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COMMON FIXED POINT THEOREM IN INTUITIONISTIC FUZZY METRIC SPACES

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Abstract. In this paper, a common fixed point theorem for R-weakly commuting maps in intuitionistic fuzzy metric spaces is proved.

1. Introduction and Preliminaries

In this section, using the idea of intuitionistic fuzzy metric spaces introduced by Park [5] we define the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous t–representable.

Definition 1.1. A complete lattice is a partially ordered set in which every nonempty subset admits supremum and infimum.

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Lemma 1.2. ([2]) *Consider the set* L^* *and operation* $\leq_{L^*}$ *defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \iff x_1 \leq y_1$ *and* $x_2 \geq y_2$*, for every* $(x_1, x_2), (y_1, y_2) \in L^*$*. Then* $(L^*, \leq_{L^*})$ *is a complete lattice .*

Definition 1.3. ([1]) An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where, for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1,x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0,x) = 0 \diamond x = x$, for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$ these definitions can be straightforwardly extended.

Definition 1.4. ([2]) A triangular norm (t–norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

$(\forall x \in L^*)(\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)

$(\forall (x,y) \in (L^*)^2)(\mathcal{T}(x,y) = \mathcal{T}(y,x))$, (commutativity)

$(\forall (x,y,z) \in (L^*)^3)(\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y),z))$, (associativity)

$(\forall (x,x',y,y') \in (L^*)^4)(x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x,y) \leq_{L^*} \mathcal{T}(x',y'))$. (monotonicity)

If $(L^*, \leq_{L^*}, \mathcal{T})$ is an Abelian topological monoid with unit 1_{L^*} then $\mathcal{T}$ is said to be a *continuous* t–norm.

Definition 1.5. ([2]) A continuous t–norm $\mathcal{T}$ on L^* is called continuous t*–representable* if and only if there exist a continuous t–norm $*$ and a continuous t–conorm $\diamond$ on $[0,1]$ such that, for all $x = (x_1, x_2), y = (y_1, y_2) \in L^*$,

$$\mathcal{T}(x,y) = (x_1 * y_1, x_2 \diamond y_2).$$

For example $\mathcal{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* is a continuous t–representable.

Definition 1.6. A negator on L^* is any decreasing mapping $\mathcal{N} : L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*}) = 1_{L^*}$ and $\mathcal{N}(1_{L^*}) = 0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x)) = x$, for all $x \in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N : [0,1] \longrightarrow [0,1]$ satisfying $N(0) = 1$ and $N(1) = 0$. N_s denotes the standard negator on $[0,1]$ defined as, for all $x \in [0,1]$, $N_s(x) = 1 - x$. We define $(N_s(\lambda), \lambda) = \mathcal{N}_s(\lambda)$.

Definition 1.7. Let M, N are fuzzy sets from $X^2 \times (0, +\infty)$ to $[0,1]$ such that $M(x,y,t) + N(x,y,t) \leq 1$ for all $x, y \in X$ and $t > 0$, in which, M is membership degree and N is non-membership degree of an intuitionistic fuzzy set. The triple $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous t–representable and $\mathcal{M}_{M,N}$ is a mapping $X^2 \times (0, +\infty) \rightarrow L^*$ (an intuitionistic fuzzy set, see Definition 2.4) satisfying the following conditions for every $x, y, z \in X$ and $t, s > 0$:

(a) $\mathcal{M}_{M,N}(x,y,t) >_{L^*} 0_{L^*}$;

(b) $\mathcal{M}_{M,N}(x,y,t) = 1_{L^*}$ if and only if $x = y$;

(c) $\mathcal{M}_{M,N}(x, y, t) = \mathcal{M}_{M,N}(y, x, t)$;
(d) $\mathcal{M}_{M,N}(x, y, t + s) \geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(x, z, t), \mathcal{M}_{M,N}(z, y, s))$;
(e) $\mathcal{M}_{M,N}(x, y, \cdot) : (0, \infty) \longrightarrow L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy metric*. Here,

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)).$$

Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. For $t > 0$, define the *open ball* $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$, as

$$B(x, r, t) = \{y \in X : \mathcal{M}_{M,N}(x, y, t) >_{L^*} (N_s(r), r) = \mathcal{N}_s(r)\}.$$

A subset $A \subseteq X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subseteq A$. Let $\tau_{\mathcal{M}_{M,N}}$ denote the family of all open subset of X. $\tau_{\mathcal{M}_{M,N}}$ is called the *topology induced by intuitionistic fuzzy metric*. A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a *Cauchy sequence* if for each $\varepsilon > 0$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\mathcal{M}_{M,N}(x_n, x_m, t) >_{L^*} \mathcal{N}_s(\varepsilon),$$

and for each $n, m \geq n_0$. The sequence $\{x_n\}$ is said to be *convergent* to $x \in V$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and denoted by $x_n \stackrel{\mathcal{M}_{M,N}}{\longrightarrow} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \longrightarrow 1_{L^*}$ whenever $n \longrightarrow \infty$ for every $t > 0$. An intuitionistic fuzzy metric space is said to be *complete* if and only if every Cauchy sequence is convergent (see [3, 5]).

Lemma 1.8. ([3]) *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. Then, $\mathcal{M}_{M,N}(x, y, t)$ is nondecreasing with respect to t, for all x, y in X.*

Example 1.9. ([7]) Let (X, d) be a metric space. Denote $\mathcal{T}(a, b) = (a_1b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2)$ in L^* and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x, y, t) = (M(x, y, t), N(x, y, t)) = (\frac{t}{t + md(x, y)}, \frac{d(x, y)}{t + d(x, y)}),$$

in which $m > 1$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Let $\mathcal{T}$ be a continuous t–norm on L^* in which, for every $\mu \in (0, 1)$, there exists $\lambda \in (0, 1)$ such that

$$\mathcal{T}^{n-1}(\mathcal{N}_s(\lambda), ..., \mathcal{N}_s(\lambda)) >_{L^*} \mathcal{N}_s(\mu), \tag{1.1}$$

where $\mathcal{N}_s$ is an standard negation. For more information see [6].

Definition 1.10. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. $\mathcal{M}_{M,N}$ is said to be continuous on $X \times X \times (0, \infty)$ if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(x_n, y_n, t_n) = \mathcal{M}_{M,N}(x, y, t)$$

whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X \times X \times (0, \infty)$ converges to a point $(x, y, t) \in X \times X \times (0, \infty)$ i.e., $\lim_n \mathcal{M}_{M,N}(x_n, x, t) = \lim_n \mathcal{M}_{M,N}(y_n, y, t) = 1_{L^*}$ and

$$\lim_n \mathcal{M}_{M,N}(x, y, t_n) = \mathcal{M}_{M,N}(x, y, t).$$

Lemma 1.11. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. Then $\mathcal{M}_{M,N}$ is continuous function on $X \times X \times (0, \infty)$.*

Proof. The proof is same as fuzzy metric spaces (see Proposition 1 of [4]). □

2. The Main Results

Definition 2.1. Let f and g be maps from an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. The maps f and g are said to be weakly commuting if

$$\mathcal{M}_{M,N}((fog)(x),(gof)(x),t) \geq_{L^*} \mathcal{M}_{M,N}(f(x),g(x),t)$$

for each x in X and $t > 0$.

Definition 2.2. Let f and g be maps from an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ into itself. The maps f and g are said to be R-weakly commuting if there exists some positive real number R such that

$$\mathcal{M}_{M,N}((fog)(x),(gof)(x),t) \geq_{L^*} \mathcal{M}_{M,N}(f(x),g(x),t/R)$$

for each x in X and $t > 0$.

Weak commutativity implies R-weak commutativity in intuitionistic fuzzy metric space. However, R-weak commutativity implies weak commutativity only when $R \leq 1$.

Example 2.3. Let $X = \mathbf{R}$. Let $\mathcal{T}(a,b) = (a_1b_1, \min(a_2+b_2,1))$ for all $a = (a_1,a_2), b = (b_1,b_2) \in L^*$ and let $\mathcal{M}_{M,N}$ be the intuitionistic fuzzy set on $X \times X \times\,]0,+\infty[$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = \left((\exp(\frac{|x-y|}{t}))^{-1}, \frac{\exp(\frac{|x-y|}{t})-1}{\exp(\frac{|x-y|}{t})} \right),$$

for all $t \in \mathbf{R}^+$. Then $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space. Define $f(x) = 2x-1$ and $g(x) = x^2$. Then,

$$\mathcal{M}_{M,N}((fog)(x),(gof)(x),t) - \left((\exp(2\frac{|x-1|^2}{t}))^{-1}, \frac{\exp(2\frac{|x-1|^2}{t})-1}{\exp(2\frac{|x-1|^2}{t})} \right)$$

$$\left((\exp(\frac{|x-1|^2}{t/2}))^{-1}, \frac{\exp(\frac{|x-1|^2}{t/2})-1}{\exp(\frac{|x-1|^2}{t/2})} \right) = \mathcal{M}_{M,N}(f(x),g(x),t/2)$$

$$<_{L^*} \left((\exp(\frac{|x-1|^2}{t}))^{-1}, \frac{\exp(\frac{|x-1|^2}{t})-1}{\exp(\frac{|x-1|^2}{t})} \right) = \mathcal{M}_{M,N}(f(x),g(x),t)$$

Therefore, for $R = 2$, f and g are R-weakly commuting. But f and g are not weakly commuting since exponential function is strictly increasing.

Theorem 2.4. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a complete intuitionistic fuzzy metric space and let f and g be R-weakly commuting self-mappings of X satisfying the following conditions:*

(a) *$f(X) \subseteq g(X)$;*

(b) *f or g is continuous;*

(c) *$\mathcal{M}_{M,N}(f(x),f(y),t) \geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}(g(x),g(y),t))$, where $\mathcal{C} : L^* \longrightarrow L^*$ is a continuous function such that $\mathcal{C}(a) >_{L^*} a$ for each $a \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$.*

Then f and g have a unique common fixed point.

Proof. Let x_0 be an arbitrary point in X. By (a), choose a point x_1 in X such that $f(x_0) = g(x_1)$. In general choose x_{n+1} such that $f(x_n) = g(x_{n+1})$. Then for $t > 0$

$$\begin{aligned}\mathcal{M}_{M,N}(f(x_n), f(x_{n+1}), t) &\geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}(g(x_n), g(x_{n+1}), t))\\ &= \mathcal{C}(\mathcal{M}_{M,N}(f(x_{n-1}), f(x_n), t))\\ &>_{L^*} \mathcal{M}_{M,N}(f(x_{n-1}), f(x_n), t)\end{aligned}$$

Thus $\{\mathcal{M}_{M,N}(f(x_n), f(x_{n+1}), t); n \geq 0\}$ is increasing sequence in L^*. Therefore, tends to a limit $a \leq_{L^*} 1_{L^*}$. We claim that $a = 1_{L^*}$. For if $a <_{L^*} 1_{L^*}$ on making $n \longrightarrow \infty$ in the above inequality we get $a \geq_{L^*} \mathcal{C}(a) >_{L^*} a$, a contradiction. Hence $a = 1_{L^*}$, i.e.,

$$\lim_n \mathcal{M}_{M,N}(f(x_n), f(x_{n+1}), t) = 1_{L^*}.$$

If we define

$$c_n(t) = \mathcal{M}_{M,N}(f(x_n), f(x_{n+1}), t) \tag{2.1}$$

then $\lim_{n\to\infty} c_n(t) = 1_{L^*}$. Now, we prove that $\{f(x_n)\}$ is a Cauchy sequence in $f(X)$. Suppose that $\{f(x_n)\}$ is not a Cauchy sequence in $f(X)$. For convenience, let $y_n = fx_n$ for $n = 1, 2, 3, \cdots$. Then there is an $\epsilon \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$ such that for each integer k, there exist integers $m(k)$ and $n(k)$ with $m(k) > n(k) \geq k$ such that

$$d_k(t) = \mathcal{M}_{M,N}(y_{n(k)}, y_{m(k)}, t) \leq_{L^*} \mathcal{N}_s(\epsilon) \quad \text{for} \quad k = 1, 2, \cdots . \tag{2.2}$$

We may assume that

$$\mathcal{M}_{M,N}(y_{n(k)}, y_{m(k)-1}, t) >_{L^*} \mathcal{N}_s(\epsilon), \tag{2.3}$$

by choosing $m(k)$ be the smallest number exceeding $n(k)$ for which (2.2) holds. Using (2.1), we have

$$\begin{aligned}\mathcal{N}_s(\epsilon) &\geq_{L^*} d_k(t)\\ &\geq_{L^*} \mathcal{T}(\mathcal{M}_{M,N}(y_{n(k)}, y_{m(k)-1}, t/2), \mathcal{M}_{M,N}(y_{m(k)-1}, y_{m(k)}, t/2))\\ &\geq_{L^*} \mathcal{T}(c_k(t/2), \mathcal{N}_s(\epsilon))\end{aligned}$$

Hence, $d_k(t) \longrightarrow \mathcal{N}_s(\epsilon)$ for every $t > 0$ as $k \longrightarrow \infty$.

$$\begin{aligned}&d_k(t) = \mathcal{M}_{M,N}(y_{n(k)}, y_{m(k)}, t)\\ &\geq_{L^*} \mathcal{T}^2(\mathcal{M}_{M,N}(y_{n(k)}, y_{n(k)+1}, t/3), \mathcal{M}_{M,N}(y_{n(k)+1}, y_{m(k)+1}, t/3), \mathcal{M}_{M,N}(y_{m(k)+1}, y_{m(k)}, t/3)\\ &\geq_{L^*} \mathcal{T}^2(c_k(t/3), \mathcal{C}(\mathcal{M}_{M,N}(y_{n(k)}, y_{m(k)}, t/3)), c_k(t/3))\\ &\mathcal{T}^2(c_k(t/3), \mathcal{C}(d_k(t/3)), c_k(t/3)).\end{aligned}$$

Thus, as $k \longrightarrow \infty$ in the above inequality we have

$$\mathcal{N}_s(\epsilon) \geq_{L^*} \mathcal{C}(\mathcal{N}_s(\epsilon)) >_{L^*} \mathcal{N}_s(\epsilon)$$

which is a contradiction. Thus, $\{f(x_n)\}_n$ is Cauchy and by the completeness of X, $\{f(x_n)\}_n$ converges to z in X. Also $\{g(x_n)\}_n$ converges to z in X. Let us suppose that the mapping f is continuous. Then $\lim_n(fof)(x_n) = f(z)$ and $\lim_n(fog)(x_n) = f(z)$. Further we have since f and g are R-weakly commuting

$$\mathcal{M}_{M,N}((fog)(x_n), (gof)(x_n), t) \geq_{L^*} \mathcal{M}_{M,N}(f(x_n), g(x_n), t/R).$$

On letting $n \to \infty$ in the above inequality we get $\lim_n (gof)(x_n) = f(z)$, by Lemma 1.11. We now prove that $z = f(z)$. Suppose $z \neq f(z)$ then $\mathcal{M}_{M,N}(z, f(z), t) <_{L^*} 1_{L^*}$. By (c)

$$\mathcal{M}_{M,N}(f(x_n), (fof)(x_n), t) \geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}(g(x_n), (gof)(x_n), t)).$$

On making $n \to \infty$ in the above inequality we get

$$\mathcal{M}_{M,N}(z, f(z), t) \geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}(z, f(z), t)) >_{L^*} \mathcal{M}(z, f(z), t),$$

a contradiction. Therefore, $z = f(z)$. Since $f(X) \subseteq g(X)$ we can find z_1 in X such that $z = f(z) = g(z_1)$. Now,

$$\mathcal{M}((fof)(x_n), f(z_1), t) \geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}((gof)(x_n), g(z_1), t)).$$

Taking limit as $n \to \infty$ we get

$$\mathcal{M}_{M,N}(f(z), f(z_1), t) \geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}(f(z), g(z_1), t)) = 1_{L^*}$$

since $\mathcal{C}(1_{L^*}) = 1_{L^*}$, which implies that $f(z) = f(z_1)$, i.e., $z = f(z) = f(z_1) = g(z_1)$. Also for any $t > 0$,

$$\mathcal{M}_{M,N}(f(z), g(z), t) = \mathcal{M}((fog)(z_1), (gof)(z_1), t) \geq_{L^*} \mathcal{M}_{M,N}(f(z_1), g(z_1), t/R) = 1_{L^*}$$

which again implies that $f(z) = g(z)$. Thus z is a common fixed point of f and g.

Now to prove uniqueness let if possible $z' \neq z$ be another common fixed point of f and g. Then there exists $t > 0$ such that $\mathcal{M}(z, z', t) <_L 1_{\mathcal{L}}$, and

$$\begin{aligned}\mathcal{M}_{M,N}(z, z', t) &= \mathcal{M}_{M,N}(f(z), f(z'), t) \\ &\geq_{L^*} \mathcal{C}(\mathcal{M}_{M,N}(g(z), g(z'), t)) = \mathcal{C}(\mathcal{M}_{M,N}(z, z', t)) \\ &>_{L^*} \mathcal{M}_{M,N}(z, z', t)\end{aligned}$$

which is contradiction. Therefore, $z = z'$, i.e., z is a unique common fixed point of f and g. □

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