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TRACES OF ENTIRE FUNCTIONS ON ALGEBRAIC SUBVARIETIES

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Abstract. We show that, to some extent, the behavior of an entire function of several complex variables is reflected by its behavior on algebraic subvarieties.

1. Introduction

Let f be a holomorphic function on $\mathbb{C}^n$; given a family of algebraic subvarieties of $\mathbb{C}^n$, is it possible to determine the order of growth of f from the order of growth of the restriction of f along a general member of the family? For linear subspaces of $\mathbb{C}^n$, this problem was intensively studied by Pierre Lelong (see for example [7], [8]). The order of growth for entire functions can be defined in the following way.

Definition 1. *Let f be an entire function $f(z)$ on $\mathbb{C}^n$. We say that f is of finite order if if there exists a positive number t such that $|f(z)| = O(\exp(|z|^t)$. If f is of finite order, then the order of growth (or simply the order) of f is defined as*

$$\rho = \inf\{t \,:\, |f(z)| = O(\exp(|z|^t)\}.$$

If f is not of finite order, we say that f is of infinite order.

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The order of growth can be defined also for meromorphic functions, using Nevanlinna theory. However, since in this paper we consider only entire functions, we shall not state the definition, and refer instead to [5]. An interesting result concerning the order of meromorphic functions was proved by Gauthier and Hengartner.

Theorem 2. *Let S^{2n-1} denote the $(2n-1)$-sphere of radius 1 in $\mathbb{C}^n$ and, for $\theta \in S^{2n-1}$, let L_θ denote the line passing through θ and the origin. Let f be a meromorphic function on $\mathbb{C}^n$. Then, $f|L_\theta$ is of the same order as f, for almost all $\theta \in S^{2n-1}$. Moreover, if there exists a set of positive measure $S \subset S^{2n-1}$ such that $f|L_\theta$ is rational for all $\theta \in S$, then f is rational.*

In the present paper we investigate to what extent the order of growth of an entire function is reflected by its behavior along members of a family of algebraic subvarieties of $\mathbb{C}^n$. Since we cannot provide a general theory, we consider only some classical examples of families (Section 2 and Section 3). In Section 4 we show, by contrast, that the behavior of an entire function f along real curves does not provide any information about the order of growth of f.

2. Traces along subvarieties passing through the origin

2.1. **Grassmannians.** Let G_k denote the Grassmannian manifold of k-dimensional planes passing through the origin in $\mathbb{C}^n$. If E is a k-plane in $\mathbb{C}^n$, we denote by $[E]$ the corresponding point in G_k. Gauthier and Hengartner [5] showed that the order of a meromorphic function on $\mathbb{C}^n$ is determined by the order of the restriction of f along lines (or k-planes more generally) through the origin. In fact the following result, from which Theorem 2 follows as a particular case, is proved in [5].

Theorem 3. *Let f be a meromorphic function on $\mathbb{C}^n$. Then $f|X$ is of the same order as f for all $[X] \in G_k$ outside a set of measure zero.*

We wish to consider such matters with regards to topological rather than measure-theoretic genericity. To this end we introduce the following lemmas.

Lemma 4. *Let $p : Y \to X$ be a surjective map of irreducible quasi-projective varieties. If S is Zariski dense in X, then $p^{-1}(S)$ is Zariski dense in Y. If S' is Zariski dense in Y, then $p(S')$ is Zariski dense in X*

Proof. In this proof, all topological notions and notations are with respect to the Zariski topologies. Let $Z = \overline{p^{-1}(S)}$. Then, by Chevalley's theorem, its image $p(Z)$ is locally closed, that is, open in its closure. Since $p(Z)$ contains S, it follows that $p(Z)$ is open in X. Therefore the dimension on Z equals the sum of the dimension of X and the dimension of a general fiber of p, which is the same as the dimension of Y. Since Y irreducible, we must have $Y = Z$, which proves the first part.

As for the second part, if $p(S')$ is contained in a subvariety V, then $p^{-1}(V)$ is a subvariety of Y containing S'. Thus, $p^{-1}(V) = Y$ and $V = X$.

□

Lemma 5. *Let $S \subset G_k$ be a Zariski dense subset. Then $\widetilde{S} = \cup_{[E]\in S} E$ is Zariski dense in $\mathbb{C}^n$.*

Proof. Let $\mathfrak{S} \subset \mathbb{C}^n \times G_k$ be the set $\{(x, [E]) : x \in E\}$. As a product of quasi-projective varieties, it is also quasi-projective. Denote the projections p_1 (resp. p_2)

on the factors $\mathbb{C}^n$ (resp. G_k). Lemma 4 implies that $S' = p_2^{-1}(S)$ is Zariski dense in $\mathfrak{S}$. We note that $\tilde{S} = p_1(S')$, which is dense by the second part of Lemma 4.

□

Theorem 6. *Let f be an entire function on $\mathbb{C}^n$, such that $f \mid E$ is a polynomial of degree at most d, for all $[E] \in S$, where $S \subset G_k$ is Zariski dense. Then f is a polynomial of degree at most d.*

Proof. Let

$$f = \sum_{k=0}^{\infty} f_k$$

be the homogeneous expansion of f. For $k > d$, the polynomial f_k is zero on $\widetilde{S}$, which is Zariski dense. Therefore $f_k = 0$, which completes the proof.

□

Remark 7. *In Lemma 5 of [5] the authors show a similar result without assuming a uniform bound on the polynomials $f \mid E$. However, the set S in their case is non-polar. A Zariski dense set can be polar. For example, every countable subset of $\mathbb{C}$ having an accumulation point is both Zariski dense and polar. The following example shows that the conclusion of Theorem 6 fails, if we merely drop the restriction on the degrees of the polynomials.*

Example 8. *Let λ_j be a sequence of distinct non-zero complex numbers. For each j, set*

$$g_k(z, w) = \eta_k \prod_{j=1}^{k-1} (w - \lambda_j z),$$

where η_k is chosen so that $|g_k| < 2^{-k}$ on the ball centered at the origin and of radius k and moreover, $a_k = \eta_k (-1)^k \lambda_1 \lambda_2 \cdots \lambda_k > 0$. Then,

$$f(z, w) = \sum_{k=1}^{\infty} g_k(z, w)$$

is an entire function which is not a polynomial, since

$$f(z, 0) = \sum_{k=1}^{\infty} a_k z^k, \quad a_k > 0,\ k = 1, 2, \cdots .$$

Now, for fixed j we write

$$f(z, w) = \sum_{k=1}^{j-1} p_k(z, w) + \sum_{k=j}^{\infty} p_k(z, w),$$

for all (z, w). In particular, on the line $w = \lambda_j z$, we have

$$f(z, w) = \sum_{k=1}^{j-1} p_k(z, \lambda_j z) + \sum_{k=j}^{\infty} p_k(z, \lambda_j z),$$

where the first term is a polynomial in z and the second term is zero, since $p_k(z, \lambda_j z) = 0$, for $j \le k$. Thus, on the line $w = \lambda_j z$, the function $f(z, w)$ is a polynomial in z.

2.2. **Weighted projective spaces.** Let $(a_1, \dots, a_n)$ be a vector, whose components a_k are positive integers, for $k = 1, \cdots, n$. Fix $z = (z_1, \dots z_n) \in \mathbb{C}^n$ a non-zero vector, and consider the curve C parametrized by $\gamma(t) = (t^{a_1} z_1, \dots t^{a_n} z_n), t \in \mathbb{C}$. The collection of all such curves, for z ranging over all non-zero vectors , is parametrized by the weighted projective space $\mathbb{P}_{(a_1, \dots a_n)}$, which is defined as follows.

Definition 9. *Given a vector $(a_1, \dots a_n)$ of positive integers, consider the action of the multiplicative group $\mathbb{C}^*$ on $\mathbb{C}^n \setminus \{0\}$ defined by*

$$t(z_1, \dots z_n) = (t^{a_1} z_1, \dots t^{a_n} z_n).$$

The quotient space $(\mathbb{C}^n \setminus \{0\}) / \mathbb{C}^* = \mathbb{P}_{(a_1, \dots a_n)}$ is called weighted projective space with weight vector $(a_1, \dots a_n)$.

The standard projective space is therefore $\mathbb{P}_{(1, \dots 1)} = \mathbb{P}^{n-1}$. We refer to [2] for the proof of the following well known facts: the weighted projective space $\mathbb{P}_{(a_1, \dots a_n)}$ is a projective variety of dimension $(n-1)$, and $\mathbb{P}_{(a_1, \dots a_n)}$ is smooth if and only if $\mathbb{P}_{(a_1, \dots a_n)}$ is isomorphic to the projective space of dimension $(n-1)$.

Example 10. *The map*

$$\begin{array}{rccl} f: & \mathbb{P}_{1,2} & \longrightarrow & \mathbb{P}^1 \\ & [w_0 : w_1] & \longmapsto & [w_0^2 : w_1] = [z_0 : z_1] \end{array}$$

is an isomorphism since $[\sqrt{z_0} : z_1] = [-\sqrt{z_0} : z_1]$ in $\mathbb{P}_{1,2}$.

Definition 11. *Given a vector $a = (a_1, \dots a_n)$, the a-degree of a polynomial is defined by letting the a-degree of the monomial w_k be a_k, and then extending the definition according to the definition of degree function. A polynomial is said to be weighted homogeneous of a-degree d, if all its monomial terms have a-degree d.*

The common zero locus of a finite collection of weighted homogeneous polynomials defines an algebraic subvariety of $\mathbb{P}_{(a_1, \dots a_n)}$. Let $z = (z_1, \dots z_n)$ be a non-zero vector in $\mathbb{C}^n$: we denote by $C = C_z$ the image of the curve $\gamma(t) = \gamma_z(t) = (t^{a_1} z_1, \dots t^{a_n} z_n)$ and by $[C] \in \mathbb{P}_{(a_1, \dots a_n)}$ the corresponding point on the weighted projective space. We say that such a curve C (and by abuse also $[C]$) is an a-curve.

Definition 12. *If $f(z)$ is an entire function on $\mathbb{C}^n$, the restriction to $C = C_z$ is an entire function of one variable t, and we say that $f|C$ is a polynomial of degree d if f is a polynomial of degree d in the variable t. We say that $f|C$ is of finite order if $f(tz^{a_1}, \cdots tz^{a_n})$ is of finite order as a function of t.*

Note that, if f is a polynomial of a-degree d, then it is a polynomial of degree d along every a-curve $C \in \mathbb{P}_{(a_1, \dots a_n)}$. The following analogue of Theorem 6 shows conversely that, if f is a polynomial of degree d along every a-curve $C \in \mathbb{P}_{(a_1, \dots a_n)}$, then f is a polynomial of a-degree d.

Theorem 13. *Let f be an entire function in $\mathbb{C}^n$. Assume that there exists a Zariski dense subset $S \subset \mathbb{P}_{(a_1, \dots a_n)}$, such that $f|C$ is a polynomial of degree at most d, for all a-curves $[C] \in S$. Then f is a polynomial of a-degree at most d and hence of degree at most $\sum (d/a_k)$.*

Proof. Since every monomial $z^m = z_1^{m_1} \cdots z_n^{m_n}$ is weighted homogeneous of a-degree $a \cdot m$, we may write f uniquely as a weighted homogeneous expansion $f =$

$\sum f_k$, where f_k is weighted homogeneous of a-degree k. Indeed, if $\sum p_m(z)$ is the homogeneous expansion of f, then

$$f(z) = \sum p_m(z) = \sum_{k=0}^{\infty} \left(\sum_{a \cdot m = k} p_m(z) \right) = \sum_{k=0}^{\infty} f_k(z).$$

Then, for all $k > d$, f_k is identically equal to zero on a Zariski dense subset $\tilde{S} = \bigcup_{[C] \in S} C$; hence it must be identically equal to 0. □

In Nevanlinna theory, polynomial functions all have the same order, and that makes Theorem 3 uninteresting for polynomials. We shall now generalize Theorem 3 in non-trivial cases. In order to do this, we introduce some simple geometry that allows us to apply the results from [5]. Let $[w_1 : \cdots : w_n]$ be the homogeneous coordinates in $\mathbb{P}^{n-1}$ and $[z_1 : \cdots : z_n]$ be the homogeneous coordinates in $\mathbb{P}_{(a_1,\ldots a_n)}$. Let Γ_k be the group of a_k-th roots of 1, and let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_n$. Consider the action of Γ on $\mathbb{P}^{n-1}$ given by

$$(\epsilon_1, \ldots \epsilon_n)[w_1 : \cdots : w_n] = [\epsilon_1 w_1 : \cdots : \epsilon_n w_n].$$

Lemma 14. *$\mathbb{P}_{(a_1,\ldots a_n)}$ is isomorphic to the quotient $\mathbb{P}^{n-1}/\Gamma$ with respect to the action defined as above.*

Proof. Leaving the proof to the reader, we merely give the quotient map $\gamma' : \mathbb{P}^{n-1} \to \mathbb{P}_{(a_1,\ldots a_n)}$, since it will be used in the sequel:

(1) $$\gamma'([w_1 : \cdots : w_n]) = [w_1^{a_1} : \cdots : w_n^{a_n}].$$

□

Let $\gamma : \mathbb{C}^n \backslash \{0\} \to \mathbb{C}^n \backslash \{0\}$ be the equivariant mapping defined by $\gamma(w_1, \ldots w_n) = (w_1^{a_1}, \ldots w_n^{a_n})$, and denote by $\pi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$ and $\pi' : \mathbb{C}^n - \{0\} \to \mathbb{P}_{(a_1,\ldots a_n)}$ the quotient maps. Then the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}^n \setminus \{0\} & \xrightarrow{\pi} & \mathbb{P}^{n-1} \\ \Big\downarrow \gamma & & \Big\downarrow \gamma' \\ \mathbb{C}^n \setminus \{0\} & \xrightarrow{\pi'} & \mathbb{P}_{(a_1,\ldots a_n)} \end{array}$$

It follows that a curve $(t^{a_1} z_1, \ldots t^{a_n} z_n)$ naturally lifts to a union of $a_1 a_2 \ldots a_n$ lines $(t z_1^{1/a_1}, \ldots, t z_n^{1/a_n})$ (counting multiplicity).

Lemma 15. *Let f be an entire function of finite order. Then for any $[C] \in \mathbb{P}_{a_1,\cdots a_n}$, the restriction $f|C$ is of finite order.*

Proof. Let $R > 1$, A and ρ positive numbers such that if $|(z_1, \ldots z_n)| > R$, then $|f(z_1, \ldots z_n)| < A \exp |(z_1, \ldots, z_n)|^\rho$. Clearly if $w = (w_1, \ldots, w_n)$ is sufficiently large, then $|(w_1^{a_1}, \ldots, w_n^{a_n})|$ is larger than R. Let $a = \max\{a_1, \ldots, a_n\}$. We may assume $\|w\| > 1$; so

$$|(f \circ \gamma)(w_1, \ldots, w_n)| = |f(w_1^{a_1}, \ldots, w_n^{a_n})| < A \exp(|(w_1^{a_1}, \ldots, w_n^{a_n})|^\rho) \leq$$
$$A \exp\left((\sum |w_k|^{2a_k})^{\rho/2} \right) < A \exp(n^{\rho/2} \|w\|^{a\rho}) < A' \exp(\|w\|^{a\rho+1}),$$

for large w which concludes the proof.

□

The previous two lemmas show that the order of f along a twisted curve C, which is image of $(t^{a_1}z_1, \dots t^{a_n}z_n)$, is determined by the order of $f \circ \gamma$ along the inverse image of this curve under γ, which consists of a finite union of lines. Moreover, the behavior of $f \circ \gamma$ is the same on each of those lines: if L is any line that projects to C (that is $\gamma(L) = C$), then $(f \circ \gamma)|L$ behaves like $f \mid C$.

We shall define a measure on the weighted projective spaces. Let μ be the standard measure on $\mathbb{P}^{n-1}$, induced by the Fubini-Study metric. If U is a measurable set, we define the average measure of U:

$$\hat{\mu}(U) = \sum_{g \in \Gamma} g(U).$$

Define a subset $O \subset \mathbb{P}_{(a_1,\dots a_n)}$ to be measurable if and only if the inverse image $\gamma'^{-1}(O)$ is measurable in $\mathbb{P}^{n-1}$. The collection of such measurable subsets forms a sigma-algebra and the average measure $\hat{\mu}$ descends to a measure on $\mathbb{P}_{(a_1,\dots a_n)}$. Since Γ is a finite group, a subset $O \subset \mathbb{P}_{(a_1,\dots a_n)}$ has measure zero if and only if $\gamma'^{-1}(O) \subset \mathbb{P}^{n-1}$ has measure zero.

Theorem 16. *Let f be a a holomorphic function on $\mathbb{C}^n$ of order ρ. Then for all $[C] \in \mathbb{P}_{a_1,\cdots a_n}$ outside a set of measure zero, $f|C$ is of order ρ.*

Proof. Consider the pull-back $\tilde{f} = f \circ \gamma$. By the previous Lemma, $\tilde{f}$ is of finite order $\tilde{\rho}$. Hence, by Theorem 3 $\tilde{f}|L$ is of finite order $\tilde{\rho}$ for almost all lines. Moreover, $\tilde{f}|L$ is of order $\tilde{\rho}$ if and only if $\tilde{f}|gL$ is of order ρ for all $g \in \Gamma$. Fix a non zero vector $(z_1, \dots z_n)$ and let $(w_1, \dots w_n)$ be such that $\gamma(w_1, \dots w_n) = (z_1, \dots z_n)$. Let C be parametrized by $\gamma(t) = (t^{a_1}z_1, \dots t^{a_n}z_n)$. Since $\tilde{f}(tw_1, \dots tw_n) = f \circ \gamma(tw_1, \dots tw_n) = f(t^{a_1}z_1, \dots t^{a_n}z_n)$, it follows that f has order $\tilde{\rho}$ along the curve C. Since $\tilde{f}$ has order $\tilde{\rho}$ outside a set of lines of measure zero, f has the same order ρ outside a set of curves of measure zero. □

3. Traces along parallel translations of subvarieties

Let $V_0 \subset \mathbb{C}^n$ be a subvariety of $\mathbb{C}^n$ and $u \in \mathbb{C}^n$ be a fixed vector. In this section we study the traces of a holomorphic function along a family $\{V_c\}$, for $c \in \mathbb{C}$, where the varieties $V_c = V_0 + cu$ are obtained by translation in the direction u.. The space that parametrizes such a family is not compact.

3.1. **Parallel hyperplanes.** The analogue of Theorem 6 does not hold, as the function $f(x, y) = e^x - y$ shows. In the next subsection we shall show, however, that something can be said even in this case. The following example is an analogue of Example 8.

Example 17. *Given an increasing sequence $\{n_j : j = 1, 2, \cdots\}$ of positive integers, and given a sequence of distinct points $e_j \in \mathbb{C}$, there is an entire function $f(z, w)$ having the property that, for each j the function $f(e_j, \cdot)$ is a polynomial of degree n_j. Moreover, if $z \neq e_j$ for all j, then $f(z, \cdot)$ is not a polynomial.*

For each $k = 1, 2, \cdots$, we define a polynomial $g_k(z)$ as follows: set $g_1 = 1$ and for $k > 1$, set

$$g_k(z) = a_k \prod_{j=1}^{k-1} (z - e_j),$$

where $a_k \neq 0$ is chosen so small that

$$\max_{|z|\leq k} |g_k(z)| \leq \frac{1}{2^k k^{n_k}}.$$

The series

$$\sum_{k=1}^{\infty} g_k(z) w^{n_k}$$

converges uniformly on compacta. Indeed, writing

$$\sum_{k=1}^{\infty} g_k(z) w^{n_k} = \sum_{k=1}^{m} g_k(z) w^{n_k} + \sum_{k=m+1}^{\infty} g_k(z) w^{n_k},$$

and noting that, for $|z| \leq m, |w| \leq m$, and $k > m$, we have the estimate

$$|g_k(z) w^{n_k}| \leq \frac{1}{2^k k^{n_k}} k^{n_k} = \frac{1}{2^k},$$

we see that on $|z| \leq n, |w| \leq n$, the series is a polynomial plus a uniformly convergent series. Hence, the series converges uniformly on compacta and represents an entire function $f(z,w)$. Note that, for $z = e_m$,

$$f(e_m, w) = \sum_{k=1}^{m-1} g_k(e_k) w^{n_k} + a_m \prod_{j=1}^{m} (e_m - e_j) w^{n_m} + \sum_{k=m+1}^{\infty} g_k(e_m) w^{n_k},$$

is a polynomial of degree n_m (and no less), since each g_k is a polynomial of degree n_k and $g_k(e_m) = 0$, for $k > m$.

The second part of the statement follows immediately by looking at the construction of f. Indeed, if z is different from all e_j, then $g_k(z) \neq 0$ for all k. Thus,

$$f(z,w) = \sum_{k=1}^{\infty} g_k(z) w^k = \sum_{k=1}^{\infty} a_k w^k,$$

where $a_k \neq 0, k = 1, 2, \cdots$.

In the preceding example, we may choose the sequence $\{e_j\}$ to be dense so as to obtain a function $f(z,w)$ which is a polynomial in w for a dense set of z but for each n the set of z for which $f(z,\cdot)$ is a polynomial of degree n has no accumulation point.

3.2. **Tube domains.** Most studies on the order of growth of entire functions are concerned only with functions of finite order. Let $M_f(r) = \max_{|z \leq r|} |f(z)|$. Then the order of an entire function can be expressed as follows [7, page 9]

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}.$$

If f is an entire function of a single complex variable, then the order of f can also be expressed [7, Theorem 1.9 a)] in terms of the MacLaurin coefficients $\{a_n\}$ of f:

$$\rho(f) = \limsup_{n\to\infty} \frac{n \log n}{\log(1/|a_n|)}.$$

Suppose $f(z,w)$ is holomorphic on a tube domain $D \times \mathbb{C}$, where D is a domain in $\mathbb{C}$. We may write

$$f(z,w) = \sum_{n=0}^{\infty} a_n(z) w^n, \quad z \in D,$$

where $a_n \in \mathcal{O}(D), n = 0, 1, \cdots$. For a compact subset $K \subset D$, we denote

$$M_n(K) = \max_{z \in K} |a_n(z)|.$$

Definition 18. *A holomorphic function $f(z, w)$ defined on a tube domain $D \times \mathbb{C}$ is said to be of order ρ_K in the w direction over K if*

$$\rho_K(f) = \limsup_{n \to \infty} \frac{n \log n}{\log(1/M_n(K))}.$$

A holomorphic function $f(z, w)$ in a tube domain $D \times \mathbb{C}$ is locally of finite (respectively infinite) order in the w direction if it is of finite (respectively infinite) order in the w direction over every relatively compact open subset of D.

From the Cauchy inequalities, it follows that $M_n(K) \to 0$, as $n \to \infty$. Thus, if $K_1 \subset K_2$, we have for large n that $0 \leq \log M_n(K_2)/\log M_n(K_1) \leq 1$. Consequently, if $K_1 \subset K_2$, then $\rho_{K_1}(f) \leq \rho_{K_2}(f)$. The following Harnack type principle is due to Pierre Lelong [8, Th. 33].

Lemma 19. *If f is holomorphic in a tube domain $D \times \mathbb{C}$ and D_1, D_2 are relatively compact open subsets of D, then for sufficiently large n, the ratio*

$$\frac{\log M_n(\overline{D}_1)}{\log M_n(\overline{D}_2)}$$

is bounded from below and from above by two positive numbers (we agree to disregard those n for which $M_n(\overline{D}_j) = 0$, since $M_n(\overline{D}_1) = 0$ if and only if $M_n(\overline{D}_2) = 0$).

Recall that a subset Y of a topological space S is said to be *residual* if $S \setminus Y$ is of first Baire category. If S is of second category, then we may think of a residual subset as topologically representing the 'majority' of points of S in the sense that, not only Y is 'large' but also $S \setminus Y$ is 'small'. As a consequence of the previous lemma we have the following.

Theorem 20. *If f is holomorphic in a 2-dimensional tube domain $D \times \mathbb{C}$, then either the order of f in the w direction is locally finite or it is locally infinite. In the first case, f_z is of course of finite order for every $z \in D$. In the second case, the order of f_z is infinite for most z, in the sense that the set of such z is residual in D.*

Proof. The first assertion in an immediate consequence of the previous lemma. The second assertion is trivial. To prove the third assertion, suppose that the order of f in the w direction is locally infinite. Let E the set of $z \in D$ for which $\rho(f_z) < +\infty$. Then

$$E = \cup_{j,k}\{z \in D : |f(z, w)| \leq k \exp(|w|^j)\} = \cup_{j,k} E_{j,k}.$$

Clearly, each $E_{j,k}$ is closed and it is nowhere dense since f is locally of infinite order in the w direction. This concludes the proof. □

Example 21. *Given an increasing sequence $\{n_j : j = 1, 2, \cdots\}$ of positive integers, and given a sequence of distinct points $e_j \in \mathbb{C}$, there is an entire function $F(z, w)$ having the property that, for each j the function $F(e_j, \cdot)$ is of order n_j (and no less).*

Proof. In Example 17, we construct an entire function $f(z, w)$, such that, for each e_j, the function $f(e_j, \cdot)$ is a polynomial of degree n_j (and no less). Put $F = e^f$. □

The following theorem is a consequence of Theorem 20.

Theorem 22. *Let f be an entire function. Then f is either of finite order, or it is of infinite order on a residual set of lines through the origin.*

Proof. Let $\sigma X :\to \mathbb{C}^2$ be the blow up of $\mathbb{C}^2$ at the origin. It is well known X is the total space of a line bundle $\pi : X \to \mathbb{P}^1$, and it follows that for all $u \in \mathbb{P}^1$ $X^0 = X - \pi^{-1}(u) \cong \mathbb{C}^2$. Therefore the the theorem follows if we apply Theorem 20 to $f \circ \sigma_{|X^0}$. □

3.3. **Traces along translations of submanifolds.** In this section we consider the case of families that are obtained by translating a given subvariety. The difficulty that arises is that members of the family can have non-empty intersection.

Lemma 23. *Let $p : X \to Y$ be a map of Stein manifolds, where $\dim X = m$ and $\dim Y = 1$. Assume further that, for each $y \in Y$, $p^{-1}(y)$ is a Stein manifold. Let f is a holomorphic function on Y. If there exists a subset S of Y with an accumulation point , such that $f|p^{-1}(y)$ is constant for all $y \in S$, then f is constant on each fiber of p.*

Proof. Given a point $y \in S$, there exists an open neighborhood O of y, and a system of local coordinates on O centered at y, say $(z, t_1, \dots t_{m-1})$, such that $p^{-1}(y) \cap O = \{z = 0\}$ (see [6]). We can construct the locally defined vector fields, for $k = 1, \dots m - n$:

$$v_k = \frac{\partial}{\partial t_k}$$

Hence $v_k(f)$ is identically zero on $p^{-1}(y) \cap O$, for all $y \in S$. Since $p^{-1}(y) \cap O$ is open in $p^{-1}(y)$, it follows that $v_k(f)(x) = 0$ for all $x \in p^{-1}(y)$ for all $y \in S$. The zero locus of $v_k(p)$ contains infinitely many irreducible components accumulating at $p^{-1}(y)$, which is possible only if $v_k(p)$ is identically zero. It follows that f is constant along each fiber over $p(O)$, and f descends to a function $\tilde{f}$ on $p(O)$. Consider now a maximal open set $U \subset Y$ over which $\tilde{f}$ extends as a holomorphic function. Suppose that $U \neq Y$: since $\tilde{f} \circ p = f$ on $p^{-1}(U)$, f itself would not be an entire function. Hence $U = Y$, and f descends to an entire function on Y. Therefore f descends to a function on Y.

□

Lemma 24. *Let f be an entire function on $\mathbb{C}^n$, and V_0 be a proper embedding of $\mathbb{C}^{n-1}$ into $\mathbb{C}^n$ containing the origin. For $z \in \mathbb{C}^n$, we write $z = (z_1, z')$, with $z_1 \in \mathbb{C}$ and $z' \in \mathbb{C}^{n-1}$. For a vector $(c, 0')$ let $V_c = V_0 + (c, 0')$. Suppose that there exists a sequence of vectors $(c_k, 0')$ converging to $(0, 0')$ such that f is bounded along the translate V_{c_k}, for all k. Then f is constant on each V_c*

Proof. First of all, by Liouville's theorem, f is constant on each V_{c_k}. Since V_0 is an analytic submanifold of $\mathbb{C}^n$, there exists an open polydisk D_0 containing $(0, 0')$ and a holomorphic function g on D_0 such that $V_0 \cap D_0$ coincides with the zero locus of g. Therefore, if we let $D_c = D_0 + (c, 0')$, $V_c \cap D_c$ is given by the zero locus of $g(z_1 - c, z')$. Consider the open set $O = \cup_c O_c \subset \mathbb{C}^{n+1}$, where $O_c = D_c \times \{c\}$. We may consider O as the image of the polydisc D_0 by the automorphism of $\mathbb{C}^{n+1}$ given by$(z, z', c) \mapsto (z + c, z', c)$. Hence, O is Stein. Consider the holomorphic function G defined on O by $G(z_1, z', c) = g(z_1 - c, z')$. Let $q : \mathbb{C}^{n+1} \to \mathbb{C}^n$ be the projection map $q(z_1, z', c) = (z_1, z')$, and p be the projection $p(z_1, z', c) = c$. Moreover, let

$\mathbb{V}$ be the n-dimensional Stein submanifold of O given by $G = 0$, and restrict the projections p and q to $\mathbb{V}$. The pull back $\tilde{f} = f \circ q$ is a holomorphic function on $\mathbb{V}$ with is constant on the fibers $p^{-1}(c_k) = (V_{c_k} \cap D_{c_k}) \times \{c_k\}$. Hence, by Lemma 23, $\tilde{f}$ is constant on each fiber of p. This implies that f is constant on $V_c \cap D_c$, for all $c \in \mathbb{C}$: by the identity principle, it must be constant on the entire submanifold V_c, for all c. □

Remark 25. *If $V_c \cap V_{c'} \neq \emptyset$ for all (c, c') belonging to an open set of $\mathbb{C}^2$, then f must be a constant function.*

4. Traces along real curves

In the previous sections we established that in many interesting situations, the behavior of an entire function is dictated by the behavior of its restriction along complex subvarieties. In the following section we show that this is not true if we replace complex subvarieties by real subvarieties.

Definition 26. *For E a subset of $\mathbb{C}$, we denote by $A(E)$ the family of continuous functions on E which are holomorphic on the interior of E. We say that E is an approximation set if, for each function $g \in A(E)$ and each $\epsilon > 0$ on E, there exists an entire function f, such that $|f - g| < \epsilon$ on E.*

Approximation sets have been completely characterized by Norair U. Arakelian (see [3]). In fact, on approximation sets, we can do better than uniform approximation as the following lemma shows (see [3, p. 161]).

Lemma 27. *Let E be an approximation set in $\mathbb{C}$. Then, for each $g \in A(E)$ and each $\epsilon > 0$, there is an entire function f such that on E, not only $|f(z) - g(z)| < \epsilon$ but also $|f(z) - g(z)| \to 0$ as $z \to \infty$ on E.*

An asymptotic path in $\mathbb{C}^n$ is a continuous curve $\gamma : [0, +\infty) \to \mathbb{C}^n$, such that $\gamma(t) \to \infty$, as $t \to +\infty$. We assume that $\gamma(0) = 0$. An asymptotic path is said to be simple if γ is injective; it is said to be strictly monotonic if $|\gamma(t)|$ is strictly increasing.

Theorem 28. *(a) In $\mathbb{C}$, for every simple asymptotic path γ, there exist entire functions of arbitrarily fast growth which tend to zero along γ.*
(b) In $\mathbb{C}^2$, there exists a simple asymptotic path γ such that, if an entire function f tends to zero along γ, then $f \equiv 0$.

Proof. (a) Let γ be a simple asymptotic path in $\mathbb{C}$. We may construct another simple asymptotic path σ disjoint from γ. The union $E = \gamma \cup \sigma$ is an approximation set and so, by Lemma 27, if $\epsilon > 0$ and $\varphi : [0, +\infty)$ is an arbitrary continuous function, setting $g(z) = 0$ on γ and $g(z) = \varphi(|z|) + \epsilon$ on σ, we obtain an entire function f for which

$$f(z) \to 0, \quad z \in \gamma; \qquad |f(z)| > \varphi(|z|), \quad z \in \sigma.$$

This proves (a).

(b) Anatoliy Georgievich Vitushkin [10] showed the existence of a compact totally disconnected set K in $\mathbb{C}^2$, whose polynomial hull contains the bidisc. We may assume that K is contained in the bidisc. Given $r > 0$, by covering the sphere S_r of center 0 and radius r by finitely many bidiscs, we see that there exists a compact totally disconnected set K_r whose polynomial hull contains the sphere S_r. Since we may cover by arbitrarily small bidiscs, given $\epsilon > 0$, we may assume that K_r is contained in the shell

$$A(r,\epsilon) = \{(z,w) :< r - \epsilon < \sqrt{|z|^2 + |w|^2} < r + \epsilon\}.$$

Since the hull of K_r contains the sphere S_r, it also contains the closed ball $\overline{B}_r$.

In this fashion we may construct a sequence K_j, $j = 1, 2, \cdots$ of compact totally disconnected sets, such that for each j, the polynomial hull of K_j contains the ball $\overline{B}_j$ of radius j, and K_j is contained in the shell $A(j,\epsilon)$. Now, by a theorem of Louis Antoine [1], for each j, there exists a simple arc γ_j, which passes through each point of K_j and we may assume that $\gamma_j \subset A(j,\epsilon)$. It is easy to construct a simple asymptotic path γ, which contains each γ_j. Suppose f is an entire function which tends to zero on γ. Fix $r > 0$ and $\delta > 0$. Choose j so large that $|f| < \delta$ on $\gamma \cap K_j$ and the hull of K_j contains $\overline{B}_r$. Since the polynomial hull is the same as the holomorphic hull, $|f| < \delta$ on B_r. Since r and δ were arbitrary positive numbers, $f = 0$. This completes the proof of (b). □

If γ is an asymptotic path in $\mathbb{C}$ and θ is a rotation of $\mathbb{C}^n$, we denote by γ_θ the asymptotic path obtained by the corresponding rotation of γ.

Theorem 29. *Let γ be a strictly monotonic simple asymptotic path in $\mathbb{C}$, and let φ be a positive continuous function on $[0,+\infty)$. Then, there exists an entire function f on $\mathbb{C}$ such that, for every γ_θ, $f(z) \to 0$ as $z \to \infty$ on γ_θ and moreover, $\max_{|z|=r} |f(z)| > \varphi(r)$, for each $r \geq 0$.*

Proof. Thus, there exist entire functions tending to zero on every rotation of γ and having arbitrarily fast growth. To obtain such a function f we may first construct a strip U containing γ such that $E = \overline{\mathbb{C}} \setminus U$ is connected and locally connected. Thus E is a set of uniform approximation. We may construct U close enough to γ that every rotation γ_θ of γ is eventually in E. We may construct a simple asymptotic path $\sigma \subset U \setminus \gamma$. The set $F = E \cup \sigma \cup \gamma$ is also a set of approximation. We define a function $g \in A(F)$ by setting $g = 0$ on E and γ and g some continuous function on σ which grows so quickly that $\varphi(|z|) = o(g(z))$ as $z \to \infty$ on σ. Since F is a set of approximation, there exists an entire function f such that $|f(z) - g(z)| \to 0$, as $z \to \infty$ on F. In particular, for $\epsilon > 0$ and sufficiently large r, if $z \in \sigma$, with $|z| = r$

$$|f(z)| > |g(z)| - o(1) > \frac{\varphi(|z|)}{\epsilon} - o(1) > \varphi(r).$$

□

Corollary 30. *Let φ be a positive continuous function on $[0,+\infty)$. Then, there exists an entire function f on $\mathbb{C}^2$ such that, $f(z) \to 0$ along every real ray from the origin and moreover,*

$$\max_{|z|^2+|w|^2=r^2} |f(z,w)| > \varphi(r), \quad \forall r \geq 0.$$

Proof. Set $\varphi_1(r) = \sqrt{\varphi(r\sqrt{2})}$. By Theorem 29 , there is an entire function f_1 of one complex variable tending to zero on each ray and such that $\max_{|z|=r} |f_1(z)| > \varphi_1(r)$. Set $f(z,w) = f_1(z)f_1(w)$. Let X_ζ be the real ray in $\mathbb{C}^2$ passing through a point $\zeta \in S^3$, where $\zeta = (r_1e^{i\theta_1}, r_2e^{i\theta_2})$, $r_1^2+r_2^2 = 1$. We may assume $r_1 \neq 0$. A point of X_ζ has the form $(\rho r_1e^{i\theta_1}, \rho r_2e^{i\theta_2})$ and $f(z,w) = f_1(\rho r_1e^{i\theta_1})f_1(\rho r_2e^{i\theta_2})$. As $\rho \to +\infty$, $\rho r_je^{i\theta_j}$ remain respectively on the rays $\arg z = \theta_j, j = 1, 2$. Thus, $f_1(z) \to 0$ and

$f_2(w)$ remains bounded. Hence $f(z,w) \to 0$. We have shown that f tends to zero along each real ray from the origin.

Now we check the growth of f.

$$\max_{|z|^2+|w|^2=r^2} |f(z,w)| \geq \max_{|z|^2=|w|^2=r^2/2} |f_1(z)||f_1(w)|$$
$$\geq \max_{|u|=r/\sqrt{2}} |f_1(u)|^2 > (\varphi_1(r/\sqrt{2}))^2 = \varphi(r).$$

Thus, f has the required growth. □

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