J. DIFFERENTIAL GEOMETRY
**114** (2020) 193-242

# EXPLICIT AMBIENT METRICS AND HOLONOMY

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## Abstract

We present three large classes of examples of conformal structures whose Fefferman–Graham ambient metrics can be found explicitly. Our method for constructing these examples rests upon a set of sufficiency conditions under which the Fefferman–Graham equations are assured to reduce to a system of inhomogeneous linear partial differential equations. Our examples include conformal pp-waves and, more importantly, conformal structures that are defined by generic co-rank 3 distributions in dimensions 5 and 6. Our examples illustrate various aspects of the ambient metric construction.

The holonomy algebras of our ambient metrics are studied in detail. In particular, we exhibit a large class of metrics with holonomy equal to the exceptional non-compact Lie group $\mathbf{G}_2$ as well as ambient metrics with holonomy contained in $\mathbf{Spin}(4,3)$.

## 1. Introduction

Let $g_0$ be a smooth semi-Riemannian metric on an $n$-dimensional manifold $M$, with local coordinates $(x^i)$ and $n \geq 2$. A ***pre-ambient metric*** for $g_0$ is a smooth semi-Riemannian metric $\tilde{g}$ of the form

$$\tilde{g} = \varrho\, dt^2 + t\, dtd\varrho + t^2 g(x^i, \varrho),$$

defined on the ambient space $\widetilde{M} = (0, +\infty) \times M \times (-\epsilon, \epsilon), \epsilon > 0$, with coordinates $(t, x^i, \varrho)$ and such that $g(x^i, \varrho)|_{\varrho=0} = g_0(x^i)$. We call a pre-ambient metric an ***ambient metric*** if

$$\mathrm{Ric}(\tilde{g}) = 0. \tag{1.1}$$

*Mathematics Subject Classification.* Primary 53C29, 53A30, Secondary 53C50.

*Key words and phrases.* Conformal structures, Fefferman–Graham ambient metric, exceptional holonomy, generic distributions, pp-waves.

Support for this research was provided by National Science Foundation (ACI 1148331SI2-SSE); by the Australian Research Council (FT110100429 and DP120104582); the Polish Ministry of Research and Higher Education (NN201 607540 and NN202 104838); and by the Polish National Science Center (NCN) via DEC-2013/09/B/ST1/01799.

Received May 12, 2015.

The questions of existence and uniqueness of ambient metrics were fully answered by Fefferman and Graham [11, 12]. The importance of the ambient metric is that the semi-Riemannian invariants of $\tilde{g}$ give rise to conformal invariants of $g_0$.

The Fefferman–Graham equations (1.1) are a complicated non-linear system of partial differential equations for $g(x^i, \varrho)$ which, in general, cannot be solved explicitly. Indeed, prior to the results of [18, 19], the only known explicit examples of ambient metrics arose from Einstein metrics $g_0$ or certain products of Einstein metrics.

The goal of this paper is to present some interesting new classes of examples of *explicit* ambient metrics, to calculate the holonomy of these ambient metrics, and to exhibit the parallel forms which define the holonomy reductions. Our examples are derived from the general results of [2] wherein sufficiency conditions are presented which ensure that *the Fefferman–Graham equations reduce to linear PDEs*. We shall solve these linear equations in either closed form or by standard power series methods. For $n$ odd, we give examples of non-analytic ambient metrics while, for $n$ even, we give ambient metrics containing the so-called Fefferman–Graham ambiguity.

In Section 2, we review some basic facts regarding the ambient metric construction; we summarize the results of [2]; and we describe the Nurowski conformal class associated with any generic rank 2 distribution in 5 dimensions. Let us simply remark here that while the Fefferman–Graham equations are of second order, they have a singular point at $\varrho = 0$ and, hence, the first $\varrho$-derivative of $g(x, \varrho)$ at $\varrho = 0$ cannot be chosen freely. Indeed, one readily finds that $\dfrac{\partial g}{\partial \varrho}|_{\varrho=0} = 2P$, where $P$ is the Schouten tensor of $g_0$ (see (2.7)). An essential common feature of all our examples is that the Schouten tensor of $g_0$ is 2-step nilpotent (when considered as a 1-1 tensor). Accordingly, it is natural that to postulate that

$$g(x^i, \varrho) = g_0 + h(x^i, \varrho), \quad h|_{\varrho=0} = 0, \tag{1.2}$$

where $h(x^i, \varrho)$ (also considered as $(1, 1)$ tensor) has the *same image and kernel as* $P$ for all $\varrho$.

The ambient metrics we consider arise from three different types of conformal structures. In Section 3, we consider conformal pp-waves; in Section 4 conformal structures defined by generic rank 2 distributions in dimension 5 are studied [23]; and, in Section 5, Bryant conformal structures [6], defined for certain rank 3 distributions in 6 dimensions, are examined. Because the underlying manifold dimensions are odd in Section 4 and even in Section 5, different aspects of the ambient metric construction can be explored.

The conformal pp-waves serve as a simple prototype for conformal structures for which the Fefferman–Graham equations become linear.

We extend our results from [18] to other signatures and by determining all solutions to (1.1). When $n$ is odd, we find ambient metrics which are defined for $\varrho \geq 0$ but are only $(n-1)/2$ times differentiable in $\varrho$ at $\varrho = 0$. When $n$ is even, the general ambient metrics contain logarithmic terms in $\varrho$ and are only defined on the domain $\varrho > 0$. An upper bound for the holonomy algebras of the ambient metrics is determined.

The second class of examples arise from conformal structures, of signature $(3, 2)$, defined by rank 2 distributions

(1.3)
$$\mathcal{D} = \operatorname{span}\{\partial_q, \partial_x + p\,\partial_y + q\,\partial_p + F(x, y, z, p, q)\,\partial_z\}, \quad \text{with} \quad \frac{\partial^2 F}{\partial q \partial q} \neq 0.$$

The ambient metrics for these conformal structures have been studied in the earlier work of Leistner and Nurowski [19] and in Graham and Willse [16]. The most remarkable aspect of the associated conformal structures is that their normal conformal Cartan connection reduces to the exceptional 14-dimensional, simple, non-compact Lie group $\mathbf{G}_2$ (as shown in [23]) and, hence, have their conformal holonomy[1] contained in this group. In [19] it was shown that for $F = q^2 + \sum_{i=0}^{6} a_i p^i + bz$, with $a_i$ and $b$ constant, the holonomy of the associated analytic ambient metric is contained in, and generically equal to, $\mathbf{G}_2$. These results were subsequently generalized in [16] to *all* conformal structures defined by analytic distributions (1.3) where it is established that the holonomy of the ambient metric is contained in $\mathbf{G}_2$. A sufficient criterion was given for the holonomy of the ambient metric being equal to $\mathbf{G}_2$. However, the examples in [19] with holonomy equal to $\mathbf{G}_2$ do *not* satisfy this criterion, and in general the criterion is difficult to check.

In §4.1 we show that the most general class of distributions (1.3) for which the associated conformal class satisfies the aforementioned criteria for linear Fefferman–Graham equations is given by

$$F = q^2 + f(x, y, p) + h(x, y)z. \tag{1.4}$$

In §4.2 we then present the general formal solutions (in terms of power series) to the Fefferman–Graham equations, including those that are only twice differentiable in $\varrho$ at $\varrho = 0$. We derive a linear *first order* system that has the same (analytic) solutions as the Fefferman–Graham equations. We believe that this reduction in order of the Fefferman–Graham is a novel feature of ambient metrics with reduced holonomy that merits further investigation. In §4.3 we use this first order system to determine the explicit formula for the parallel 3-form. We give a simple,

[1] We point out that all of our considerations are *local* in the sense that we work on simply connected manifolds. This implies that the holonomy groups we encounter are connected and, hence, can be identified with their Lie algebras. Moreover, throughout the paper $\mathfrak{g}_2$ and $\mathbf{G}_2$ refer to the *split real form* of the exceptional simple Lie algebra of type $G_2$ and to the corresponding connected Lie group in $\mathbf{SO}(4, 3)$.

effective criterion that ensures that the ambient metric has holonomy *equal* to $\mathbf{G}_2$. Closed form examples of analytic and non-analytic ambient metrics are presented. The holonomy of certain non-analytic ambient metrics are seen not to be sub-representations of $\mathbf{G}_2$.

The third class of examples is derived from Bryant's conformal structures for rank 3 distributions defined in $\mathbb{R}^6$ by

$$\mathcal{D} = \mathrm{span}\{ \frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial y^1}, \frac{\partial}{\partial x^1} - F(x^a, y^a)\,\frac{\partial}{\partial y^2}, \frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial y^3}\}, \tag{1.5}$$

with $\frac{\partial F}{\partial x^3} \neq 0$, and where $(x^1, x^2, x^3, y^1, y^2, y^3)$ are coordinates on $\mathbb{R}^6$. According to [6] the *conformal holonomy* is contained in $\mathbf{Spin}(4,3)$. As in the previous section, we first classify those distributions (1.5) whose associated conformal class satisfies the hypothesis of Theorem 2.1 and, hence, give rise to linear Fefferman–Graham equations for the ambient metric. We find that

$$F(x^1, x^2, x^3, y^1, y^2, y^3) = f(x^1, x^3, y^2). \tag{1.6}$$

The conformal classes and ambient metrics for these distributions have many remarkable properties.

In §5.2 we prove that the conformal structure defined by (1.6) has vanishing Fefferman–Graham obstruction tensor for any function $f$ (and is not, generically, conformally Einstein). We use this fact to further simplify the Fefferman–Graham equations and subsequently find all smooth ambient metrics. To our knowledge, these are the first examples of ambient metrics for Bryant's conformal classes.

In §5.3 we illustrate one of the fundamental properties of the ambient metric construction by calculating the diffeomorphism relating the two ambient metrics constructed from different representatives of the conformal class. We are unaware of any such explicit examples of this Fefferman–Graham diffeomorphism.

The holonomy of the ambient metrics is studied in some detail in §5.4. We discover that the algebraic properties of the covariant derivative of a certain 2-form $\alpha$ (see (5.25)) plays a decisive role in determining the possible reductions in holonomy. See Propositions 5.2 and 5.3. The square of $\alpha$ is always a recurrent 4-form which implies that holonomy is a sub-representation of $\mathfrak{gl}_4(\mathbb{R}) \ltimes \Lambda^2(\mathbb{R}^4)$ (see (5.32)). If $\alpha$ itself is recurrent, then the holonomy is a sub-representation of the Poincare algebra $\mathfrak{po}(3,2)$ and there are exactly three parallel 4-forms (See Table 1). We are surprised once more that it is possible to obtain closed form expressions for these parallel 4-forms for all ambient metrics arising from equation (1.6). A variety of possible holonomy algebras, coming from specific choices of the function $f(x^1, x^3, y^2)$, is given in Table 2.

Many of the calculations for this paper were performed using the Maple *DifferentialGeometry* package. Sample worksheets, illustrating the results of this paper, can be found at http://digitalcommons.usu.edu/dg_applications/.

**Acknowledgments.** We thank Travis Willse for comments on the first version of this article.

## 2. Preliminaries

In this section, we provide some background material which helps to put our computations and results in perspective. Here and throughout the paper we use the convention to denote the symmetric product of two one forms $\phi$ and $\psi$ by

$$\phi\,\psi := \tfrac{1}{2}\left(\phi \otimes \psi + \psi \otimes \phi\right).$$

**2.1. Invariant description of the Fefferman–Graham construction.** We begin by giving a coordinate-free and conformally invariant description of the Fefferman–Graham ambient metric construction for a conformal manifold $(M, [g_0])$. This construction is a curved generalization of the flat setting wherein the conformal $n$-sphere is viewed as the projectivized light-cone in an ambient $n+2$ dimensional Minkowski space.

In the curved setting we first have to define the space on which the ambient metric is defined. For this, consider the ***ray bundle*** $\pi : \mathcal{C} \to M$, defined in terms of the conformal structure $c = [g_0]$ on $M$, by

$$\mathcal{C} = \bigcup_{p \in M} \{\, g|_p \mid g \in c \,\} \ \overset{\pi}{\longrightarrow}\ M.$$

The ray bundle $\mathcal{C}$ carries a natural $\mathbb{R}^+$ action, namely, $\varphi_t(g|_p) = (t^2 g|_p)$, $t > 0$, and is, therefore, often referred to as the cone bundle for the conformal structure $(M, [g_0])$. The ***tautological tensor*** $G$ on $\mathcal{C}$ is defined by

$$G|_{(g|_p)}(X, Y) = g|_p(d\pi(X), d\pi(Y)) \quad \text{for} \quad X, Y \in T_{g|_p}\mathcal{C}.$$

Note that $G$ is homogeneous of degree 2 with respect to the action $\varphi_t$, that is, $\varphi_t^* G = t^2 G$. Note also that fixing a metric $g \in c$ trivializes $\mathcal{C}$ to $\mathbb{R}^+ \times M$ by $t^2 g|_p \mapsto (t, p)$.

Next consider the product $\mathcal{C} \times \mathbb{R}$ and denote the $\mathbb{R}$-coordinate by $\varrho$. The cone $\mathcal{C}$ is embedded into this product via $\iota : \mathcal{C} \hookrightarrow \mathcal{C} \times \mathbb{R}$ as $\iota(\mathcal{C}) = \mathcal{C} \times \{0\}$. The action $\varphi$ is trivially extended to $\mathcal{C} \times \mathbb{R}$.

With these notions in place we define a ***pre-ambient space*** for the conformal manifold $(M, c)$ to be a semi-Riemannian manifold $(\widetilde{M}, \tilde{g})$ such that **[i]** $\widetilde{M}$ is a $\varphi_t$-invariant neighborhood of $\mathcal{C} \times \{0\}$ in $\mathcal{C} \times \mathbb{R}$; **[ii]** $\tilde{g}$ is homogeneous of degree 2 with respect to $\varphi_t$; and **[iii]** $\iota^* \tilde{g} = G$. Evidently, the pre-ambient metric $\tilde{g}$ is not uniquely determined by the conformal class $[g_0]$. The idea, proposed in [11, 12], is to require the pre-ambient metric $\tilde{g}$ to satisfy, in addition to **[i]**–**[iii]**, the condition of Ricci flatness

$$\operatorname{Ric}(\tilde{g}) = 0, \tag{2.1}$$

and then to establish that this condition ensures uniqueness. Ricci flatness, however, turns out to be too strong a condition—in even dimensions it is possible to exhibit cases where Ricci-flat pre-ambient metrics do not exist. Accordingly, the definition of an ambient metric $\tilde{g}$ for the conformal class $[g_0]$ will depend on the parity of $n = \dim(M)$.

When $n$ is odd, a pre-ambient space $(\widetilde{M}, \tilde{g})$ is called an ***ambient space*** if

$$\operatorname{Ric}(\tilde{g}) \in O(\varrho^\infty), \quad \text{i.e.,} \quad \partial_\varrho^k \operatorname{Ric}(\tilde{g})|_{\varrho=0} = 0 \quad \text{for } k = 0, 1, \ldots\ . \tag{2.2}$$

When $n$ is odd the ambient metric is unique to all orders in $\varrho$, apart from diffeomorphisms of $\widetilde{M}$ that restrict to the identity on $\mathcal{C}$.

When $n > 2$ is even, one can only find pre-ambient metrics satisfying the condition (2.2) for $k = 0, \ldots, \frac{n}{2} - 1$. Therefore, an ***ambient space*** for $(M, c)$ is a pre-ambient space $(\widetilde{M}, \tilde{g})$ such that

$$\operatorname{Ric}(\tilde{g}) \in O(\varrho^{\frac{n}{2}-1}), \tag{2.3}$$

and such that, at each $g|_p \in \mathcal{C}$,

$$\iota^*(\varrho^{1-\frac{n}{2}} \operatorname{Ric}(\tilde{g}))|_{g|_p} = \pi^* S,$$

with $S \in \otimes^2 T_p M$ symmetric and $\operatorname{tr}_{g|_p}(S) = 0$.

For (2.2) to be satisfied beyond order $k = \frac{n}{2} - 1$ a certain symmetric tensor $\mathcal{O}_{ij}(g_0)$, referred to as the ***Fefferman–Graham obstruction tensor***, must vanish. The obstruction tensor is trace and divergence-free and conformally invariant. Fefferman and Graham show that the vanishing of $\mathcal{O}_{ij}(g_0)$ is equivalent to the existence of a pre-ambient metric satisfying condition (2.2) to all orders $k$ and, hence, for analytic metrics $g_0$, equivalent to the existence of a Ricci-flat analytic metric $\tilde{g}$. In this situation, that is, when $n$ is even, $\mathcal{O}_{ij}(g_0)$ vanishes and $g_0$ is analytic, analytic ambient metrics are still only unique to order $\frac{n}{2}$.

It is also shown in [12] that for a fixed metric $g_0$ from the conformal class $c$, every ambient metric can be brought into the local normal form

$$\tilde{g} = 2\, dt\, d(\varrho t) + t^2 g(x^i, \varrho), \quad \text{with} \quad g(x^i, \varrho)|_{\varrho=0} = g_0(x^i), \tag{2.4}$$

and where $(t, x^i, \varrho)$ are local coordinates on $\widetilde{M} = \mathbb{R}^+ \times M \times (-\epsilon, \epsilon)$, $\epsilon > 0$. This is the local form we will use with throughout this paper when searching for Ricci-flat ambient metrics.

With respect to the normal form (2.4), the components of $\operatorname{Ric}(\tilde{g})$ may be computed in terms of the Levi-Civita connection $\nabla$ and the

Ricci tensor $R_{ij}$ of the unknown metric $g(x^k,\varrho) = g_{ij}(x^k,\varrho)\,dx^i dx^j$. Equations (2.2) and (2.3) then read (see [12, Eq. 3.17])

(2.5a)

$$\rho g''_{ij} - \rho g^{kl} g'_{ik} g'_{jl} + \tfrac{1}{2}\rho g^{kl} g'_{kl} g'_{ij} - \tfrac{n-2}{2} g'_{ij} - \tfrac{1}{2} g^{kl} g'_{kl} g_{ij} + R_{ij} = O(\rho^m),$$

$$g^{kl}\left(\nabla_k g'_{il} - \nabla_i g'_{kl}\right) = O(\rho^m), \tag{2.5b}$$

$$g^{kl} g''_{kl} + \tfrac{1}{2} g^{kl} g^{pq} g'_{pk} g'_{ql} = O(\rho^m). \tag{2.5c}$$

Here $m = \infty$ when $n$ is odd and $m = \frac{n-2}{2}$ when $n$ is even and the prime denotes the partial derivative of $g_{ij}$ with respect to $\rho$. Henceforth, we shall refer to these equations as the ***Fefferman–Graham equations***.

We remark that if $g_0$ is an Einstein metric, that is, $Ric(g_0) = \Lambda g_0$, then an ambient metric is given by

$$\tilde{g} = 2dtd(\varrho t) + t^2\Big(1 + \frac{\Lambda\varrho}{2(n-1)}\Big)^2 g_0. \tag{2.6}$$

**2.2. Linear Fefferman–Graham Equations.** The Fefferman–Graham equations are a second order system of PDE for the components of $g(x,\rho)$. However, since they have a singular point at $\varrho = 0$, the first $\varrho$-derivative of $g(x,\varrho)$ at $\varrho = 0$ cannot be chosen freely. Indeed, it is determined that

$$g'|_{\varrho=0} = 2\,P, \quad \text{where} \quad P = \frac{1}{n-2}\Big(\operatorname{Ric}(g_0) \;-\; \frac{\operatorname{S}(g_0)}{2(n-1)}\; g_0\Big) \tag{2.7}$$

is the Schouten tensor of the metric $g_0$ and $S$ the scalar curvature. This implies that

$$g(x,\varrho) = g_0(x) + \varrho P + O(\varrho^2).$$

In our early work on explicit solutions to the Fefferman–Graham equations [19, 20], the Schouten tensor of $g_0$ enjoyed some striking properties. As a (1,1) tensor, $P^2 = 0$ and the directional covariant derivatives $\nabla_Y P = 0$ for all directions $Y$ in the image of P. This suggested that we consider the ansatz

$$g(x,\varrho) = g_0(x) + h(x,\varrho),$$

where $h$ has the same aforementioned properties of $P$. This ansatz resulted in inhomogeneous linear Fefferman–Graham equations for $h(x,\varrho)$ although, at that time, it was unclear precisely why this was the case. These linear Fefferman–Graham equations were themselves particularly simple in form and exact solutions could be found. Shortly thereafter, we observed that the Schouten tensor in these examples was divergence-free and that our ansatz also enjoyed this property.

Based upon this preliminary work, a detailed study of the algebraic structure of the Fefferman–Graham equations was undertaken [2]. It is clear that if $h$ is two-step nilpotent, then the left-hand side of (2.5c) vanishes identically. If, in addition, $h$ is divergence-free, then a calculation of modest length shows that the left-hand side of (2.5b) also vanishes identically. Consequently, *the question of when the Fefferman–Graham equations* (2.5) *reduce to second order, inhomogeneous linear equations for* $h(x,\varrho)$ *reduces to the question of when the Ricci tensor* $\mathrm{Ric}(g_0)$ *is affine linear in* $h$. One answer to this question is given in Section 3 of [2] and is re-stated here.

**Theorem 2.1.** *Let* $g_0(x)$ *be a semi-Riemannian metric and let*

$$\tilde{g} = 2\,dt\,d(\varrho t) + t^2(g_0 + h(x,\varrho)), \quad h|_{\varrho=0} = 0 \tag{2.8}$$

*be an associated pre-ambient metric. Suppose that*

- **[i]** *$h$ is 2-step nilpotent and that there is a totally null distribution $\mathcal{N}$ with* $\mathrm{im}(h) \subset \mathcal{N}$*;*
- **[ii]** *the distribution $\mathcal{N}^\perp$ is integrable; and*
- **[iii]** *$h$ is divergence free.*

*Then the Fefferman–Graham equations* (2.5) *reduce to the system*

$$\varrho\ddot{h}_{ij} - (\frac{n}{2}-1)\dot{h}_{ij} - \frac{1}{2}\overset{0}{\Box}h_{ij} + \overset{0}{R}{}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}{}^k{}_{(i}h_{j)k} + \overset{0}{R}_{ij} + Q^{(2)}_{ij}(h) = 0, \tag{2.9}$$

*where* $\overset{0}{\Box}h_{ij} = \overset{0}{\nabla}{}^k\overset{0}{\nabla}_k h_{ij}$*, and where the quadratic terms in $h$ are*

$$\begin{aligned} Q^{(2)}_{ij}(h) \;=\; & \tfrac{1}{2}h^{kl}\overset{0}{\nabla}_k\overset{0}{\nabla}_l h_{ij} - h^{kp}h^l{}_{(i}\overset{0}{R}_{j)klp} + \\ & + \overset{0}{\nabla}_k h_{l(i}\overset{0}{\nabla}_{j)}h^{kl} - \tfrac{1}{4}\overset{0}{\nabla}_i h^{kl}\overset{0}{\nabla}_j h_{kl} - \overset{0}{\nabla}_{[k}h_{l]i}\overset{0}{\nabla}{}^k h_j{}^l. \end{aligned}$$

*Furthermore, if*

- **[iv]** $\overset{0}{\nabla}_Z Y \in \mathcal{N}$ *for all* $X, Y \in \mathcal{N}$*;*
- **[v]** $\overset{0}{\nabla}_X Y \in \mathcal{N}^\perp$ *for all* $X \in TM$ *and* $Y \in \mathcal{N}$*; and*
- **[vi]** $\mathcal{L}_Y h = 0$ *for all* $Y \in \mathcal{N}$*;*

*then the quadratic terms* $Q^{(2)}_{ij}(h)$ *vanish and the Fefferman–Graham equations* (2.9) *further reduce to the inhomogeneous linear system of PDE*

$$\varrho\ddot{h}_{ij} - (\frac{n}{2}-1)\dot{h}_{ij} - \frac{1}{2}\overset{0}{\Box}h_{ij} + \overset{0}{R}{}^k{}_{ij}{}^l h_{kl} + \overset{0}{R}{}^k{}_{(i}h_{j)k} + \overset{0}{R}_{ij} = 0. \tag{2.10}$$

Perhaps it is not too surprising that such a long list of conditions would be needed to arrive at inhomogeneous linear Fefferman–Graham equations. It is, therefore, quite interesting that these conditions can all be satisfied in the various nontrivial settings presented in Sections 3–5.

For the applications of Theorem 2.1 that we shall give in this paper, it will be useful to express the hypothesis of this theorem in terms of a null frame $\{e_i\}$, with dual basis $\{\Theta^i\}$, for the metric $g_0$. With index conventions

$$i, j, k, \ldots \in \{1, \ldots, n\}, \qquad a, b, c, \ldots \in \{1, \ldots, p\},$$
$$A, B, C, \ldots \in \{p+1, \ldots, n-p\}, \quad \bar{a}, \bar{b}, \bar{c} \ldots \in \{n-p+1, \ldots, n\},$$

the metric for this frame is

$$g_0 = \overset{0}{g}_{ij}\Theta^i\Theta^j = 2\overset{0}{g}_{a\bar{c}}\Theta^a\Theta^{\bar{c}} + \overset{0}{g}_{AB}\Theta^A\Theta^B. \tag{2.11}$$

Let

$$h = h_{\bar{a}\bar{c}}\,\Theta^{\bar{a}}\Theta^{\bar{c}} \quad \text{so that} \quad h^\sharp = h_{\bar{a}}{}^b\Theta^{\bar{a}} \otimes \mathrm{e}_b. \tag{2.12}$$

Then $h$ is 2-step nilpotent,

$$\mathcal{K}^\perp = \mathcal{N} = \{e_a\} = \{e_1, e_2, \ldots, e_p\} \tag{2.13}$$

is a total null distribution and

$$\mathcal{K} = \mathcal{N}^\perp = \{e_a, e_A\} = \{e_1, e_2, \ldots, e_{n-p}\}. \tag{2.14}$$

Theorem 2.1 now implies that the Fefferman–Graham equations for the pre-ambient metric $g_0$ are homogeneous linear PDE for $h$ provided that **[i]** the components of $h$ are invariant along $\mathcal{N}$, that is,

$$\mathcal{L}_{e_a}(h_{\bar{a}\bar{c}}) = 0, \tag{2.15a}$$

and **[ii]** the following structure equations for the null frame $\{e_i\}$ hold:

$$\begin{array}{rrcl} \mathbf{S_1}: & [e_a, e_b] & = & 0, \\ \mathbf{S_2}: & [e_a, e_A] & \in & \mathcal{K}, \\ \mathbf{S_3}: & [e_a, e_{\bar{b}}] & \in & \mathcal{K}, \\ \mathbf{S_4}: & [e_A, e_B] & \in & \mathcal{K}. \end{array} \tag{2.15b}$$

See [2] for details. It will be a simple matter to verify the conditions (2.15) in the examples which we consider in Sections 3–5.

**2.3. Conformal geometries associated to Pfaffian systems.** A Pfaffian system on a manifold $M$ is a collection of 1-forms $\mathcal{I} \subset \Omega^1(M)$. One assumes that $\mathcal{I}$ is regular in the sense that there is a sub-bundle $I \subset T^*M$ such that, locally, $\mathcal{I}$ coincides with the sections of $I$. Similarly, a regular vector distribution is a collection of vector fields $\mathcal{D} \subset \mathcal{X}(M)$ defined as the local sections of a sub-bundle of $TM$. The annihilator of a regular Pfaffian system is a regular vector distribution and conversely.

Two Pfaffian systems $(\mathcal{I}, M)$ and $(\mathcal{J}, N)$ are said to be locally equivalent if about each point in $M$ there is an open set $U \subset M$; an open set $V \subset N$; and a local diffeomorphism $\phi : U \to V$ such that $\mathcal{I}|_U = \phi^*(\mathcal{J}|_V)$. The equivalence problem for Pfaffian systems is to determine a set of local invariants of $\mathcal{I}$ which characterize $\mathcal{I}$ up to local equivalence. The Cartan equivalence method [8, 13, 25] is a systematic approach to the solution of the equivalence problem. For certain classes of Pfaffian systems (those whose symbol algebras coincided with the nilpotent part of a graded semi-simple Lie algebra) the equivalence problem can also be studied from the viewpoint of parabolic geometry [7].

In [23], Nurowski showed how certain, rather special, classes of differential equations (or their related Pfaffian systems $\mathcal{I}$) define various conformal structures. The curvature invariants of the associated Fefferman–Graham metrics are, thus, diffeomorphism invariants of $\mathcal{I}$. Accordingly, in these situations, one then has a very novel approach to the equivalence problem. Moreover, the Fefferman–Graham metrics for these conformal structures can often be explicitly computed by the method of the previous section. Consequently, the Nurowski construction also provides new insights into the Fefferman–Graham ambient metric construction.

Surely the most interesting example of the conformal structures in [23] arises in the study of generic rank 3 Pfaffian systems on a 5 dimensional manifold $M$—for such systems there is a local coframe $\{\theta^1, \theta^2, \theta^3, \theta^4, \theta^5\}$ with $\mathcal{I} = \text{span}\{\theta^1, \theta^2, \theta^3\}$ and with structure equations

$$\begin{aligned} d\,\theta^1 &\equiv \theta^3 \wedge \theta^4 \quad \text{mod } \{\theta^1, \theta^2\}, \\ d\,\theta^2 &\equiv \theta^3 \wedge \theta^5 \quad \text{mod } \{\theta^1, \theta^2\}, \\ d\,\theta^3 &\equiv \theta^4 \wedge \theta^5 \quad \text{mod } \{\theta^1, \theta^2, \theta^3\}. \end{aligned}$$

The structure group for these equations is the 12-parameter group

$$A = \begin{bmatrix} \delta t_1 & \delta t_2 & 0 & 0 & 0 \\ \delta t_3 & \delta t_4 & 0 & 0 & 0 \\ t_5 & t_6 & \delta & 0 & 0 \\ t_7 & t_8 & t_9, & t_1 & t_2 \\ t_{10} & t_{11} & t_{12} & t_3 & t_4 \end{bmatrix}, \quad \delta = t_1 t_4 - t_2 t_3. \tag{2.16}$$

By employing either the Cartan approach or the parabolic geometry approach one then constructs a 12-dimensional bundle $\pi : P \to M$ with a local co-frame $\{\Theta^i, \pi^i\}$ such that $\text{span}\{\theta^i\} = \text{span}\{\Theta^i\}$. The structure equations are

$$\begin{aligned} d\,\Theta^1 &= \Theta^3 \wedge \Theta^4 + \Theta^1 \wedge (2\pi^1 + \pi^4) + \Theta^2 \wedge \pi^2, \\ d\,\Theta^2 &= \Theta^3 \wedge \Theta^5 + \Theta^1 \wedge \pi^3 + \Theta^2 \wedge (\pi^1 + 2\pi^4), \\ d\,\Theta^3 &= \Theta^4 \wedge \Theta^5 + \Theta^1 \wedge \pi^5 + \Theta^2 \wedge \pi^6 + \Theta^3 \wedge (\pi^1 + \pi^4), \\ d\,\Theta^4 &= \Theta^1 \wedge \pi^7 + 4/3\pi^3 \wedge \pi^6 + \Theta^4 \wedge \pi^1 + \Theta^5 \wedge \pi^2, \\ d\,\Theta^5 &= \Theta^4 \wedge \Theta^5 + \Theta^2 \wedge \pi^7 - 4/3\Theta^3 \wedge \pi^5 + \Theta^4 \wedge \pi^3 + \Theta^5 \wedge \pi^4. \end{aligned} \tag{2.17}$$

One finds the structure group for (2.17) to be a 9-parameter matrix group of the form

$$\begin{bmatrix} A & 0 \\ B_{7\times 5} & C_{7\times 7} \end{bmatrix}. \tag{2.18}$$

The matrix A is now given by (2.16) with

$$t_7 = \frac{t_1 t_8}{t_2} - \frac{2}{3}\frac{(t_1 t_6 - t_2 t_5)^2}{t_2 \delta^2}, \qquad t_9 = \frac{4}{3}\frac{(t_3 t_6 - t_4 t_5)}{\delta},$$
$$t_{10} = \frac{t_3 t_8}{t_2} - \frac{2}{3}\frac{t_3 t_6^2}{t_2 t_4 \delta} - \frac{2}{3}\frac{(t_3 t_6 - t_4 t_5)^2}{t_4 \delta^2}, \qquad t_{11} = \frac{t_4 t_8}{t_2} - \frac{2}{3}\frac{t_6^2}{\delta t_2},$$
$$t_{12} = \frac{4}{3}\frac{t_3 t_6 - t_4 t_5}{\delta}.$$

The form of the diagonal blocks of the matrix A are reminiscent of the formula for the standard representation of SO(3,2) (preserving the symmetric matrix with 1's along the anti-diagonal). Based on this observation one readily deduces that the structure group preserves the conformal class of the metric

$$g_{\mathcal{I}} = 2\Theta^1\Theta^5 - 2\Theta^2\Theta^4 + \frac{4}{3}\Theta^3\Theta^3. \tag{2.19}$$

Therefore, *the conformal class* $[g_{\mathcal{I}}]$ *is an invariant of the Pfaffian system* $\mathcal{I}$.

A classical result of Goursat gives a local normal form for the generic rank 3 Pfaffian system $\mathcal{I} = \{\theta^1, \theta^2, \theta^3\}$. Specifically, $\mathcal{I}$ is always locally equivalent to the standard Pfaffian system for the Monge equation

$$\frac{dz}{dx} = F(x, y, z, \frac{dy}{dx}, \frac{d^2y}{dx^2}),$$

and, therefore, one can always introduce local coordinates $(x, y, p, q, z)$ such that

$$\theta^1 = dy - p\,dx, \quad \theta^2 = dp - q\,dx, \quad \theta^3 = dz - F(x, y, z, p, q)dx.$$

Explicit formulas for the co-frame $\Theta^i$ satisfying (2.17) are given in [24].

Bryant [6] has shown that similar results hold for generic rank 3 distributions in 6 dimensions, that is, to each such distribution one can associate an invariantly defined conformal class. More details regarding the Bryant conformal class are given in Section 5.

The **DG** software commands ***StructureGroup*** and ***MatrixSubgroup*** can be used to verify the above statements regarding the structure groups (2.18). The commands ***NurowskiConformalClass*** and ***BryantConformalClass*** are used to calculate the conformal classes for specific distributions.

**2.4. Symbolic computation of infinitesimal holonomy.** Let $g$ be a semi-Riemannian metric on an $n$-dimensional manifold $M$ with Christoffel connection $\nabla$ and curvature tensor $R$. In this section, we briefly discuss the symbolic tools available in the **DG** package for the computation of the infinitesimal holonomy algebra $\mathfrak{hol}(g)$. These tools are sufficiently robust to fully analyze the infinitesimal holonomy in most situations.

Our starting point is the Ambrose–Singer theorem [5]. This theorem asserts that the infinitesimal holonomy algebra at a point $x_0 \in M$ is spanned by the curvature tensor $R(U,V)$ and its iterated directional covariant derivatives $(\nabla_X R)(U,V)$, $(\nabla_Y \nabla_X R)(U,V), \ldots$, when evaluated at $x_0$ and viewed as endomorphisms of the tangent space $T_{x_0}M$. To render this theorem as a symbolic algorithm one needs an efficient way to calculate these higher order derivatives of curvature and a proper stopping criteria. Tools which can give an independent verification of the results are essential.

Fix a local frame $\{X_i, i = 1 \ldots n\}$ for $M$. Let $\mathcal{H}_0 = \{H_{01}, H_{02}, \ldots H_{0r_1}\}$ be a basis for the curvature endomorphisms $R(X_i, X_j)$, $1 \leq i < j \leq n$. Set $\mathcal{H}_1 = \{\}$. Now calculate the directional covariant derivatives $J = \nabla_{X_i}(H_{0j})$ of the tensors in $\mathcal{H}_0$ one-by-one. As one proceeds, determine if $J$ is linearly independent from the 1-1 tensors in the set $\mathcal{H}_0 \cup \mathcal{H}_1$. If so, then redefine $\mathcal{H}_1$ as $\mathcal{H}_1 \cup J$. Next generate a new list $\mathcal{H}_2$ from the linearly independent directional covariant derivatives of $\mathcal{H}_1$. The process terminates when no new linearly independent 1-1 tensors are found. The totality of 1-1 tensors $\mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cdots$ found in this way will define a basis for the infinitesimal holonomy algebra at a generic point.

These calculations involve only basic linear algebra computations and differentiations and so are well-suited for symbolic computation. They are implemented in **DG** with the command ***InfinitesimalHolonomy***. If the coefficients of the curvature endomorphisms are all rational functions of the coordinates, then the software will correctly decide if a given curvature endomorphism belongs to the span of those previously computed. In the presence of fractional powers, exponentials or trigonometric functions in the coefficients this membership question is best resolved at a given base point $x_0$. Should the base point not be generically chosen the infinitesimal holonomy calculation may terminate prematurely. This problem will be detected in the next step of our symbolic computations. We remark that if a 1-1 tensor $J$ is found to be linearly independent from $\mathcal{H}_i$ at $x_0$, then it is independent in a neighborhood of $x_0$.

The next step is to calculate the parallel vectors, parallel symmetric tensors of rank 2 and parallel differential forms. While the internal Maple PDE solver ***pdsolve*** excels at solving over-determined systems of linear differential equations, it is usually overwhelmed by the large number of defining differential equations for a parallel tensor. This difficulty is easily overcome with the **DG** software. Suppose, for example, that one wishes to calculate the parallel $p$-forms for a fixed value of $p$. At each point $x \in M$ the infinitesimal holonomy algebra acts on the bundle of exterior forms $\Lambda^p_x(M)$ as derivations. Let $\Lambda^p_{\mathrm{inv}}(M) \subset \Lambda^p(M)$ be the sub-bundle of infinitesimal holonomy invariant $p$-forms. It is a fundamental theorem that any parallel p-form is a section of $\Lambda^p_{\mathrm{inv}}(M)$.

The bundle $\Lambda^p_{\rm inv}(M)$ is easily calculated in **DG** with the command ***InvariantTensors***. We emphasize that this requires the holonomy algebra be determined at each point $x$ and this is exactly what ***InfinitesimalHolonomy*** accomplishes. The output of this command is then passed to ***ParallelTensors*** to explicitly calculate the parallel forms. Of course, it is a simple matter to use **DG** to verify directly that the parallel forms so obtained are, indeed, parallel.

Once the parallel tensor fields $\mathcal{T}$ are determined, the command ***MatrixSubalgebra*** will calculate the sub-algebra $\mathrm{stab}(\mathcal{T})$ of $\mathfrak{gl}(n)$ which leaves these parallel tensors infinitesimal invariant. The infinitesimal holonomy algebra and this stabilizer sub-algebra (all calculated at an arbitrary point) must coincide—this insures the correctness of the holonomy computations. Another check is provided by the command ***LieAlgebraData*** which verifies that the holonomy endomorphisms define a Lie algebra. In the future, the command ***HolonomyAdvisor*** will match the results of these computations (in the case of Riemannian metrics) against the Berger classification.

## 3. Generalized pp-waves: ambient metrics and their holonomy

**3.1. Fefferman–Graham equations for generalized pp-waves.** In this section, we provide explicit formulae for ambient metrics for conformal structures on certain pseudo-Riemannian manifolds. In Lorentzian signature, these are known as pp-waves. The formulae presented here generalize our results on conformal pp-waves in [18].

Let $1 \leq p \leq n/2$ be a natural number, let

$$\mathcal{U} \subset \mathbb{R}^n \ni (x^1, \dots, x^{n-2p}, u^1, \dots, u^p, v^1, \dots, v^p) \tag{3.1}$$

be an open set, and $H_{ac} = H_{ca}$ smooth functions on $\mathcal{U}$ satisfying

$$\frac{\partial}{\partial v^b}(H_{ac}) = 0. \tag{3.2}$$

Then we define a ***generalized pp-wave*** in signature $(p, n-p)$ (for short pp-wave) as

$$g_H = 2du^a(\delta_{ab}dv^b + H_{ab}du^b) + \delta_{AB}dx^A dx^B. \tag{3.3}$$

Here and in what follows the indices $A, B, C, \ldots \in \{1, \ldots n-2p\}$ and $a, b, c, \ldots \in \{1, \ldots, p\}$. For $p = 1$ these are the usual Lorentzian pp-waves.

For the metric (3.3) we have the adapted co-frame

$$\Theta^a = du^a, \quad \bar{\Theta}^b = dv^b + H_a{}^b du^a, \quad \Theta^A = du^a,$$

with dual frame

$$e_a = \frac{\partial}{\partial v^a}, \quad \bar{e}_b = \frac{\partial}{\partial u^b} - H_b{}^c \frac{\partial}{\partial v^c} \quad E_A = \frac{\partial}{\partial u^A}.$$

The distribution $\mathcal{N} = \mathrm{span}\{e_a\}$ is totally null and $\mathcal{N}^\perp = \mathrm{span}\{e_a, E_A\}$ (our indexing conventions here are slightly different from those in Section 2.2). The relevant structure equations are

$$[e_a, e_b] = 0, \quad [e_a, \bar{e}_b] = 0, \quad [e_a, e_A] = 0, \quad [e_A, e_B] = 0.$$

With $h = h_{ab}\,\bar{\theta}^a\,\bar{\theta}^b$ and $h_{ab} = h_{ab}(\varphi, x^A, u^c)$, we conclude that the generalized pp-wave metric (3.3) satisfies the conditions of Theorem 2.1. The Fefferman–Graham equations, therefore, reduce to the inhomogeneous linear PDE (2.10) which, in this case, become

$$2\varrho h'' + (2-n)h' - \Delta(H+h) = 0. \tag{3.4}$$

In the next section we provide a general existence result for these equations.

**3.2. Analytic and non-analytic ambient metrics for pp-waves.** In [18] we have seen how equation (3.4) can be solved by standard power series expansion, noting that its indicial exponents are $s = 0$ and $s = n/2$. Here we give a more general existence result which includes non-analytic solutions. This result reflects the non-uniqueness of the ambient metric in even dimensions and explicitly shows the Fefferman–Graham ambiguity at order $n/2$.

**Theorem 3.1.** *Let $\Delta = \delta^{AB}\partial_A\partial_B$ be the flat Laplacian in $(n-2p)$ dimensions and $H = H(x^A, u^b)$.*

**[i]** *When $n$ is odd, the most general solutions $h = h(\varrho, x^A, u^b)$ to (3.4) with $h(\varrho) \to 0$ as $\varrho \downarrow 0$ are*

$$h = \sum_{k=1}^{\infty} \frac{\Delta^k H}{k!\prod_{i=1}^{k}(2i-n)}\varrho^k + \varrho^{n/2}\Big(\alpha + \sum_{k=1}^{\infty}\frac{\Delta^k\alpha}{k!\prod_{i=1}^{k}(2i+n)}\varrho^k\Big). \tag{3.5}$$

*Here $\alpha = \alpha(x^A, u^b)$ is an arbitrary smooth function of its variables. In particular, when $H$ is analytic, a unique solution that is analytic in $\varrho$ in a neighborhood of $\varrho = 0$ with $h(0) = 0$ is given by $\alpha \equiv 0$.*

**[ii]** *When $n = 2s$ is even, the most general solutions $h$ to equation (3.4) with $h(\varrho) \to 0$ when $\varrho \downarrow 0$ are*

$$\begin{aligned} h \;=\; & \textstyle\sum_{k=1}^{s-1} \frac{\Delta^k H}{k!\prod_{i=1}^{k}(2i-n)}\varrho^k + \varrho^s\Big(\alpha + \sum_{k=1}^{\infty}\frac{\Delta^k\alpha}{k!\prod_{i=1}^{k}(2i+n)}\varrho^k\Big) \\ & + c_n\varrho^s\Big(\textstyle\sum_{k=0}^{\infty}(\log(\varrho) - q_k)\frac{\Delta^{s+k}H}{k!\prod_{i=1}^{k}(2i+n)}\varrho^k\Big) \\ & + c_n\varrho^s Q * \textstyle\sum_{k=0}^{\infty}\frac{\Delta^{s+k}H}{k!\prod_{i=1}^{k}(2i+n)}\varrho^k, \end{aligned} \tag{3.6}$$

*where $\alpha = \alpha(x^A, u^b)$ and $Q = Q(x^A, u^b)$ are arbitrary smooth functions of their variables, $*$ denotes the convolution of two functions* *with respect to the $x^A$-variables, and the constants are*

$$c_n = -\frac{1}{(s-1)!\prod_{i=0}^{s-1}(2i-n)}, \quad q_0 = 0, \quad q_k = \sum_{i=1}^{k}\frac{n+4i}{i(n+2i)}, \quad \text{for } k=1,2,\ldots. \tag{3.7}$$

*In particular, only when $\Delta^s H \equiv 0$ are there solutions that are analytic in $\varrho$ in a neighborhood of $\varrho = 0$ and with $h(0) = 0$. These solutions are not unique but are parameterized by analytic functions $\alpha$.*

*Proof.* Let $h$ be a solution to (3.4) with initial conditions $h(\varrho, x^A, u^b)_{|\varrho=0} \equiv 0$. Then $F = H + h$ is a solution to the homogeneous equation

$$\varrho F'' + (1-\tfrac{n}{2})F' - \tfrac{1}{2}\Delta F = 0, \tag{3.8}$$

with the initial condition $F(\varrho, x^A, u^b)_{|\varrho=0} = H(x^A, u^b)$. Let $\hat{F}$ and $\hat{H}$ denote the Fourier transforms with respect to the $x^A$-variables of $F$ and $H$. Recalling that $\widehat{\Delta F} = -\|x\|^2\hat{F}$, we easily find the Fourier transform of equation (3.8) to be

$$\varrho\hat{F}'' + (1-\tfrac{n}{2})\hat{F}' + \frac{\|x\|^2}{2}\hat{F} = 0, \tag{3.9}$$

with initial condition $\hat{F}(\varrho, x^A, u^b)_{|\varrho=0} = \hat{H}$. For each point $(x^A, u^b) \in \mathbb{R}^{n-p}$, which we fix for the moment, this is an ODE in $\varrho$ whose solutions can be determined by the Frobenius method (see, for example, [9, Section 6.3.3]). The roots of the indicial polynomial of (3.9) are 0 and $n/2$ so that we must distinguish the cases when $n$ is odd and $n$ is even.

For $n$ odd the general solution to (3.9) is given by

$$\hat{F} = a\varrho^{n/2}F_+ + bF_-, \tag{3.10}$$

for arbitrary constants $a$ and $b$, and with $F_1$ and $F_2$ given by

$$F_\pm(x^A, \varrho) = 1 + \sum_{k=1}^{\infty}\frac{(-1)^k\|x\|^{2k}}{k!\prod_{i=1}^{k}(2i \pm n)}\varrho^k. \tag{3.11}$$

Hence, when we vary $(x^A, u^b)$ we get that the general solution to (3.9) is given by

$$\hat{F} = \alpha\varrho^{n/2}F_+ + \beta F_-, \tag{3.12}$$

where $\alpha = \alpha(x^A, u^b)$ and $\beta = \beta(x^A, u^b)$ are arbitrary functions of their variables. The initial conditions $\hat{F}|_{\varrho=0} = \hat{H}$, or more precisely $\hat{F} \to \hat{H}$ if $\varrho \downarrow 0$, imposes $\beta = \hat{H}$. Hence, we obtain

$$\hat{F} = \sum_{k=0}^{\infty}\frac{\widehat{\Delta^k H}}{k!\prod_{i=1}^{k}(2i-n)}\varrho^k + \sum_{k=0}^{\infty}\frac{\widehat{\Delta^k \check{\alpha}}}{k!\prod_{i=1}^{k}(2i+n)}\varrho^k,$$

taking into account that $(-1)^k\|x\|^{2k}\hat{H} = \widehat{\Delta^k H}$ and denoting by $\check{\alpha}$ the inverse Fourier transform of $\alpha$. Applying the inverse Fourier transform to this expression gives us $F$, and, hence, the formula (3.5) for $h$ when $n$ is odd.

If $n = 2s$ is even, the general solution to (3.9) is given by

$$\hat{F} = \alpha\varrho^s F_+ + \beta\Big(c_n(-1)^s\|x\|^n\varrho^s\log(\varrho)F_+ + F_{c_n}\Big), \tag{3.13}$$

where $\alpha = \alpha(x^A, u^b)$ and $\beta = \beta(x^A, u^b)$ are arbitrary functions of their variables, $c_n$ is the constant defined in the statement of the theorem, $F_+$ is defined in (3.11) and $F_{c_n}$ can be computed as

$$\begin{aligned} F_{c_n} &= 1 + \sum_{k=1}^{s-1}\frac{(-1)^k\|x\|^{2k}}{k!\prod_{i=1}^{k}(2i-n)}\varrho^k \\ &\quad - c_n(-1)^s\|x\|^n\varrho^s\left(\sum_{k=1}^{\infty}\frac{(-1)^k\|x\|^{2k}}{k!\prod_{i=1}^{k}(2i+n)}\hat{q}_k\varrho^k\right). \end{aligned}$$

Here $\hat{q}_0 = \hat{q}_0(x^A, u^b)$ is an arbitrary function of the $x^A$'s and the $u^c$'s and the remaining $\hat{q}_k$ are defined as

$$\hat{q}_k = \hat{q}_0 + \sum_{i=1}^{k}\frac{n+4i}{i(n+2i)}, \quad \text{for } k = 1, 2, \dots.$$

Again, the initial condition $\hat{F} \to \hat{H}$ when $\varrho \downarrow 0$ imposes $\beta = \hat{H}$. Applying the inverse Fourier transform to our solutions $\hat{F}$, and taking into account that $(-1)^k\|x\|^{2k}\hat{H} = \widehat{\Delta^k H}$ and that the convolution satisfies $\widehat{Q * \Delta H} = 2\pi^{\frac{n-2}{2}}\widehat{Q}\ \widehat{\Delta H}$, yields the formula in the theorem with the function $Q$ in the theorem given by the inverse Fourier transform of $2\pi^{\frac{n-2}{2}}\hat{q}_0$. q.e.d.

For $n$ odd, the solutions $h_{ac}$ given in Theorem 3.1 by analytic functions $H_{ac}$ defining the pp-wave metric $g_H$ in (3.3) provide us with the unique analytic ambient metric

$$\widetilde{g}_H = 2d(\varrho t)dt + t^2\Big(g + 2\Big(\sum_{k=1}^{\infty}\frac{\Delta^k H_{ac}}{k!\prod_{i=1}^{k}(2i-n)}\varrho^k\Big)du^a du^c\Big), \tag{3.14}$$

for the conformal class of $g_H$. Theorem 3.1 also shows that, when $n$ is odd, the ambiguity introduced by the arbitrary function $\alpha$ gives only *non-analytic* solutions, as guaranteed by the uniqueness statement in the Fefferman–Graham result. In contrast, when $n$ is even, the ambiguity coming from the function $\alpha$ adds an analytic part to a solution and, in case of $\Delta^{n/2}H_{ac} = 0$, gives new *analytic* ambient metrics.

We conclude this section by giving an example of a Lorentzian pp-wave (i.e., with $p = 1$), which, in even dimensions, does not satisfy the sufficient condition $\Delta^{n/2}H = 0$ for our ansatz to give an analytic ambient metric, but which, however, admits non-analytic ambient metrics.

**Example 3.1.** For $k \in \mathbb{R}$ we consider the $n$-dimensional Lorentzian pp-wave

$$g = 2dudv + \mathrm{e}^{2kx^1}du^2 + \sum_{A=1}^{n-2}(dx^A)^2, \tag{3.15}$$

that is, with $H = \frac{1}{2}\mathrm{e}^{2kx^1}$. This metric is a product of a 3-dimensional pp-wave and the flat Euclidean space of dimension $n-3$. To exclude the flat case, we assume $k \neq 0$ from now on. Note that, when $n$ is even, this metric does not satisfy the condition $\Delta^{n/2}H = 0$, which ensures the existence of an analytic ambient metric. However, to find an ambient metric (non-analytic when $n$ is even) we make the ansatz with $h = \frac{1}{2}\left(\phi(\varrho) - 1\right)\mathrm{e}^{2kx^1}$ for a function $\phi$ of $\varrho$ with $\phi(0) = 1$ and set

$$\begin{aligned} \widetilde{g} &= 2d(\varrho t)dt + t^2(g + 2hdu^2) \\ &= 2d(\varrho t)dt + t^2\left(2dudv + \phi(\varrho)\mathrm{e}^{2kx^1}du^2 + \sum_{i=1}^{n-2}(dx^i)^2\right). \end{aligned}$$

According to Theorem 2.1 this metric is Ricci flat if

$$2\varrho\phi'' + (2-n)\phi' - 4k^2\phi = 0,$$

and the initial condition is $\phi(0) = 1$. Of course, this equation can be solved using the power series techniques in Theorem 3.1 exhibiting the difference between $n$ odd and even. But, for example, when $n = 4$ this equation becomes

$$\varrho\phi'' - \phi' - 2k^2\phi = 0.$$

The general solution with $\phi(0) = 1$ is given in closed form using the modified Bessel functions $I_2$ and $K_2$ as

$$\phi(\varrho) = 4k^2\varrho K_2(2k\sqrt{2\varrho}) + C\varrho I_2(2k\sqrt{2\varrho}),$$

where $C$ is an arbitrary constant. This solution is not analytic at $\varrho = 0$, since the function $\varrho \mapsto \varrho K_2(2k\sqrt{2\varrho})$ fails to have a bounded second derivative at $\varrho = 0$. The function $\varrho \mapsto \varrho I_2(2k\sqrt{2\varrho})$ is analytic and the constant $C$ is a remnant of the non-uniqueness. In contrast, when $n = 5$ the equation becomes

$$\varrho\phi'' - \frac{3}{2}\phi' - 2\phi = 0.$$

The general solution with $\phi(0) = 1$ is given by

$$\begin{aligned} \phi(\varrho) &= \cosh(2k\sqrt{2\varrho}) - 2k\sqrt{2}\sinh(2k\sqrt{2\varrho}) + \tfrac{8k^2}{3}\,\varrho\cosh(2k\sqrt{2\varrho}) \\ &\quad + 3\,C\left(\sinh(2k\sqrt{2\varrho}) - 2k\sqrt{2\varrho}\cosh(2k\sqrt{2\varrho}) + \tfrac{8k^2}{3}\varrho\sinh(2k\sqrt{2\varrho})\right), \end{aligned}$$

with an arbitrary constant $C$. The unique analytic solution is obtained by $C = 0$.

**3.3. On the ambient holonomy for conformal pp-waves.** In this and some of the following sections, we will determine the holonomy algebras of semi-Riemannian manifolds $(\widetilde{M}, \widetilde{g})$ at some point $p \in \widetilde{M}$. However, since the holonomy algebras at different points are conjugated to each other in $\mathbf{SO}(r, s)$, we will not make the point explicit in our notation. For a parallel tensor field $\Phi$, i.e., with $\widetilde{\nabla}\Phi = 0$, we will use the terminology *stabilizer of* $\Phi$ by which we mean all linear maps $H \in \mathfrak{so}(T_p\widetilde{M}, g_p)$ that act trivially on $\Phi|_p$, i.e., $H{\cdot}\Phi|_p = 0$. If $\widetilde{\nabla}\Phi = 0$, then the holonomy algebra is contained in the stabilizer of $\Phi$. Moreover, we say that a distribution $\mathcal{V}$ is parallel if it is invariant under parallel transport, or equivalently if $\widetilde{\nabla}_X V \in \Gamma(\mathcal{V})$ for all $X \in T\widetilde{M}$ and all $V \in \Gamma(\mathcal{V})$. The *stabilizer of* $\mathcal{V}$ consists of all linear maps in $\mathfrak{so}(T_p\widetilde{M}, g_p)$ that leave $\mathcal{V}|_p$ invariant. Again, if there is a parallel distribution, then the holonomy algebra is contained in its stabilizer.

To describe the possible holonomy for ambient metrics of conformal pp-waves, we first note that a pp-wave $(M, g)$ as defined in (3.3) admits a parallel null distribution $\mathcal{V}$ spanned by the parallel vector fields $\frac{\partial}{\partial v^a}$, for $a = 1, \dots, p$. Moreover, the image of the Ricci endomorphism is contained in $\mathcal{V}$. Then, on the other hand, the results in [20, Theorem 1.1], in the case $p = 1$, and the generalization in [22, Theorem 1] imply that the normal conformal tractor bundle admits a totally null subbundle of rank $p + 1$ that is parallel for the normal conformal tractor connection of $[g]$, i.e., its fibres are invariant under the conformal holonomy. On the other hand, in the case of odd-dimensional analytic pp-waves, the general theory in [16] ensures that parallel objects of the tractor connection carry over to parallel objects of the analytic ambient metric. Hence, for an odd dimensional analytic pp-wave, the holonomy of the analytic ambient metric in (3.14) admits an invariant totally null plane of rank $p+1$. However, in the next theorem we strengthen this result by directly determining the holonomy of general metrics of the form (3.3). We will prove the following result in the case of $p = 1$ and in Lorentzian signature but it clearly generalizes to larger $p$ and other signatures in an obvious way.

**Theorem 3.2.** *Let* $F = F(\varrho, u, x^1, \dots, x^{n-2})$ *be a smooth function on* $\mathbb{R}^n$. *Then the metric*

$$\widetilde{g}_F = 2d(t\varrho)dt + t^2\Big(2du(dv + F\,du) + \delta_{AB}dx^A\,dx^B\Big) \tag{3.16}$$

*on* $\widetilde{M} = \mathbb{R}^{n+1} \times \mathbb{R}_{>0}$ *satisfies the following properties.*

**[i]** *The distribution* $\widetilde{\mathcal{V}} = \mathrm{span}(\partial_v, \partial_\varrho)$ *is parallel, i.e., it is invariant under parallel transport.*

**[ii]** *Let* $\mathcal{V} = \mathbb{R}\cdot\partial_v$ *be the distribution of null lines spanned by* $\partial_v$*, and* $\mathcal{V}^\perp$ *be the distribution of vectors in* $T\widetilde{M}$ *that are orthogonal to* $\partial_v$*, i.e.,* $\mathcal{V}^\perp = \mathrm{span}(\partial_v, \partial_\varrho, \partial_1, \ldots, \partial_{n-2}, \partial_t)$*. Then the curvature* $\widetilde{R}$ *of* $\widetilde{g}_F$ *satisfies* $\widetilde{R}(U,V)Y = 0$*, for all* $U, V \in \mathcal{V}^\perp$ *and* $Y \in T\widetilde{M}$*.*

**[iii]** *The holonomy algebra* $\mathfrak{hol}(\widetilde{g}_F)$ *of* $\widetilde{g}_F$ *is contained in*

$$\left(\mathfrak{sl}_2\mathbb{R} \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2})\right) \oplus \mathbb{R} = \left\{ \begin{pmatrix} X & Z & aJ \\ 0 & 0 & -Z^\top \\ 0 & 0 & -X^\top \end{pmatrix} \;\middle|\; \begin{array}{l} X \in \mathfrak{sl}_2\mathbb{R}, \\ Z \in \mathbb{R}^2 \otimes \mathbb{R}^{n-2}, \\ a \in \mathbb{R} \end{array} \right\},$$

*where* $J$ *is the* $2\times 2$*-matrix* $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

*Proof.* Let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{g}_F$. Then direct computation reveals that

(3.17)

$$\begin{aligned} \widetilde{\nabla}\partial_v &= \tfrac{1}{t}\partial_v \otimes dt - \partial_\varrho \otimes du, \\ \widetilde{\nabla}\partial_\varrho &= \tfrac{1}{t}\partial_\varrho \otimes dt + F_\varrho\,\partial_v \otimes du, \\ \widetilde{\nabla}\partial_A &= \tfrac{1}{t}\partial_A \otimes dt + F_A\,\partial_v \otimes du - \partial_\varrho \otimes dx^A, \quad A = 1, \ldots, n-2, \end{aligned}$$

in which $F_A = \partial_A(F)$, and $F_\varrho = \partial_\varrho(F)$. The first two equations show that the distribution $\widetilde{\mathcal{V}}$ is invariant under parallel transport, and also that, in general, there is no invariant null line in $\widetilde{\mathcal{V}}$. Moreover, it allows us to show that the curvature satisfies $\widetilde{R}(X,Y)\partial_v = 0$ for all $X, Y \in T\widetilde{M}$,

$$\begin{aligned} \widetilde{R}(\partial_A, \partial_u)\partial_\varrho &= \widetilde{R}(\partial_\varrho, \partial_u)\partial_i = F_{A\varrho}\,\partial_v, \\ \widetilde{R}(\partial_\varrho, \partial_u)\partial_\varrho &= F_{\varrho\varrho}\,\partial_v, \\ \widetilde{R}(\partial_u, \partial_B)\partial_A &= -(\delta_{AB}F_\varrho + F_{AB})\,\partial_v, \end{aligned}$$

and that all other terms of the form $\widetilde{R}(X,Y)\partial_\varrho$ and $\widetilde{R}(X,Y)\partial_i$ are zero, unless the symmetry of $\widetilde{R}$ prevents this. This shows that the image of $\widetilde{\mathcal{V}}^\perp$ under $\widetilde{R}(X,Y)$ is contained in $\mathbb{R}\cdot\partial_v$. The symmetries of the curvature then imply part (ii) of the Theorem.

For the last part, first we note that, since $\widetilde{\mathcal{V}}$ is parallel, the holonomy algebra of $\widetilde{g}$ is contained in the stabilizer in $\mathfrak{so}(2,n)$ of the totally null plane $\widetilde{\mathcal{V}}$, which is equal to

$$(3.18)\qquad \begin{aligned} &(\mathfrak{gl}_2\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2} \oplus \mathbb{R}) \\ &= \left\{ \begin{pmatrix} X & Z & aJ \\ 0 & S & -Z^\top \\ 0 & 0 & -X^\top \end{pmatrix} \;\middle|\; \begin{array}{l} X \in \mathfrak{gl}_2\mathbb{R},\ a \in \mathbb{R} \\ Z \in \mathbb{R}^2 \otimes \mathbb{R}^{n-2}, \\ S \in \mathfrak{so}(n-2) \end{array} \right\}. \end{aligned}$$

Moreover, the equations (3.17) show that the 2-form $\mu = tdt \wedge du$ is parallel with respect to the Levi-Civita connection. Hence, the projection of the holonomy to $\mathfrak{gl}_2\mathbb{R}$ actually lies in $\mathfrak{sl}_2\mathbb{R}$, i.e., $X \in \mathfrak{sl}_2\mathbb{R}$. Note that $\mathbb{R}\cdot J$ commutes with all of $(\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2})$, so the holonomy reduces to

$$\Big((\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{so}(n-2)) \ltimes (\mathbb{R}^2 \otimes \mathbb{R}^{n-2})\Big) \oplus \mathbb{R}.$$

Finally, to show that the elements in the holonomy algebra have no $\mathfrak{so}(n-2)$ component, i.e., that $S = 0$ in (3.18), we fix a point $p \in \widetilde{M}$ and use the Ambrose–Singer Holonomy Theorem to show that $\mathfrak{hol}_p(\widetilde{g})$ maps $\widetilde{\mathcal{V}}^{\perp}|_p$ to $\widetilde{\mathcal{V}}|_p$. Indeed, let $V \in \widetilde{\mathcal{V}}^{\perp}|_p$ be in the fibre of $\widetilde{\mathcal{V}}^{\perp}$ at $p$. Let $V_q = \mathcal{P}_\gamma(V)$ denote the parallel transport of $V$ along a curve that ends at $q \in \widetilde{M}$. By the invariance of the distribution $\widetilde{\mathcal{V}}^{\perp}$ we have $V_q \in \widetilde{\mathcal{V}}^{\perp}|_q$, however, applying the curvature at $q$, $\widetilde{R}_q(X,Y)$ for $X, Y \in T_q\widetilde{M}$, gives

$$\widetilde{R}_q(X,Y)V_q \;\in\; \mathcal{V}|_q \;\subset\; \widetilde{\mathcal{V}}|_q,$$

by the above formulae. Since $\widetilde{\mathcal{V}}$ is also invariant under parallel transport, we obtain

$$\mathcal{P}_\gamma^{-1} \circ R(X,Y) \circ \mathcal{P}_\gamma(V) \in \widetilde{\mathcal{V}}|_p,$$

for every $V \in \widetilde{\mathcal{V}}^{\perp}|_p$. Then the Ambrose–Singer theorem implies the third statement. q.e.d.

For a conformal pp-wave defined by a function $H$, this theorem for $F = H + h$ gives an upper bound for the holonomy algebra of the analytic ambient metric $\widetilde{g}$ in (3.14) in the cases when it exists, e.g., when $n$ is odd or when $\Delta^{\frac{n}{2}} H = 0$. This upper bound is an improvement of the result in [18, Corollary 2]. The special structure of the ambient metric, in particular, its Ricci flatness, might reduce the holonomy further. Indeed, for special conformal pp-waves in Lorentzian signature, such as plane waves or Cahen–Wallach spaces, the ambient holonomy reduces further:

**Proposition 3.1.** *Let $(S_{AB})_{i,j=1}^{n-2}$ be a symmetric matrix of functions $S_{AB} = S_{AB}(u)$ with trace $S = S(u)$ and let*

$$g = 2du\,dv + 2\left(S_{AB}(u)\,x^A x^B\right) du^2 + \delta_{AB} dx^A\, dx^B \tag{3.19}$$

*be the corresponding Lorentzian plane wave metric on $\mathbb{R}^n$. If $f = f(u)$ is a solution to the ODE*

$$f'' = (f')^2 - \tfrac{2}{n-2} S,$$

*then the metric $\hat{g} = \mathrm{e}^{2f} g$ is Ricci-flat. Moreover, the metric*

$$\widetilde{g} = 2d(t\varrho)dt + t^2 \mathrm{e}^{2f(u)} g$$

*is an ambient metric for the conformal class $[g]$ and admits two parallel null vector fields $\frac{1}{t}\partial_\varrho$ and $\frac{1}{t}(\partial_v + h\,\partial_\varrho)$, where $h = h(u)$ is a solution* *to* $h' = \mathrm{e}^{2f}$*. Consequently, the holonomy algebra of* $\widetilde{g}$ *is contained in* $\mathbb{R}^2 \otimes \mathbb{R}^{n-2}$*.*

This proposition is a generalization of the corresponding statements in [17] about the conformal holonomy of plane waves. Its proof follows from straightforward computations and the well-known formula (2.6) for the ambient metric of Einstein metrics in the case of $\Lambda = 0$. A class of examples to which this proposition applies are the symmetric Cahen–Wallach spaces for which the metric (3.19) is defined by a *constant* symmetric matrix $S_{AB}$.

## 4. Some generic rank 3 pfaffian systems in five dimensions: ambient metrics and their holonomy

In Section 2.2, we described the construction of conformal structures on 5-dimensional manifolds from a generic rank 3 Pfaffian systems, or from the under-determined differential equation

$$z' = F(x, y, p, q, z). \tag{4.1}$$

We begin this section by showing that conditions of Theorem 2.1 are satisfied precisely when

$$F = q^2 + f(x, y, p) + h(x, y)z, \tag{4.2}$$

where $f$ and $h$ are smooth functions of their variables. The explicit *general* power series solution to the Fefferman–Graham equations is obtained in §4.2. The parallel 3-form is determined in Proposition 4.1. Simple sufficiency conditions are then presented which insure that the holonomy of the ambient metric is $\mathbf{G_2}$. For particular choices of the functions $f$ and $g$, closed form solutions to Fefferman–Graham equations are found. Some examples of non-smooth ambient metrics and their holonomy are presented.

**4.1. Rank 3 pfaffian systems in 5 dimensions and conformal classes with linear Fefferman–Graham equations.** The formula for the metric defining the conformal class for the general Monge equation (4.1) is given [24] as

$$\begin{aligned}\tilde{g}_{\mathcal{D}} \;=\; & g_{11}\,(\tilde{\omega}^1)^2 + g_{12}\,\tilde{\omega}^1\tilde{\omega}^2 + g_{13}\,\tilde{\omega}^1\tilde{\omega}^3 + g_{14}\,\tilde{\omega}^1\tilde{\omega}^4 + g_{15}\,\tilde{\omega}^1\tilde{\omega}^5 \\ & + g_{22}\,(\tilde{\omega}^2)^2 + g_{23}\,\tilde{\omega}^2\tilde{\omega}^3 + g_{25}\,\tilde{\omega}^2\tilde{\omega}^5 + g_{33}\,\tilde{\omega}^3\tilde{\omega}^3,\end{aligned}$$

where

$$\begin{aligned}&\tilde{\omega}^1 = dy - p\,dx, \quad \tilde{\omega}^2 = dz - F\,dx - F_q\,(dp - q\,dx), \\ &\tilde{\omega}^3 = dp - q\,dx, \quad \tilde{\omega}^4 = dq, \quad \tilde{\omega}^5 = dx.\end{aligned}$$

The coefficients $g_{ij}$ are given by equation (1.3) in [24]. As noted in this paper there is an error in the formula for the conformal class in the original paper [23]—the coefficient formulas are correct but the co-frame is improperly defined. We note that $g_{33} = -20(F_{qq})^4$ so that $g_{33} < 0$.

For our purposes, it is advantageous to use a null co-frame for the metric $\tilde{g}$. Such a co-frame is defined by

$$
\begin{aligned}
&\tilde{\theta}^1 = \tilde{\omega}^1, \quad \tilde{\theta}^2 = \tilde{\omega}^2, \quad \tilde{\theta}^3 = \sqrt{|g_{33}|}\,\tilde{\omega}^3,\\
&\tilde{\theta}^4 = g_{25}\,\tilde{\omega}^5 + g_{23}\,\tilde{\omega}^3 + g_{22}\,\tilde{\omega}^2,\\
&\tilde{\theta}^5 = g_{14}\,\tilde{\omega}^4 + g_{11},\tilde{\omega}^1 + g_{12}\,\tilde{\omega}^2 + g_{13}\,\tilde{\omega}^3 + g_{15}\,\tilde{\omega}^5,
\end{aligned}
\tag{4.3}
$$

with respect to which the metric $\tilde{g}_{\mathcal{D}}$ becomes $g_F = \tilde{\theta}^1\tilde{\theta}^5 + \tilde{\theta}^2\tilde{\theta}^4 - (\tilde{\omega}^3)^2$.

**Theorem 4.1.** *Let $\mathcal{F} = \{e_1, e_2, e_3, e_4, e_5\}$ be the frame dual to the null co-frame (4.3). Then $\mathcal{F}$ satisfies the conditions $\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{S_4}$ of Theorem 2.1, with $\mathcal{N} = \{e_2, e_5\}$, if and only if, after a change of variables, $F$ is of the form*

$$F = q^2 + b(x, y, p, z)\,q + c(x, y, p, z). \tag{4.4}$$

*In addition, if the Schouten tensor of the metric $g_F$ is required to take values in the span of $\{(\tilde{\theta}^1)^2,\ \tilde{\theta}^1\tilde{\theta}^4,\ (\tilde{\theta}^4)^2\}$, then $F$ is further simplified to*

$$F = q^2 + f(x, y, p) + h(x, y)z. \tag{4.5}$$

*Proof.* In terms of the notation (2.13)–(2.14), the totally null 2-plane distribution $\mathcal{N}$ is

$$\mathcal{N} = \{e_a\} = \operatorname{span}\{e_2, e_5\},$$

with

$$\mathcal{N}^\perp = \{e_a, e_A\} = \operatorname{span}\{e_2, e_5, e_3\} \text{ and } \operatorname{span}\{e_{\bar{b}}\} = \{e_1, e_4\}.$$

The sufficiency conditions of Theorem 2.1 then read ($S_4$ is trivially satisfied)

$$
\begin{aligned}
&\mathbf{S_1}: \quad [e_2, e_5] = 0,\\
&\mathbf{S_2}: \quad [e_2, e_3] \equiv [e_5, e_3] \equiv 0 \mod \{e_2, e_3, e_5\},\\
&\mathbf{S_3}: \quad [e_2, e_1] \equiv [e_2, e_4] \equiv [e_5, e_1] \equiv [e_5, e_4] \equiv 0 \mod \{e_2, e_3, e_5\}.
\end{aligned}
\tag{4.6}
$$

The conditions $[e_2, e_4] \equiv 0$ and $[e_5, e_4] \equiv 0 \mod \{e_2, e_3, e_5\}$ immediately give $F_{qqq} = 0$ and $F_{qqz} = 0$ which imply that

$$F(x, y, p, q, z) = a(x, y, p)\,q^2 + b(x, y, p, z)\,q + c(x, y, p, z).$$

Under the change of variables $\bar{z} = \dfrac{z}{a(x, y, p)}$ the Monge equation (4.1) transforms to

$$
\begin{aligned}
\bar{z}' &= \frac{z'}{a} - \frac{z}{a^2}(a_x + a_y p + a_p q) = \frac{1}{a}(aq^2 + bq + c) - \frac{z}{a^2}(a_x + a_y\,p + a_p q)\\
&= q^2 + (\frac{b}{a} - \frac{a_p z}{a^2})q + \frac{c}{a} - \frac{z}{a^2}(a_x + a_y\,p)
\end{aligned}
$$

$$= q^2 + A(x,y,p,z)\,q + B(x,y,p,z)$$
$$= q^2 + \bar{A}(x,y,p,\bar{z})\,q + \bar{B}(x,y,p,\bar{z}).$$

Accordingly, we may assume that $F$ takes the form (4.4). The conditions $\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}$ are now all satisfied.

We now look to the consequences of the hypothesis on the Schouten tensor $P$ of the metric $g_F$. First, from $P(e_4, e_5) = 0$, we find that $b_z = 0$ and our Monge equation becomes

$$z' = q^2 + b(x,y,p)\,q + c(x,y,p,z).$$

We now use the change of variables $\bar{z} = z + \beta(x,y,p)$, where $\beta_q = -b$, to transform this differential equation to

$$z' = q^2 + c(x,y,p,z).$$

The conditions $P(e_1, e_5) = 0$ and $P(e_3, e_4) = 0$ lead to $c_{zz} = 0$ and $c_{zp} = 0$, as required. q.e.d.

For the special class of the Monge equations defined by (4.2) one readily finds the non-zero metric coefficients to be

$$g_{11} = -72 f_{pp} + 48 h_x + p48 h_y - 32 h^2, \quad g_{13} = -160 h, \quad g_{14} = 480,$$
$$g_{15} = -240 f_p - 480 q h, \quad g_{25} = 240 \quad g_{33} = -320,$$

where $f_p$ denotes the partial derivative $\partial_p(f)$, $h_x = \partial_x(h)$ and so on. The null co-frame we shall use is a scaled version of (4.3), namely

(4.7)
$$\theta^1 = \tilde{\theta}^1 = \; dy - p\,dx, \quad \theta^2 = \tilde{\theta}^2 \; = \; dz + (q^2 - f - hz)\,dx - 2q\,dp,$$
$$\theta^3 = \frac{1}{\sqrt{40}}\,\tilde{\theta}^3 \; = \; 2\sqrt{2}\,(dp - q dx), \quad \theta^4 = \frac{1}{80}\,\tilde{\theta}^4 \; = \; 3\,dx,$$
$$\begin{aligned}\theta^5 &= \frac{1}{80}\,\tilde{\theta}^5 \\ &= 6\,\tilde{\omega}^4 + \tfrac{1}{10}(6p h_y + 6 h_x - 9 f_{pp} - 4h^2)\,\tilde{\omega}^1 - 2h\,\tilde{\omega}^3 - (6qh + 3f_p)\,\tilde{\omega}^5 \\ &= (\tfrac{9}{10} p f_{pp} + \tfrac{2}{5} p h^2 - \tfrac{3}{5} p^2 h_y - \tfrac{3}{5} p h_x - 4qh - 3 f_p)\,dx \\ &\quad + (\tfrac{3}{5} p h_y + \tfrac{3}{5} h_x - \tfrac{9}{10} f_{pp} - \tfrac{2}{5} h^2)\,dy - 2h\,dp + 6\,dq.\end{aligned}$$

The conformal class becomes

$$g_{\mathcal{D}_{f,h}} = -2\theta^1\theta^5 - 2\theta^2\theta^4 + \theta^3\theta^3. \tag{4.8}$$

Thus, if we take

$$h = A\,\theta^1\theta^1 + 2B\theta^1\theta^4 + C\,\theta^4\theta^4, \tag{4.9}$$

where $A = A(x,y,p,\varrho)$, $B = B(x,y,p,\varrho)$ and $C = C(x,y,p,\varrho)$, then the conditions of Theorem 2.1 are all satisfied. The Fefferman–Graham equations for the conformal class (4.8) are, therefore, linear equations.

The Ricci tensor for $g_{\mathcal{D}_{f,h}}$ and the Laplace operator $\Box_{g_{\mathcal{D}_{f,h}}} h_{ij}$ are

$$\begin{aligned}\mathrm{Ric}(g_{\mathcal{D}_{f,h}}) &= (-\tfrac{9}{80}f_{pppp})\theta^1\theta^1 - 2(\tfrac{9}{40}h_y + \tfrac{3}{80}f_{ppp})\theta^1\theta^4 \\ &\quad -(\tfrac{1}{10}h_x + \tfrac{1}{10}ph_y + \tfrac{1}{60}f_pp - \tfrac{1}{15}h^2)\,\theta^4\theta^4, \\ \Box_{g_{\mathcal{D}_{f,h}}}(h) &= (\tfrac{1}{8}A_{pp})\theta^1\theta^1 + (\tfrac{1}{4}B_{pp} - \tfrac{1}{18}A_p)\theta^1\theta^4 \\ &\quad +(\tfrac{1}{8}C_{pp} - \tfrac{1}{18}B_p + \tfrac{1}{324}A)\,\theta^4\theta^4.\end{aligned}$$

The curvature terms in (2.10) all vanish and so it follows that the Fefferman–Graham equations for the conformal class $g_{D_{f,h}}$, associated with the Pfaffian system for (4.2), are the linear inhomogeneous equations

$$\begin{aligned}&L(A) = \tfrac{9}{40}f_{pppp}, \quad \text{where} \quad L = 2\varrho\frac{\partial^2}{\partial\varrho^2} - 3\frac{\partial}{\partial\varrho} - \frac{1}{8}\frac{\partial^2}{\partial p^2}, \\ &L(B) = -\tfrac{1}{36}A_p + \tfrac{3}{40}f_{ppp} + \tfrac{9}{20}h_y, \quad \text{and} \\ &L(C) = -\tfrac{1}{18}B_p + \tfrac{1}{324}A + \tfrac{1}{30}f_{pp} - \tfrac{2}{15}h^2 + \tfrac{1}{5}(ph_y + h_x).\end{aligned} \tag{4.10}$$

with initial conditions $A|_{\rho=0} = B|_{\rho=0} = C|_{\rho} = 0 = 0$.

One can check directly that the ambient metric

$$\tilde{g}_{\mathcal{D}_{f,h}} = 2dtd(\varrho t) + t^2\big(g_{\mathcal{D}_{f,h}} + A\,(\theta^1)^2 + 2B\,\theta^1\theta^4 + C\,(\theta^4)^2\big) \tag{4.11}$$

is, indeed, Ricci flat by virtue of (4.10).

**4.2. Solutions to the Fefferman–Graham equations.** In this section, we find *all* formal solutions to the linear system (4.10), that is, including those that are not smooth in $\varrho$. We first observe, by analogy with Theorem 3.1, that the two independent solutions to $L(\varrho^k) = 0$ are given by $k = 0$ and $k = \frac{5}{2}$. Thus, the most general solution to the system (4.10) can be obtained by writing each coefficient $A$, $B$, $C$ in (4.11) as a power series of the form

$$\sum_{k=1}^{\infty} a_k(x,y,p)\varrho^k + \varrho^{5/2}\sum_{k=0}^{\infty}\alpha_k(x,y,p)\varrho^k.$$

**Theorem 4.2.** *The general solution to the linear system* (4.10)*, vanishing at* $\varrho = 0$*, is*

(4.12)

$$\begin{aligned}A &= 3\textstyle\sum_{k=1}^{\infty} a_k \frac{\partial^{(2k+2)}f}{\partial p^{(2k+2)}}\,\varrho^k + 60\,\varrho^{5/2}\sum_{k=0}^{\infty} q_k\,\frac{\partial^{2k}\alpha_0}{\partial p^{2k}}\,\varrho^k, \\ B &= -\tfrac{3}{20}\varrho h_y - \tfrac{1}{3}\textstyle\sum_{k=1}^{\infty}(2k-5)a_k\,\frac{\partial^{(2k+1)}f}{\partial p^{(2k+1)}}\,\varrho^k \\ &\quad + \tfrac{20}{3}\varrho^{5/2}\textstyle\sum_{k=0}^{\infty} q_k\left(9\frac{\partial^{2k}\beta_0}{\partial p^{2k}} - 2k\frac{\partial^{(2k-1)}\alpha_0}{\partial p^{(2k-1)}}\right)\varrho^k, \\ C &= \tfrac{2}{45}\varrho h^2 - \tfrac{1}{15}\varrho(ph_y + h_x) + \tfrac{2}{27}\textstyle\sum_{k=1}^{\infty}(k-3)(2k-5)a_k\,\frac{\partial^{2k}f}{\partial p^{2k}}\,\varrho^k \\ &\quad + \tfrac{20}{3}\,\varrho^{5/2}\textstyle\sum_{k=0}^{\infty} q_k\left(9\frac{\partial^{2k}\gamma_0}{\partial p^{2k}} - 4k\frac{\partial^{(2k-1)}\beta_0}{\partial p^{(2k-1)}} + \frac{2k(2k-1)}{9}\frac{\partial^{(2k-2)}\alpha_0}{\partial p^{(2k-2)}}\right)\varrho^k.\end{aligned}$$

*The constants* $a_k$ *and* $q_k$ *are given by*

$$a_k = \frac{1}{5}\frac{(2k-1)(2k-3)}{2^{2k}(2k)!}, \quad q_k = \frac{(k+2)(k+1)}{2^{2k}(2k+5)!}, \tag{4.13}$$

*and* $\alpha_0$*,* $\beta_0$ *and* $\gamma_0$ *are arbitrary smooth functions of the variables* $x$*,* $y$ *and* $p$*.*

*When* $f$ *and* $h$ *are analytic, the unique analytic solution to* (4.10) *is given by* (4.12) *with* $\alpha_0 = \beta_0 = \gamma_0 = 0$ *and this determines the unique ambient metric* (4.11) *that is analytic in* $\varrho$*.*

Note that the analytic solutions are totally determined by the distribution, that is, by the functions $f$ and $h$. On the other hand, the non-smooth part of the general solution is determined only by the functions $\alpha_0$, $\beta_0$ and $\gamma_0$ and, hence, do not depend on the distribution $\mathcal{D}_{f,h}$ at all.

Our next goal is to determine explicitly the parallel 3-form that gives the reduction to $\mathbf{G}_2$ of the holonomy of the ambient metric in (4.11) for the *analytic* solutions of (4.10). Such a holonomy reduction imposes conditions on the connection and, hence, implies first order conditions on the metric coefficients. It turns out that this first order system actually implies the second order equations (4.10) as their integrability conditions.

**Theorem 4.3.** *Let* $f = f(x,y,p)$ *and* $h = h(x,y)$ *be smooth functions. Then the analytic solutions of the second order system* (4.10) *solve the first order system*

$$\begin{aligned} B_p &= \tfrac{5}{9}A - \tfrac{2}{9}\varrho A_\varrho, \\ C_p &= \tfrac{1}{324}\varrho A_p + \tfrac{2}{3}B + \tfrac{1}{180}\varrho f_{ppp} + \tfrac{1}{30}\varrho h_y, \\ B_\varrho &= -\tfrac{1}{72}A_p - \tfrac{1}{40}f_{ppp} - \tfrac{3}{20}h_y, \\ C_\varrho &= \tfrac{1}{648}A - \tfrac{1}{72}B_p - \tfrac{1}{90}f_{pp} + \tfrac{2}{45}h^2 - \tfrac{1}{15}(ph_y + h_x). \end{aligned} \tag{4.14}$$

*Moreover, the twice differentiable solutions* $A$*,* $B$ *and* $C$ *of* (4.14) *are solutions of* (4.10)*.*

*Proof.* It is a simple matter to check that the analytic solutions of the second order system (4.10) given in Theorem 4.2 with $\alpha_0 = \beta_0 = \gamma_0 = 0$ satisfy the first order system (4.14).

To derive the second order equations from the first order system, we first use the integrability condition arising from the equations (4.14) for the derivatives of $B$. We immediately compute

$$0 = B_{\varrho p} - B_{p\varrho} = \tfrac{1}{9}\left(L(A) - \tfrac{9}{40}f_{pppp}\right), \tag{4.15}$$

which is precisely the first equation in (4.10) The integrability condition arising from the derivatives of $C$ is

$$C_{p\varrho} - C_{\varrho p} = \tfrac{1}{648}A_p + \tfrac{1}{324}\varrho A_{p\varrho} + \tfrac{1}{60}f_{ppp} + \tfrac{1}{10}h_y + \tfrac{1}{72}B_{pp} + \tfrac{2}{3}B_\varrho.$$

The right-hand side of this equation vanishes upon substituting for the derivatives of $B$ from (4.14).

The second and third equations in (4.10) can now be derived directly from (4.14). q.e.d.

We remark that the non-smooth solutions (4.12) to the second order system (4.10) given $f = h = 0$ and $\alpha_0 = \alpha_0(x, y)$, do *not* satisfy the first order system (4.14). Indeed, in this case we get $A \equiv 0$, the first two equations of (4.14) become $B_p = 0$ and $B_\varrho = 0$, and these do not hold when choosing, for example, $\frac{\partial^2 \beta_0}{\partial p^2} \neq 0$.

For each solution $A$, $B$, $C$ of (4.14), note also that the functions $B$ and $C$ are uniquely determined by the function $A$ and the initial conditions.

**4.3. The holonomy of the ambient metrics.** The first order system (4.14) enables us to find the parallel 3-form that defines the $\mathbf{G}_2$-reduction of the ambient metric. We first define a null co-frame for the ambient metric (4.11) by

$$\begin{aligned}
\omega^1 &= 9\sqrt{2}dt + t\sqrt{2}h\,\theta^4,\\
\omega^2 &= \theta^1,\\
\omega^3 &= -\frac{1}{9}t\,h(\varrho dt + td\varrho) - t^2\theta^2 + \frac{1}{2}\,t^2\,C\,\theta^4,\\
\omega^4 &= t\theta^3,\\
\omega^5 &= \theta^4,\\
\omega^6 &= \frac{1}{2}\,t\,A\,\theta^1 + t^2 B\,\theta^4 - t^2\theta^5,\\
\omega^7 &= \tfrac{\sqrt{2}}{18}(\varrho dt + td\varrho),
\end{aligned} \tag{4.16}$$

with the $\theta^i$'s given by (4.7). The metric (4.11) becomes

$$\tilde{g}_{\mathcal{D}_{f,h}} = 2\omega^1\omega^7 + 2\omega^2\omega^6 + 2\omega^3\omega^5 + (\omega^4)^2. \tag{4.17}$$

A direct computation proves the following

**Proposition 4.1.** *Let $\mathcal{D}_{f,h}$ be the distribution on $\mathbb{R}^5$ defined by (4.5). Let $A = A(x, y, p, \varrho)$, $B = B(x, y, p, \varrho)$, and $C = C(x, y, p, \varrho)$ be twice differentiable solutions of the first order system (4.14) associated to $f$ and $h$. Then a parallel 3-form for the ambient metric (4.17) is*

$$\Upsilon = 2\omega^{123} - \omega^{147} - \omega^{246} - \omega^{345} + \omega^{567}, \tag{4.18}$$

*where $\omega^{ijk} = \omega^i \wedge \omega^j \wedge \omega^k$. The holonomy representation of (4.17) is, therefore, contained in the standard (split-form) representation of $\mathfrak{g}_2$.*

Note that this theorem also applies to non-analytic solutions to the first order system (4.14).

Let $X_1, \ldots, X_7$ be the vector fields dual to the 1-forms $\omega^i$ in (4.16). Then, for future use, we note that standard representation of $\mathfrak{g}_2$ is given in terms of the 1-1 tensors $E_i^j = X_i \otimes \omega^j$ by

$$
\begin{aligned}
&h_1 = E_3^2 - E_6^5, && h_2 = E_3^1 - E_7^5,\\
&h_3 = E_3^4 - E_4^5 + E_6^1 - E_7^2, && h_4 = E_2^5 - E_3^6 + 2E_4^1 - 2E_7^4,\\
&h_5 = E_1^5 - E_3^7 - 2E_4^2 + 2E_6^4, && h_6 = E_2^1 - E_7^6,\\
&h_7 = E_1^1 - 2E_2^2 + E_3^3 - E_5^5 && h_8 = E_2^2 - E_3^3 + E_5^5 - E_6^6,\\
&\qquad + 2E_6^6 - E_7^7, && \\
&h_9 = E_1^2 - E_6^7, && h_{10} = E_2^4 - E_4^6 - E_5^1 + E_7^3,\\
&h_{11} = E_1^4 - E_4^7 + E_5^2 - E_6^3, && h_{12} = E_1^6 - E_2^7 + 2E_4^3 - 2E_5^4,\\
&h_{13} = E_2^3 - E_5^6, && h_{14} = E_1^3 - E_5^7.
\end{aligned}
\tag{4.19}
$$

Our next theorem gives a simple sufficiency condition under which the ambient metric has holonomy *equal to* $\mathfrak{g}_2$.

**Theorem 4.4.** *Let $A$, $B$ and $C$ be solutions to the first order system (4.14) and suppose that $A_\varrho \neq 0$. Then the holonomy of the ambient metric $\tilde{g}_{\mathcal{D}_{f,h}}$ equals $\mathfrak{g}_2$.*

*Proof.* Let $R$ be the curvature tensor of the metric (4.17). To prove the theorem, we need only check that the $h_i$ are all in the span of the curvature endomorphisms $R(X, Y)$ and their directional covariant derivatives. To this end, set $\mathcal{H}_i = \operatorname{span}\{h_1, h_2, \ldots, h_i\}$. Then one easily calculates ($\dot{A} = A_\varrho$)

(4.20)

$$
\begin{aligned}
&R(X_3, X_4) = \tfrac{1}{2}\dot{A}\, h_1, && \nabla_{X_2} h_1 = -\tfrac{\sqrt{2}}{18t}\, h_2,\\
&\nabla_{X_5} h_1 = \tfrac{\sqrt{2}}{18t}\, h_3 \bmod \mathcal{H}_2, && \nabla_{X_5} h_3 = \tfrac{\sqrt{2}}{18t}\, h_4 \bmod \mathcal{H}_3,\\
&\nabla_{X_2} h_4 = -9t\tfrac{\sqrt{2}}{2}\dot{A}\, h_5 \bmod \mathcal{H}_4, && \nabla_{X_5} h_4 = -\tfrac{\sqrt{2}}{6t}\, h_6 \bmod \mathcal{H}_5,\\
&\nabla_{X_5} h_5 = -\tfrac{\sqrt{2}}{18t}\, h_7 \bmod \mathcal{H}_6, && \nabla_{X_2} h_6 = -9t\tfrac{\sqrt{2}}{2}\dot{A}\, h_8 \bmod \mathcal{H}_7,\\
&\nabla_{X_2} h_7 = 27t\tfrac{\sqrt{2}}{2}\dot{A}\, h_9 \bmod \mathcal{H}_8, && \nabla_{X_5} h_8 = \tfrac{\sqrt{2}}{18t}\, h_{10} \bmod \mathcal{H}_9,\\
&\nabla_{X_2} h_1 0 = \tfrac{\sqrt{2}}{18t}\, h_{11} \bmod \mathcal{H}_{10}, && \nabla_{X_5} h_{11} = -\tfrac{\sqrt{2}}{18t}\, h_{12} \bmod \mathcal{H}_{11},\\
&\nabla_{X_5} h_{12} = -\tfrac{\sqrt{2}}{6t}\, h_{13} \bmod \mathcal{H}_{12}, && \nabla_{X_2} h_{13} = -9t\tfrac{\sqrt{2}}{2}\dot{A}\, h_{14} \bmod \mathcal{H}_{13}.
\end{aligned}
$$

This establishes the theorem. q.e.d.

As an immediate consequence of Theorem 4.4 we have the following generalization of Theorem 1.1 in [19].

**Corollary 4.1.** *If $f = f(x, y, p)$ is a polynomial function of $p$ of order $\geq 4$, then the functions $A$, $B$, $C$ in (4.12) (with $\alpha_0 = \beta_0 = \gamma_0$)*

*are polynomial functions of $p$ and $\varrho$ and the ambient metric $\tilde{g}_{\mathcal{D}_{f,h}}$ has full holonomy $\mathfrak{g}_2$.*

Here are two explicit examples of ambient metrics with $\mathfrak{g}_2$ holonomy which are not polynomials in $\varrho$.

**Example 4.1.** First, consider the distribution $\mathcal{D}_{f,h}$ with $f(x,y,p) = \sin(p)$ and $h(x,y) = 0$. To solve the equation $L(A) = f_{pppp}$ for $A$, we first remark that if $f = f(x,y,p)$ satisfies a constant coefficient linear ODE with respect to the variable $p$ whose fundamental solutions are $f_1(x,y,p),\dots,f_N(x,y,p)$, then the analytic solutions of $L(A) = f_{pppp}$ take the form

$$A = a_1(\varrho)f_1(x,y,p) + a_2(\varrho)f_2(x,y,p) + \ldots + a_N(\varrho)f_N(x,y,p),$$

where the $a_1(\varrho),\dots,a_N(\varrho)$ satisfy a system of linear second order ODE.

For the problem at hand, this implies that $A = a_1(\varrho)\sin(p) + a_2(\varrho)\cos(p)$, where

$$2\varrho a_{1,\varrho\varrho} - 3a_{1,\varrho} + \tfrac{1}{8}a_1 = \tfrac{9}{40}, \quad 2\varrho a_{2,\varrho\varrho} - 3a_{2,\varrho} + \tfrac{1}{8}a_2 = 0, \quad a_1(0) = a_2(0) = 0.$$

The analytic solutions to these equations are

$$a_1(\varrho) = \tfrac{3}{20}\varrho\cos(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{9}{10}\sqrt{\varrho}\sin(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{9}{5}(\cos(\tfrac{\sqrt{\varrho}}{2}) - 1) \quad \text{and} \quad a_2(\varrho) = 0.$$

Then we find that $B = b(\varrho)\cos(p)$ and $C = c(\varrho)\sin(p)$, where

$$\begin{aligned} b(\varrho) &= -\tfrac{1}{120}\varrho^{\frac{3}{2}}\sin(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{1}{10}\varrho\cos(\tfrac{\sqrt{\varrho}}{2}) + \tfrac{1}{2}\sqrt{\varrho}\sin(\tfrac{\sqrt{\varrho}}{2}) + \cos(\tfrac{\sqrt{\varrho}}{2}) - 1, \\ c(\varrho) &= \tfrac{1}{2160}\varrho^2\cos(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{1}{120}\varrho^{\frac{3}{2}}\sin(\tfrac{\sqrt{\varrho}}{2}) \\ &\quad - \tfrac{13}{180}\varrho\cos(\tfrac{\sqrt{\varrho}}{2}) + \tfrac{1}{3}\sqrt{\varrho}\sin(\tfrac{\sqrt{\varrho}}{2}) + \tfrac{2}{3}(\cos(\tfrac{\sqrt{\varrho}}{2}) - 1). \end{aligned}$$

Note that these functions are analytic in $\varrho$.

**Example 4.2.** Likewise, for $f(x,y,p) = \mathrm{e}^p$ and $h = 0$, the solutions of equations (4.10) are of the form $A = a(\varrho)\,\mathrm{e}^p$, $B = b(\varrho)\,\mathrm{e}^p$, and $C = c(\varrho)\,\mathrm{e}^p$, with

$$\begin{aligned} a(\varrho) &= \tfrac{3}{20}\varrho\cosh(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{9}{10}\sqrt{\varrho}\sinh(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{9}{5}(\cosh(\tfrac{\sqrt{\varrho}}{2}) - 1), \\ b(\varrho) &= -\tfrac{1}{120}\varrho^{\frac{3}{2}}\sinh(\tfrac{\sqrt{\varrho}}{2}) + \tfrac{1}{10}\varrho\cosh(\tfrac{\sqrt{\varrho}}{2}) \\ &\quad - \tfrac{1}{2}\sqrt{\varrho}\sinh(\tfrac{\sqrt{\varrho}}{2}) + \cosh(\tfrac{\sqrt{\varrho}}{2}) - 1, \\ c(\varrho) &= \tfrac{1}{2160}\varrho^2\cosh(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{1}{120}\varrho^{\frac{3}{2}}\sinh(\tfrac{\sqrt{\varrho}}{2}) \\ &\quad + \tfrac{13}{180}\varrho\cosh(\tfrac{\sqrt{\varrho}}{2}) - \tfrac{1}{3}\sqrt{\varrho}\sinh(\tfrac{\sqrt{\varrho}}{2}) + \tfrac{2}{3}(\cosh(\tfrac{\sqrt{\varrho}}{2}) - 1). \end{aligned}$$

**Example 4.3.** It is easy to produce examples of ambient metrics with holonomy equal to $\mathbf{G}_2$ where the sufficiency condition $A_\rho \neq 0$ of Theorem 4.4 fails. One example, taken from [19], is given by $f = p^3$, $h = 0$ and another is given by $f = 0$ and $h = y$.

**Example 4.4.** We close this section with some simple examples of holonomy for the non-smooth ambient metrics derived from the Hilbert-Cartan equation $z' = q^2$, (that is, $f = h = 0$) with one or more of the functions $\alpha_0$, $\beta_0$, $\gamma_0$ in (4.12) being non-zero. We restrict to the domain where $\varrho > 0$. With $\alpha_0 = 1, \beta_0 = p, \gamma_0 = 0$, the holonomy is $\mathfrak{so}(4,3)$. For $\alpha_0 = p$, $\beta_0 = 0, \gamma_0 = 0$, the holonomy is again $\mathfrak{g}_2$. Finally, for $\alpha_0 = \beta_0 = 0, \gamma_0 = c(x)$ the holonomy is $\mathfrak{sl}_2(\mathbb{R}) \ltimes \mathfrak{he}_7$, where $\mathfrak{he}_7$ is the 7-dimensional Heisenberg algebra. In this case the parallel forms are

$$t\omega^{15}, \quad 18t^2\omega^{125} - 3\sqrt{2}xt\,\omega^{145} + \frac{x^2}{2}\omega^{156}, \quad -3\sqrt{2}\omega^{145} + x\omega^{156}, \quad \omega^{156},$$

and, therefore, the holonomy algebra is *not* a subalgebra of $\mathfrak{g}_2$.

## 5. Generic 3-distributions in dimension 6 and their ambient metrics

**5.1. Generic 3-distributions in dimension 6 and Bryant conformal classes with linear Fefferman–Graham equations.** In this section, we construct explicit examples of ambient metrics for conformal structures with metrics of signature $(3,3)$ which are naturally associated to generic rank 3 distributions in dimension 6. The associated conformal structures were introduced by Bryant in [6] as structures that encode local invariants of such distributions. We will call them *Bryant's conformal structures.*

Since our purpose is to find new examples of ambient metrics, we consider a special class of rank 3 distributions which will lead us to linear Fefferman–Graham equations. This class of distributions is defined on open set $\mathcal{U} \subset \mathbb{R}^6$, by

$$\mathcal{D}_f = \mathrm{Span}\left( \frac{\partial}{\partial x^3} - x^2\frac{\partial}{\partial y^1},\ \frac{\partial}{\partial x^1} - f\frac{\partial}{\partial y^2},\ \frac{\partial}{\partial x^2} - x^1\frac{\partial}{\partial y^3} \right), \tag{5.1}$$

where $f = f(x^1,x^2,x^3,y^1,y^2,y^3)$ is a smooth function of the coordinates $(x^1,x^2,x^3,y^1,y^2,y^3)$ on $\mathbb{R}^6$. The genericity condition $[\mathcal{D},\mathcal{D}]+\mathcal{D} = TM^6$ requires that $w = f_{x^3} - x^2 f_{y^1} \neq 0$ and this will be always assumed in what follows.

For a ***one-adapted*** co-frame ([6], equation (2.3)), we take

$$\begin{aligned} &\theta^1 = dy^1 + x^2\,dx^3, \quad &&\theta^2 = w\,(dy^2 + f\,dx^1),\\ &\theta^3 = dy^3 + x^1\,dx^2, \quad &&\sigma^1 = w\,dx^1,\\ &\sigma^2 = \frac{1}{w}dx^2, &&\sigma^3 = w\,dx^3 + (f_{x^2} - x^1 f_{y^3})dx^2. \end{aligned} \tag{5.2}$$

By definition, a ***two-adapted co-frame*** $\{\theta_1,\theta_2,\theta_3,\omega^1,\omega^2,\omega^3\}$ is one that satisfies the structure equations ([6], equation (2.15))

$$d\,\theta_i = -\alpha^k_k \wedge \theta_i + \alpha^j_i \wedge \theta_j + \frac{1}{2}\epsilon_{ijk}\,\omega^j \wedge \omega^k \quad \text{and}$$
$$d\,\omega_i = -\epsilon^{ikj}\beta_k \wedge \theta_j - \alpha^i_j \wedge \omega^j.$$

One can check that there are uniquely defined functions $p_i$ such that the forms

$$\begin{aligned} &\omega^1 = \sigma^1, \quad \omega^2 = \sigma^2 - \frac{f_{y^1y^1}(x^2)^2 - 2x^2 f_{x^3y^1} + f_{x^3x^3}}{3((f_{y^1})^2(x^2)^2 - 2f_{x^3}f_{y^1}x^2 + (f_{x^3})^2)}\,\theta^1, \\ &\omega^3 = \sigma^3 + p_1\,\theta^1 + p_2\,\theta^2 + p_3\,\theta^3 \end{aligned} \tag{5.3}$$

define a 2-adapted co-frame. In what follows we denote this co-frame, consisting of the first set of forms in (5.2) and the forms in (5.3), by

$$\mathcal{B} = \{\theta^1,\ \theta^2,\ \theta^3,\ \theta^4 = \omega^1,\ \theta^5 = \omega^2,\ \theta^6 = \omega^3\}. \tag{5.4}$$

In terms of the co-frame $\mathcal{B}$, the Bryant's conformal class $[g_{\mathcal{D}_f}]$ is represented by the metric

$$g_{\mathcal{D}_f} = 2\theta^1\theta^4 + 2\theta^2\theta^5 + 2\theta^3\theta^6. \tag{5.5}$$

We begin our analysis by characterizing those distributions (5.1) for which the Fefferman–Graham equations for the ambient metric for the Bryant metric (5.5) are, by virtue of Theorem 2.1, inhomogeneous linear equations.

**Theorem 5.1.** *Let $\mathcal{F} = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ be the frame dual to the 2-adapted coframe (5.4). Then $\mathcal{F}$ satisfies the conditions $\mathbf{S_1}, \mathbf{S_2}, \mathbf{S_3}, \mathbf{S_4}$ of Theorem 2.1, with $\mathcal{N} = \{e_1, e_3, e_5\}$, if and only if $f$ is independent of $y^1$ and $y^3$. Moreover, after a change of variables, $f$ will be independent of $x^2$, that is,*

$$f(x^1, x^2, x^3, y^1, y^2, y^3) = F(x^1, x^3, y^2). \tag{5.6}$$

*Proof.* With $\mathcal{N} = \{e_1, e_3, e_5\}$, the required structure equations $\mathbf{S_1}, \mathbf{S_2}$, $\mathbf{S_3}, \mathbf{S_4}$ reduce to

$$\begin{aligned} &\mathbf{S_1}: \quad [e_1, e_3] = [e_1, e_5] = [e_3, e_5] = 0, \\ &\mathbf{S_3}: \quad [e_a, e_{\bar{b}}] = 0 \text{ mod } \mathcal{N}, \text{ for } \ a = 1, 3, 5, \quad \bar{b} = 2, 4, 6. \end{aligned}$$

In terms of the structure equations $[e_i, e_j] = F^k_{ij} e_k$, these conditions become

$$F^c_{ab} = F^{\bar{c}}_{ab} = F^{\bar{c}}_{a\bar{b}} = 0, \qquad \text{for} \quad a, b, c = 1, 3, 5, \quad \bar{b}, \bar{c} = 2, 4, 6. \tag{5.7}$$

One has that $F^1_{15} + F^2_{14} = -f_{y^1}$ so that (5.7) immediately implies that $f$ is independent of $y^1$. One then finds that

$$f_{x^3x^3} f_{y^3} = 3(f_{x^3})^2 F^1_{13} - f_{x^3x^3} F^2_{34} = 0. \tag{5.8}$$

and, consequently, either $f_{y^3} = 0$ or $f_{x^3x^3} = 0$.

With $f_{y^3} = 0$, the equations $F^1_{15} = F^1_{35} = F^6_{14} = 0$ yield

$$(5.9)\qquad \begin{array}{lrcl} [a] & f_{x^1x^2} f_{x^3} - f_{x^1x^3} f_{x^2} & = & 0, \\ [b] & f_{x^2x^3} f_{x^3} - f_{x^3x^3} f_{x^2} & = & 0, \\ [c] & f_{x^2y^2} f_{x^3} - f_{x^3y^2} f_{x^2} & = & 0. \end{array}$$

These equations imply that the ratio of partial derivatives $\dfrac{f_{x^2}}{f_{x^3}}$ is a function of the variable $x^2$ alone. We may, therefore, write

$$(5.10)\qquad f(x^1, x^2, x^3, y^2) = F(x^1, \tilde{x}^3, y^2) \quad \text{where} \quad \tilde{x}^3 = x^3 + L(x^2).$$

With $\tilde{y}^1 = y^1 - \int x^2 L'(x^2)\, dx^2$, we find that

$$\tilde{\theta}^1 = d\tilde{y}^1 + x^2\, d\tilde{x}^3 = dy^1 + x^2\, dx^3 = \theta^1 \quad \text{and}$$
$$\tilde{\sigma}^3 = w\, d\tilde{x}^3 + (f_{x^2} - x^1 f_{y^3})\, dx^2 = w\, dx^3 - x^1 f_{y^3} dx^2 = \sigma^3,$$

so that the 1-adapted co-frame (5.2) is unaffected by this change of coordinates and (5.6) holds.

If $f_{x^3x^3} = 0$, we find that $F^1_{15} = F^5_{35} = 0$ yield $f_{x^2x^3} = f_{x^3y^2} = 0$ and (5.9)[b] holds. The combination $F^1_{35} + F^2_{34} = 0$ then proves that $f_{y^3} = 0$. The conditions $F^1_{35} = F^6_{36} = 0$ now reduce directly to (5.9). q.e.d.

With $f = f(x^1, x^3, y^2)$, the defining formulas for the last three 1-forms for the 2-adapted co-frame (5.4) simplify to

$$\theta^4 = \sigma^1, \quad \theta^5 = \sigma^2 - \frac{f_{x^3x^3}}{3(f_{x^3})^2}\, \theta^1,$$
$$\theta^6 = \sigma^3 + \frac{f_{x^1x^3} - f f_{x^3y^2} + f_{x^3} f_{y^2}}{3(f_{x^3})^2}\, \theta^2.$$

This co-frame will be used for the remainder of this section and the next.

In accordance with our ansatz (2.12) for the ambient metric we take

$$h = S_{22}\,(\theta^2)^2 + 2S_{24}\theta^2\theta^4 + 2S_{26}\,\theta^2\theta^6 + S_{44}\,(\theta^4)^2$$
$$+ 2S_{46}\,\theta^4\theta^6 + S_{66},(\theta^6)^2,$$

where $S_{ab} = S_{ab}(x^1, x^3, y^2, \varrho)$. The Fefferman–Graham equations for the Bryant conformal class (5.5) will now reduce to the inhomogeneous linear equations (2.10)

In the next section we shall use some general results about the ambient metric construction to find a *particular* solution to the Fefferman–Graham equations which will, in turn, allow us to further reduce the Fefferman–Graham equations to a very simple homogeneous linear system.

**5.2. Fefferman–Graham equations for the Bryant conformal class.** Recall that in even dimensions $n$, not every conformal class $[g]$ admits a smooth ambient metric. When proving their results in [11, 12], Fefferman and Graham fixed a metric $g_0$ in the conformal class and performed the *power series expansion* of $g(x^i, \varrho)$ (see (2.4)) in the variable $\varrho$

$$g(x^i, \varrho) = g_0 + 2\varrho P + \varrho^2 \mu + \dots. \tag{5.11}$$

It is shown that $P$ is the Schouten tensor for $g_0$ and, if $n > 4$, that $\mu$ is given by

$$\mu_{ij} = \frac{1}{4-n} B_{ij} + P_i{}^k P_{kj}. \tag{5.12}$$

Here $B_{ij}$ is the ***Bach tensor*** of $g_0$, defined in terms of the Weyl tensor $W_{ijkl}$ and the ***Cotton tensor*** $C_{ijk}$ by

$$B_{ij} = \nabla^k C_{ijk} - P^{kl} W_{kijl} \quad \text{and} \quad C_{ijk} = \nabla_k P_{ij} - \nabla_j P_{ik}. \tag{5.13}$$

In dimension $n = 6$, for the power expansion of $g(x, \varrho)$ to continue beyond the $\varrho^2$ term, the metric $g_0$ representing the conformal class $[g_0]$ must satisfy $\mathcal{O}_{ij} = 0$, where

(5.14)

$$\begin{aligned} \mathcal{O}_{ij} \; = \; & \nabla^k \nabla_k B_{ij} - 2W_{kijl} B^{kl} 8P^{kl} \nabla_l C_{(ij)k} - 4C^k_{\ i}{}^l C_{ljk} \\ & -4P_k{}^k B_{ij} + 2C_i{}^{kl} C_{jkl} + 4C_{ij}{}^l \nabla_l P^k_{\ k} - 4W_{kijl} P^k_{\ m} P^{ml} \end{aligned}$$

is the Fefferman–Graham ***obstruction tensor***. If the obstruction tensor vanishes, then the substitution of $h_0(x, \varrho) = g_0 + 2P\varrho + \mu\varrho^2$, into (2.4) for $\tilde{g}$, provides a particular ambient metric

$$\tilde{g} = 2d(\varrho t)dt + t^2(g_0 + 2P\varrho + \mu\varrho^2). \tag{5.15}$$

In other words, $\mathrm{Ric}(\tilde{g}) = 0$ and $h_0$ is a particular solution to the Fefferman–Graham equations.

Properties of the Bryant conformal class $[g_{\mathcal{D}_f}]$ are given in the next proposition.

**Proposition 5.1.** *With $f = f(x^1, x^3, y^2)$ the conformal classes $[g_{\mathcal{D}_f}]$ defined by the metric (5.5) have the following properties:*

**[i]** *They are Bach-flat and their Schouten tensor squares to zero, $P_i{}^k P_{kj} = 0$. In particular, the quadratic terms in $\rho$ in the expansions (5.11) and (5.15) vanishes.*

**[ii]** *Generically they are not conformally Cotton, and, hence, not conformally Einstein.*

**[iii]** *Each term in the formula (5.14) for the Fefferman–Graham obstruction tensor $\mathcal{O}_{ij}$ vanishes separately and, thus, $\mathcal{O}_{ij} = 0$.*

**[iv]** *An ambient metric for all such conformal classes is given by*

$$\tilde{g}_f = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\,\varrho\, P). \tag{5.16}$$

*Proof.* The fundamental observation is that the Schouten tensor $P$ of $g_{\mathcal{D}_f}$ satisfies

$$\begin{aligned} P &\in \operatorname{span}_{\mathcal{A}}\{\theta^i \otimes \theta^j \mid i,j \in \{2,4,6\}\}, \\ \nabla P &\in \operatorname{span}_{\mathcal{A}}\{\theta^i \otimes \theta^j \otimes \theta^k \mid i,j,k \in \{2,4,6\}\}. \end{aligned} \tag{5.17}$$

where $\mathcal{A}$ is the ring of smooth functions in the variables $x^1, x^3, y^2$. Property (5.17) implies that the Schouten tensor is 2-step nilpotent, $P_{ik}P^k_{\ j} = 0$. Moreover, the Weyl tensor $W$ satisfies

$$W(X,.,.,.) \in \operatorname{span}\{\theta^i \otimes \theta^j \otimes \theta^k \mid i,j,k \in \{2,4,6\}\},$$

whenever $X \in \operatorname{span}\{e_1, e_3, e_5\}$.

These properties for the Schouten and the Weyl tensors imply the vanishing of the Bach tensor and of all other terms in the formula (5.14) for $\mathcal{O}_{ij}$.

That the metrics generically are not conformally Cotton-flat can be seen by checking that generically there is no vector $\Upsilon^i$ such that

$$C_{ijk} + \Upsilon^\ell W_{\ell ijk} = 0,$$

which is a necessary condition for a metric to be conformal to a Cotton-flat metric [15].

The formula (5.16) for an ambient metric then follows immediately from (5.11). It is a straightforward computation with **DG** to directly verify that $\tilde{g}_f$ is, indeed, Ricci-flat. q.e.d.

Proposition 5.1 and Theorem 2.1 now imply that the Fefferman–Graham equations for the metric

$$\tilde{g}_{f,S} = 2d(\varrho t)dt + t^2(g_{\mathcal{D}_f} + 2\,\varrho\, P + S(x^1, x^3, y^2, \varrho)), \tag{5.18}$$

with

$$\begin{aligned} S(x^1,x^3,y^2,\varrho) &= S_{22}\,(\theta^2)^2 + S_{44}\,(\theta^4)^2 + S_{66}\,(\theta^6)^2 \\ &\quad + 2S_{24}\,\theta^2\theta^4 + 2S_{26}\,\theta^2\theta^6 + 2S_{46}\,\theta^4\theta^6 \end{aligned}$$

will be *homogeneous* linear equations for the functions $S_{ij}$, namely,

$$\rho\ddot{S}_{ij} - 2\dot{S}_{ij} - \frac{1}{2}\overset{0}{\Box} S_{ij} + \overset{0}{R}{}^k{}_{ij}{}^l S_{kl} + \overset{0}{R}{}^k{}_{(i} S_{j)k} = 0. \tag{5.19}$$

The arguments used in the Proposition 5.1 now show that the Laplacian terms and the curvature terms vanish. Therefore, $\tilde{g}_{f,S}$ is an ambient metric for the conformal class of Bryant's metrics $[g_{\mathcal{D}_f}]$ if and only if the functions $S_{ij} = S_{ij}(x^1, x^3, \varrho)$ satisfy the same linear PDE

$$\varrho\frac{\partial^2 S_{ij}}{\partial \varrho^2} - 2\frac{\partial S_{ij}}{\partial \varrho} = 0. \tag{5.20}$$

The most general smooth solution $S_{ij}$ satisfying the initial conditions $S_{ij}(x^1, x^3, y^2, 0) = 0$, is given by $S = \varrho^3 Q$ with

$$Q(x^1, x^3, y^2) = Q_{22}\,(\theta^2)^2 + 2Q_{24}\theta^2\theta^4 + 2Q_{26}\,\theta^2\theta^6 + Q_{44}\,(\theta^4)^2 + 2Q_{46}\,\theta^4\theta^6 + Q_{66}\,(\theta^6)^2, \tag{5.21}$$

where $Q_{ij} = Q_{ij}(x^1, x^3, y^2)$. *Every such $Q$ gives an ambient metric for $g_{\mathcal{D}_f}$ by*

$$\tilde{g}_{f,Q} = 2d(\varrho t)dt + t^2(\, g_{\mathcal{D}_f} + 2\,\varrho\, P + \varrho^3\, Q(x^1, x^3, y^2)\,), \tag{5.22}$$

*with $P$ the Schouten tensor of $g_{\mathcal{D}_f}$.*

The tensor $Q$, which is trace-free with respect to $g_{D_f}$, is responsible for the ***Fefferman–Graham ambiguity***, that is, for the ambiguity of ambient metrics for conformal classes in even dimensions. As we shall see, the Fefferman–Graham ambiguity contributes to the holonomy of the metric $\tilde{g}_{f,Q}$ and, therefore, cannot be gauged away by a coordinate transformation.

**Remark 5.1** (Non-vanishing obstruction)**.** We close this section with the observation that *not all* of Bryant's conformal structures have a vanishing obstruction tensor. This happens, for example, for the rank 3 distribution associated with the function $f = x^3 + x^1x^2 + (x^2)^2 + (x^3)^2$. Consequently, the conformal class for this $f$ does not admit an analytic ambient metric.

**5.3. The Fefferman–Graham diffeomorphism.** If $g_1$ and $g_2$ are conformally equivalent metrics, then there exists a local diffeomorphism $\chi$ such that the corresponding Fefferman–Graham metrics satisfy

$$\tilde{g}_1 - \chi_*(\tilde{g}_2) \in O(\rho^N), \tag{5.23}$$

where $N = \infty$ for $n$ odd and $N = n/2 - 1$ for $n$ even. The Fefferman–Graham diffeomorphism $\chi$ is constructed by ODE methods and, hence, it is quite remarkable that we are able to explicitly find this map for the Bryant conformal classes under consideration.

Let $\tilde{g}_1$ be the metric (5.22) and let

$$\tilde{g}_2 = 2d(\varrho t)dt + t^2(\, \kappa^2 g_{\mathcal{D}_f} + 2\,\varrho\, P_{\kappa^2} + \varrho^3\, R(x^1, x^3, y^2)\,). \tag{5.24}$$

Here $\kappa = \kappa(x^1, x^3, y^2)$, our conformal factor is $\kappa^2$, $P_{\kappa^2}$ is the Schouten tensor for the scaled metric, and $R$ another choice for the Fefferman–Graham ambiguity (as in (5.21)).

The diffeomorphism $\chi$ is then found to be

$$\chi(t, \rho, x^1, x^2, x^3, y^1, y^2, y^3) = \\ = \left(\frac{t}{\kappa}, \kappa^2\rho, x^1, x^2 + \mu\rho, x^3, y^1 + \frac{\kappa_{x^1} - \kappa_{y^2} f}{\kappa f_{x^3}}\rho, y^2, y^3 + \mu x^1 \rho + \frac{\kappa_{x^3}}{\kappa f_{x^3}}\rho\right),$$

where

$$\mu = \frac{f_{x^3x^3}}{3\kappa(f_{x^3})^2}\,\kappa_{x^1} - \frac{f_{x^1x^3} + f_{x^3}f_{y^2} - f f_{x^3y^2}}{3\kappa(f_{x^3})^2}\kappa_{x^3} - \frac{f f_{x^3x^3} - 3(f_{x^3})^2}{3\kappa(f_{x^3})^2}\kappa_{y^2},$$

and we find that

$$\chi_*(\tilde{g}_2) - \tilde{g}_1 = \rho^3(\kappa^4 R - Q).$$

In short, the effect on the Fefferman–Graham metric (5.22) of choosing a different representative for the Bryant conformal class is diffeomorphically equivalent to a scaling of the Fefferman–Graham ambiguity $Q$. Therefore, any non-zero component of $Q$ may be normalized to $\pm 1$.

**5.4. The holonomy of the ambient metrics.** We now will study the holonomy of ambient metrics $\tilde{g}_{f,Q}$ for conformal classes $[g_{\mathcal{D}_f}]$. The 2-form

$$(5.25)\qquad \begin{aligned}\alpha = t\,dt\wedge\theta^2 &+ \frac{t^2(2f_{x^3y^2}f + f_{x^3}f_{y^2} - 2f_{x^1x^3})}{6f_{x^3}^2}\,\theta^2\wedge\theta^4\\ &- \frac{t^2 f_{x^3x^3}}{3f_{x^3}^2}\,\theta^2\wedge\theta^6 - \frac{t^2}{2}\,\theta^4\wedge\theta^6\end{aligned}$$

plays a distinguished role in our analysis. Accordingly, we introduce another co-frame $\{\omega^i\}$ so that the Fefferman–Graham metric becomes

$$(5.26)\qquad \tilde{g}_{f,Q} = \omega^1\omega^5 + \omega^2\omega^6 + \omega^3\omega^7 + \omega^4\omega^8,$$

(that is, $\{\omega^i\}$ is a null co-frame) and the 2-form $\alpha$ becomes

$$(5.27)\qquad \alpha = \omega^5\wedge\omega^8 + \omega^6\wedge\omega^7.$$

There is a unique such co-frame of the form

$$(5.28)\qquad \begin{aligned}\omega^1 &= \frac{\rho}{t}\,dt + d\rho + c^1_4\,\theta^4 + c^1_6\,\theta^6,\\ \omega^2 &= c^2_\rho\,d\rho + c^2_1\,\theta^3 + c^2_4\,\theta^4\,\theta^6,\\ \omega^3 &= c^3_\rho d\rho + c^3_1\,\theta^1 + c^3_4\,\theta^4 + c^3_6\,\theta^6,\\ \omega^4 &= c^4_2\,\theta^2 + c^4_4\,\theta^4 + c^4_5\,\theta^5 + c^4_6\theta^6,\\ \omega^5 &= t\,dt + c^5_4\,\theta^4 + c^5_6\,\theta^6,\\ \omega^6 &= t\,\theta^4,\\ \omega^7 &= c^7_6\,\theta^6,\\ \omega^8 &= \theta^2.\end{aligned}$$

The formulas for the coefficients $c^i_j$ are easily determined but are rather lengthy and will not be given here. The dual frame is denoted by $\{E_i\}$.

A straight-forward **DG** calculation shows that the covariant derivative of $\alpha$ satisfies

(5.29)

$$\nabla\alpha = \alpha\otimes\lambda_1 + 3\rho^2(\lambda_6\otimes\omega^6 + \lambda_7\otimes\omega^7 + \lambda_8\otimes\omega^8), \quad \text{where}$$

$$\lambda_1 = -\frac{2f f_{x^3y^2} + f_{x^3} f_{y^2} - 2f_{x^1x^3}}{2t(f_{x^3})^2}\,\omega^6 + \frac{3f_{x^2y^2} f_{x^3} - f_{x^3x^3} f_{y^2}}{3(f_{x^3})^3}\,\omega^8,$$
$$\lambda_6 = \frac{1}{2} Q_{44}\,\omega^6 \wedge \omega^8 - \frac{1}{2} Q_{46}\,\omega^7 \wedge \omega^8,$$
$$\lambda_7 = -\frac{1}{2} S_{46}\,\omega^4 \wedge \omega^6 + 2Q_{66}\,\omega^7 \wedge \omega^8,$$
$$\lambda_8 = \frac{t}{4} Q_{24}\,\omega^4 \wedge \omega^6 - \frac{t}{2} Q_{26}\,\omega^7 \wedge \omega^8.$$

The exterior derivative of $\lambda_1$ takes the form

$$d\lambda_1 = \kappa_1\,\omega^6 \wedge \omega^8 + \kappa_2\,\omega^7 \wedge \omega^8, \quad \text{with}$$

(5.30)
$$\kappa_1 = \frac{1}{3t(f_{x3})^2} \frac{\partial}{\partial y^2}\big(f_{y^2} - \frac{f\, f_{x^3y^2}}{f_{x^3}} + \frac{f_{x^1x^3}}{f_{x^3}}\big), \quad \text{and}$$
$$\kappa_2 = -\frac{2}{3t(f_{x^3})^2} \frac{\partial^2 \log(f_{x^3})}{\partial y^2 \partial x^3}.$$

Let

(5.31)
$$\begin{aligned} \beta_1 &= \tfrac{1}{2}\alpha \wedge \alpha = \omega^5 \wedge \omega^6 \wedge \omega^7 \wedge \omega^8, \\ \beta_2 &= \omega^{1567} - \omega^{2568} - \omega^{3578} + \omega^{4678}, \quad \text{and} \\ \beta_3 &= \omega^{1256} + \omega^{1357} - \omega^{1458} - 2\,\omega^{1467} - 2\,\omega^{2358} \\ &\quad - \omega^{2367} + \omega^{2468} + \omega^{3478}. \end{aligned}$$

Equations (5.29) shows that $\beta_1$ is recurrent, that is, $\nabla\beta_1 = 2\lambda_1 \otimes \beta_1$. It, therefore, follows that the holonomy representation for $\tilde{g}_{f,Q}$ is, with respect to the co-frame (5.28), a sub-representation of

(5.32)
$$\mathfrak{gl}_4\mathbb{R} \ltimes \Lambda_4\mathbb{R} = \left\{ \begin{pmatrix} X & Z \\ 0 & -X^\top \end{pmatrix} \mid X \in \mathfrak{gl}_4\mathbb{R},\ Z + Z^\top = 0 \right\} \subset \mathfrak{so}(4,4),$$

where $\Lambda_4\mathbb{R}$ denotes the skew symmetric $4 \times 4$ matrices. The representation of $\mathfrak{gl}_4\mathbb{R}$ on the Abelian ideal $\Lambda_4\mathbb{R}$ is given by $X \cdot Z = XZ - (XZ)^\top$.

To describe the holonomy algebras of the Fefferman–Graham metrices $\tilde{g}_{f,Q}$ in greater detail, we introduce various sub-algebras of $\mathfrak{gl}_4\mathbb{R} \ltimes \Lambda_4\mathbb{R}$. These are given in Table 1.

**Table 1.** Sub-algebras of $\mathfrak{so}(4,4)$

| Name | Dim | Matrices | Invariants |
|---|---|---|---|
| $\mathfrak{gl}_4\mathbb{R} \ltimes \Lambda_4\mathbb{R}$ | 22 | $X \in \mathfrak{gl}_4\mathbb{R},\ Z + Z^\top = 0$ | $\langle\beta_1\rangle$ |
| $\mathfrak{sl}_4\mathbb{R} \ltimes \Lambda_4\mathbb{R} \simeq \mathfrak{po}(3,3)$ | 21 | $\mathrm{tr}(X) = 0$ | $\beta_1$ |
| $(\mathbb{R} \oplus \mathfrak{sp}_2\mathbb{R}) \ltimes \Lambda_4\mathbb{R}$ | 17 | $XJ + JX^\top = zJ$ | $\langle\alpha\rangle$ |
| $\mathfrak{sp}_2\mathbb{R} \ltimes \Lambda_4\mathbb{R}$ | 16 | $XJ + JX^\top = 0$ | $\langle\alpha\rangle$, $\beta_3$ |
| $\mathfrak{sp}_2\mathbb{R} \ltimes \mathbb{R}^{3,2} \simeq \mathfrak{po}(3,2)$ | 15 | $XJ + JX^\top = 0$, $Z \in \{J\}^\perp$ | $\alpha$, $\beta_1$, $\beta_2$, $\beta_3$ |

The $4\times 4$ matrix $J$ is defined by $J = \begin{pmatrix} 0 & -\mathrm{I}_2 \\ \mathrm{I}_2 & 0 \end{pmatrix}$. The representation of $\mathfrak{sl}_4\mathbb{R}$ on $\Lambda^2\mathbb{R}^4 \simeq \Lambda_4\mathbb{R}$ carries an $\mathfrak{sl}_4\mathbb{R}$-invariant bilinear form of signature $(3,3)$ defined by the relation $\sigma\wedge\ \xi = \langle\sigma,\xi\rangle\, e_1\wedge e_2\wedge e_3\wedge e_4$. This leads to the isomorphism $\mathfrak{sl}_4\mathbb{R}\simeq\mathfrak{so}(3,3)$. Hence, $\mathfrak{sl}_4\mathbb{R}\ltimes\Lambda_4\mathbb{R}\ \simeq\ \mathfrak{po}(3,3)$, where $\mathfrak{po}(3,3)=\mathfrak{so}(3,3)\ltimes\mathbb{R}^{3,3}$ denotes the Poincaré algebra in signature $(3,3)$.

A similar isomorphism exists for $\mathfrak{sp}_2\ltimes\Lambda_4\mathbb{R}$. First, $\mathfrak{sp}_2\mathbb{R}$ acts trivially on $J$ by its very definition. Secondly, $J$ is non-degenerate with respect to the above signature $(3,3)$ scalar product and, hence, $\Lambda_4\mathbb{R}$ splits invariantly under $\mathfrak{sp}_2\mathbb{R}$ as

$$\Lambda_4\mathbb{R} = (\mathbb{R}\cdot J)^\perp\oplus\mathbb{R}\cdot J = \mathbb{R}^{3,2}\oplus\mathbb{R}. \tag{5.33}$$

This furnishes the isomorphism $\mathfrak{sp}_2\mathbb{R}\simeq\mathfrak{so}(3,2)$ and so $\mathfrak{sp}_2\ltimes\Lambda_4\mathbb{R}\simeq\mathfrak{po}(3,2)\oplus\mathbb{R}$, where $\mathfrak{po}(3,2)=\mathfrak{so}(3,2)\ltimes\mathbb{R}^{3,2}$ denotes the Poincaré algebra in signature $(3,2)$. The summand $\mathbb{R}$ is the center of $\mathfrak{sp}_2\ltimes\Lambda_4\mathbb{R}$ and is given by the matrix $Z=J$ in (5.32)

We note that the representations $\mathfrak{po}(3,3)$ and $\mathfrak{po}(3,2)$ given here are *not* equivalent to the more familiar representation of $\mathfrak{po}(3,3)$ and $\mathfrak{po}(3,2)$ in $\mathfrak{so}(4,4)$ and $\mathfrak{so}(3,3)$ as the stabilizers of a null vector.

Finally, we remark that our representation of $\mathfrak{po}(3,2)$ is contained in $\mathfrak{spin}(4,3)$. To verify this we introduce the co-frame

$$\begin{aligned}
\xi^1 &= -\frac{a}{64}\,\omega^1 + a\,\omega^5, & \xi^2 &= \frac{a}{16}\,\omega^2 - \frac{a}{4}\,\omega^6, & \xi^3 &= \frac{a}{8}\,\omega^3 - \frac{a}{8}\,\omega^7,\\
\xi^4 &= \frac{a}{2}\,\omega^4 - \frac{a}{32}\,\omega^8, & \xi^5 &= \frac{a}{16}\,\omega^2 + \frac{a}{4}\,\omega^6, & \xi^6 &= -\frac{a}{64}\,\omega^1 - a\,\omega^5,\\
\xi^7 &= -\frac{a}{2}\,\omega^4 - \frac{a}{32}\,\omega^8, & \xi^8 &= -\frac{a}{8}\,\omega^3 - \frac{a}{8}\,\omega^7, & &\text{with } a=\sqrt{32}.
\end{aligned}$$

In this co-frame the metric $\tilde{g}_{f,Q}$ and the 4-form $\beta_3$ become

$$\tilde{g}_{f,Q} =\ -(\xi^1)^2-(\xi^2)^2-(\xi^3)^2-(\xi^4)^2+(\xi^5)^2+(\xi^6)^2+(\xi^7)^2+(\xi^8)^2,$$

and

$$\begin{aligned}
\beta_3 = \xi^{1234} &- \xi^{1256} + \xi^{1278} - \xi^{1357} - \xi^{1368} - \xi^{1458} + \xi^{1467}\\
&+ \xi^{2358} - \xi^{2367} - \xi^{2457} - \xi^{2468} + \xi^{3456} - \xi^{3478} + \xi^{5678}.
\end{aligned}$$

Thus, $\beta_3$ is precisely the 4-form whose stabilizer defines the representation $\mathfrak{spin}(4,3)\subset\mathfrak{so}(4,4)$, where we use the conventions in [4].

We have seen that the holonomy of $\tilde{g}_{f,Q}$ is contained in (5.32). The holonomy is further reduced when the 1-form $\lambda_1$ is closed or when the 2-forms $\lambda_6$, $\lambda_7$ and $\lambda_8$ in (5.29) all vanish.

**Proposition 5.2.** *The 1-form $\lambda_1$, defined by* (5.30)*, is closed if and only if*

$$f(x^1,x^3,y^2) = A(x^1,x^3). \tag{5.34}$$

*In this case the 4-form* $\tilde{\beta}_1 = \zeta^2 \beta_1$, *with* $\zeta = \dfrac{1}{(A_{x^3})^{\frac{2}{3}}}$, *is parallel and the holonomy of* $\tilde{g}_{f,Q}$ *is reduced to a sub-representation of* $\mathfrak{sl}_4\mathbb{R} \ltimes \Lambda_4(\mathbb{R}) \simeq \mathfrak{po}(3,3)$.

*Proof.* The 4-form $\beta_1$ may be scaled to a parallel form $\tilde{\beta}_1$ if and only if the 1-form $\lambda_1$ is closed, that is, $\kappa_1 = \kappa_2 = 0$. The equation $\kappa_2 = 0$ is easily integrated to yield

$$f(x^1, x^3, y^2) = P(x^1, y^2)A(x^1, x^3) + Q(x^1, y^2).$$

Since $f_{x^3} \neq 0$, we must have $P \neq 0$. Then, with $B = B(x^1, y^2)$, $P = \dfrac{1}{B_{y^2}}$ and $Q = \dfrac{B_{x1} + C(x^1, y^2)}{B_{y^2}}$, the equation $\kappa_1 = 0$ reduces to

$$\frac{\partial}{\partial y^2} \frac{C_{y^2}}{B_{y^2}} = 0,$$

which implies

$$C = k(x^1)B + \ell(x^1).$$

The function $\ell$ can immediately be absorbed into the function $A(x^1, x^3)$. A simultaneous scaling of $A$ and $B$ by a function of $x^1$ can be used to translate $k(x^1)$ to 0. We conclude that

$$f(x^1, x^3, y^2) = \frac{A + B_{x^1}}{B_{y^2}}.$$

Making the change of variables $\tilde{y}^2 = B(x^1, y^2)$ and noting that

$$\tilde{\theta}^2 = d\tilde{y}^2 + A\,dx^1 = B_{y^2}dy^2 + (A + B_{x^1})\,dx^1 = B_{y^2}(dy^1 + F dx^1) = B_{y^2}\theta^2$$

finishes the proof. q.e.d.

Since $\mathfrak{po}(3,3)$ and $\mathfrak{spin}(4,3)$ have the same dimension, but are different, we obtain the following.

**Corollary 5.1.** *Ambient metrics* (5.22) *for Bryant's conformal structures* $g_{\mathcal{D}_f}$, *with* $f = f(x^1, x^3, y^2)$, *cannot have holonomy equal to* $\mathfrak{spin}(4,3)$.

We now describe the reduction of holonomy in the case where $\lambda_6 = \lambda_7 = \lambda_8 = 0$. Let $\mathcal{B}$ be the sub-bundle of degree 4-forms spanned by the forms (5.31).

**Proposition 5.3.** *Assume* $\lambda_6 = \lambda_7 = \lambda_8 = 0$.
**[i]** *Then* $\alpha$ *is a recurrent 1-form and the holonomy is reduced to a sub-representation of* $(\mathbb{R} \oplus \mathfrak{sp}_2\mathbb{R}) \ltimes \Lambda_4(\mathbb{R})$. *The bundle* $\mathcal{B}$ *is parallel and, hence, holonomy invariant.*
**[ii]** *If, in addition,* $\kappa_2 = 0$ *then there is a least 1 parallel 4-form in* $\mathcal{B}$ *and the holonomy reduces to a sub-representation of* $\mathfrak{sp}_2\mathbb{R} \ltimes \Lambda_4(\mathbb{R})$.

**[iii]** *If, in addition,* $\kappa_1 = \kappa_2 = 0$ *(so that* $d\lambda_1 = 0$*), then a multiple* $\tilde\alpha$ *of* $\alpha$ *is parallel and there are three parallel 4-forms* $\tilde\beta_1, \tilde\beta_2, \tilde\beta_3 \in \mathcal{B}$*. The holonomy is reduced to a sub-representation of* $\mathfrak{sp}_2\mathbb{R} \rtimes \mathbb{R}^{3,2} \simeq \mathfrak{po}(3,2) \subset \mathfrak{spin}(4,3)$.

*Proof.* The first statement in **[i]** is immediate from (5.29). The 4-form bundle $\mathcal{B}$ is parallel by explicit computation,—one finds that $\nabla\beta_a = \beta_b \otimes \Lambda^b_a$, where

$$\Lambda = \begin{bmatrix} 2\lambda_1 & \lambda_2 & 0 \\ 0 & \lambda_1 & -\lambda_2 \\ 0 & 0 & 0 \end{bmatrix},$$

and

$$\Upsilon = d\Lambda + \Lambda\wedge\Lambda = \begin{bmatrix} 2d\,\lambda_1 & d\,\lambda_2 + \lambda_1\wedge\lambda_2 & 0 \\ 0 & d\,\lambda_2 & -d\,\lambda_2 - \lambda_1\wedge\lambda_2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here $\lambda_2$ is a 1-form whose explicit formula we shall not need. The coefficients of any parallel 4-form $\tilde\beta = b_1\beta_1 + b_2\beta_2 + b_3\beta_3$ must satisfy $\Upsilon\cdot[b_1, b_2, b_3]^\top = 0$ in which case, generically, $b_1 = b_2 = b_3 = 0$. However, if $\kappa_2 = 0$, one finds that $\Upsilon$ takes the simpler form

$$\Upsilon = \kappa_1 \begin{bmatrix} 2\,\omega^6\wedge\omega^8 & \nu\,\omega^6\wedge\omega^8 & 0 \\ 0 & \omega^6\wedge\omega^8 & -\nu\,\omega^6\wedge\omega^8 \\ 0 & 0 & 0 \end{bmatrix}.$$

The coefficients of a parallel form $\beta$ must now satisfy $2b_1 + \nu b_2 = 0$ and $b_2 - \nu b_3 = 0$ and, indeed, one finds that $\tilde\beta_3 = \beta_3 + \nu\beta_2 - \frac{1}{2}\nu^2\beta_1$ is a parallel form. The coefficient $\nu$ (taking into account that $\kappa_2 = 0$) is given by

$$\nu = \rho\left(\frac{f_{x^1x^3x^3}}{(f^3_x)} + (10f f_{x^3y^2} + 5f_{x^3}f_{y^2} - 19f_{x^1x^3})\frac{f_{x^3x^3}}{9(f_{x^3})^4} - \frac{f_{x^3y^2}}{(f_{x^3})^2}\right).$$

To prove **[iii]**, we take $f$ to be of the form (5.34) in which case we find that

$$\lambda_1 = -d\,\ln(\zeta) = -\frac{d\zeta}{\zeta} \quad\text{and}\quad \lambda_2 = \zeta\,d(\frac{\nu}{\zeta}). \tag{5.35}$$

The 4-forms

$$\tilde\beta_1 = \zeta^2\beta_1, \quad \tilde\beta_2 = \zeta\,(\beta_2 - \nu\beta_1), \quad\text{and}\quad \tilde\beta_3$$

are then parallel. Under the $SO(4,4)$ rotation $\omega^1 \to \omega^1 - \dfrac{\nu}{2}\omega^8$, $\omega^4 \to \omega^4 + \dfrac{\nu}{2}\omega^5$ the forms $\alpha$ and $\tilde\beta_a$ become

$$\hat\alpha = \alpha, \quad \hat\beta_1 = \zeta^2\beta_1, \quad \hat\beta_2 = \zeta\beta_2, \quad \hat\beta_3 = \beta_3.$$

In other words, after this simple change of frame the original forms (5.31) are parallel after the appropriate scaling. This proves the statements regarding the holonomy in parts **[ii]** and **[iii]** in the statement of the proposition. Again, we find it rather surprising that the parallel forms for this broad class of Fefferman–Graham metrics can be explicitly computed. q.e.d.

**Remark 5.2.** Since the conformal holonomy is contained in the ambient holonomy, the result in [22, Theorem 1] implies that a conformal class $[g_{\mathcal{D}_f}]$ with $f = f(x^1, x^3)$ contains a certain preferred metric $g_0$. This metric admits a parallel totally null rank 3 distribution which contains the image of the Ricci-tensor (or equivalently, of the Schouten tensor). We have already established in the proof of Proposition 5.1 that the metric $g_{\mathcal{D}_f}$ is equal to this metric $g_0$, which shows that we have chosen a suitable conformal factor.

Note that the null 4-plane $\widetilde{\mathcal{V}}$ of vectors $X$ such that $X \lrcorner \alpha = 0$ is the only subspace that is invariant under the holonomy. In particular, it does not admit any invariant lines, reflecting the fact that the conformal classes defined by $f = f(x^1, x^3)$ generically are not conformally Einstein.

Table 2 gives a few examples of the holonomy for various explicit Fefferman–Graham metrics.

**Remark 5.3.** In case **6** in Table 2, the holonomy is the 12 dimensional matrix algebra defined by (see (5.32)) $X = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, where $a \in \mathfrak{sl}_2(\mathbb{R})$ and $b \in \mathfrak{gl}_2(\mathbb{R})$, and with $Z_{38} = 0$. This algebra is the sub-algebra of $\mathfrak{gl}_4(\mathbb{R}) \ltimes \Lambda_4(\mathbb{R})$ which stabilizes the forms $\{\omega^5 \wedge \omega^6, \omega^{356}, \omega^{456}, \omega^{567}, \omega^{568}\}$. There is one parallel 2-form, three parallel 3-forms and six parallel 4-forms.

**Remark 5.4.** For case **7** in Table 2 the holonomy algebra is spanned by the endomorphisms

$$
\begin{aligned}
h_1 &= E_1 \otimes \omega^4 - E_8 \otimes \omega^5, \\
h_5 &= -E_1 \otimes \omega^3 - E_2 \otimes \omega^4 + E_7 \otimes \omega^5 + E_8 \otimes \omega^6, \\
h_2 &= E_2 \otimes \omega^2 - E_7 \otimes \omega^6, \\
h_6 &= E_1 \otimes \omega^7 - E_3 \otimes \omega^6, \\
h_3 &= \tfrac{1}{2}(E_1 \otimes \omega^8 + E_2 \otimes \omega^7 - E_3 \otimes \omega^6 - E_4 \otimes \omega^5), \\
h_4 &= E_1 \otimes \omega^6 - E_2 \otimes \omega^5, \\
h_7 &= -E_2 \otimes \omega^8 + E_4 \otimes \omega^6.
\end{aligned}
$$

This algebra is the sub-algebra of $\mathfrak{gl}_4(\mathbb{R}) \ltimes \Lambda_4(\mathbb{R})$ which stabilizes the forms $\{\omega^5, \omega^6, \omega^3 \wedge \omega^5 - \omega^4 \wedge \omega^6\}$. The non-trivial structure equations are $[h_1, h_6] = [h_2, h_5] = [h_3, h_4] = e_7$ and, hence, the holonomy algebra is isomorphic to the Heisenberg algebra $\mathfrak{he}_3(\mathbb{R})$ (but not with its standard representation).

There are two parallel 1-forms, three parallel 2-forms, six parallel 3-forms and nine parallel 4-forms. The parallel 1-forms are null and

**Table 2.** Explicit holonomy for metrics for $\tilde{g}_{f,Q}$

| | $f$ | $Q$ | $\lambda_*$ | $[\kappa_1, \kappa_2]$ | rk. $\Upsilon$ | forms | Holonomy | dim. |
|---|---|---|---|---|---|---|---|---|
| **1.** | $(x^3)^2 + (y^2)^2$ | $Q_{44} = 1$ | $\lambda_6 \neq 0$ | $[\frac{1}{6(x^3)^2}, 0]$ | na | $\langle \beta_1 \rangle$ | $\mathfrak{gl}_4\mathbb{R} \ltimes \Lambda_4\mathbb{R}$ | 22 |
| | | $Q_{46} = 1$ | $\lambda_6 \neq 0$ $\lambda_7 \neq 0$ | $\kappa_2 = 0$ | | | | |
| **2.** | $x^1(x^3)^2$ | $Q_{66} = 1$ | | $\kappa_1 = 0$ $\kappa_2 = 0$ | na | $\tilde{\beta}_1$ | $\mathfrak{sl}_4\mathbb{R} \ltimes \Lambda_4\mathbb{R}$ $\simeq \mathfrak{po}(3,3)$ | 21 |
| **3.** | $(x^1)^2 + (x^3)^2$ $+(y^2)^2$ | $Q_{22} = 1$ | 0 | $[\frac{1}{6(x^3)^2}, 0]$ $\kappa_2 = 0$ | 3 | $\langle \alpha \rangle$, $\langle \beta_1 \rangle$ | $(\mathbb{R} \oplus \mathfrak{sp}_2\mathbb{R})$ $\ltimes \Lambda_4\mathbb{R}$ | 17 |
| **4.** | $(x^1)^2 + (x^3)^2$ | $Q_{22} = 1$ | 0 | $[\frac{1}{6(x^3)^2}, 0]$ | 2 | $\langle \tilde{\alpha} \rangle$, $\tilde{\beta}_3$ | $\mathfrak{sp}_2\mathbb{R} \ltimes (\mathbb{R} \ltimes \mathbb{R}^{3,2})$ $\simeq \mathfrak{po}(3,2) \oplus \mathbb{R}$ | 16 |
| **5.** | $(x^3)^2$ | 0 | 0 | $\kappa_1 = 0$ $\kappa_2 = 0$ | 0 | $\tilde{\alpha}$, $\tilde{\beta}_1$ $\tilde{\beta}_2$, $\tilde{\beta}_3$ | $\mathfrak{sp}_2\mathbb{R} \ltimes \mathbb{R}^{3,2}$ $\simeq \mathfrak{po}(3,2)$ | 15 |
| **6.** | $x^3$ | $Q_{44} = 1$ | $\lambda_6 \neq 0$ | $\kappa_1 = 0$ $\kappa_2 = 0$ | na | Remark 5.3 | $(\mathfrak{sl}_2\mathbb{R} \ltimes \mathfrak{gl}_2\mathbb{R})$ $\ltimes \mathbb{R}^{3,2}$ | 12 |
| **7.** | $(x^1)^s x^3$ | 0 | 0 | $\kappa_1 = 0$ $\kappa_2 = 0$ | 0 | Remark 5.4 | $\mathfrak{he}_7\mathbb{R}$ | 7 |

mutually orthogonal. Since the conformal holonomy is contained in the ambient holonomy [3], the parallel 1-forms define two conformal standard tractors that are null and parallel for the normal conformal tractor connection. It is well-known (see, for example, [17, 21]) that parallel tractors, on a dense open set, correspond to local Einstein scales (or Ricci-flat scales in case of null tractors).

For the value $s = -\frac{3}{2}$ the parallel 1-forms take the simple form

$$\{\frac{2}{t}\omega^5 + \sqrt{x^1}\omega^6, \frac{1}{t\sqrt{x^1}}\omega^5\}.$$

The conformal class $[g_{\mathcal{D}_f}]$ contains the two parameter family of Ricci-flat metrics

$$\frac{1}{\Big((at+b)\sqrt{x^1}+ct\Big)^2}\ \tilde{g}_{f,Q}, \tag{5.36}$$

where $a, b, c$ are constants and $a^2 + b^2 + c^2 = 1$.

**Remark 5.5.** It is possible to determine various simple lower bounds on the holonomy dimension. For example, if $Q_{24} \neq 0$, then from the curvature endomorphisms $R(E_3, E_6)$, $R(E_3, E_8)$, $R(E_4, E_8)$, $R(E_5, E_6)$ one finds that the holonomy algebra contains

$$\{E_2\otimes\omega^7 - E_3\otimes\omega^6,\ E_2\otimes\omega^8 - E_4\otimes\omega^6,\ E_3\otimes\omega^8 - E_4\otimes\omega^7,\ E_4\otimes\omega^1 - E_5\otimes\omega^8\}.$$

The iterated directional covariant derivatives of these 4 tensors with respect to $E_6$, $E_7$, $E_8$ show that the dimension of the holonomy algebra is $\geq 15$.

## 6. Maple worksheets

In this section, we present the Maple worksheets used to derive the results of Section 3. The version of DifferentialGeometry software and the worksheets themselves can be obtained from the DifferentialGeometry website [1].

### 6.1. Worksheet 1.

| The conformal class associated to a Monge Equation |
|---|

▶ Synopsis

In this worksheet we shall derive the conformal class associated to the Monge equation (4.1), (4.2). We calculate the Fefferman–Graham equations for this class.

▶ Getting Started

We begin by loading the DG packages to be used.

```
> with(DifferentialGeometry): with(Tensor):
> with(ExteriorDifferentialSystems):
```

▶ The Conformal Class for a Rank 3 Pfaffian System

In this section, we shall construct the co-frame (4.7) used to define the conformal class (4.8). We begin by creating a coordinate chart.

```
> DGEnvironment[Coordinate]([x, y, p, q, z], M0):
```

For ease of notation, we suppress the printing of the arguments in the functions $f$ and $h$.

```
> DGSuppress(f(x, y, p), h(x, y));
```

The formulas used to define the conformal class are implemented in the program `NurowskiConformalClass`. With the keyword argument `version = 2`, the null coframe (4.3) is calculated.

```
> Omega, G := NurowskiConformalClass(F, M0, version = 2);
```

We scale the co-frame `Omega` as in (4.7).

```
> th1 := Omega[1]; th2 := Omega[2];
```

$$th1 := -p\,dx + dy \quad th2 := (q^2 - f - hz)\,dx + dz - 2q\,dy$$

```
> th3 := (1/sqrt(40)) &mult Omega[3]; th4 := (1/80) &mult Omega[4];
```

$$th3 := -2\sqrt{2}(-q\,dx + dp) \quad th4 := 3dx$$

```
> th5 := (1/80) &mult Omega[5];
```

$$th5 := (-4qh - 3f_p + \tfrac{9}{10}pf_{pp} + \tfrac{2}{5}ph^2 - \tfrac{3}{5}p^2h_y - \tfrac{3}{5}ph_x)\,dx + (-\tfrac{9}{10}f_{pp} - \tfrac{2}{5}h^2 + \tfrac{3}{5}ph_y + \tfrac{3}{5}h_x)\,dy - 2h\,dp + 6\,dq$$

```
> CoFrame := [th1, th2, th3, th4, th5]:
```

Now create an anholonomic co-frame environment.

```
> DGEnvironment[AnholonomicCoFrame](CoFrame, M5, vectorlabels = [E],
formlabels = [theta]):
```

The underlying coordinates are still $[x, y, z, p, q]$ but now tensorial quantities are expressed in terms of the coframe (4.7) and its dual (denoted here by $E_i$). Thus, the conformal class (4.8) is written as

```
> gN := evalDG(-2*theta1 &s theta5 - 2*theta2 &s theta4 + theta3 &t theta3):
```

Now calculate the structure equations (4.6). For example, we find that

```
> LieBracket(E5, E4);
```

$$\tfrac{\sqrt{2}}{9}E_3 + \tfrac{2}{9}hE_5$$

From these structure equations we deduced that the Fefferman–Graham equations for the ambient metric for (4.8) are the linear equations (2.10).

▶ The Linear Fefferman–Graham Equations

In this section, we evaluate each term in the linear Fefferman–Graham equations (2.10).

```
>  DGSuppress([A(x, y, p, rho), B(x, y, p, rho), C(x, y, p,rho)]);
```

Our ansatz (4.9) is

```
> H := evalDG(A(x, y, p, rho)*theta1 &t theta1 + B(x, y, p, rho)*2*theta1
&s theta4 + C(x, y, p,rho)*theta4 &t theta4)
```

The first two terms in (2.10) are calculated using `DGmap` to map the differentiation command `diff` onto the coefficients of `H`.

```
> FG1 := rho &mult DGmap(1, diff, H, rho, rho);
```

$$FG1 := \rho A_{\rho,\rho}\,\theta^1 \otimes \theta^2 + \rho B_{\rho,\rho}\,(\theta^1 \otimes \theta^4 + \theta^4 \otimes \theta^1) + \rho C_{\rho,\rho}\,\theta^4 \otimes \theta^4$$

```
> FG2 := (-3/2)&mult DGmap(1, diff, H, rho);
```

$$FG2 := -\tfrac{3}{2}\rho A_\rho\,\theta^1\otimes\theta^2 - \tfrac{3}{2}\rho B_\rho\,(\theta^1\otimes\theta^4+\theta^4\otimes\theta^1) - \tfrac{3}{2}\,\rho C_\rho\,\theta^4\otimes\theta^4$$

For the Laplacian term we need the inverse metric and the Christoffel connection of gN.

```
> InversegN := InverseMetric(gN):
> nabla := Christoffel(gN):
```

The Laplacian is given by the trace of the second covariant derivatives.

```
> FG3a := CovariantDerivative(CovariantDerivative(H, nabla), nabla);
> FG3 := (-1/2) &mult ContractIndices(InversegN, FG3a, [[1, 3], [2, 4]])
```

$$-\tfrac{1}{16}A_{p,p}\,\theta^1\otimes\theta^1 + (-\tfrac{1}{16}B_{p,p}+\tfrac{1}{72}A_p)\,(\theta^1\otimes\theta^4+\theta^4\otimes\theta^1) + (-\tfrac{1}{16}C_{p,p} - \tfrac{1}{36}B_p + \tfrac{1}{648}A)\,\theta^4\otimes\theta^4$$

The curvature term is computed next.

```
> Riem := CurvatureTensor(gN):
> RiemUp := RaiseLowerIndices(InversegN, R, [4]):
> FG4 := ContractIndices(RiemUp, H, [[1, 1], [4, 2]]);
```

$$0\,\theta^1\otimes\theta^1$$

Finally, here are the terms involving the Ricci tensor.

```
> Ricci := RicciTensor(gN);
```

$$-\tfrac{9}{80}f_{p,p,p,p}\,\theta^1\otimes\theta^1 - (\tfrac{9}{40}h_y+\tfrac{3}{80}f_{p,p,p})\,(\theta^1\otimes\theta^4+\theta^4\otimes\theta^1) - (\tfrac{1}{10}h_x + \tfrac{1}{10}h_y p - \tfrac{1}{15}h^2 + \tfrac{1}{60}f_{p,p})\,\theta^4\otimes\theta^4$$

```
> RicciUp := RaiseLowerIndices(Inverse_gN, Ricci, [1]):
> FG5 := ContractIndices(RicciUp, H, [[1, 1]]);
```

$$0\,\theta 1\otimes\theta 1$$

The sum of these terms gives the Fefferman–Graham equations.

```
> FG := evalDG(FG1 + FG2 + FG3 + FG4 + FG5 + Ricci):
```

The command `DGinformation` allows us to extract the individual components of the tensor `FG`, for example,

```
> 2*DGinformation(FG, "CoefficientList", [[1, 4]])[1];
```

$$2\rho B_{\rho,\rho} - 3B_\rho - \tfrac{1}{8}B_{p,p} + \tfrac{1}{36}A_p - \tfrac{9}{20}h_y - \tfrac{3}{40}f_{p,p,p}$$

See equation (4.10).

### 6.2. Worksheet 2.

| The Ambient Metric and Its Infinitesimal Holonomy |
| --- |

► Synopsis

In this worksheet we study the infinitesimal holonomy of the Fefferman–Graham metric for the Nurowski conformal class. Proposition 4.1 is established and then the calculations for Theorem 4.4 are illustrated. The holonomy for the first metric in Example 4.1 is computed. The holonomy algebra is shown to be the split real form of $\mathfrak{g}_2$

▶ Getting Started

We begin by loading the **DG** packages to be used.

```
> with(DifferentialGeometry): with(Tensor): with(LieAlgebras):
>  with(ExteriorDifferentialSystems):
```

This time we also load the small package `ExplicitAmbientMetricsAndHolonomy` which was specifically created to support the computations for this paper. This package is loaded separately with a `read` statement.

```
> read "C:\\Users\\Ian\\Desktop\\ExplicitAmbientMetricsAndHolonomy.mm";
> with(ExplicitAmbientMetricsAndHolonomy);
> DGSuppress(f(x,y,p), h(x,y), A(x, y, p, rho), B(x, y, p, rho), C(x, y,
p, rho));
```

▶ The Ambient Metric

The command `NurowskiFeffermanGrahamMetric` calculates the Fefferman–Graham metric for a given choice of functions $f$ and $h$ and solutions $A$, $B$, $C$ to the linear Fefferman–Graham equations.

We use `infolevel` to obtain information regarding the various frames which are defined during execution.

```
> infolevel[ NurowskiFeffermanGrahamMetric] := 2:
> NFG := NurowskiFeffermanGrahamMetric(M, f(x,y,p), h(x,y), A(x,y,p,rho),
B(x,y,p,rho), C(x,y,p,rho))
```

*Calculating the Nurowski coframe (dim. 5) in the coordinate frame M0*
*Initializing anholonomic frame M1(dim. 5) from the Nurowski coframe*
*vectorlabels = [E], formlabels = [theta]*
*Initializing anholonomic frame M2(dim. 5) for the scaled Nurowski coframe*
*vectorlabels = [E], formlabels = [theta]*
*Initializing anholonomic frame M3(dim. 7) for Fefferman–Graham normal metric*
*vectorlabels = [E], formlabels = [Tau, Rho, theta]*
*Initializing anholonomic frame M4(dim. 7) for Fefferman–Graham null metric*
*vectorlabels = [X], formlabels = [omega]*
*Initializing anholonomic frame M5(dim. 7) for Fefferman–Graham orthonormal metric*
*vectorlabels = [Z], formlabels = [sigma]*

The output of `NurowskiFeffermanGrahamMetric` is a record giving all the various coframes and metrics.

The export `NurowskiScaledCoframe` of the record `NFG` gives the co-frame (4.7), for example,

```
> NFG:-NurowskiScaledCoframe[5];
```

$$(-4qh - 3f_p + \tfrac{9}{10}pf_{pp} + \tfrac{2}{5}ph^2 - \tfrac{3}{5}p^2h_y - \tfrac{3}{5}ph_x)\,dx + (-\tfrac{9}{10}f_{pp} - \tfrac{2}{5}h^2 + \tfrac{3}{5}ph_y + \tfrac{3}{5}h_x)\,dy - 2h\,dp + 6\,dq$$

Various forms of the Fefferman–Graham metric are available.

```
> FGmetricNormal := NFG:-FeffermanGrahamNormalMetric;
```

$$2\rho dT\otimes dT + tdT\otimes dP + tdP\otimes dT + t^2A\theta^1\otimes\theta^1 + t^2B\theta^1\otimes\theta^4 - t^2\theta^1\otimes\theta^5 - t^2\theta^2\otimes\theta^4 + \cdots$$

```
> FGmetricNull := NFG:-FeffermanGrahamNullMetric;
```

$$\theta^1\otimes\theta^7 + \theta^2\otimes\theta^6 + +\theta^3\otimes\theta^5 + \theta^4\otimes\theta^4 + \theta^5\otimes\theta^3 + \theta^5\otimes\theta^2 + \theta^6\otimes\theta^1$$

```
> FGmetricOrtho := NFG:-FeffermanGrahamOrthonormalMetric;
```

$$\sigma^1 \otimes \sigma^1 + \sigma^2 \otimes \sigma^2 + \sigma^3 \otimes \sigma^3 + \sigma^4 \otimes \sigma^4 - \sigma^5 \otimes \sigma^5 - \sigma^6 \otimes \sigma^6 - \sigma^7 \otimes \sigma^7$$

The linear Fefferman–Graham equations are computed by

```
> FGEquations := FeffermanGrahamEquations(f(x,y,p), h(x,y), A(x,y,p,ρ),
B(x,y,p,ρ), C(x,y,p,ρ))
```

giving, for example,

```
> FGEquations[1];
```

$$2A_{\rho\rho} - 3A_{\rho} - \frac{1}{8}A_{pp} - \frac{9}{40}f_{pppp}$$

We check that the metric FGmetricNull is Ricci flat by virtue of the Fefferman–Graham equations.

```
> Ric := RicciTensor(FGmetricNull):
> DGsimplify(simplify(Ric, FGEquations));
```

$$0\,\omega^1 \otimes \omega^1$$

▶ The Parallel Three Form

The parallel 3-form for the Fefferman–Graham metric in the null co-frame is:

```
> Upsilon := NFG:-ThreeFormNull;
```

$$\Upsilon = 2\,\omega^1 \wedge \omega^2 \wedge \omega^3 - \omega^1 \wedge \omega^4 \wedge \omega^7 - \omega^2 \wedge \omega^4 \wedge \omega^6 - \omega^3 \wedge \omega^4 \wedge \omega^5 + \omega^5 \wedge \omega^6 \wedge \omega^7$$

We check that $\Upsilon$ is parallel by calculating its covariant derivative and simplifying the result using the first order system, as given by:

```
> FirOrdSys := FirstOrderSystem(f(x, y, p), h(x, y), [A(x, y, p, rho),
B(x, y, p, rho), C(x, y, p, rho)]);
> FirOrdSys[1]
```

$$B_p = \tfrac{5}{9}A - \tfrac{2}{9}\varrho A_{\varrho}$$

Calculate the Christoffel connection of the Fefferman–Graham metric.

```
> nabla := Christoffel(FGmetricNull):
```

Calculate the covariant derivative of the 3-form.

```
> nablaUpsilon := CovariantDerivative(Upsilon, nabla):
```

Simplify using the first order system.

```
> DGsimplify(DEtools[dsubs](FirOrdSys, nablaUpsilon));
```

$$0\,\omega^1 \otimes \omega^1 \otimes \omega^1 \otimes \omega^1$$

▶ The Holonomy of the Fefferman–Graham Metric with $A_\rho \neq 0$

The command G2Holonomy retrieves the 14 holonomy tensors $h_i$ listed in (4.19).

```
> Hol := G2Holonomy():
```

Let us check one of the equations in the proof of Theorem 4.4.

```
> Hol[3];
```

$$X_3 \otimes \omega^4 - X_3 \otimes \omega^5 + X_6 \otimes \omega^1 - X_7\omega^2$$

Calculate the directional derivative and simplify using the first order system.

```
> Dh3 := DirectionalCovariantDerivative(X5, Hol[3], nabla):
> Dh3 := DGsimplify(DEtools[dsubs](FirOrderSystem, Dh3));
```

$$\frac{\sqrt{2}}{18t}X_2 \otimes \omega^5 - (\cdots)X_3 \otimes \omega^2 - \frac{h}{9}X_3 \otimes \omega^4 - \frac{\sqrt{2}}{18t}X_3 \otimes \omega^6 \cdots$$

This reduces, modulo the previous holonomy tensors, to

```
> DGmod(Dh3, Hol[1 ..  3])
```

$$\frac{\sqrt{2}}{18t}X_2 \otimes \omega^5 - \frac{\sqrt{2}}{18t}X_4 \otimes \omega^6 + \frac{\sqrt{2}}{9t}X_4 \otimes \omega^1 - \frac{\sqrt{2}}{9t}X_7 \otimes \omega^4$$

which is, up to a scalar multiple, the next holonomy tensor

```
> Hol[4];
```

$$X_2 \otimes \omega^5 - X_3 \otimes \omega^6 + 2\, X_4 \otimes \omega^1 - 2\, X_7 \otimes \omega^4$$

▶ Example 4.3

We use `PolynomialSolutionsForAmbientMetricCoefficients` to find the polynomial solutions to the Fefferman–Graham equations from (4.12), with $f = p^3$, $h = 0$, $\alpha_0 = \beta_0 = \gamma_0 = 0$.

```
> A, B, C := PolynomialSolutionsForAmbientMetricCoefficients(p^3, 0, 0,
0,0);
```

$$A, B, C = 0, -\frac{3\rho}{20} - \frac{p\rho}{15}$$

Define the corresponding Fefferman–Graham metric

```
> NFG4_3 := NurowskiFeffermanGrahamMetric(M, p^3, 0, [A, B, C]);
> FGmetric4_3 := NFG4_3:-FeffermanGrahamNullMetric:
```

We set the infolevel to see the number of the curvature endomorphisms at each order.

```
> infolevel[InfinitesimalHolonomy] := 2
```

Calculate the holonomy.

```
> Hol4_3 := InfinitesimalHolonomy(FGmetric4_3, [t = 1, rho = 1, p = 0],
output = "tensor"):
```

*The dimension of the holonomy algebra at order 0 is 2*
*The dimension of the holonomy algebra at order 1 is 6*
⋮
*The dimension of the holonomy algebra at order 4 is 13*
*The dimension of the holonomy algebra at order 5 is 14*

The span of the holonomy tensors `Hol4_3` coincide with that previously defined.

```
> DGequal(Hol, Hol4_3);
```

*true*

▶ The Holonomy Algebra

We check that the holonomy algebra is, indeed, $\mathfrak{g}_2$. Convert the holonomy tensors to matrices, calculate the structure equations and initialize the resulting Lie algebra as `holalg`.

```
> HolMatrices := convert(Hol, DGMatrix):
> DGEnvironment[LieAlgebra](HolMatrices, holalg);
```

The holonomy algebra is semi-simple.

```
> Query(holalg, "SemiSimple");
```

*true*

Find a Cartan subalgebra

```
> CSA := CartanSubalgebra(holalg)
```

$$[e_7, e_8]$$

Calculate the root space decomposition and its associated properties.

```
> P := SimpleLieAlgebraProperties(CSA):
```

```
> P:-RootType;
```

$$\text{“}G2\text{”}$$

▶ EXAMPLE 4.4
We calculate the parallel 3-forms for the last metric. Begin with the solutions to the Fefferman–Graham equations.

```
> A, B, C := PolynomialSolutionsForAmbientMetricCoefficients(0, 0, 0, 0, c(x));
```

$$A, B, C := 0, 0, \rho^{5/2} c(x)$$

Here is the Fefferman–Graham metric.

```
> FGmetric4_4 := NFG4_4:-FeffermanGrahamNullMetric:
```

Find the holonomy.

```
> Hol4_4 := InfinitesimalHolonomy(FGmetric4_4, [t= 1, rho = 1, p = 1]):
```

It is 10-dimensional.

```
> nops(Hol4_4)
```

$$10$$

Simplify the holonomy matrices.

```
> Hol4_4 := CanonicalBasis(Hol4_4):
```

Find the holonomy invariant 3-forms.

```
> HolInv3Forms := InvariantTensors(Hol4_4, GenerateDGobjects[DGforms](3));
```

$$[\omega^1 \wedge \omega^2 \wedge \omega^5, \quad \omega^1 \wedge \omega^4 \wedge \omega^5, \quad \omega^1 \wedge \omega^5 \wedge \omega^6]$$

Find the parallel 3 forms.

```
> CovariantlyConstantTensors(FGmetric4_4, HolInv3Forms);
```

$$[18t^2\, \omega^1 \wedge \omega^2 \wedge \omega^5 - 3\sqrt{2}xt\, \omega^1 \wedge \omega^4 \wedge \omega^5 + \frac{x^2}{2}\, \omega^1 \wedge \omega^5 \wedge \omega^6, \quad -3\sqrt{2}\, \omega^1 \wedge \omega^4 \wedge \omega^5 + x\, \omega^1 \wedge \omega^5 \wedge \omega^6, \quad \omega^1 \wedge \omega^5 \wedge \omega^6]$$

We remark that the package `ExplicitAmbientMetricsAndHolonomy` also provides functionality for calculating the Fefferman–Graham metrics for the Bryant conformal class and for checking all the results in §5.

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