On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups

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Abstract. We discuss the existence of an orthogonal basis consisting of decomposable vectors for some symmetry classes of tensors associated with certain subgroups of the full symmetric group. The dimensions of these symmetry classes of tensors are also computed.

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§1. Introduction

Denote by $S_n$ the symmetric group on $\{1,2,\ldots,n\}$. Let $V$ be a unitary complex vector space of dimension $m$. Suppose $n$ is an integer $\geq 2$. Let $^nV$ be the $n$-th tensor power of $V$, and write $v := v_1 \cdot v_2 \cdots v_n$ for the tensor product of the indicated vectors.

For $\sigma \in S_n$, there is a (unique) linear operator $P(\sigma^{-1})$ on $^nV$ which has the effect $P(\sigma^{-1})(v_1 \cdot v_2 \cdots v_n) := v_{\sigma(1)} \cdot v_{\sigma(2)} \cdots v_{\sigma(n)}$, for all $v_1, v_2, \ldots, v_n \in V$. Let $G$ be a subgroup of $S_n$ and $\lambda$ be an irreducible complex character of $G$. We define $T(G, \lambda)$ as a linear operator on $^nV$ with the following definition

\begin{equation}
T(G, \lambda) := \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma)P(\sigma).
\end{equation}

With respect to the induced inner product in $^nV$, $T(G, \lambda)$ is an orthogonal projection onto its range $V^n_\lambda(G)$, (see [3], [8]). Let $I(G)$ be the set of all the irreducible complex characters of $G$. It follows from the orthogonality relations
for characters that \{T(G, \lambda) \mid \lambda \in I(G)\} is a set of annihilating idempotents which sum to the identity.

The image of \( v := v_1 \cdot v_2 \cdot \cdots \cdot v_n \) under \( T(G, \lambda) \) is denoted by \( v^\lambda := v_1^* \cdot v_2^* \cdots v_n^* \) and it is called a decomposable tensor. \( V^n(G) \) is called the symmetry class of tensors associated with \( G \) and \( \lambda \), and the dimension of \( V^n(G) \) is

\[
\dim V^n(G) = \frac{\lambda(1)}{|G|} \sum_{\sigma \in G} \lambda(\sigma) m^c(\sigma)
\]

where \( c(\sigma) \) is the number of cycles, including cycles of length one, in the disjoint cycle decomposition of \( \sigma \), (see [7]). With respect to the induced inner product in \( ^nV \), and the orthogonal relations for characters we have

\[
^nV = \bigoplus_{\chi \in I(G)} V^n_\chi(G)
\]

which is an orthogonal direct sum.

Let \( \Gamma_m^n \) be the set of all sequences \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \), 1 \( \cdot \alpha_i \cdot m \), so \( \alpha \) is a mapping from a set of \( n \) elements into a set of \( m \) elements. Then the group \( G \) acts on \( \Gamma_m^n \) by \( \sigma \cdot \alpha := \alpha \circ \sigma^{-1} \), \( \sigma \in G \), which is a composition of two functions. Let \( G_\alpha := \{ \sigma \in G \mid \sigma \cdot \alpha = \alpha \} \) be the stabilizer of \( \alpha \), and \( O(\alpha) = \{ \sigma \in G \mid \sigma \cdot \alpha = \alpha \} \) be the orbit with representative \( \alpha \). In this setting we have \( G_{\sigma \cdot \alpha} = \sigma G_\alpha \sigma^{-1} \), for all \( \sigma \in G \).

Let \( \Delta \) be a system of distinct representatives of the orbits of \( G \) acting on \( \Gamma_m^n \) and define

\[
\overline{\Delta} = \{ \alpha \in \Delta \mid \sum_{\sigma \in G_\alpha} \lambda(\sigma) \neq 0 \}.
\]

Let \( \{e_1, e_2, \ldots, e_m\} \) be an orthonormal basis of \( V \). With respect to the induced inner product, one easily obtains the condition \( e^\gamma_\alpha := e_{\gamma_1} \cdot e_{\gamma_2} \cdots e_{\gamma_n} \neq 0 \) if and only if \( \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \overline{\Delta} \). Moreover we have:

\[
< e^\lambda_\alpha | e^\beta_\beta > = \begin{cases} 
\frac{\lambda(1)}{|G|} \sum_{\sigma \in G_\beta} \lambda(\sigma \tau^{-1}), & \text{if } \alpha = \tau \cdot \beta \text{ for some } \tau \in G, \\
0, & \text{if } O(\alpha) \neq O(\beta).
\end{cases}
\]

For \( \gamma \in \overline{\Delta} \), \( V^\gamma_\lambda = \langle e^\lambda_\sigma \mid \sigma \in G \rangle \) is called the orbital subspace of \( V^n_\lambda(G) \). In [3], Freese proved that

\[
\dim V^\gamma_\lambda = \frac{\lambda(1)}{|G_\gamma|} \sum_{\sigma \in G_\gamma} \lambda(\sigma).
\]
In particular, if $\lambda$ is a linear character of $G$, then $\dim V^\lambda_\gamma = 1$ for all $\gamma \in \Delta$. By the definition of $V^\lambda_\gamma$, it follows that

\begin{equation}
V^\lambda_n(G) = \bigoplus_{\gamma \in \Delta} V^\lambda_\gamma
\end{equation}

is an orthogonal direct sum.

If $\alpha = \sigma \cdot \gamma$ and $\beta = \tau \cdot \gamma$, then $\sigma \tau^{-1} \cdot \beta = \alpha$, therefore using formula (1.5), we have:

\begin{equation}
\langle e^\lambda_{\sigma \cdot \gamma} \mid e^\lambda_{\tau \cdot \gamma} \rangle = \frac{\lambda(1)}{|G|} \sum_{\pi \in \sigma G, \sigma^{-1}} \lambda(\pi).
\end{equation}

An orthogonal basis of the form \( \{ e^\lambda_\gamma \mid \gamma \in S \} \), where $S$ is a subset of $\Gamma^G_m$, is called an orthogonal basis of decomposable symmetrized tensor for $V^\lambda_n(G)$, in this case we say that $V^\lambda_n(G)$ has an $O$-basis. By (1.7) $V^\lambda_n(G)$ has an $O$-basis if an only if $V^\lambda_\gamma$ has an $O$-basis for all $\gamma \in \Delta$. In particular, if $\lambda$ is a linear character, since $\dim V^\lambda_\gamma = 1$, for all $\gamma \in \Delta$, then $V^\lambda_\gamma$ has an $O$-basis which implies that $V^\lambda_n(G)$ has an $O$-basis.

Several papers are devoted in investigation of the existence of an $O$-basis for $V^\lambda_n(G)$, for example [9]. In [5] a necessary and sufficient condition for the existence of an $O$-basis for $V^\lambda_n(G)$ is given, where $G$ is a cyclic or a dihedral group. Also in [1] a necessary and sufficient condition for the existence of an $O$-basis for $V^\lambda_n(G)$ is given, where $G$ is the dicyclic group, i.e. a group generated by two elements $a$ and $b$ such that $a^{2n} = 1$, $b^2 = a^n$, $b^{-1}ab = a^{-1}$ and denoted by $T_{4n}$ in [6]; and in [2] a necessary and sufficient condition for the existence of an $O$-basis for the symmetry classes of tensors associated with the direct and central product of some permutation groups is given. In this paper we study the symmetry classes of tensors associated with the groups $U_{6n}$ and $V_{8n}$, which are defined by generators and relations in [6]. We investigate the problem of finding necessary and sufficient conditions for the existence of an $O$-basis for the above mentioned groups. We also find the dimensions of the symmetry classes of tensors associated with them.

\section{The Group $U_{6n}$}

The group $U_{6n}$, $n \geq 1$, is defined in [6] as a group generated by the elements $a$ and $b$ such that $a^{2n} = b^3 = 1$, $a^{-1}ba = b^{-1}$, i.e., $U_{6n} := \langle a, b \mid a^{2n} = b^3 = 1, a^{-1}ba = b^{-1} \rangle$. It is obvious that $\langle b \rangle$ is a normal subgroup of $U_{6n}$ and $U_{6n} = \langle b \rangle : \langle a \rangle \cong \mathbb{Z}_3 : \mathbb{Z}_{2n}$, which is isomorphism to the semi-direct product of a cyclic group of order 3 by a cyclic group of order $2n$. This group is of order
6n, and its elements are of the form \( U_{6n} = \{ a^r, a^rb, a^rb^2 \mid 0 \leq r < 2n \} \). It is not hard to see that \( U_{6n} \) has 3n conjugacy classes which are

\[
\{ a^{2r} \}, \{ a^{2r}b, a^{2r}b^2 \}, \{ a^{2r+1}, a^{2r+1}b, a^{2r+1}b^2 \}, \quad r = 0, 1, \ldots, n-1,
\]

and the character table of \( U_{6n} \) is:

<table>
<thead>
<tr>
<th>( \chi_j )</th>
<th>( \psi_k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma )</td>
<td>( 6n )</td>
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<tr>
<td></td>
<td>( 6n )</td>
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<td>( 3n )</td>
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<td>( 2n )</td>
</tr>
</tbody>
</table>

\[
\begin{array}{c|cccccc}
\chi_j & 1 & a^{2r} & b & a^{2r}b & a & a^{2r+1} \\
\psi_k & 1 & \omega^{2jr} & 1 & \omega^{2jr} & \omega^j & \omega^{(2r+1)j} \\
\end{array}
\]

\[
\omega = \exp\left(\frac{2\pi i}{2n}\right), \quad 1 \cdot r \cdot n-1, \quad 0 \cdot j \cdot 2n-1, \quad 0 \cdot k \cdot n-1.
\]

From the above table we see that \( U_{6n} \) has 2n linear characters \( \chi_j, 0 \cdot j \cdot 2n-1 \), and n non-linear irreducible characters \( \psi_k, 0 \cdot k \cdot n-1 \) of degree 2. Now we will embed this group in a suitable symmetric group. If \((1 2 \cdots 2n)(2n+1 2n+2)\) and \((2n+1 2n+2 2n+3)\) are permutations in \( S_{2n+3} \), then it can be verified that the mapping \( a \mapsto (1 2 \cdots 2n)(2n+1 2n+2), b \mapsto (2n+1 2n+2 2n+3) \), embeds \( U_{6n} \) in \( S_{2n+3} \). Now considering \( U_{6n} \) as a subgroup of \( S_{2n+3} \) we find the dimensions of the symmetry classes of tensors associated with the group \( U_{6n} \).

**Theorem 1.** Let \( G = U_{6n}, \ n \geq 1 \) and let \( V \) be an \( m \)-dimensional inner product space. Then considering \( G \) as a subgroup of \( S_{2n+3} \) as above, we have the following formulae for the dimensions of the symmetry classes of tensors associated with \( U_{6n} \).

\[
\dim V_{\chi_j}^{2n+3}(G) = \frac{m}{6n} \left[ (m^2 + 2) \sum_{l=0}^{n-1} \omega^{2lj} m(2l, 2n) + 3m \sum_{l=0}^{n-1} \omega^{(2l+1)j} m(2l+1, 2n) \right],
\]

\[
\dim V_{\psi_k}^{2n+3}(G) = \frac{2m(m^2-1)}{3n} \sum_{l=0}^{n-1} \omega^{2kl} m(2l, n),
\]

\( 0 \cdot j \cdot 2n-1, \ 0 \cdot k \cdot n-1, \)

where \((0, n) := n\), and \((k, n)\) denotes the greatest common divisor of \( k \) and \( n \), and \( \omega = \exp\left(\frac{2\pi i}{2n}\right) \).

**Proof.** Recall that for a permutation \( \tau \) we let \( c(\tau) \) denote the number of cycles in the cycle structure of \( \tau \) including cycles of length one. Note that if \( \tau \) is a cycle of length \( s \) and \((t, s) = d, \) then \( \tau^t \) has \( d \) cycles of length \( s/d \) and therefore \( c(\tau^t) = d + c(\tau) - 1 \) so \( c(1) = 2n + 3, \ c(a^{2\tau}) = (2r, 2n) + 3, \)
Now we discuss the existence of an $O$-basis associated with the group $U_{6n}$. Let $V$ be an $m$-dimensional unitary space over the complex field, if $m = 1$, then $\dim \ V^{2n+3}(U_{6n}) = 0$ or $1$, therefore it is trivial that in the case of $\dim V = 1$ an $O$-basis for every $\lambda \in I(U_{6n})$ exists. Therefore we assume that $m \geq 2$. As before, if $\chi$ is a linear character of $G$, since the orbital subspaces have dimension $\cdot 1$, then the symmetry class of tensors associated with $G$ and $\chi$ has an $O$-basis. Therefore we will consider non-linear irreducible complex characters of $U_{6n}$, i.e. the characters $\psi_k \ 0 \cdot k \cdot n - 1$.

Note that if $n = 1$, then $U_6 \cong S$ where $- = \{3, 4, 5\}$ and $S = (a_1 = (3 \ 4), b_1 = (3 \ 4 \ 5))$. In this case, we can consider $\psi$ given by $\psi(a_i^*b_i^*) := \psi_0(a_i^*b_i^*)$ as a nonlinear irreducible character of $S$. We have

\[
V^{5}_\psi(U_6) = V \quad V^{3}(S).
\]

Since $S$ is 2-transitive by [4], $V^{3}(S)$ does not have an $O$-basis, therefore $V^{5}_\psi(U_6)$ does not have an $O$-basis.

**Theorem 2.** Let $G = U_{6n}$, $n \geq 1$, and $\psi = \psi_k$, $0 \cdot k \cdot n - 1$ and let $m = \dim V \geq 2$. Then $V^{2n+3}(G)$ is non zero and does not have an $O$-basis.

**Proof.** Take $\gamma := (1, \underbrace{2, 2, \ldots, 2}_m, 1) \in \Gamma^{2n+3}$. Since $G$ is generated by the permutations $a = (1 \ 2 \ \ldots \ 2n)(2n + 1 \ 2n + 2)$ and $b = (2n + 1 \ 2n + 2 \ 2n + 3)$ we can conclude that $G_\gamma = 1$, and by equation (1.6),

\[
\dim V^{\gamma}_\psi = \frac{2}{1} \cdot 2 = 4.
\]

Therefore $V^{2n+3}(G)$ is non zero. Let $\{e_1, e_2, \ldots, e_m\}$ be an orthogonal basis of $V$. Now, by the equation (1.8), we have:

\[
< e_\mu, e_\tau > = \frac{2}{6n} \psi(\tau \mu^{-1}).
\]

Since $\tau, \mu \in U_{6n}$, we have $\tau = a^j b^s$ and $\mu = a^k b^t$, for some $j, k, s$ and $t$ in $\mathbb{Z}$, then using the formula (1.8), we obtain:

\[
< e_\mu, e_\tau > = 0 \iff j + k \text{ is odd and } s \equiv t \pmod{3}.
\]

Therefore from the set $\{e^\psi_\sigma | \sigma \in G\}$ we can choose at most two orthogonal vectors, but $\dim V^\psi_\gamma = 4$, hence $V^\psi_\gamma$ does not have an $O$-basis. Whence $V^{2n+3}(U_{6n})$ does have an $O$-basis. \quad \square
§3. The Group $V_{8n}$

In this section we define the group $V_{8n}$, and we will study the existence of an $O$-basis for the symmetry classes of tensors associated with this group and irreducible characters of $V_{8n}$. The dimensions of these symmetrized tensor spaces are also given.

Let $n$ be a positive integer. The group $V_{8n}$ is defined in [6] for $n$ odd. But one can define it for arbitrary $n$ as follows

$$V_{8n} := \langle a, b \mid a^{2n} = b^4 = 1, aba = b^{-1}, ab^{-1}a = b \rangle.$$

This group has order $8n$ and in the following we will discuss its conjugacy classes and irreducible complex characters of $V_{8n}$. Since our discuss in the cases of $n$ even or odd differs, therefore first we will assume that $n$ is odd. To describe the conjugacy classes and the irreducible characters of $V_{8n}$ from [6] we see that $V_{8n}$ has $2^n + 3$ conjugacy classes which are

- $\{1\}, \{b^2\}, \{a^{2r+1}, a^{-2r-1}b^2\}$, $r = 0, \ldots, n - 1$
- $\{a^s, a^{-2s}\}, \{a^{2s}b^2, a^{-2s}b^2\}$, $s = 1, \ldots, \frac{n-1}{2}$
- $\{a^jb^k : j \text{ even, } k = 1 \text{ or } 3\}$, and
- $\{a^jb^k : j \text{ odd, } k = 1 \text{ or } 3\}$.

The irreducible complex character table of $V_{8n}$ has four linear characters $\chi_1, \chi_2, \chi_3, \chi_4$, and $n$ characters $\psi_j$, $0 \cdot j \cdot n - 1$, of degree 2, and a further $n - 1$ characters $\phi_j$, $1 \cdot j \cdot n - 1$, of degree 2 as follows:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$V_{8n}([\sigma])$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi_1$</td>
<td>$1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1$</td>
</tr>
<tr>
<td>$\chi_2$</td>
<td>$1 \ 1 \ 1 \ 1 \ 1 \ -1 \ -1 \ -1$</td>
</tr>
<tr>
<td>$\chi_3$</td>
<td>$1 \ 1 \ -1 \ 1 \ 1 \ 1 \ -1 \ -1$</td>
</tr>
<tr>
<td>$\chi_4$</td>
<td>$1 \ 1 \ -1 \ 1 \ 1 \ -1 \ -1 \ -1$</td>
</tr>
<tr>
<td>$\psi_j$, $0 \cdot j \cdot n - 1$</td>
<td>$2 \ -2 \ \omega^{2(2r+1)i} - \omega^{-2(2r+1)i} \ \omega^{4s}j + \omega^{-4s}j \ -\omega^{4s}j - \omega^{-4s}j \ 0 \ 0$</td>
</tr>
<tr>
<td>$\phi_j$, $1 \cdot j \cdot n - 1$</td>
<td>$2 \ \omega^{(2r+1)i} + \omega^{-(2r+1)i} \ \omega^{2s}j + \omega^{-2s}j \ \omega^{2s}j + \omega^{-2s}j \ 0 \ 0$</td>
</tr>
</tbody>
</table>

$$\omega = \exp\left(\frac{2\pi i}{2n}\right), \ 0 \cdot r \cdot n - 1, \ 1 \cdot s \cdot \frac{n-1}{2}.$$

Now we embed $V_{8n}$ in a suitable symmetric group. It is easy to see that...
dim \rightarrow (1 \ 2 \ \cdots \ 2n)(2n+1 \ 2n+2 \ \cdots \ 4n) \text{ and }
\begin{align*}
b \rightarrow (1 \ 2 \ 2n+1 \ 2n+2) \prod_{k=2}^{n+1} 
((2k-1 \ 2(n-k)+4 \ 2(n+k)-1 \ 2(2n-k)+4) \\
(2k \ 2(2n-k)+3 \ 2(2n-k)+2(n-k)+3))
\end{align*}
gives an embedding of $V_{8n}$ in $S_{8n}$ so we assume that $V_{8n}$ is a subgroup of $S_{8n}$. We need the following observation for the proof of the next theorem. Suppose that $t$ is an odd positive number and consider the disjoint sets $A = \{a_1, a_2, \ldots, a_t\}$ and $B = \{b_1, b_2, \ldots, b_t\}$. Let $x = (a_1 \ a_2 \ \cdots \ a_t)$ and $y = (b_1 \ b_2 \ \cdots \ b_t)$ be two cycles permuting elements of $A$ and $B$ respectively, and let $z = (a_1 \ b_1)(a_2 \ b_2) \cdots (a_t \ b_t)$ be a permutation on $A \cup B$. Then the permutation $xyz$ is a cycle of length $2t$, and

$$xyz = (a_1 \ b_2 \ a_3 \ \cdots \ a_t \ b_1 \ a_2 \ b_3 \ \cdots \ b_t).$$

In the following theorem we find the dimensions of the symmetry classes of tensors associated with the group $V_{8n}$.

**Theorem 3.** Let $G = V_{8n}$, $n$ odd, and let $V$ be an $m$-dimensional inner product space. Then considering $G$ as a subgroup of $S_{8n}$, we have the following:

\[
\dim V_{\chi_1}^{4n}(V_{8n}) = \frac{1}{4n} \left[ \sum_{k=0}^{n-1} m^{2(2k+1,2n)} + \sum_{k=0}^{n-1} (m^{2(2k,2n)} + m^{2k,2n}) \right] \\
+ nm^n + nm^{2n+1} - \frac{m^{4n} + m^{2n}}{2},
\]

\[
\dim V_{\chi_2}^{4n}(V_{8n}) = \frac{1}{4n} \left[ \sum_{k=0}^{n-1} m^{2(2k+1,2n)} + \sum_{k=0}^{n-1} m^{2(2k,2n)} + m^{(2k,2n)} \right] \\
- nm^n - nm^{2n+1} - \frac{m^{4n} + m^{2n}}{2},
\]

\[
\dim V_{\chi_3}^{4n}(V_{8n}) = \frac{1}{4n} \left[ \sum_{k=0}^{n-1} -m^{2(2k+1,2n)} + \sum_{k=0}^{n-1} (m^{2(2k,2n)} + m^{2k,2n}) \right] \\
+ nm^n - nm^{2n+1} - \frac{m^{4n} + m^{2n}}{2},
\]

\[
\dim V_{\chi_4}^{4n}(V_{8n}) = \frac{1}{4n} \left[ \sum_{k=0}^{n-1} -m^{2(2k+1,2n)} + \sum_{k=0}^{n-1} (m^{2(2k,2n)} + m^{2k,2n}) \right] \\
- nm^n + nm^{2n+1} - \frac{m^{4n} + m^{2n}}{2},
\]
dim } V^4_n (V_8^n) = 
\sum_{k=1}^{n-1} \frac{m_{2k+2n}(m_{2k+2n} - 1)}{n} \cos \left( \frac{4k\pi j}{n} \right),
\quad 0 \cdot j \cdot n - 1.

\dim V^4_n (V_8^n) = \frac{1}{n} \left[ \frac{m_{2n}(2m_{2n} - 1)}{2} \sum_{k=1}^{n-1} \left( \frac{m_{2k+1}}{m_{2k+1}} \cos \left( \frac{(2k+1)\pi j}{n} \right) \right) \right.
\left. + \sum_{k=1}^{n-1} \left( m_{2k+2n} m_{2k+2n} \cos \left( \frac{2k\pi j}{n} \right) \right) \right],
\quad 1 \cdot j \cdot n - 1.

Here, \((0,n) := n\), and \((k,n)\) denotes the greatest common divisor \(k\) and \(n\).

**Proof.** As before, we know that if \(\tau\) is a cycle of length \(s\), then \(\tau^t\) has \((t,s)\) cycles and therefore \(c(\tau^t) = (t,s) + c(\tau) - 1\), where \(c(\tau)\) denotes the number of cycles in the cycle structure of \(\tau\) including cycles of length one. So \(c(1) = 4n\), \(c(b^2) = 2n\), \(c(a^{2k+1}) = 2(2k + 1, 2n)\), \(c(a^{2k}) = 2(2k, 2n)\) and \(c(b) = n\). Since the order of \(ab\) is 2 by calculation we obtain the only fixed points of \(ab\) are \(n + 1\) and \(3n + 1\), hence \(c(ab) = \frac{2n-2}{2} + 2 = 2n + 1\). Also we have \(b^2 = (1 2 n + 1) (2 2 n + 2) \cdots (2 n 4 n)\) and the permutation \(a\) is a product of two disjoint cycles \((1 2 \cdots 2 n)\) and \((2 n + 1 2 n + 2 \cdots 4 n)\). Since \(n\) is odd, by previous observation, one can show that \(c(a^{2k+2}) = \frac{1}{2} c(a^{2k}) = (2k, 2n)\). Using the character table of \(V_8^n\), and the formula (1.2) the theorem follows. \(\square\)

Now we discuss the existence of an \(O\)-basis associated with the group \(V_8^n\) \(n\) odd. As before, let \(V\) be an \(m\)-dimensional unitary space and \(m \geq 2\). If \(n = 1\), \(V_8 \cong D_8\), the dihedral group, and by [5] the symmetry classes of tensor associated \(V_8\) has an \(O\)-basis.

**Theorem 4.** Let \(G = V_8^n\), \(n\) odd, \(n \neq 1\), \(\phi = \phi_j\), \(1 \cdot j \cdot n - 1\). Assume that \(m = \dim V \geq 2\). Then \(V_8^{2n}\) \((G)\) is non-zero and does not have an \(O\)-basis.

**Proof.** Take \(\gamma := (1, 2, 2, \ldots, 2, 1, 2, 2, \ldots, 2) = (\gamma_1, \gamma_2, \ldots, \gamma_{4n}) \in \Gamma_{4n}^m\), by the structure of permutations \(a\) and \(b\) in \(G\), i.e., \(a = (1 2 \cdots 2 n)(2 n + 1 2 n + 2 \cdots 4 n)\) and

\[
\begin{align*}
\begin{pmatrix}
1 & 2 & 2n + 1 & 2n + 2 \\
3 & 2n & 2n + 3 & 4n \\
4 & 4n - 1 & 2n + 4 & 2n - 1 \\
5 & 2n - 2 & 2n + 5 & 4n - 2 \\
& & & \\
& & & \\
n - 1 & 3n + 4 & 3n - 1 & n + 4 \\
n + 3 & 3n & 3n + 3 \\
n + 1 & 3n + 2 & 3n + 1 & n + 2
\end{pmatrix}
\end{align*}
\]
one can conclude that $\langle a \rangle \cap G_{\gamma} = 1$. Since $n \neq 1$, therefore $\langle b \rangle \cap G_{\gamma} = 1$. Since $(a^rb^{-1})(1) \neq 1, 2n + 1, 2n + 2$ for $r = 1, 2, \ldots, 2n - 2$ and $(a^{-1}b)^{-1}(2n + 2) = 2n \neq 1, 2n + 1, 2n + 2$, hence $a^rb \notin G_{\gamma}$, $\forall r \in \mathbb{Z}$. Also $(a^rb^{-1})(1) \neq 1, 2n + 1, 2n + 2$ for $r = 1, 2, \ldots, 2n - 2$ and $(a^{-1}b^2)^{-1}(2n + 1) = b^2a(2n + 1) = b^2(2n + 2) = 2 \neq 1, 2n + 1, 2n + 2$, hence $a^rb^2 \notin G_{\gamma}$, $\forall r \in \mathbb{Z}$.

Conclude that $G_{\gamma} = 1$. Therefore $\dim V_{\phi}^n = 4$. Hence $V_{\phi}^{4n}(G)$ is non zero.

Now we let $\{e_1, e_2, \ldots, e_m\}$ be an orthonormal basis of $V$ and calculate the inner product $\langle e_{\mu, \gamma}^\phi, e_{\nu, \gamma}^\phi \rangle$ for all $\mu, \tau \in V_{\phi}$. Let $\mu = a^rb^s$ and $\tau = a^rb'$, then we have

$$\langle e_{\mu, \gamma}^\phi, e_{\nu, \gamma}^\phi \rangle = \frac{2}{8n} \phi(\tau \mu^{-1}) = \frac{1}{4n} \phi(a^kb^{s-t}a^{-r})$$

therefore from table II we get $\langle e_{\mu, \gamma}^\phi, e_{\nu, \gamma}^\phi \rangle = 0 \iff |s-t|$ is odd. Hence we can choose at most two orthogonal vector from the set $\{e_{\phi, \gamma}^\phi | \gamma \in G\}$, therefore $V_{\phi}^n$ does not have an $O$-basis. Hence $V_{\phi}^{4n}(V_{\phi})$ does not have an $O$-basis.

**Theorem 5.** Let $G = V_{8n}$, $n$ odd, and let $\psi = \psi_j = 0 \cdot j \cdot n - 1$. Assume that $m = \dim V \geq 2$, then $V_{\psi}(G)$ has an $O$-basis.

**Proof.** Let $H$ be a subgroup of $G$, $\psi = \psi_j = 0 \cdot j \cdot n - 1$. Since $\psi_j(1) = 2$, we have $\psi \downarrow H \mid 1_H = 0$ or $2$. If $\psi \downarrow H \mid 1_H = 1$, then there is a linear non identity character $\chi$ of $H$ such that $\chi = \psi \downarrow H$. Since $\chi$ is a linear character, $|\chi(h)| = 1$, for all $h \in H$. First we find the general form of the elements of $H$ using only the condition $|\chi(h)| = |\psi(h) - 1| = 1$.

Since $\psi(b^2 - 1) = -2 - 1 = -3$, we have $b^2 \notin H$, i.e. $H \cap \{b\} = 1$. As before $(n, j)$ denotes the greatest common divisor of $n$ and $j$. Let $(n, j) = d$, one can conclude that the elements of $H$ can only be chosen from the set:

$$\{ a^t n / d, a^{2t+1} n / d b^2, a^t b, a^t b^{-1} \mid t \in \mathbb{Z} \}.$$ 

Because from table II we see that $\psi(a^{(2t+1)n}d) = \psi(a^{2t+1}n/db^2) = \psi(a^t b^{t+1}) = 0$ and $\psi(a^{2tn}d) = 2$. Since $\sum_{\sigma \in H} \psi(\sigma) = |H|$, therefore half elements of $H$ must be of the form $a^{2tn}d$, where $t \in \mathbb{Z}$. Hence the group $H$ is a subgroup of $\langle a^{n/d} \rangle$ or $\langle a^{2n/d}, a^{(2k+1)b^{t+1}} \rangle$, for some $k \in \mathbb{Z}$.

Now instead of $H$ we consider $G_{\gamma}$ as a subgroup of $G$. If $\psi \downarrow G_{\gamma} \mid 1_G = 1$, we have $\dim V_{\phi}^\psi = 2$ and by the above remark $G_{\gamma} \cdot \langle a^{n/d} \rangle$, then $\{ e_{\gamma}, e_{b \gamma} \}$ is an $O$-basis for $V_{\phi}^\psi$. If $G_{\gamma} \cdot \langle a^{2n/d}, a^{(2k+1)b} \rangle$ or $G_{\gamma} \cdot \langle a^{2n/d}, a^{(2k+1)b^{-1}} \rangle$, then $\{ e_{\gamma}, e_{b \gamma} \}$ is an $O$-basis for $V_{\phi}^\psi$.

If $\psi \downarrow H \mid 1_H = 2$, then $H$ is a subgroup of $\ker \psi$. Note that $\cos(\pi 4sj) = \pm 1$ if and only if $\pi 4sj = k\pi$ for some $k \in \mathbb{Z}$, and if only if $\pi 4s$, therefore $\ker \psi \cdot \langle a^{4n/d} \rangle$. In this case $\dim V_{\phi}^\psi = 4$ and the set $\{ e_{\gamma}, e_{a^{n/d} \gamma}, e_{b \gamma}, e_{a^{n/d} b \gamma} \}$
is an $O$-basis of $V_\alpha^n$. The theorem now follows. Note that the order of $a^{2r}$ is $d$ and $\psi(a^{2r+1}) = -\psi(a^{-(2r+1)})$, $r \in \mathbb{Z}$.

Now we assume that $n$ is even. In this case the group $G = V_{8n}$ has $2n + 6$ conjugacy classes which are:

\[
\begin{align*}
&\{1\}, \{b^n\}, \{a^n\}, \{a^n b^2\}, \{a^{2r+1}, a^{-(2r+1)} b^2\}, r = 0, \ldots, n - 1 \\
&\{a^{2r}, a^{-2s}\}, \{a^{2s} b^2, a^{-2s} b^2\}, s = 1, \ldots, n/2 - 1 \\
&\{a^{2k} b^{(-1)^k} | 0 \cdot k \cdot n - 1\}, k = 0, 1, \ldots, n/2 - 1 \\
&\{a^{2k} b^{(-1)^k} | 0 \cdot k \cdot n - 1\}, k = 0, 1, \ldots, n/2 - 1 \\
&\{a^{2k+1} b^{(-1)^k} | 0 \cdot k \cdot n - 1\}, k = 0, 1, \ldots, n/2 - 1 \\
&\{a^{2n+1} b^{(-1)^k} | 0 \cdot k \cdot n - 1\}, k = 0, 1, \ldots, n/2 - 1.
\end{align*}
\]

The derived subgroup of $G$ is $\langle a^2 b^2 \rangle$, hence $G$ has eight linear characters $\chi_1, \chi_2, \ldots, \chi_8$. Since $H = \langle b^2 \rangle$ is a normal subgroup of $G$ and $G/H \cong D_{2n}$, we obtain that $n - 1$ irreducible characters $\psi_j$ with $j \cdot n - 1$, of degree 2. Since $b^2$ is not in the derived subgroup and $(b^2)^2 = 1$, there exists a linear character $\chi_2$ such that $\chi_2(b^2) = -1$. The product of the linear character $\chi_2$ with $\psi_j$, gives further $n - 1$ irreducible characters $\psi_j \cdot \chi_2$, $1 \cdot j \cdot n - 1$, of degree 2. Since character values in cases $n \equiv 0$ (mod 4) and $n \equiv 2$ (mod 4) differ, therefore we distinguish these cases and give the character table of $V_{8n}$ in Table III and Table IV respectively.

The embedding of $G$ in $S_{4n}$, $n = \text{even}$, is different from the case $n = \text{odd}$. In this case if we take the following permutations in $S_{4n}$, $a \mapsto (1 \ 2 \ \ldots \ 2n)(2n + 1 \ 2n + 2 \ \ldots \ 4n)$, and 

\[
b \mapsto (1 \ 2 \ 2n + 1 \ 2n + 2)(\prod_{k=2}^{n/2} (2k - 1 \ 2(n-k) + 4 \ 2(n+k) - 1 \ 2(n-k) + 4))
\]

\[
(2k - 1 \ 2(n-k) + 3 \ 2(n+k) - 2(n-k) + 3)|(n+1 \ n+2 \ n+3 | n+2 \ n+3 \ n+2),
\]

then we see that we have a monomorphism of $G$ into $S_{4n}$. So we assume that $G$ is a subgroup of $S_{4n}$. Take $- = \{\{1,2n+1\}, \{2,2n+2\}, \ldots, \{2n,4n\}\}$. The group $G/H$ acts on $- \cdot \{\sigma \cdot i, \sigma \cdot (2n+i) \}$ and this action is faithful. We put $i : = \{i,2n+i\}$, $1 \cdot i \cdot 2n$, and consider $G/H$ as a subgroup of $S_{2n}$, therefore the cycle structure of $aH$ and $bH$ on $-$ are as follows:

\[
aH := (1 \ 2 \ \ldots \ 2n) \text{ and } bH := (1 \ 2)(3 \ 2n)(4 \ 2n-1) \cdots (n \ n+3)(n+1 \ n+2).
\]

Therefore we consider $D_{4n} := \langle aH, bH \rangle$ as a subgroup of $S_{2n}$.

Let $G = V_{8n}$, $n$ even, and $H = \{1, b^2\}$, $\psi = \psi_j$, $1 \cdot j \cdot n - 1$. Since $H \cdot \ker \psi$ the character $\overline{\psi}(\sigma H) = \psi(\sigma)$ is an irreducible character of $G/H = D_{4n}$. Let $W$ be a $p$-dimensional inner product space, $p \geq 2$. Let $\gamma = (1, 1, 2, \ldots, 2)$ be in $\Gamma_p^2$ and note that $bH \in (D_{4n})_\gamma$ special one can conclude that $(D_{4n})_\gamma =
\{1, bH\}. Similar to the proof of (Theorem 3.1, [5]), if \(W_{\gamma}^{\psi_j}(D_{4n})\) has an \(O\)-basis, then \(\psi_j(a^k) = 2 \cos \frac{2\pi j k}{2n} = 0\) for some \(k\) in \(\mathbb{Z}\). In other words \(\frac{2\pi j k}{2n} = (2l + 1)\frac{\pi}{2}\) for some integer \(l\). This implies \(2j_2\) divide \(n\), where \(j = j_2j_2\), \(j_2\) odd and \(j_2\) a power of 2, i.e.,

\[(3.1) \quad W_{\gamma}^{\psi_j}(D_{4n})\) has an \(O\)-basis \(\Rightarrow 2n \equiv 0 \pmod{4j_2}\).

**Theorem 6.** Let \(G = V_{8n}\) and assume that \(\dim V \geq 2\). Then \(4^n V\) has an \(O\)-basis if and only if \(n\) is a power of 2.

**Proof.** For \(\psi = \psi_j, 1 \cdot j \cdot n - 1\) and \((\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \beta_1, \beta_2, \ldots, \beta_{2n}) \in \Gamma_{4n}^m\), we may assume that \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2n})\) and \(\beta = (\beta_1, \beta_2, \ldots, \beta_{2n})\) are elements of \(\Gamma_{4n}^m\) and therefore we will set \((\alpha, \beta) := (\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \beta_1, \beta_2, \ldots, \beta_{2n})\).

In this setting we have:

\[
e_{(\alpha, \beta)}^{\psi} = \frac{\psi(1)}{|G|} \sum_{\sigma \in G} \psi(\sigma)e_{\sigma^{-1}(\alpha, \beta)}
\]

\[
= \frac{\psi(1)}{|G|} \sum_{\sigma \in G} \psi(\sigma)e_{\sigma^{-1}(\alpha, \beta)} + \sum_{\sigma \in G} \psi(\sigma_b^2)e_{\sigma b^2(\alpha, \beta)}
\]

\[
= \frac{\psi(1)}{|G|} \sum_{\sigma \in G} \psi(\sigma)e_{\sigma^{-1}(\alpha, \beta)} + \sum_{\sigma \in G} \psi(\sigma)e_{\sigma^{-1}(\alpha, \beta)} + \sum_{\sigma \in G} \psi(\sigma)e_{\sigma^{-1}(\alpha, \beta)}
\]

\[
= \frac{\psi(1)}{|G/H|} \sum_{\sigma \in G/H} \psi(\sigma) \frac{1}{2} \left[ e_{\sigma^{-1}(\alpha, \beta)} + e_{\sigma^{-1}(\alpha, \beta)} + e_{\sigma^{-1}(\alpha, \beta)} \right]
\]

specially, when \(\alpha = \beta\), we have

\[
e_{(\alpha, \alpha)}^{\psi} = \frac{\psi(1)}{|G/H|} \sum_{\sigma \in G/H} \psi(\sigma)e_{\sigma^{-1}(\alpha, \alpha)}
\]

Hence \(e_{(\alpha, \alpha)}^{\psi} \mid e_{(\beta, \beta)}^{\psi} \geq 0\) if and only if \(\langle e_{\alpha}^{\psi} \rangle \mid e_{\beta}^{\psi} \geq 0\), moreover

\[
\dim V_{(\alpha, \alpha)}^{\psi} = \frac{\psi(1)}{|G_{(\alpha, \alpha)}|} \sum_{\sigma \in G_{(\alpha, \alpha)}} \psi(\sigma) = \frac{\psi(H)}{|(G/H)_{(\alpha, \alpha)}|} \sum_{\sigma \in (G/H)_{(\alpha, \alpha)}} \psi(\sigma) = \dim V_{(\alpha, \alpha)}^{\psi}.
\]

Therefore \(V_{(\alpha, \alpha)}^{\psi}\) has an \(O\)-basis if and only if \(V_{(\alpha, \alpha)}^{\psi}\) has an \(O\)-basis. Note that \(b^2 \in G_{(\alpha, \alpha)}\) and the stabilizer of \(\alpha\) under \(G/(b^2)\) is \(G_{(\alpha, \alpha)}/(b^2)\).

If \(V_{(\gamma, \gamma)}^{2n}(G)\) has an \(O\)-basis, then for every \(\gamma \in \Gamma_{2n}^m\), \(V_{(\gamma, \gamma)}^{2n}\) has an \(O\)-basis.

Hence by the above remark we obtain an \(O\)-basis for \(V_{(\gamma, \gamma)}^{2n}(G/H)\). So \(V_{2n}^{2n}(G/H)\) has
an $O$-basis. If $\psi = \psi_j$ and $j = j_2j_2'$ where $j_2'$ is the odd part of $j$ and $j_2$ is a power of 2, then by formula (3.1) we have $2n \equiv 0 \pmod{4j_2}$, which holds all $j$ from $1, 2, \ldots, n - 1$. This implies that $n$ is a power of 2.

Conversely, assume that $n$ is a power of 2, and $\psi = \psi_j$ (or $\psi_j \cdot \chi_2$). Let $\gamma = (\alpha, \beta) \in \Delta$, where $\alpha, \beta \in \Gamma_{2n}$. If $\alpha = \beta$, by the above remark $V_{\chi_1}^\psi$ has an $O$-basis and

$$e_{\psi, \chi_2}^{(\alpha, \alpha)} = \frac{\psi(1)\chi_2(1)}{|G|} \sum_{\sigma \in G} \psi(\sigma)\chi_2(\sigma)e_{\sigma, (\alpha, \alpha)}$$

$$= \frac{\psi(1)}{|G|} \left[ \sum_{\sigma \in G} \psi(\sigma)\chi_2(\sigma)e_{\sigma, (\alpha, \alpha)} + \sum_{\sigma \in G} \psi(\sigma b^2)\chi_2(\sigma b^2)e_{\sigma b^2, (\alpha, \alpha)} \right]$$

$$= \frac{\psi(1)}{|G|} \sum_{\sigma \in G} \psi(\sigma)\chi_2(\sigma) \left[ e_{\sigma - \alpha} - e_{\sigma - \alpha} \right] = 0,$$

hence $V_{\chi}^\psi = 0$.

If $\alpha \neq \beta$, then $b^2 \in G_\gamma$ which implies that $\langle b \rangle \cap G_\gamma = 1$. As in the proof of Theorem 5, $< \psi \mid_{G_\gamma} | 1_{G_\gamma} > = 0, 1$ or 2.

If $< \psi \mid_{G_\gamma} | 1_{G_\gamma} > = 2$, then $G_\gamma \cdot \ker \psi$ and by the formula (1.6), dim$V_{\chi}^\psi = 4$. Using the character table of $V_{\delta_n}$, Tables III and IV, we obtain $G \cdot \langle (a^2)^{\frac{n}{2}} \rangle$. In this case, using formula (1.8) and Tables III and IV one can show that the set $\{ e_{\gamma}, e_{b \cdot \gamma}, e_{\frac{n}{2}}a^{-1} \gamma, e_{\frac{n}{2}}a^{-1}b^{-1} \gamma \}$ is an orthogonal basis for $V_{\chi}^\psi(G)$. Hence $V_{\chi}^\psi(G)$ has an $O$-basis.

If $< \psi \mid_{G_\gamma} | 1_{G_\gamma} > = 1$, then $\psi \mid_{G_\gamma}$ is a non-identity linear character of $G_\gamma$, and the norm $|\psi(x)| = 1$, i.e. $\psi(x) = 0$ or 2 for all $x \in G_\gamma$. By formula (1.6), dim$V_{\chi}^\psi = 2$. From here on we must deal with the cases $\psi = \psi_j$ and $\psi_j \cdot \chi_2$ separately. First assume that $\psi = \psi_j$, $1 \cdot j \cdot n - 1$. The elements of $G_\gamma$ are of the form $\langle a^{2(n+1)^{\frac{n}{2}}} b^{2} \rangle$, $\langle \frac{n}{2} \rangle \gamma$ on which the value of $\psi$ is zero, and $\langle a^{2(n+1)^{\frac{n}{2}}} b^{2} \rangle$, $\langle \frac{n}{2} \rangle \gamma$ on which the value of $\psi$ is 2. The equality $\sum_{\sigma \in G_\gamma} \psi(\sigma) = |G_\gamma|$ implies that the values of $\psi$ on exactly half of elements of $G_\gamma$ must be zero. If $a^{b \pm n} \in G_\gamma$, then $(a^{b \pm n})^2 = 1$ or $b^2$, and since $b^2 \notin G_\gamma$ therefore $t$ must be odd. Therefore $G_\gamma \cdot \langle a^{\frac{n}{2}} \rangle \cdot K$ or $\langle a^{\frac{n}{2}} b^2 \rangle \cdot K$, where $K = \langle a^{2\frac{n}{2}} \rangle$ or $\langle a^{\frac{n}{2}} b^2 \rangle$ or $\langle a^{2(n+1)b^{\pm 1}} \rangle$. In the case $K = \langle a^{2(n+1)b^{\pm 1}} \rangle$, the set $\{ e_{\gamma}, e_{\frac{n}{2}}a^{-1} \gamma \}$ and in other cases $\{ e_{\gamma}, e_{b \cdot \gamma} \}$ is an orthogonal for $V_{\chi}^\psi$. Hence $V_{\chi}^\psi$ has an $O$-basis.

If $\psi = \psi_2 \cdot \chi_2$, similar to the previous case, $G_\gamma$ is a subgroup of the form $\langle a^{\frac{n}{2}} \rangle$ where $K = \langle a^{2(n+1)b^{\pm 1}} \rangle$ or $\langle a^{2(n+1)b^{\pm 1}} \rangle$ or $\langle a^{2(n+1)b^{\pm 1}} \rangle$. If $K = \langle a^{2(n+1)b^{\pm 1}} \rangle$ take the set $\{ e_{\gamma}, e_{\frac{a}{2\gamma}} \}$ and the other cases the set $\{ e_{\gamma}, e_{b \cdot \gamma} \}$ are the orthogonal basis for $V_{\chi}^\psi$. 

\[ \square \]
Theorem 7. Let $G = V_{8n}$, $n$ even. Assume $m = \dim V \geq 2$. Then the dimensions of symmetry classes of tensors associated with $G$ and the irreducible characters of $G$ are:

$$
\dim V_{\chi_1}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} + nm^{2n+2} + (3 + n)m^{2n} + 2nm^n \\
+ 2 \sum_{k=0}^{n-1} m^{2(2k+1,2n)} + 4 \sum_{s=1}^{n/2-1} m^{2(2s,2n)} \right\},
$$

$$
\dim V_{\chi_2}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} - nm^{2n+2} + (n - 1)m^{2n} \right\},
$$

$$
\dim V_{\chi_3}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} + nm^{2n+2} + (3 + n)m^{2n} - 2nm^n \\
+ 4 \sum_{s=1}^{n/2-1} m^{2(2s,2n)} - 2 \sum_{k=0}^{n-1} m^{2(2k+1,2n)} \right\},
$$

$$
\dim V_{\chi_4}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} - nm^{2n+2} + (n - 1)m^{2n} \right\},
$$

$$
\dim V_{\chi_5}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} - nm^{2n+2} + (3 - n)m^{2n} - 2nm^n \\
+ 4 \sum_{s=1}^{n/2-1} m^{2(2s,2n)} + 2 \sum_{k=0}^{n-1} m^{2(2k+1,2n)} \right\},
$$

$$
\dim V_{\chi_6}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} + nm^{2n+2} - (n + 1)m^{2n} \right\},
$$

$$
\dim V_{\chi_7}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} - nm^{2n+2} - (n - 3)m^{2n} + 2nm^n \\
+ 4 \sum_{s=1}^{n/2-1} m^{2(2s,2n)} - 2 \sum_{k=0}^{n-1} m^{2(2k+1,2n)} \right\},
$$

$$
\dim V_{\chi_8}^{4n}(G) = \frac{1}{8n} \left\{ m^{4n} + nm^{2n+2} - (n + 1)m^{2n} \right\}.
$$
\[
\dim V_{\psi_j}^{4n}(G) = \frac{1}{4n} \left\{ 2m^{4n} + 2(1 + 2(-1)^j)m^{2n} + 8 \sum_{s=1}^{n/2-1} m^{2(2s,2n)} \cos \frac{2\pi sj}{n} \right.
+ 4 \sum_{k=0}^{n-1} m^{2(2k+1,2n)} \cos \frac{\pi j(2k + 1)}{n} \left\},
\]

\[
\dim V_{\psi_j,\chi_2}^{4n}(G) = \frac{1}{4n} \left\{ (2m^{4n} - 2m^{2n}) \right\},
\]

where \(1 \cdot j \cdot n - 1\).

**Proof.** Similarly to the proof of the Theorem 3, note that \(c(1) = 4n\), \(c(b^2) = c(a^n) = c(a^n b^2) = 2n\), \(c(a^r) = 2(r, 2n)\), \(c(a^{4s} b^2) = 2(4s, 2n)\), \(c(a^{4t+2} b^2) = 2(4t + 2, 2n)\), \(c(b^{1}) = n\), \(c(ab) = 2n\) and \(c(ab^{1}) = 2n + 2\). \(\square\)

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Table III
The character table of \( V_{8n}, \ n \equiv 0 \pmod{4} \).

| \(|Cl(\sigma)|\) | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | n | n | n | n | n |
|------------------|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \(\sigma\)      |   |   |   |   |   |   |   |   |   |   |   |   |   |   |
| \(\chi_1\)      | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(\chi_2\)      | 1 | -1 | 1 | -1 | i | i | -1 | -1 | -1 | i | i | 1 | 1 | 1 |
| \(\chi_3\)      | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 |
| \(\chi_4\)      | 1 | -1 | 1 | -1 | -i | i | 1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 |
| \(\chi_5\)      | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 |
| \(\chi_6\)      | 1 | -1 | 1 | -1 | i | i | 1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 |
| \(\chi_7\)      | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(\chi_8\)      | 1 | -1 | 1 | -1 | -i | i | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| \(\psi_j\)      | 2 | 2 | 2(-1)^j | 2(-1)^j | \(\alpha^{j(4k+1)}\) | \(\alpha^{j(4k+3)}\) | \(\alpha^{j(4s)}\) | \(\alpha^{j(4t+2)}\) | \(\alpha^{(4s)}\) | \(\alpha^{(4t+2)}\) | 0 | 0 | 0 | 0 |
| \(\psi_j \cdot \chi_2\) | 2 | -2 | 2(-1)^j | -2(-1)^j | \(i\alpha^{j(4k+1)}\) | \(-i\alpha^{j(4k+3)}\) | \(\alpha^{j(4s)}\) | \(-\alpha^{j(4t+2)}\) | \(\alpha^{j(4s)}\) | \(-\alpha^{j(4t+2)}\) | 0 | 0 | 0 | 0 |

\[ \alpha^{j^r} = \omega^{j^r} + \omega^{-j^r} = 2 \cos \left( \frac{\pi j^r}{n} \right), \ \omega = \exp \left( \frac{2\pi i}{2n} \right); \]

\[ 0 \cdot k \cdot n/2 - 1, \ 1 \cdot s \cdot n/4 - 1, \ 0 \cdot t \cdot n/4 - 1, \ 1 \cdot j \cdot n - 1. \]
Table IV
The character table of $V_{5n}$, $n \equiv 2 \pmod{4}$.

| $|Cl(\sigma)|$ | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | n | n | n | n | n |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\sigma$ | 1 | $b^2$ | $a^n$ | $a^n b^2$ | $a^{4k+1}$ | $a^{4k+3}$ | $a^{4s}$ | $a^{4t+2}$ | $a^{4s} b^2$ | $a^{4t+2} b^2$ | $b$ | $b^{-1}$ | $a b$ | $a b^{-1}$ |
| $\chi_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2$ | 1 | -1 | -1 | 1 | $i$ | -$i$ | 1 | -1 | -1 | 1 | -$i$ | $i$ | 1 | -1 |
| $\chi_3$ | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 |
| $\chi_4$ | 1 | -1 | -1 | 1 | -$i$ | $i$ | 1 | -1 | -1 | 1 | $i$ | -$i$ | 1 | -1 |
| $\chi_5$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | 1 | -1 |
| $\chi_6$ | 1 | -1 | -1 | 1 | $i$ | -$i$ | 1 | -1 | -1 | 1 | $i$ | -$i$ | 1 | -1 |
| $\chi_7$ | 1 | 1 | 1 | 1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | -1 | 1 | -1 |
| $\chi_8$ | 1 | -1 | -1 | 1 | -$i$ | $i$ | 1 | -1 | -1 | 1 | -$i$ | $i$ | 1 | -1 |
| $\psi_j$ | 2 | 2 | $2(-1)^j$ | $2(-1)^j$ | $\alpha_j^{(4k+1)}$ | $\alpha_j^{(4k+3)}$ | $\alpha_j^{(4s)}$ | $\alpha_j^{(4t+2)}$ | $\alpha_j^{(4s)}$ | $\alpha_j^{(4t+2)}$ | 0 | 0 | 0 | 0 |
| $\psi_j \cdot \chi_2$ | 2 | -2 | $-2(-1)^j$ | $2(-1)^j$ | $i\alpha_j^{(4k+1)}$ | $-i\alpha_j^{(4k+3)}$ | $\alpha_j^{(4s)}$ | $-\alpha_j^{(4t+2)}$ | $-\alpha_j^{(4s)}$ | $\alpha_j^{(4t+2)}$ | 0 | 0 | 0 | 0 |

$\alpha_j^{ir} = \omega^{jr} + \omega^{-jr} = 2 \cos(\frac{\pi jr}{n})$, $\omega = \exp \left(\frac{2\pi i}{2n}\right)$;

$0 \cdot k \cdot n/2 - 1$, $1 \cdot s \cdot n/4 - 1$, $0 \cdot t \cdot n/4 - 1$, $1 \cdot j \cdot n - 1$. 
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