)\|_E^2| \\ & \leq 2 \int_{t_1}^{t_2} (1+\tau) \|w(\tau)\|_E^2 d\tau + b_1 \int_{t_1}^{t_2} (1+\tau)^2 \|w_t(\tau)\|_{L^2}^2 d\tau. \end{aligned}$$

Using (1.9), we find the right hand side of (2.7) tends to 0 as t_1 and $t_2 \rightarrow +\infty$ and conclude (1.10).

Finally we shall show (1.11). Lemma 2 gives

$$\liminf_{t \rightarrow +\infty} (1+t) \|w(t)\|_{L^2}^2 = 0.$$

From

$$\frac{d}{dt} \{ (1+t) \|w(t)\|_{L^2}^2 \} = \|w(t)\|_{L^2}^2 + 2(1+t)(w(t), w_t(t))_{L^2},$$

where $(\cdot, \cdot)_{L^2}$ denotes the inner product in L^2 , we obtain

$$\begin{aligned} & |(1+t_2) \|w(t_2)\|_{L^2}^2 - (1+t_1) \|w(t_1)\|_{L^2}^2| \\ & \leq 2 \int_{t_1}^{t_2} \|w(\tau)\|_{L^2}^2 d\tau + \int_{t_1}^{t_2} (1+\tau)^2 \|w_t(\tau)\|_{L^2}^2 d\tau \rightarrow 0 \end{aligned}$$

as t_1 and $t_2 \rightarrow +\infty$ by Lemma 2 and (1.9), where we have used

$$|2(1+t)(w(t), w_t(t))_{L^2}| \leq \|w(t)\|_{L^2}^2 + (1+t)^2 \|w_t(t)\|_{L^2}^2.$$

3. Proof of Theorem 3 (2)

Firstly we shall show (1.12) and (1.13). We put $\varphi(t) = (\frac{13}{b_0} + t)^4$ and $\psi(t) = 3(\frac{13}{b_0} + t)^3$. Easy computations give

$$\begin{aligned} b(x)\varphi(t) - \frac{\varphi_t(t)}{2} - \psi(t) & \geq C_{18}(1+t)^4, \\ \psi(t) - \frac{\varphi_t(t)}{2} & \geq C_{19}(1+t)^3, \\ \frac{\psi_{tt}(t) - b(x)\psi_t(t)}{2} & \geq -C_{20}(1+t)^2 \end{aligned}$$

for some positive constants C_{18} , C_{19} and C_{20} . Thus we have

$$(3.1) \quad \begin{aligned} & \int_{\Omega} Z(t, x) dx \\ & \geq C_{21} \{ (1+t)^4 \|w_t(t)\|_{L^2}^2 + (1+t)^3 \|\nabla w(t)\|_{L^2}^2 - (1+t)^2 \|w(t)\|_{L^2}^2 \} \end{aligned}$$

for some positive constant C_{21} .

Similarly as in section 2, there exists a positive constant C_{22} such that

$$\frac{b(x)\psi(t) - \psi_t(t)}{2} - \frac{\psi(t)^2}{2\eta\varphi(t)} \geq -C_{22}(1+t)^2$$

holds. From this, we have

$$(3.2) \quad \int_{\Omega} X(t, x) dx \geq C_{23} \{ (1+t)^4 \|w(t)\|_E^2 - (1+t)^2 \|w(t)\|_{L^2}^2 \}.$$

Integrating (2.1) over $[0, t]$ and using (3.1), (3.2) and (2.6), we obtain

$$(3.3) \quad \begin{aligned} & (1+t)^4 \|w(t)\|_E^2 + \int_0^t \{ (1+\tau)^4 \|w_t(\tau)\|_{L^2}^2 + (1+\tau)^3 \|\nabla w(\tau)\|_{L^2}^2 \} d\tau \\ & \leq C_{24} \left(\|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2 + (1+t)^2 \|w(t)\|_{L^2}^2 + \int_0^t (1+\tau)^2 \|w(\tau)\|_{L^2}^2 d\tau \right). \end{aligned}$$

Put

$$u(t, x) = \int_0^t w(\tau, x) d\tau.$$

Noting the assumption $w_1(x) + b(x)w_0(x) = 0$, we find u satisfies (1.1) – (1.3) with $u(0, x) = 0$ and $u_t(0, x) = w_0(x)$. Applying Theorem 3 (1), (1.9) and noting $u_t(t, x) = w(t, x)$, we obtain

$$(3.4) \quad (1+t)^2 \|w(t)\|_{L^2}^2 + \int_0^t (1+\tau)^2 \|w(\tau)\|_{L^2}^2 d\tau \leq C_{25} \|(w_0, w_1)\|_{H^1 \cap L^{2,1} \times L^{2,1}}^2.$$

(3.3) and (3.4) give (1.12) and (1.13).

Next, we shall show (1.14). (1.12) gives

$$\liminf_{t \rightarrow +\infty} (1+t)^4 \|w(t)\|_E^2 = 0.$$

Integration the both sides of

$$\begin{aligned} & \frac{d}{dt} \{ (1+t)^4 \|w(t)\|_E^2 \} \\ & = 4(1+t)^3 \|w(t)\|_E^2 - (1+t)^4 \int_{\mathbb{R}^N} b(x) w_t(t, x)^2 dx \end{aligned}$$

over $[t_1, t_2]$ ($0 \leq t_1 \leq t_2 < \infty$) gives

$$\begin{aligned} & \left| ((1+t_2)^4 \|w(t_2)\|_E^2 - (1+t_1)^4 \|w(t_1)\|_E^2) \right| \\ & \leq 4 \int_{t_1}^{t_2} (1+\tau)^3 \|w(\tau)\|_E^2 d\tau + b_1 \int_{t_1}^{t_2} (1+\tau)^4 \|w_t(\tau)\|_{L^2}^2 d\tau \rightarrow 0 \end{aligned}$$

as t_1 and $t_2 \rightarrow +\infty$ by (1.12) and conclude (1.14).

Next we shall show (1.15). Put $\varphi(t) = 0$ and $\psi(t) = (1+t)^3$. Then we have

$$\begin{aligned} |\psi(t)(w_t(t), w(t))_{L^2}| & \leq C_{26} \{ (1+t)^4 \|w_t(t)\|_{L^2}^2 + (1+t)^2 \|w(t)\|_{L^2}^2 \}, \\ \frac{b(x)\psi(t) - \psi_t(t)}{2} & \geq C_{27}(1+t)^3 - C_{28}(1+t)^2. \end{aligned}$$

From these we obtain

$$\begin{aligned} & \int_{\Omega} X(t, x) dx \\ (3.5) \quad & \geq C_{29}(1+t)^3 \|w(t)\|_{L^2}^2 - C_{30} \{ (1+t)^4 \|w_t(t)\|_{L^2}^2 + (1+t)^2 \|w(t)\|_{L^2}^2 \} \end{aligned}$$

Moreover,

$$(3.6) \quad \int_{\Omega} Z(t, x) dx \geq -C_{31} \{ (1+t)^3 \|w_t(t)\|_{L^2}^2 + (1+t)^2 \|w(t)\|_{L^2}^2 \}$$

Integrating the both sides of (2.1) over $[0, t]$ and using (3.5), (3.6) and (2.6), we obtain

$$\begin{aligned} & (1+t)^3 \|w(t)\|_{L^2}^2 \\ & \leq C_{32} \{ (1+t)^4 \|w_t(t)\|_{L^2}^2 + (1+t)^2 \|w(t)\|_{L^2}^2 \\ & + \int_0^t (1+\tau)^3 \|w_t(\tau)\|_{L^2}^2 d\tau + \int_0^t (1+\tau)^2 \|w(\tau)\|_{L^2}^2 d\tau + \|w_0\|_{H^1}^2 + \|w_1\|_{L^2}^2 \}. \end{aligned}$$

Noting (1.12) and (1.13), we have (1.15).

Finally, we shall show (1.16). By (1.13), we find

$$\liminf_{t \rightarrow +\infty} (1+t)^3 \|w(t)\|_{L^2}^2 = 0.$$

Integration the both sides of

$$\begin{aligned} & \frac{d}{dt} \{ (1+t)^3 \|w(t)\|_{L^2}^2 \} \\ & = 3(1+t)^2 \|w(t)\|_{L^2}^2 + 2(1+t)^3 (w_t(t), w(t))_{L^2} \end{aligned}$$

over $[t_1, t_2]$ ($0 \leq t_1 \leq t_2 < +\infty$) gives

$$\begin{aligned} & \left| (1+t_2)^3 \|w(t_2)\|_{L^2}^2 - (1+t_1)^3 \|w(t_1)\|_{L^2}^2 \right| \\ & \leq 4 \int_{t_1}^{t_2} (1+\tau)^2 \|w(\tau)\|_{L^2}^2 d\tau + \int_{t_1}^{t_2} (1+\tau)^4 \|w_t(\tau)\|_{L^2}^2 d\tau \rightarrow 0 \end{aligned}$$

as t_1 and $t_2 \rightarrow +\infty$ by (1.12) and (1.13).

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