ALBANIAN JOURNAL
OF MATHEMATICS
Volume 3, Number 2, Pages 63–74
ISSN 1930-1235: (2009)

NEAR-EXTREMES AND RELATED POINT PROCESSES

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Abstract. Let $X_i, i \geq 1$ be a sequence of random variables with continuous distribution functions and let $\{N(t), t \geq 0\}$ be a random counting process. Denote by $X_{i:N(t)}, i \leq N(t)$ the i-th lower order statistics of $X_1, \ldots, X_{N(t)}, t \geq 0$ and define a point process in $\mathbb{R}$ by $\mathbf{M}_{t,m}(\cdot) := \sum_{i=1}^{N(t)} \mathbf{1}(X_{N(t)-m+1:N(t)} - X_i \in \cdot), m \in \mathbb{N}$. In this paper we derive distributional and asymptotical results for $\mathbf{M}_{t,m}(\cdot)$. For special marginals of the point process we retrieve some general results for the number of m-th near-extremes.

1. Introduction

Let $X_i, i \geq 1$ be a sequence of random variables with continuous distribution functions and let $\{N(t), t \geq 0\}$ be a random counting process independent of $X_n, n \geq 1$. Denote by $X_{N(t)-i+1:N(t)}$ the i-th largest order statistic of $X_1, \ldots, X_{N(t)}$, if $N(t) \geq i$. For any positive constant a and $m \in \mathbb{N}$ define the discrete random variable $\mathbf{K}_t(a, m)$ by

$$\mathbf{K}_t(a,m) \quad := \quad \sum_{i=1}^{N(t)} \mathbf{1}(X_{N(t)-m+1:N(t)} - X_i \in [0,a)), \quad \text{if } N(t) \geq m,$$

and 0, otherwise. $\mathbf{K}_t(a, m)$ counts the number of sample points X_i which fall in the random window $W_{t,a,m} := (X_{N(t)-m+1:N(t)} - a, X_{N(t)-m+1:N(t)}]$ ($\mathbf{1}(\cdot)$ stands for the indicator function).

Basic asymptotic properties of $\mathbf{K}_t(a, m)$ are obtained in Hashorva (2003), Hashorva and Hüsler (2008). The motivation for considering the random variable $\mathbf{K}_t(a, m)$ comes from the fact that for some applications the randomly indexed order statistics are of direct interest, for instance when dealing with claim sizes in an insurance context. Statistical applications can be found in Hashorva and Hüsler (2005).

Distributional and asymptotical results in connection with the number of sample points X_i such that $X_{n-m+1:n} - X_i \in B$, where $B = [0, a)$ or $B = (-a, 0]$ are derived in Balakrishnan and Stepanov (2004, 2005) and Dembinska et al. (2007).

In an asymptotic context it is of some interest to allow the length of the random window $W_{t,a,m}$ to depend directly on t. This can be achieved for instance if $a = a(t)$ or $a = a(N(t))$. In Balakrishnan and Stepanov (2004, 2005) fixed length random windows are dealt with in detail.

In this paper we have following objectives in mind:
a) Instead of defining different random variables (like $\mathbf{K}_t(a, m), a > 0$), we choose a

2000 *Mathematics Subject Classification.* Primary 60F15; Secondary 60G70.
Key words and phrases. Near-extremes; extreme value theory; point processes; asymptotic results.

more general approach utilising point processes. In fact a point processes approach (considering only the maxima) is suggested in Hashorva and Hüsler (2000). One advantage of the point process approach is that several previously studied random quantities are retrieved when the Borel set B is an interval $[a, b) \subset \mathbb{R}$.

b) We allow the random window to grow/shrink with t by considering a scaling of the Borel set B.

c) Besides the iid case (independence and common distribution assumption on $X_i, i \geq 1$) we consider the general setup where $X_i, i \geq 1$ can be dependent. From the statistical point of view this extension is interesting since dependence is often observed in practical situations.

Explicitly, define a family of point processes $\mathbf{M}_{t,m}(\cdot)$ by

$$\mathbf{M}_{t,m}(B) \quad := \quad \sum_{i=1}^{N(t)} \mathbf{1}(X_{N(t)-m+1:N(t)} - X_i \in B).$$

The random variable $\mathbf{M}_{t,m}(B)$ counts the number of sample points X_i in the Borel set $B \subset \mathbb{R}$ near the m-th randomly indexed order statistics. We suppose without loss of generality that $N(0) = m$ almost surely. Further, we assume throughout this paper that $\{N(t), t \geq 0\}$ has almost surely non-decreasing sample paths. If $m = 1$ and $B \subset (-\infty, 0]$ put $\mathbf{M}_{t,1}(B) := \mathbf{1}(0 \in B)$. Hashorva and Hüsler (2008) derive distributional and asymptotic properties of the point process

$$M_{n,m}(B) \quad := \quad \sum_{i=1}^{n} \mathbf{1}(X_{n-m+1:n} - X_i \in B), \quad n > 1, \quad B \subset [0, \infty)$$

assuming that $X_n, n \geq 1$ possess a common continuous distribution function F.
In this paper we deal with distributional and asymptotic properties of $\mathbf{M}_{t,m}(\cdot)$. For $m > 1$ we consider in some detail also the interesting case $B \subset (-\infty, 0]$.
When $B = [0, a)$, or $B = (-a, 0], a > 0$ Balakrishnan and Stepanov (2005), and Dembinska et al. (2007) derive some distributional and asymptotical properties of $\mathbf{M}_{t,m}(B)|N(t) = n$.

Outline of the rest of the paper: In the next section we provide few preliminary results. In Section 3 we begin with some asymptotic results for the iid case. Then we focus on the situation where the sample points $X_i, i \geq 1$ can be dependent. Two illustrating examples are presented in Section 4. The proofs of all the results are relegated to Section 5.

2. Preliminaries

Let $X_i, i \geq 1$ be independent random variables with common continuous distribution function F and let $N(t)$ be as defined above independent of $X_i, i \geq 1$.
In this section we derive the probability generating function (p.g.f.) of marginals of the point process $\mathbf{M}_{t,m}(\cdot)$. Then we give a preliminary result on the joint weak convergence of randomly indexed upper order statistics.

Write in the following $\stackrel{d}{=}$ to mean equality of distribution functions of two given random variables. In the next lemma we derive the p.g.f. of the point process of interest.

Lemma 1. *Let $\{X_i, i \geq 1\}$ be independent random variables with common continuous distribution function F. Let $x \in \mathbb{R}$ and m, n be two integers such that*

$F(x) \in (0,1)$, and $1 \le m < n, m, n \in \mathbb{N}$. Then we have for any Borel set $B \subset [0,\infty)$ or $B \subset (-\infty, 0]$

$$\Big(M_{n,m}(B) | X_{n-m+1:n} = x\Big) \stackrel{d}{=} \mathbf{1}(0 \in B) + \sum_{i=1}^{n_{B,m}} \mathbf{1}(x - \eta_i^{[x]} \in B), \tag{1}$$

where $n_{B,m} := n-m, \eta_i^{[x]} \stackrel{d}{=} X_1 | X_1 \le x, i \ge 1$ if $B \subset [0,\infty)$, and $n_{B,m} := m-1, \eta_i^{[x]} \stackrel{d}{=} X_1 | X_1 > x, i \ge 1$ if $B \subset (-\infty, 0]$ and $m > 1$, with $\eta_i^{[x]}, i \ge 1$ independent random variables.
Furthermore, if $N(t), t \ge 0$ is a counting process such that $N(0) = m$ almost surely being further independent of $X_i, i \ge 1$, then we have

$$\begin{aligned} \boldsymbol{E}\Big\{s^{\mathbf{M}_{t,m}(B) - \mathbf{1}(0 \in B)}\Big\} &= \sum_{n=m}^{\infty} \boldsymbol{P}\{N(t) = n\} \int_{\mathbb{R}} \Big[1 - (1-s)\boldsymbol{P}\Big\{x - \eta_1^{[x]} \in B\Big\}\Big]^{n_{B,m}} \\ &\quad \times dF_{n-m+1:n}(x), \quad \forall s \in (0,1), \end{aligned} \tag{2}$$

with $F_{n-m+1:n}$ the distribution function of $X_{n-m+1:n}$.

By the above lemma for $n > m > 1$ we obtain (suppose $0 \not\in B$)

$$\boldsymbol{E}\{\mathbf{M}_{t,m}(B) | N(t) = n\} = n_{B,m} \int_{\mathbb{R}} \boldsymbol{P}\{x - \eta_1^{[x]} \in B\}\, dF_{n-m+1:n}(x), \tag{3}$$

where the distribution function $F_{n-m+1:n}$ of $X_{n-m+1:n}$ has F-density (cf. Theorem 1.5.1 of Reiss (1989))

$$\frac{n! F^{n-m}(x)(1-F(x))^{m-1}}{(n-m)!(m-1)!}, \quad x \in \mathbb{R}. \tag{4}$$

Consequently $\boldsymbol{E}\{\mathbf{M}_{t,m}(B)\} < \infty$ if $\boldsymbol{E}\{N(t)\} < \infty, t > 0$.

Remark 1. If $B = B_1 \cup B_2$ with B_1, B_2 two disjoint Borel sets such that $B_1 \subset (-\infty, 0], B_2 \subset [0, \infty)$ we have

$$M_{n,m}(B) = M_{n,m}(B_1) + M_{n,m}(B_2), \quad n > m > 1.$$

For any $x \in \mathbb{R}$ such that $F(x) \in (0,1)$ the random variable $M_{n,m}(B_1)$ and $M_{n,m}(B_2)$ are conditionally independent given $X_{n-m+1:n}$. This fact is important and leads to general results for general Borel sets $B \subset \mathbb{R}$.

A common asymptotic assumption on the counting process $\{N(t), t \ge 0\}$, which we want to impose for our asymptotic results is the convergence in probability

$$N(t)/t \stackrel{p}{\to} Z, \quad t \to \infty, \tag{5}$$

where Z is a non-negative random variable such that $\boldsymbol{P}\{Z \in (0,\infty)\} = 1$.

The second important asymptotic assumption concerns the asymptotic behaviour of the sample maxima of the random sequence $X_i, i \ge 1$. Specifically, we assume that the underlying distribution function F is in the max-domain of attraction of a univariate extreme value distribution function H (write this as $F \in MDA(H)$), i.e.

$$\lim_{t \to \infty} \sup_{x \in \mathbb{R}} \Big| F^t(q(t)x + r(t)) - H(x) \Big| = 0, \tag{6}$$

with $q(t) > 0, r(t)$ two measurable functions. For further details on extreme value theory we refer the reader to the following monographs: Leadbetter et al. (1983), Resnick (1987), Reiss (1989), Falk et al. (2004), De Haan and Ferreira (2006).

Denote by α_H, x_H the lower and the upper endpoint of the distribution function H. The univariate extreme value distribution function H is either the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$, the Weibull distribution $\Psi_\alpha(x) = \exp(-|x|^\alpha), x < 0, \alpha > 0$, or the Fréchet distribution $\Phi_\alpha(x) = \exp(-x^{-\alpha}), x > 0, \alpha > 0$. Note in passing that if

$$\lim_{x\to\infty} \frac{1-F(x+a)}{1-F(x)} = \beta(a) \in (0,1), \quad \forall a > 0, \tag{7}$$

then F is in the Gumbel max-domain of attraction with $q(t) := 1, t > 0$.

The asymptotic condition on F in (6) is equivalent with the joint convergence in distribution (see e.g. Falk et al. (2004))

$$\Big(\frac{X_{n:n} - r(n)}{q(n)}, \dots, \frac{X_{n-k+1:n} - r(n)}{q(n)}\Big) \overset{d}{\to} (Y_1, \dots, Y_k), \quad \forall k \ge 2 \tag{8}$$

as $n \to \infty$, with $(Y_1, \dots, Y_k)$ a random vector in $\mathbb{R}^k$ with density function

$$h_k(x_1, \dots, x_k) = H(x_k) \prod_{i=1}^k \frac{H'(x_i)}{H(x_i)} \tag{9}$$

which is positive for $\alpha_H < x_l < x_{l-1} < \dots < x_1 < x_H$. If $X_i, i \ge 1$ are dependent, then the convergence in distribution of the upper order statistics follows under additional asymptotic restrictions. Indeed, several results for asymptotic behaviour of univariate sample extremes of non-iid sequences are available in the literature. For instance, if $X_i, i \ge 1$ is a stationary random sequence such that (6) holds and the weak distributional mixing conditions $D_l(\boldsymbol{u}_n), l \in \mathbb{N}, D'(u_n)$ (see Falk et al. (2004)) are satisfied with $u_n = q(n) + r(n), \boldsymbol{u}_n = (u_n, \dots, u_n) \in \mathbb{R}^k$, then (8) holds with $l = k$. The definitions of $D_k(\boldsymbol{u}_n)$ and $D'(u_n)$ are given in Leadbetter et al. (1983). The mixing conditions are satisfied if $X_n, n \ge 1$ are independent with distribution function F such that (6) holds.

3. Main Results

In this section we shall derive several asymptotic results for the scaled point process

$$\widetilde{\mathbf{M}}_{t,m}(B) := \mathbf{M}_{t,m}(\widetilde{q}(t)B), \quad B \subset \mathbb{R},$$

where $\widetilde{q}(t)$ is a positive measurable scaling function. In our asymptotic results we relate $\widetilde{q}(t)$ with the scaling function $q(t)$ (provided that assumption (6) is valid) by the following relation

$$\lim_{t\to\infty} \frac{\widetilde{q}(t)}{q(t)} = Q \in [0, \infty). \tag{10}$$

We consider briefly the iid setup, i.e., we assume that $X_i, i \ge 1$ are independent with common continuous distribution function F. As previously shown in Hashorva and Hüsler (2008), Hashorva (2003), Hashorva and Hüsler (2005), under this assumption a convenient approach to derive asymptotic results for the scaled point process of interest is to utilise (2).

If both (5) and (6) hold, then Proposition 2.1 of Hashorva (2003) implies for any $k \ge 1$ as $t \to \infty$

$$(11)\qquad \Big(\frac{X_{N(t):N(t)} - r(t)}{q(t)}, \dots, \frac{X_{N(t)-k+1:N(t)} - r(t)}{q(t)}\Big) \stackrel{d}{\to} (Y_1^*, \dots, Y_k^*),$$

where for any $i \ge 1$ we have if $H = \Lambda$

$$Y_i^* \stackrel{d}{=} Y_i + \ln Z$$

and

$$Y_i^* \stackrel{d}{=} Z^{-1/\alpha} Y_i, \quad Y_i^* \stackrel{d}{=} Z^{1/\alpha} Y_i$$

holds if $H = \Psi_\alpha$ or $H = \Phi_\alpha$, respectively. Furthermore, Z is independent of $Y_i, i \ge 1$.

When $H = \Lambda$ or $H = \Psi_\alpha, \alpha > 0$ we obtain with similar arguments as in Hashorva (2004) using (2) and (11) for all $s \in (0,1)$

$$(12)\ \lim_{t\to\infty} \boldsymbol{E}\{s^{\widetilde{\mathbf{M}}_{t,m}(B)}\} = \boldsymbol{E}\Big\{\exp(-(1-s)Z\ln\Big(H(Y_m^* - Qb)/H(Y_m^* - Qa)\Big)\Big\},$$

with $B := [a,b), 0 < a < b < \infty$. The above limiting expression is the p.g.f. of a mixed Poisson random variable. Hence convergence in distribution for $\widetilde{\mathbf{M}}_{t,m}(B)$ follows. If $H = \Lambda$, then the limiting p.g.f. in (12) does not depend on Z. This fact is mentioned in Corollary 2.7 of Hashorva (2003) (only for the case a, b positive and $m = 1$). $F \in MDA(\Lambda)$ follows if for instance F satisfies (7) so that we can choose the scaling function $q(t)$ as a positive constant. Consequently, (12) implies the result of Theorem 2.1 of Balakrishnan and Stepanov (2005) (for $\beta(a) \in (0,1)$). If (10) holds with $Q = 0$, then by (12)

$$\lim_{t\to\infty} \boldsymbol{E}\{s^{\widetilde{\mathbf{M}}_{t,m}(B)}\} \quad = \quad \boldsymbol{E}\{\exp(-(1-s)Z\ln(H(Y_m^*)/H(Y_m^*))\} = 1, \quad \forall s \in (0,1)$$

implying the convergence in probability

$$\widetilde{\mathbf{M}}_{t,m}(B) \stackrel{p}{\to} 0, \quad t \to \infty.$$

In case that $H = \Phi_\alpha$ the sequence $\mathbf{M}_{t,m}(B), t > 0$ is not tight.

Similarly, if $F \in \mathrm{MDA}(H)$ we obtain for any $m > 1, B := [a,b), -\infty < a < b < 0$

$$(13)\lim_{t\to\infty} \boldsymbol{E}\{s^{\widetilde{\mathbf{M}}_{t,m}(B)}\} \quad = \quad \boldsymbol{E}\Big\{\Big[1-(1-s)\frac{1}{\ln H(Y_m^*)}\ln\Big(\frac{H(Y_m^* - Qb)}{H(Y_m^* - Qa)}\Big)\Big]^{m-1}\Big\}$$

for all $s \in (0,1)$. The limiting expression in (13) is the p.g.f. of a mixed binomial random variable. It is interesting, that again if $H = \Lambda$, the limiting p.g.f. does not depend on Z. Explicitly, for any $s \in (0,1)$ we have

$$(14)\qquad \lim_{t\to\infty} \boldsymbol{E}\{s^{\widetilde{\mathbf{M}}_{t,m}(B)}\} \quad = \quad \Big[1-(1-s)[\exp(Qb) - \exp(Qa)]\Big]^{m-1}.$$

The above convergence holds if F satisfies (7) implying thus the result of Theorem 3.1 in Balakrishnan and Stepanov (2005).

The iid case is tractable due to the fact that we have a compact formula for the p.g.f. of the marginals of the point process given in (2). As in Hashorva and Hüsler (2008) we discuss next the asymptotic behaviour of the scaled point process dropping the independence assumption on $X_n, n \ge 1$.

Our next results are motivated by the following observation (see Pakes and Steutel

(1997)): If ξ, ξ^* are two positive constants and i, i^*, m, m^* are given integers such that $m-1 \ge i \ge 1, m^*-1 \ge i^* \ge 1$, then we may write using (5) and (11)

$$\begin{aligned}
&\boldsymbol{P}\Big\{\widetilde{\mathbf{M}}_{t,m}([0,\xi)) > i, \widetilde{\mathbf{M}}_{t,m^*}((-\xi^*,0]) > i^*\Big\} \\
&= \boldsymbol{P}\Big\{\widetilde{\mathbf{M}}_{t,m}([0,\xi)) > i, \widetilde{\mathbf{M}}_{t,m^*}((-\xi^*,0]) > i^*,\, N(t) > \max(i+m, m^*)\Big\} \\
&= \boldsymbol{P}\Big\{X_{N(t)-m+1:N(t)} - X_{N(t)-i-m+1:N(t)} \le \xi q(t), \\
&\qquad X_{N(t)-(m^*-i^*)+1:N(t)} - X_{N(t)-m^*+1:N(t)} \le \xi^* q(t), N(t) > \max(i+m, m^*)\Big\}.
\end{aligned}$$

Consequently, as $t \to \infty$

$$\begin{aligned}
&\boldsymbol{P}\Big\{\widetilde{\mathbf{M}}_{t,m}([0,\xi)) > i, \widetilde{\mathbf{M}}_{t,m^*}((-\xi^*,0]) > i^*\Big\} \\
&\to \quad \boldsymbol{P}\{Y_m^* - Y_{m+i}^* \le \xi, Y_{m^*-i^*}^* - Y_{m^*}^* \le \xi^*\}. \qquad (15)
\end{aligned}$$

In the above derivation we do not use explicitly the fact that $X_i, i \ge 1$ are iid. In the following we suppose that (11) holds, dropping thus the independence assumption on $X_i, i \ge 1$. We consider next the joint weak convergence of the point processes $\widetilde{\mathbf{M}}_{t,1}(\cdot), \ldots, \widetilde{\mathbf{M}}_{t,m}(\cdot)$.

Theorem 2. *Let $X_i, i \ge 1$ be random variables with common continuous distribution function F, and let $\{N(t), t \ge 0\}$ be a counting stochastic process. Assume that (11) holds with $q(t) > 0$ and $r(t)$ two real functions. If the convergence in probability $N(t) \xrightarrow{p} \infty$ as $t \to \infty$ holds, then for indices $\boldsymbol{j} := \{j_{i,k}\}_{i \le I, k \le K}, \boldsymbol{j}^* := \{j^*_{i,k}\}_{i \le I, k \le K}, I, K \in I\!N, j^*_{i,k} < K, i \le I, k \le K$ and positive constants $\boldsymbol{\xi} := \{\xi_{i,k}\}_{i \le I, k \le K}, \boldsymbol{\xi}^* := \{\xi^*_{i,k}\}_{i \le I, k \le K}$ we have*

$$\begin{aligned}
&\lim_{t\to\infty} \boldsymbol{P}\{\widetilde{\mathbf{M}}_{t,k}([0,\xi_{i,k})) \le j_{i,k}, \widetilde{\mathbf{M}}_{t,k}((-\xi^*_{i,k},0]) \le j^*_{i,k},\ \text{for all } 1 \le i \le I, 1 \le k \le K\} \\
&= \boldsymbol{P}\{Y_k^* - Y^*_{k+j_{i,k}} > \xi_{i,k}, Y^*_{k-j^*_{i,k}} - Y_k^* > \xi^*_{i,k},\ \text{for all } 1 \le i \le I, 1 \le k \le K\} \\
&:= G_K(\boldsymbol{\xi}, \boldsymbol{\xi}^*, \boldsymbol{j}, \boldsymbol{j}^*) \qquad (16)
\end{aligned}$$

and $\{\widetilde{\mathbf{M}}_{t,k}((-\xi^*_{i,k},0])\}_{1 \le i \le I, 1 \le k \le K}$ converge in distribution to a random vector in $\mathbb{R}^{IK}$. If in addition

$$Y_n^* \xrightarrow{p} -\infty, \quad n \to \infty \qquad (17)$$

holds, then we have the weak convergence

$$(\widetilde{\mathbf{M}}_{t,1}(\cdot), \ldots, \widetilde{\mathbf{M}}_{t,K}(\cdot)) \xrightarrow{w} \boldsymbol{L}(\cdot), \quad t \to \infty, \qquad (18)$$

with $\boldsymbol{L}(\cdot)$ a point processes defined on $\mathbb{R}^K$.

The joint distribution function of the marginals of $\boldsymbol{L}(\cdot)$ can be obtained using (16). If we impose some additional assumptions on the dependence of $X_i, i \ge 1$ and further assume (5) it is possible to obtain a more explicit description of the limiting point process.

Corollary 3. *Under the assumptions of Theorem 2 if (5) holds with $X_i, i \ge 1$ independent of $N(t), t > 0$ and further (6) holds and conditions $D_{2K}(\boldsymbol{u}_n), D'(u_n)$*

are satisfied with $u_n = q(n) + r(n), \boldsymbol{u}_n = \mathbf{1}u_n, K \in \mathbb{N}$, *then we have the stochastic representation*

$$(Y_1^*, \ldots, Y_{2K}^*) \stackrel{d}{=} (Z^\gamma Y_1 + \beta \ln Z, \ldots, Z^\gamma Y_{2K} + \beta \ln Z), \tag{19}$$

where $\gamma := 0, 1/\alpha, -1/\alpha$ *if* $H = \Lambda, \Phi_\alpha, \Psi_\alpha, \alpha > 0$, *respectively, and* $\beta := 1$ *if* $H = \Lambda$ *and 0 otherwise. Furthermore,*

$$G_K(\boldsymbol{\xi}, \boldsymbol{\xi}^*, \boldsymbol{j}, \boldsymbol{j}^*) = \boldsymbol{P}\{Z^\gamma[Y_k - Y_{k+j_{i,k}}] > \xi_{i,k}, Z^\gamma[Y_{k-j^*_{i,k}} - Y_k] > \xi^*_{i,k}, 1 \le i \le I, 1 \le k \le K\}. \tag{20}$$

In the case $H = \Lambda$ above we have a stochastic representation (see Hashorva (2006))

$$(Y_1, \ldots, Y_m) \stackrel{d}{=} \Big(E_1 + \sum_{l=2}^{\infty} \frac{E_l - 1}{l} + C_1, \ldots, E_m + \sum_{l=m+1}^{\infty} \frac{E_l - 1}{l} + C_m\Big), \quad 2 \le m \le K, \tag{21}$$

where $E_i, i \ge 1$ are independent random variables with unit exponential distribution and $C_i := C - \sum_{l=1}^{i} \frac{1}{l}, i \ge 1$ where $C == 0.5772$ is the Euler constant. Remark that (19) is initially obtained in Hashorva (2003).

Making use of Corollary 3 weak convergence of $\widetilde{\mathbf{M}}_{t,m}(\cdot)$ for $H = \Lambda$ or $H = \Phi_\alpha, \alpha > 0$ can be easily established utilising further (12) and (13). We discuss below briefly two special cases. Next, write $Y \sim Nb(m, q)$ if the random variable Y has a negative binomial distribution function with parameters m, q and probability density function

$$\frac{\Gamma(m+k)}{\Gamma(m)\Gamma(k+1)} q^m p^k, \quad k \ge 0, \quad q \in (0,1), p := 1 - q,$$

where $\Gamma(\cdot)$ is the Gamma function.

Corollary 4. *Under the assumption of Corollary 3 if further* $H = \Lambda$, *then we have:*
i) For any $m \ge 2$ *and* $-\infty < a < b \le 0$ *two negative constants*

$$\widetilde{\mathbf{M}}_{t,m}((a,b]) \stackrel{d}{\to} U_m(a,b) + \mathbf{1}(b \in \{0\}), \quad t \to \infty, \tag{22}$$

where $U_m(a,b)$ *is a Binomial random variable with parameters* $m-1, p := e^b - e^a \in (0,1)$.
ii) For any $0 \le a < b \le \infty$ *and* $m \in \mathbb{N}$

$$\widetilde{\mathbf{M}}_{t,m}((a,b]) \stackrel{d}{\to} V_m(a,b) + \mathbf{1}(a \in \{0\}), \quad t \to \infty, \tag{23}$$

with $V_m(a,b) \sim Nb(m, [1 + e^b - e^a]^{-1})$. *iii) Furthermore, for any* $k \ge 2, r \ge 1$ *and* a_i, b_i *such that* $-\infty < a_i < b_i \le 0, i = 1, \ldots, k$ *we have* $(U_2(a_2, b_2), \ldots, U_k(a_k, b_k))$ *is independent of* $V_k(a_k, b_k), \ldots, V_{k+r}(a_{k+r}, b_{k+r})$ *with* $0 \le a_{k+i} < b_{k+i} < \infty, i = 0, \ldots, r$.

Remark 2. *a*) As noted above Z and $Y_1, \ldots, Y_m, m \ge 2$ in Corollary 3 are independent random variables.

b) By Corollary 4 we have that under iii) both $\widetilde{\mathbf{M}}_{t,m}((a,b])$ and $\widetilde{\mathbf{M}}_{t,m}((a',b'])$ with $0 \le a < b < \infty, -\infty < b' < a' \le 0$ are asymptotically independent. Balakrishnan and Stepanov (2005) show in Theorem 4.2 this fact for the case $a = 0, a' = 0$

and assuming that F satisfies (7) . The result of Corollary 4 subsumes that of Theorem 4.2 of Balakrishnan and Stepanov (2005).

4. Examples

To illustrate the results we choose two special distribution functions F.

Example 1. Assume that $N(t)$ is independent of $\{X_i, i \geq 1\}$ which are independent standard exponential random variable, i.e. $F(x) = 1 - \exp(-x)\mathbf{1}(x > 0)$. It is well-known that

$$X_{n:n} - \ln n \xrightarrow{d} Y, \quad n \to \infty,$$

with Y a unit Gumbel random variable. A convenient representation for order statistics exists in the exponential case (cf. Reiss (1989))

$$\{X_{n-m+1:n}\}_{m=1,\dots,n} \stackrel{d}{=} \Big\{\sum_{i=m}^{n} E_i/i\Big\}_{m=1,\dots,n,n\in\mathbb{N}}.$$

Using further (4) we may write for $m, j \in \mathbb{N}, m + j \leq n$

$$\begin{aligned}
\boldsymbol{P}\{\mathbf{M}_{t,m}([0,a)) > j | N(t) = n\} &= \boldsymbol{P}\{X_{n-m+1:n} - X_{n-m+1-j:n} \leq a\} \\
&= \boldsymbol{P}\Big\{\sum_{i=m}^{m+j-1} E_i/i \leq a\Big\} \\
&= \boldsymbol{P}\{X_{j:(m+j-1)} \leq a\} \\
&= \frac{(m+j-1)!}{(j-1)!(m-1)!}\int_0^a e^{-ms}(1-e^{-s})^{j-1}\, ds \\
&= \frac{(m+j-1)!}{(j-1)!(m-1)!}\int_0^{1-e^{-a}} y^{j-1}(1-y)^{m-1} dy.
\end{aligned}$$

Note that the above probability does not dependent of n, if n is sufficiently large $(n \geq j + m)$.

So we get for t large assuming further that $N(t) \xrightarrow{p} \infty$ as $t \to \infty$

$$\begin{aligned}
\boldsymbol{P}\{\mathbf{M}_{t,m}([0,a)) > j\} &= \boldsymbol{P}\{\mathbf{K}_t(a,m) > j, N(t) \geq j + m\} \\
&= \sum_{n=j+m}^{\infty} \boldsymbol{P}\{X_{j:m+j-1} \leq a\}\boldsymbol{P}\{N(t) = n\} \\
&= \boldsymbol{P}\{X_{j:m+j-1} \leq a\}\boldsymbol{P}\{N(t) \geq j + m\},
\end{aligned}$$

hence Corollary 4 yields

$$\lim_{t\to\infty} \boldsymbol{P}\{\mathbf{M}_{t,m}([0,a)) > j\} = \boldsymbol{P}\{V > j - 1\},$$

with V a negative binomial random variable with parameters m and $q := e^{-a}$. Consequently we obtain for any $x \in (0, 1)$ and $1 \leq j \leq m, j, m \in \mathbb{N}$ the following Taylor expansion

$$(24)\quad (1-x)^{-m}\frac{(m+j-1)!}{(j-1)!(m-1)!}\int_0^x y^{j-1}(1-y)^{m-1}dy = \sum_{k=j}^{\infty}\frac{\Gamma(m+k)}{\Gamma(m)k!}x^k.$$

For $\mathbf{M}_{t,m}((-a,0]), m>1, 1\le j\le m-1$

$$\begin{aligned}
\boldsymbol{P}\{\mathbf{M}_{t,m}((-a,0])>j|N(t)=n\} &= \boldsymbol{P}\{X_{n-m+j+1:n}-X_{n-m+1:n}\le a\}\\
&= \boldsymbol{P}\Big\{\sum_{i=m-j}^{m-1} E_i/i\le a\Big\}\\
&= \boldsymbol{P}\{X_{j:m-1}\le a\}\\
&= \frac{(m-1)!}{(j-1)!(m-j-1)!}\int_0^{1-e^{-a}} y^{j-1}(1-y)^{m-j-1}dy.
\end{aligned}$$

Example 2. Another tractable instance is when F is the uniform distribution on $(0,1)$. Corollary 1.6.10 in Reiss (1989) implies the following stochastic representation (Renyi representation)

$$\{X_{n-m+1:n}\}_{m=1,\dots,n}\overset{d}{=}\Big\{\frac{\sum_{i=1}^{n-m+1}E_i}{\sum_{i=1}^{n+1}E_i}\Big\}_{m=1,\dots,n},$$

with $E_1,\dots,E_{n+1}$ iid standard exponential random variables. Thus for $1\le j\le n-m$ and $a>0$

$$\begin{aligned}
\boldsymbol{P}\{\mathbf{M}_{t,m}([0,a))>j|N(t)=n\} &= \boldsymbol{P}\{X_{n-m+1:n}-X_{n-m+1-j:n}\le a\}\\
&= \boldsymbol{P}\Big\{\frac{\sum_{i=n-m-j+2}^{n-m+1}E_i}{\sum_{i=1}^{n+1}E_i}\le a\Big\}\\
&= \boldsymbol{P}\{X_{j:n}\le a\}\\
&= \frac{n!}{(j-1)!(n-j)!}\int_0^a s^{j-1}(1-s)^{n-j}\,ds
\end{aligned}$$

depending on n but not on m, if $j\le n-m$. Hence again

$$\begin{aligned}
\boldsymbol{P}\{\mathbf{M}_{t,m}([0,a))>j\} &= \boldsymbol{P}\{\mathbf{K}_t(a,m)>j, N(t)\ge j+m\}\\
&= \sum_{n=j+m}^{\infty}\boldsymbol{P}\{X_{j:n}\le a\}\boldsymbol{P}\{N(t)=n\}\\
&= \boldsymbol{P}\{X_{j:N(t)}\le a, N(t)\ge j+m\}.
\end{aligned}$$

Similarly we get for $1\le j\le m-1$

$$\begin{aligned}
\boldsymbol{P}\{\mathbf{M}_{t,m}((-a,0])>j|N(t)=n\} &= \boldsymbol{P}\{X_{n-m+j+1:n}-X_{n-m+1:n}\le a\}\\
&= \boldsymbol{P}\{X_{j:n}\le a\}.
\end{aligned}$$

The asymptotics of $\mathbf{M}_{t,m}([0,a)$ and $\mathbf{M}_{t,m}((-a,0])$ follows now easily by the properties of the order statistics $X_{j:n}$.

5. Proofs

Proof of Lemma 1 Let B be a Borel set of $[0,\infty)$. Rearranging the terms we may write

$$M_{n,m}(B)=\mathbf{1}(0\in B)+\sum_{i=1}^{n-m}\mathbf{1}(X_{n-m+1:n}-X_{i:n}\in B),\quad n>m, n, m\in\mathbb{N}.$$

For any $x \in \mathbb{R}$ such that $F(x) \in (0,1)$ the stochastic representation (cf. Reiss (1989) p. 52)

$$\Big((X_{1:n},\dots,X_{n-m:n})|X_{n-m+1:n}=x\Big) \stackrel{d}{=} (Y_{1:n-m},\dots,Y_{n-m:n-m})$$

holds, where the random variables $(Y_{1:n-m},\dots,Y_{n-m:n-m})$ are the order statistics of iid random variables $\eta_1^{[x]},\dots,\eta_{n-m}^{[x]}$ with distribution function $F_x(y) := F(y)/F(x), \forall y \le x$. If $B \subset (-\infty,0]$ we have

$$M_{n,m}(B) = \mathbf{1}(0 \in B) + \sum_{i=1}^{m-1} \mathbf{1}(X_{n-m+1:n} - X_{n-m+i+1:n} \in B),$$

hence (1) follows by conditioning on $X_{n-m+1:n}$. Since $\{N(t), t \ge 0\}$ is independent of the random sequence $X_i, i \ge 1$ the expression of the p.g.f. follows easily using (1). □

Proof of Theorem 2 The result in (16) follows along the same arguments as in (15). The limit of $G_K(\boldsymbol{\xi},\boldsymbol{\xi}^*,\boldsymbol{j},\boldsymbol{j}^*)$ as $j_{ik} \to \infty$ is a proper distribution function in $\mathbb{R}^{IK}$, hence $\{\widetilde{\mathbf{M}}_{t,k}((-\xi^*_{i,k},0])\}_{1\le i\le I, 1\le k\le K}$ converge in distribution. If (17) holds, then it follows that $G_K(\boldsymbol{\xi},\boldsymbol{\xi}^*,\cdot,\cdot)$ is a proper distribution function in $\mathbb{R}^{2IK}$. Hence by the continuous mapping theorem the convergence of the point process in (18) follows, thus the proof is complete. □

Proof of Corollary 3 By the assumptions (11) is satisfied. Since $N(t)$ is independent of $X_i, i \ge 1$ and (5) then Proposition 2.2 of Hashorva (2003) implies (11) with $k = 2K$, and furthermore

$$(Y_1^*,\dots,Y_k^*) \stackrel{d}{=} (Z^\gamma Y_1 + \beta \ln Z, \dots, Z^\gamma Y_k + \beta \ln Z),$$

where $\gamma = 0, 1/\alpha, -1/\alpha$ if $H = \Lambda, \Phi_\alpha, \Psi_\alpha, \alpha > 0$, respectively and $\beta = 1$ if $H = \Lambda$ and 0 otherwise. Hence the result follows from Theorem 2. □

Proof of Corollary 4 In view of (9) we have

$$\boldsymbol{P}\{Y_i^* \le x\} = \boldsymbol{E}\{\Gamma_i(-Z \ln H(x))\}$$

for $x \in (\alpha_H, x_H)$ where $\Gamma_i(s) = \int_s^\infty t^{i-1}e^{-t}\,dt/\Gamma(i)$ is the upper tail of the standard Gamma distribution. Since for all $s > 0$

$$\Gamma_n(s) = \sum_{r=0}^{n-1} e^{-s}s^r/\Gamma(r+1) \to 1, \quad n \to \infty$$

the dominated convergence theorem yields

$$\lim_{n\to\infty} \boldsymbol{P}\{Y_n^* \le x\} = 1.$$

Consequently, if $H = \Lambda$ or $H = \Psi_\alpha$ we get the almost sure convergence

$$Y_n^* \stackrel{a.s.}{\to} \alpha_H = -\infty.$$

Thus the weak convergence follows by Theorem 2 using further Corollary 3.

Both (22) and (23) follow then using further (12), (13) and (19). In the following we show directly (23) for $H = \Lambda$ using (2). We may write for $s \in (0,1)$ and

$b > 0, b > a \geq 0$ (set $q := e^b - e^a$)

$$
\begin{aligned}
\lim_{t\to\infty} \boldsymbol{E}\{s^{\widetilde{\mathbf{M}}_{t,m}((a,b])-\mathbf{1}(a\in\{0\})}\}
&= \int_{\mathbb{R}} \Big(\frac{H(x-b)}{H(x-a)}\Big)^{1-s} d(\boldsymbol{P}\{Y_m \leq x\})\\
&= \int_{\mathbb{R}} e^{-(1-s)qe^{-x}} d(\boldsymbol{P}\{Y_m \leq x\})\\
&= -\int_0^\infty e^{-(1-s)qt} d\Big(\sum_{r=0}^{m-1} t^r e^{-t}/\Gamma(r+1)\Big)\\
&= \sum_{r=0}^{m-1} \frac{1}{\Gamma(r+1)} \int_0^\infty e^{-(1+(1-s)q)t} t^r\, dt - \sum_{r=1}^{m-1} \frac{1}{\Gamma(r)} \int_0^\infty e^{-(1+(1-s)q)t} t^{r-1}\, dt\\
&= \sum_{r=0}^{m-1} \Big(\frac{1}{1+(1-s)q}\Big)^{r+1} - \sum_{r=1}^{m-1} \Big(\frac{1}{1+(1-s)q}\Big)^{r}\\
&= \Big(\frac{1}{1+(1-s)q}\Big)^{m} = \Big(\frac{[1+q]^{-1}}{1-sq/[1+q]}\Big)^{m}.
\end{aligned}
$$

The last claim on the independence of the random vectors $(U_2(a_2,b_2),\dots,U_k(a_k,b_k))$ and $V_k(a_k,b_k),\dots,V_{k+r}(a_{k+r},b_{k+r}))$ follows easily by calculating the limit of the joint probability density function of the corresponding marginals, or using the stochastic representation (21) together with (20). □

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