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ON MACBEATH'S FORMULA FOR HYPERBOLIC MANIFOLDS

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In memory of Kay Magaard

Abstract. Around 1973, Macbeath provided a formula for the number of fixed points of an element in a group G of conformal automorphisms of a closed Riemann surface S of genus at least two. Such a formula was initially used to obtain the character of the representation associated to the induced action of G on the first homology group of S, and later turned out to be extremely useful in many other contexts. By using a simple counting procedure, we provide a similar formula for the number of connected components of an element in a finite group of isometries of a hyperbolic manifold.

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1. Introduction

Let S be a closed Riemann surface of genus at least two and let G be a (necessarily finite) group of conformal automorphisms of S. In 1973, Macbeath [14] found a formula for the number of fixed points of each $g \in G$ in terms of the topological action of G (see Section 4). Later, in [5, 6, 7, 12], a generalization of this formula to count the number of connected components of fixed points, has been found for the cases of anti-conformal automorphisms of compact Riemann surfaces and also for dianalytic automorphisms of bordered and unbordered compact Klein surfaces (both in orientable and non-orientable cases). Let us note that if H is a finite group

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of orientation-preserving homeomorphisms of a closed orientable surface X, then it is well known that on X there is the structure of a Riemann surface making H to act as a group of conformal automorphisms; so actually Macbeath's formulas can be applied for arbitrary periodic self-homeomorphisms. Furthermore, due to well known description of discrete cocompact groups of isometries of the hyperbolic plane, the formulas in the 2-dimensional case have more explicit character.

Now, let us assume we are given a pair (M, G), where M is an $(n+1)$-dimensional hyperbolic manifold, where $n \geq 1$, and G is a finite group of its isometries. Then there is a triple $(\mathcal{F}, \mathcal{K}, \theta)$, where $\mathcal{K}$ is a group of isometries of the hyperbolic $(n+1)$-dimensional space $\mathcal{H}^{n+1}$ and $\theta : \mathcal{K} \to G$ is a surjective homomorphism with a torsion free kernel $\mathcal{F}$, so that $M = \mathcal{H}^{n+1}/\mathcal{F}$ and $M/G = \mathcal{H}^{n+1}/\mathcal{K}$. Under the assumption that $\mathcal{K}$ is finitely generated and it has a finite number of conjugacy classes of finite order elements (we say that it is of finite type), by using an elementary counting method on G, we may obtain a formula, similar to Macbeath's one, to count the number of connected components of a non-trivial element $g \in G$ (see Theorem 3.3). It seems that a similar formula is not provided in the literature. Examples of Kleinian groups $\mathcal{K}$ of finite type are the geometrically finite ones [4, 13] and, for $n = 2$, the locus of geometrically finite Kleinian groups is dense on the space of finitely generated Kleinian groups [3, 18, 19]; two examples are provided in the last section.

We should remark that, by suitable modification of the proof of Theorem 3.3, it can be shown that the provided formula to count the number of connected components of isometries still valid for G being a finite group of isometries of a Riemannian manifold M for which its universal Riemannian cover space $\widetilde{M}$ has the property that its finite order isometries have non-empty and connected set of fixed points. Examples of these are for $\widetilde{M}$ being either (i) the Teichmüller space $\mathcal{T}_g$ of genus $g \geq 1$ Riemann surfaces or (ii) the Siegel space $\mathfrak{H}_g$ parametrizing principally polarized abelian varieties.

2. Preliminaries

In this section we recall some concepts and facts concerning isometries of hyperbolic spaces, (extended) Kleinian groups and its associated manifolds, which shall be used in the rest of this paper. Good references on these are, for instance, the classical books [16, 17].

2.1. **Hyperbolic space.** For $n \geq 1$, we shall use as a model of the $(n+1)$-dimensional hyperbolic space $\mathcal{H}^{n+1}$ the $(n+1)$-dimensional upper-half space $\{x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\}$ equipped with the Riemannian metric $ds = \|dx\|/x_{n+1}$. Its conformal boundary is the n-dimensional sphere $\mathcal{S}^n = \mathbb{R}^n \cup \{\infty\}$. Each $(n-1)$-dimensional sphere $\Sigma \subset \mathcal{S}^n$ (for $n = 1$, Σ is understood as two different points) has associated a reflection $\sigma = \sigma_\Sigma$ having Σ as its locus of fixed points. By the Poincaré extension theorem, the reflection σ extends naturally to an order two orientation reversing isometry of $\mathcal{H}^{n+1}$; this being the reflection on the half-n-dimensional sphere inside $\mathcal{H}^{n+1}$ induced by Σ. A Möbius (respectively, extended Möbius) transformation of $\mathcal{S}^n$ is the composition of an even (respectively odd) number of reflections. By the classical complex analysis, for $n = 2$, and the Liouville Theorem, for $n \geq 3$, the group $\widehat{\mathcal{M}}^n$, composed by all Möbius and extended Möbius transformations, is the full group of conformal automorphisms of

$\mathcal{S}^n$. We shall denote denote by $\mathcal{M}^n$ its canonical subgroup of index two consisting of all Möbius transformations. Again, by the Poincaré extension theorem, every Möbius (respectively, extended Möbius) transformation extends to an orientation-preserving (respectively, orientation-reversing) isometry of $\mathcal{H}^{n+1}$ and all isometries of $\mathcal{H}^{n+1}$ are obtained in this way. This allows us to identify the group $\widehat{\mathcal{M}}^n$ with the group $\mathrm{Isom}(\mathcal{H}^{n+1})$ of all isometries of $\mathcal{H}^{n+1}$ and $\mathcal{M}^n$ with the index two subgroup $\mathrm{Isom}^+(\mathcal{H}^{n+1})$ of all its orientation-preserving isometries. An element of $\widehat{\mathcal{M}}^n$, viewed as an isometry of $\mathcal{H}^{n+1}$, may or may not have fixed points in $\mathcal{H}^{n+1}$ and if the former is the case, then it is called *elliptic* if it preserves orientation and *pseudo-elliptic* otherwise. The locus of fixed points, in the hyperbolic space, of an elliptic or pseudo-elliptic transformation is known to be either a point or a totally geodesic subspace of $\mathcal{H}^{n+1}$.

2.2. **Kleinian groups.** A *Kleinian group* is a discrete subgroup of $\mathcal{M}^n$ and an *extended Kleinian group* is a discrete subgroup of $\widehat{\mathcal{M}}^n$ not contained in $\mathcal{M}^n$. Elliptic or pseudo-elliptic transformations of a (extended) Kleinian group have necessarily finite orders. Let us note, from the definition, that a subgroup $\mathcal{K}$ of $\widehat{\mathcal{M}}^n$ is an extended Kleinian group if and only if $\mathcal{K}^+ = \mathcal{K} \cap \mathcal{M}^n$ is a Kleinian group.

Associated to an (extended) Kleinian group $\mathcal{F} < \mathcal{M}^n$ is a $(n+1)$-dimensional (orientable if it is Kleinian) hyperbolic orbifold $M_{\mathcal{F}} = \mathcal{H}^{n+1}/\mathcal{F}$. If, moreover, $\mathcal{F}$ is torsion free, then $M_{\mathcal{F}}$ is a $(n+1)$-dimensional hyperbolic manifold, which means that it carries a natural complete Riemannian metric of constant negative curvature inherited from the one of $\mathcal{H}^{n+1}$.

Assuming $\mathcal{F}$ to be a torsion free Kleinian group, so $\mathcal{M}_{\mathcal{F}}$ is an orientable hyperbolic manifold, by a *conformal automorphism* (respectively, *anti-conformal automorphism*) of $M_{\mathcal{F}}$ we means an orientation-preserving (respectively, orientation-reversing) self-isometry. We denote by $\mathrm{Aut}(M_{\mathcal{F}})$ the group of all conformal/anti-conformal automorphisms of $M_{\mathcal{F}}$.

A Kleinian group $\mathcal{F}$ is called *geometrically finite* if it has a finite-sided fundamental polyhedron in $\mathcal{H}^{n+1}$, in particular, it is finitely generated and the hyperbolic volume of $\mathrm{Hull}_\epsilon(\Lambda(\mathcal{F}))/\mathcal{F}$ is finite, where $\Lambda(\mathcal{F}) \subset \mathcal{S}^n$ stands for the limit set of $\mathcal{F}$ and $\mathrm{Hull}_\epsilon(\Lambda(\mathcal{F}))$ is the ϵ-neighborhood of the convex hull of $\Lambda(\mathcal{F})$ in $\mathcal{H}^{n+1}$. An extended Kleinian group is geometrically finite if its index two orientation-preserving half Kleinian group is so. Finite index extensions of geometrically finite groups are still geometrically finite. Another properties of geometrically finite groups can be found, for instance, in [1, 16].

2.3. **A finiteness property.** Let us consider an (extended) Kleinian group $\mathcal{K} < \mathcal{M}^n$. We will say that $\mathcal{K}$ is of *finite type* if it is finitely generated and it contains a finite number of conjugacy classes of elements of finite order.

Remark 2.1. If $n \in \{1, 2\}$ and $\mathcal{K}$ is finitely generated, then it is of finite type [4], but for $n \geq 3$, the finitely generated condition does not always ensure the finite type property [13], but it holds true if $\mathcal{K}$ is known to be geometrically finite.

So, if $\mathcal{K} < \mathcal{M}^n$ is of finite type, then we are able to find a collection of finite order elements $\{\kappa_1, \ldots, \kappa_r\} \subset \mathcal{K}$ so that the following holds:

(ecs1) each κ_i generates a maximal cyclic subgroup of $\mathcal{K}$;
(ecs2) the cyclic subgroups generated by $\kappa_1, \ldots, \kappa_r$ are pairwise non-conjugate in $\mathcal{K}$;

(ecs3) the collection is maximal with respect to the above two properties.

In the above, following the terminology used by Maclachlan in [15] for Fuchsian groups, we will say that the collection $\{\kappa_1, \ldots, \kappa_r\}$ is an *elliptic complete system* (*e.c.s.* in short) and any of its elements is called a *canonical generating symmetry of* $\mathcal{K}$.

3. QUANTITATIVE ASPECTS OF THE SET OF FIXED POINTS: MACBEATH'S FORMULA

In this section, we let $\mathcal{K}$ be an (extended) Kleinian group of finite type, G a finite group and $\theta : \mathcal{K} \to G$ a surjective homomorphism with a torsion free (finitely generated) Kleinian group $\mathcal{F}$. Let us denote by $\pi : \mathcal{H}^{n+1} \to M_{\mathcal{F}}$ a universal covering with $\mathcal{F}$ as its group of deck transformations. The finite group $G =_\theta \mathcal{K}/\mathcal{F}$ is a subgroup of $\mathrm{Aut}(M_{\mathcal{F}})$. Next, we proceed to search for a Macbeath's formula that permits to count the number of connected components of fixed points of each non-trivial element $g \in G$.

If $\kappa \in \mathcal{K}$, $g = \theta(\kappa)$, $h \in \mathcal{H}^{n+1}$ and $x = \pi(h)$, then $g(x) = \pi(\kappa(h))$. We shall keep all these notations throughout the rest of this paper

Each finite order element $\kappa \in \mathcal{K}$ defines an automorphism $\theta(\kappa) \in G$ acting with fixed points (the set of fixed points of κ are sent by π to fixed points of $\theta(\kappa)$). The converse is clear as π is a local homeomorphism.

Lemma 3.1. *The sets of fixed points of two distinct non-trivial elements of* $\mathcal{K}$ *of finite orders inducing the same automorphisms of* $M_{\mathcal{F}}$ *are disjoint.*

Proof. Let κ, κ' be elements of finite order of $\mathcal{K}$ and let $\theta(\kappa) = \theta(\kappa')$. Let us assume they have non-disjoint connected components, say C and C', of their loci of fixed points. If $y \in C \cap C'$, then y is a fixed point of $\kappa^{-1}\kappa' \in \ker\theta = \mathcal{F}$. As $\mathcal{F}$ is torsion free and $\kappa^{-1}\kappa'$ has a fixed point, we must have that $\kappa^{-1}\kappa' = 1$. □

Since $\mathcal{K}$ is of finite type, we may find an e.c.s. $\{\kappa_1, \ldots, \kappa_r\}$ for $\mathcal{K}$, which we assume, from now on, to be fixed. Let us denote by m_j the order of κ_j.

If $g \in G$ is a non-trivial element, say of order m, with fixed points, then property ($ecs3$) ensures that $g = \theta(\kappa)$ for some elliptic element κ of $\mathcal{K}$ which is conjugated to a power of some canonical generating symmetry κ_j. Let $J(g)$ be the set of such $j \in \{1, \ldots, r\}$ for which $g = \theta(\omega_j \kappa_j^{n_j} \omega_j^{-1})$ for some $\omega_j \in \mathcal{K}$ and some $n_j \in \{1, \ldots, m_j - 1\}$. As $\ker\theta = \mathcal{F}$ is torsion free, we may see in the above equality that the order of g and that of $\kappa_j^{n_j}$ is the same, that is, $m = m_j / \gcd(m_j, n_j)$.

Remark 3.2. For $n = 1$, $\mathcal{K}$ is either a Fuchsian or an NEC group, so the set of fixed points of an elliptic element $\kappa \in \mathcal{K}$ consist in a single point and, as the only finite order orientation reversing isometries of the hyperbolic plane are reflections, the locus of fixed points in this case is a geodesic line. In particular, different elliptic elements of the same order have different sets of fixed points. Unfortunately, this is no longer true for symmetries of higher dimensional spaces and this is a one of the essential differences between Macbeath's formula for Riemann surfaces and our formula for hyperbolic manifolds of higher dimensions. For instance, Let $n \geq 4$ and take $A, B \in \mathrm{O}_n(\mathbb{R})$ generating a non-cyclic finite group $\mathcal{U}$. Assume that none of them has eigenvalue equal to 1 and so that they are non-conjugate in $\mathcal{U}$ (this can be done for $n \geq 4$). Let us consider the isometries of the hyperbolic $(n+1)$-space, in this example modeled by the unit ball in $\mathbb{R}^{n+1}$, given by

$$T_A = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix},\ T_B = \begin{bmatrix} B & 0 \\ 0 & 1 \end{bmatrix}.$$

Then, $\mathcal{K} = \langle T_A, T_B \rangle \cong \mathcal{U}$ is a finite extended Kleinian group, where T_A and T_B both share the same geodesic set of fixed points, but they are non-conjugated in $\mathcal{K}$.

As note in the above remark, although different canonical generating symmetries give rise to different connected components of their sets of fixed points, it is no longer true for their powers of the same order. In order to neutralize that deficiency while we count the number of connected components of $\mathrm{Fix}(g)$, for $g \in G - \{I\}$, we must define the following equivalence relation on $J(g)$:

$$j_1, j_2 \in J(g):\ j_1 \sim j_2 \Leftrightarrow \mathrm{Fix}(\kappa_{j_1}^{n_{j_1}}) \cap \mathrm{Fix}(\kappa_{j_2}^{n_{j_2}}) \neq \emptyset.$$

The above relation is equivalent for the sets of fixed points of elliptic elements to be projected on the same subset of $M_{\mathcal{F}}$. Let $I(g)$ be a set of representatives for the quotient set $J(g)/\!\sim$.

Finally, as the set $\mathcal{C}$ of fixed point of an isometry of finite order $\kappa \in \mathcal{K}$ is a totally geodesic subspace of $\mathcal{H}^{n+1}$, its image $\ell = \pi(\mathcal{C})$ is a connected component of the set of fixed points of $\theta(\kappa)$; this being again a totally geodesic submanifold of $M_{\mathcal{F}}$. Let G_ℓ be the subgroup of G leaving ℓ set-wise invariant (the elements of G_ℓ permutes the points on ℓ and may or may not have fixed points on it).

We have now introduced all notations and facts needed to state and prove our main result concerning the number of connected components of the locus of fixed points of isometries in G.

Theorem 3.3 (Macbeath's formula for hyperbolic manifolds)**.** *Let $\mathcal{K}$ be an (extended) Kleinian group of finite type, G a finite abstract group and $\theta : \mathcal{K} \to G$ a surjective homomorphism whose kernel $\mathcal{F}$ is torsion free. Fix an elliptic complete system $\kappa_1, \ldots, \kappa_r$ for $\mathcal{K}$ and suppose that κ_i has order m_i. Let $g \in G$ be a non-trivial element of order m with non-empty set of fixed points. Then the number of connected components of the set of fixed points of g in the hyperbolic manifold $M_{\mathcal{F}}$ is*

$$|\mathcal{N}_G\langle g\rangle| \sum_{i \in I(g)} 1/n_i,$$

where $\mathcal{N}_G\langle g\rangle$ stands for the normalizer in G of the cyclic subgroup $\langle g\rangle$ and n_i is the order of the θ-image of the $\mathcal{K}$-normalizer of $\langle \kappa_i^{m_i/m}\rangle$.

Proof. Let π be a universal covering $\mathcal{H}^{n+1} \to M_{\mathcal{F}}$ induced by $\mathcal{F}$ and let $g \in G$ be non-trivial of order m acting with fixed points on $M_{\mathcal{F}}$. For each $j \in J(g)$ we also set $s_j := m_j/m = \gcd(m_j, n_j)$. Let $\kappa \in \mathcal{K}$ so that $\theta(\kappa) = g$. Then, for $h \in \mathcal{H}^{n+1}$, we have that $x = \pi(h)$ is a fixed point of g if and only if $\pi(h) = \pi(\kappa(h))$ and so if and only if $\gamma(h) = \kappa(h)$ for some $\gamma \in \mathcal{F}$. This means that $\gamma^{-1}\kappa \in \mathcal{K}$ has a fixed point and hence it is conjugate to a power of some element κ_i of *e.c.s.* and therefore $g = \theta(\omega\kappa_i^{\alpha s_i}\omega^{-1})$ for some α coprime with m and $\omega \in \mathcal{K}$. Clearly neither i, nor α and nor ω must be unique here. So given $i \in J(g)$ consider

$$N_i(g) = \{\omega \in \mathcal{K} : g = \theta(\omega\kappa_i^{t_i}\omega^{-1}) \text{ for some } t_i\}.$$

If we let $\mathcal{C}_i$ to be the totally geodesic subspace of the set of fixed points of $\kappa_i^{t_i}$ and if we denote by $\mathcal{K}_i$ its stabilizer in $\mathcal{K}$, which is the normalizer of $\langle \kappa_i^{s_i}\rangle$, then

$x \in \pi(\omega(\mathcal{C}_i))$. Conversely, given $\omega \in N_i(g)$, $\pi(\omega(\mathcal{C}_i)) \subseteq \mathrm{Fix}(g)$. It follows that

$$\mathrm{Fix}(g) = \bigcup_{i \in I(g)} \bigcup_{\omega \in N_i(g)} \pi(\omega(\mathcal{C}_i)).$$

Let us fix ω_i in $N_i(g)$. Then for any orther $\omega \in N_i(g)$, $\omega\mathcal{K}_i = \omega_i\mathcal{K}_i$ and so $N_i(g)$ is the left coset $\omega_i\mathcal{K}_i$. Furthermore $\omega \in N_i(g)$, gives $\theta(\omega_i^{-1}\omega) \in \mathcal{N}_G\langle g\rangle$ and so $\omega_i^{-1}\omega \in \theta^{-1}(\mathcal{N}_G\langle g\rangle)$ which in turn means that $N_i(g)$ is also left coset $\omega_i\,\theta^{-1}(\mathcal{N}_G\langle g\rangle)$. Now, $\ell_i = \pi(\omega(\mathcal{C}_i))$ is a one of the connected components of the set of fixed points of g and notice that $\theta(\omega\mathcal{K}_i\omega^{-1}) = G_{\ell_i}$. Given $\nu, \nu' \in \theta^{-1}(\mathcal{N}_G\langle g\rangle)$, we have the following chain of equivalences

$$\begin{array}{llll} \pi(\omega_i\nu(\mathcal{C}_i)) \cap \pi(\omega_i\nu'(\mathcal{C}_i)) \neq \emptyset & \Leftrightarrow & \pi(\omega_i\nu(\mathcal{C}_i)) = \pi(\omega_i\nu'(\mathcal{C}_i)) & \Leftrightarrow \\ \gamma\omega_i\nu(\mathcal{C}_i)) = \omega_i\nu'(\mathcal{C}_i), \text{ for some } \gamma \in \mathcal{F} & \Leftrightarrow & \gamma'\nu'^{-1}\nu(\mathcal{C}_i) = \mathcal{C}_i, \text{ for some } \gamma' \in \mathcal{F} & \Leftrightarrow \\ \theta(\nu'^{-1}\nu) \in G_{\ell_i} & \Leftrightarrow & \nu'^{-1}\nu \in \theta^{-1}(G_{\ell_i}) & \Leftrightarrow \\ \theta(\nu'^{-1})\theta(\nu) \in \theta(\omega\mathcal{K}_i\omega^{-1}). & & & \end{array}$$

The first equivalence follows from Lemma 3.1, the third is a consequence of the normality of $\mathcal{F}$ in $\mathcal{K}$; the remainder are rather clear. Thus, each $i \in I(g)$ produces

$$[\theta^{-1}(\mathcal{N}_G\langle g\rangle) : \theta^{-1}(G_{\ell_i})] = \frac{|\mathcal{N}_G\langle g\rangle|}{|G_{\ell_i}|} = \frac{|\mathcal{N}_G\langle g\rangle|}{n_i}$$

connected components of the locus of fixed points of g. Finally, in order to get the desired formula, we need to prove that $\pi(\omega_i(\mathcal{C}_i)) \cap \pi(\omega_j(\mathcal{C}_j)) = \emptyset$, if $i, j \in I(g)$ with $i \neq j$. In fact, otherwise (by Lemma 3.1) if they intersect, then necessarily $\pi(\omega_i(\mathcal{C}_i)) = \pi(\omega_j(\mathcal{C}_j))$; so for arbitrary $c_i \in \mathcal{C}_i$ we have $\gamma\omega_i(c_i) = \omega_j(c_j)$ for some $c_j \in \mathcal{C}_j$ and some $\gamma \in \mathcal{F}$. Therefore $\omega_j^{-1}\gamma\omega_i(\mathcal{C}_i) = \mathcal{C}_j$. In other words, there is an element $\eta \in \mathcal{K}$ so that $\eta\kappa_i^{t_i}\eta^{-1}$ and $\kappa_j^{t_j}$ have the same set of fixed points, contradicting the definition of the set $I(g)$. □

It follows from the proof of the above Theorem the following upper bound.

Corollary 3.4. *Let $\mathcal{K}, \mathcal{F}, G, \theta, \pi$, κ_i and m_i be as in Theorem 3.3. Then the number of connected components of the set of fixed points of $g \in G$ does not exceed*

$$|\mathcal{N}_G\langle g\rangle| \sum_{j \in J(g)} 1/m_j.$$

Proof. Indeed $|I(g)| \leq |J(g)|$ and $m_i \leq n_i$. □

4. Comments concerning dimension two

Due to a better understanding and description of discrete cocompact groups of isometries of the hyperbolic plane $\mathcal{H}^2$, the formulas in Theorem 3.3 have a more explicit character. In fact, as the locus $\mathrm{Fix}(\kappa_i)$ of any canonical elliptic generator x_i of a Fuchsian group, is a single point p_i and $G_{\{p_i\}} = \langle x_i\rangle$, Theorem 3.3 reduces to Macbeath's counting formula in [14].

Corollary 4.1 (Macbeath's counting formula for Riemann surfaces [14])**.** *Let $\mathcal{K}$ be a finitely generated discrete group of orientation-preserving isometries of the hyperbolic plane $\mathcal{H}^2$ and let $x_1, \ldots, x_r$ be the a set of canonical elliptic generators of it of orders $m_1, \ldots, m_r$, respectively. Let G be a finite group and $\theta : \mathcal{K} \to G$ be a surjective homomorphism, whose kernel $\mathcal{F}$ is torsion free. We may consider the natural action of $G =_\theta \mathcal{K}/\mathcal{F}$, by orientation preserving automorphisms, of the*

Riemann surface $S = \mathcal{H}^2/\mathcal{F}$*. Then the number of fixed points of* $g \in G$ *is given by the formula*

$$|\mathcal{N}_G(\langle g \rangle)| \sum 1/m_i,$$

where $\mathcal{N}$ *stands for the normalizer and the sum is taken over those* i *for which* g *is conjugate to a power of the image* $\theta(x_i)$*. In particular the number of fixed points of* g *is finite.*

An anti-holomorphic automorphism of a compact Riemann surface of genus g, with fixed points, must be an involution; its locus of fixed points consist of $s \in \{1, \ldots, g+1\}$ disjoint sets, each of which is homeomorphic to a circle (ovals) by a well known result due to Harnack. A canonical elliptic generator c_i, of a NEC group, inducing an anti-holomorphic automorphisms (with fixed points) is a reflection, which is determined by its axis. In this way, we see that $G_\ell = \theta(C(\Lambda, c_i)))$ and therefore Theorem 3.3 reduces to the main result from [6].

Corollary 4.2. *Let* $\mathcal{K}$ *be an NEC-group,* G *be a finite group and let* $\theta : \mathcal{K} \to G$ *be a surjective homomorphism whose kernel is a torsion free cocompact Fuchsian group* $\mathcal{F}$*. Let us consider the action of* G*, under* θ*, on the compact Riemann surface* $S = \mathcal{H}^2/\mathcal{F}$ *as a group of conformal and anticonformal automorphisms. Then a symmetry* σ *with fixed points is conjugate to* $\theta(c)$ *for some canonical reflection* c *of* $\mathcal{K}$ *and it has*

$$\sum [\, C(G, \theta(c_i)) : \theta(C(\mathcal{K}, c_i)) \,]$$

ovals, where C *stands for thre centralizer,* c_i *run over nonconjugate canonical reflections of* $\mathcal{K}$*, whose images under* θ *belongs to the orbit of* σ *in* G*.*

Remark 4.3. An algebraic structure of the centralizers of reflections in an NEC-group was found by Singerman in [20]. There is a simple method, based on the geometry of the hyperbolic plane, to find explicit formulas for them as described in [8]. Similarly, effective formulas are also known for periodic self-homeomorphisms of non-orientable or bordered compact surfaces [5, 7].

5. A couple of examples in hyperbolic 3-dimensional case

We shall give two examples of 3-dimensional hyperbolic manifolds, to see how our formula works in practice.

Example 5.1 (Generalized Fermat 3-manifolds)**.** Let $m, k \geq 3$ be integers. A *generalized Fermat manifold of type* (m, k) is a compact hyperbolic 3-manifold N admitting a group $H \cong \mathbb{Z}_m^k$ of isometries so that the hyperbolic orbifold M/G is homeomorphic (as orbifolds) to the orbifold $\mathcal{O}$ whose underlying space is the unit 3-dimensional sphere $\mathcal{S}^3$ and the conical locus is given by k disjoint loops (each one of index m) as shown in Figure 1 (for the case $k = 10$). In this case, the group H is called a *generalized Fermat group of type* (m, k) and the pair (N, H) a *generalized Fermat pair of type* (m, k). By Mostow's rigidity theorem, up to isometry, there is only one generalized Fermat pair of type (m, k). As the 3-orbifold $\mathcal{O}$ is closed, Haken and homotopically atoroidal, it has a hyperbolic structure [2, 9], that is, there is Kleinian group $\mathcal{K}$ for which $\mathcal{O} = \mathcal{H}^3/\mathcal{K}$ (see Figure 2). We have that $\mathcal{K}$ is generated by $x_1, \ldots, x_k$ subject to the relations:

$$x_1^m = \ldots = x_k^m = 1, x_i x_{i+1}^{-1} x_i^{-1} x_{i+1} = x_{i+1} x_{i+2}^{-1} x_{i+1}^{-1} x_{i+2},$$

where i are taken modulo k. The collection $x_1, \dots, x_k$ is a elliptic complete system of $\mathcal{K}$ and the derived subgroup $\mathcal{K}'$ of $\mathcal{K}$ is torsion free [11]. So $M = \mathcal{H}^3/\mathcal{K}'$ is a closed hyperbolic 3-manifold with abelian group $G = \mathcal{K}/\mathcal{K}' \cong \mathbb{Z}_m^k$ of automorphisms. Let us now consider the canonical projection $\theta : \mathcal{K} \to G$ and set $a_i = \theta(x_i)$. By Theorem 3.3, the number of connected components of fixed points of each a_i is exactly m^{k-1}. In fact, in this example we have that $\mathcal{N}_G\langle a_i\rangle = G$, so $|\mathcal{N}_G\langle a_i\rangle| = m^k$, and $\mathcal{N}_{\mathcal{K}}\langle x_i\rangle = \langle x_i\rangle$.

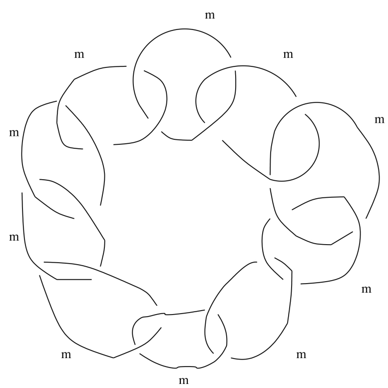

FIGURE 1. $k = 10$

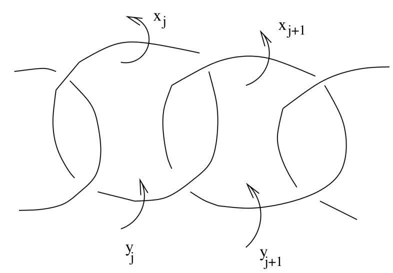

FIGURE 2. $\mathcal{O} = \mathcal{H}^3/\mathcal{K}$

Example 5.2 (Extended Schottky groups)**.** An *extended Schottky group of rank g* is an extended Kleinian group whose canonical subgroup of orientation preserving isometries is a Schottky group of rank g [10]. An example, of rank $g = 5$, is as follows. Choose three pairwise disjoint circles on the complex plane, all of them bounding a common 3-connected region. For each of these circles, we take either a reflection or an imaginary reflection that permutes both discs bounded by such a circle. Let us denote these transformations by κ_1, κ_2 and κ_3, and let $\mathcal{K}$ be the group generated by them. It happens that $\mathcal{K}$ is an extended Kleinian group isomorphic to the free product of three copies of $\mathbb{Z}_2$, $\mathcal{N}_{\mathcal{K}}\langle\kappa_i\rangle = \langle\kappa_i\rangle$ and $\{\kappa_1, \kappa_2, \kappa_3\}$ is a elliptic complete system of it. If we consider the surjective homomorphism $\theta : \mathcal{K} \to G = \mathbb{Z}_2^3 = \langle a_1, a_2, a_3\rangle$, defined by $\theta(\kappa_i) = a_i$, for $i = 1, 2, 3$, then its kernel

$$\mathcal{F} = \ker\theta = \langle\!\langle(\kappa_2\kappa_1)^2, (\kappa_2\kappa_3)^2, (\kappa_1\kappa_3)^2\rangle\!\rangle,$$

were the last stands for the normal closure, is a Schottky group of rank 5. So $M = \mathcal{H}^3/\mathcal{F}$ is homeomorphic to the interior of a handlebody of genus 5 admitting three symmetries a_1, a_2 and a_3, each one of order two. Since G is abelian, we have that $|\mathcal{N}_G\langle a_i\rangle| = 8$ and $|\theta(\mathcal{N}_{\mathcal{K}}\langle\kappa_i\rangle| = |\langle a_i\rangle| = 2$. It follows from the counting formula of Theorem 3.3 that each a_i has exactly 4 connected components of its set of fixed points; all of them being either isolated points or discs.

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