

PLURISUBHARMONIC ENVELOPES AND SUPERSOLUTIONS

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Abstract

We make a systematic study of (quasi-)plurisubharmonic envelopes on compact Kähler manifolds, as well as on domains of \mathbb{C}^n , by using and extending an approximation process due to Berman [Ber19]. We show that the quasi-plurisubharmonic envelope of a viscosity super-solution is a pluripotential super-solution of a given complex Monge–Ampère equation. We use these ideas to solve complex Monge–Ampère equations by taking lower envelopes of super-solutions.

Introduction

Weak subsolutions and supersolutions to complex Monge–Ampère equations play a central role in the development of complex analysis and geometry. These have been studied extensively, in the pluripotential sense, since the fundamental works of Bedford and Taylor [BT76, BT82].

The notions of viscosity sub/super/solutions to complex Monge–Ampère equations

$$(CMA) \quad (\theta + dd^c u)^n = e^u f dV$$

have been introduced in [HL09, EGZ11, W12, HL13]. It has been notably shown in [EGZ11, Theorem 1.9 and Proposition 1.11] that an u.s.c. function is a viscosity subsolution if and only if it is plurisubharmonic and a pluripotential subsolution. The connection between viscosity and pluripotential supersolutions has however remained mysterious so far. Our first main result gives a satisfactory answer to [DGZ16, Question 40]:

Theorem A. *Let v be a viscosity supersolution to a complex Monge–Ampère equation (CMA). The (quasi-)plurisubharmonic envelope $P(v)$ is a pluripotential supersolution to (CMA).*

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The envelope $P(v)$ depends on the context (local/global) (see Section 2). We refer the reader to section 3 for the definition of viscosity supersolutions and to Theorems 3.5 and 3.9 for more precise statements.

The proof of this result relies on an approximation scheme which is of independent interest. This method has been introduced by Berman (see [Ber19, Theorem 1.1]) when v is smooth. We need to extend it here in order to deal with functions v which are less regular, proving in particular the following:

Theorem B. *Let (X, ω) be an n -dimensional compact Kähler manifold and fix μ an arbitrary non-pluripolar positive measure. Let $v : X \rightarrow \mathbb{R}$ be a bounded Borel-measurable function. Let $\varphi_j \in \mathcal{E}^1(X, \omega)$ be the unique solution to the complex Monge–Ampère equation*

$$(\omega + dd^c \varphi_j)^n = e^{j(\varphi_j - v)} \mu.$$

Then (φ_j) converges in capacity to the (ω, μ) -envelope $P_{\omega, \mu}(v)$, where

$$P_{\omega, \mu}(v) = (\sup\{\varphi \in \text{PSH}(X, \omega) : \varphi \leq v \text{ } \mu\text{-a.e. on } X\})^*.$$

When μ is a volume form on X and u is quasi-upper-semicontinuous on X , the ω -psh upper (ω, μ) -envelope is just the usual ω -psh envelope (see Proposition 2.11).

Recall that a function v is *quasi-continuous* (resp. *quasi-usc*) if for all $\varepsilon > 0$, there exists an open subset U with capacity less than ε such that the restriction of v to the complement $X \setminus U$ is continuous (resp usc). Quasi-psh functions form a large class of quasi-continuous functions. We use here the *Monge–Ampère capacity* whose definition is recalled in Section 1.2.

It is classical that the minimum $\min(u, v)$ of two viscosity supersolutions u, v is again a viscosity supersolution. Note however that the minimum of two psh functions is no longer psh. It follows nevertheless from (an extension of) Theorem A that $P(\min(u, v))$ is both psh and a pluripotential supersolution. We extend this observation in Lemma 4.1 far beyond the viscosity frame (which deals with continuous densities), and use it to solve complex Monge–Ampère equations. We show in particular the following:

Theorem C. *Let (X, ω) be a compact Kähler manifold and μ a non-pluripolar Radon measure in some open subset of X . Assume there exists a finite energy subsolution $u_0 \in \mathcal{E}(X, \omega)$, i.e. such that*

$$(\omega + dd^c u_0)^n \geq e^{u_0} \mu, \text{ in } \Omega.$$

Then the envelope of supersolutions

$$\varphi := P\left(\inf\{\psi : \psi \in \mathcal{E}(X, \omega) \text{ and } (\omega + dd^c \psi)^n \leq \mathbf{1}_\Omega e^\psi \mu\}\right)$$

is the unique pluripotential solution of $(\omega + dd^c \varphi)^n = \mathbf{1}_\Omega e^\varphi \mu$.

The inequality $(\omega + dd^c\psi)^n \leq \mathbf{1}_\Omega e^\psi \mu$ is understood in the sense that the measure $(\omega + dd^c\psi)^n$ is concentrated in Ω where this inequality holds in the sense of positive Borel measures.

This result could be restated in the more familiar form: if there exists a subsolution, then there exists a solution. Let us emphasize here that the measure μ is not necessarily a Radon measure in all of X . A particular case of interest in complex differential geometry is when $\mu = f dV_X$ is a volume form whose density is smooth and positive in some Zariski open set Ω , but does not belong to L^1 . The existence of a subsolution insures that f does not blow up too fast near $\partial\Omega$ and is easy to check in many concrete examples: we thus provide an alternative proof of the existence of Kähler–Einstein metrics on varieties with negative first Chern class and semi log-canonical singularities, a result first obtained by different techniques by Berman–Guenancia [BG14] which generalizes the famous results of Yau [Yau78], and Aubin [Au78].

On our way to proving the above theorems, we establish several other results of independent interest: we show in particular that

- the finite energy functional I_p satisfies a quasi-triangle inequality for any $p > 0$ (Theorem 1.6 largely extends [BBEGZ19, Theorem 1.8]);
- generalized capacities are all quantitatively comparable (see Theorem 1.11 which generalizes [DNL15, Theorem 2.9]).

Organization of the paper. We introduce the main pluripotential tools in Section 1, providing simplifications and extensions of some useful results (see Theorems 1.6 and 1.11). We make a systematic study of (q)psh envelopes in Section 2 and establish Theorem B. Inspired by this convergence result we prove Theorem A in Section 3. Theorem C is proved in Section 4 while other applications are given in Section 5.

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1. Preliminaries

We review recent results in pluripotential theory and establish a few extensions of the latter that are needed in this paper. We refer the reader to [BT82, Kol98, GZ05, BEGZ10] and the references therein for more details.

In the whole paper, (X, ω) is a compact Kähler manifold of dimension $n \in \mathbb{N}^*$, and θ is a closed smooth $(1, 1)$ -form on X .

1.1. Pluripotential theory in big cohomology classes. A function u is called *quasi-plurisubharmonic* on X (*qps* for short) if in any local holomorphic coordinates it can be written as $u = \rho + \varphi$ where ρ is smooth

and φ is plurisubharmonic. It is called θ -plurisubharmonic (θ -psh for short) if it additionally satisfies

$$\theta + dd^c u \geq 0$$

in the weak sense of currents. We let $\text{PSH}(X, \theta)$ denote the set of all θ -psh functions on X which are not identically $-\infty$.

By definition the class $[\theta]$ is *big* if there exists $\psi \in \text{PSH}(X, \theta)$ such that $\theta + dd^c \psi \geq \varepsilon \omega$ for some small constant $\varepsilon > 0$.

A function u has *analytic singularities* if it can locally be written as

$$u(z) = c \log \sum_{j=1}^k |f_j(z)|^2 + h(z),$$

where the f_j 's are local holomorphic functions and h is smooth.

By the fundamental approximation theorem of Demailly [Dem92] any quasi-plurisubharmonic function u can be approximated from above by a sequence (u_j) of $(\alpha + \varepsilon_j \omega)$ -psh functions with analytic singularities, where α is a closed smooth $(1, 1)$ -form such that $\alpha + dd^c u \geq 0$. Applying this result to the potential ψ of the Kähler current $\theta + dd^c \psi$, it follows that there exists θ -psh functions with analytic singularities.

Following [Bou04, BEGZ10] the *ample locus* of $\{\theta\}$ is defined to be the set of all $x \in X$ such that there exists a θ -psh function on X with analytic singularities which is smooth in a neighborhood of x .

A θ -psh function u is said to have *minimal singularities* if it is less singular than any other θ -psh function on X , more precisely if for any $v \in \text{PSH}(X, \theta)$ there exists a constant $C = C(u, v)$ such that $u \geq v - C$. The function

$$V_\theta := \sup\{\varphi : \varphi \in \text{PSH}(X, \theta) \text{ and } \varphi \leq 0\}$$

has minimal singularities.

If u_1, \dots, u_k are θ -psh functions with minimal singularities on X , then they are locally bounded in the ample locus $\Omega := \text{Amp}(\{\theta\})$. By the seminal work of Bedford–Taylor [BT76, BT82] the current $\theta_{u_1} \wedge \dots \wedge \theta_{u_k}$ is well defined, and positive in Ω . As the total mass is bounded, one can extend it trivially to the whole of X . It was proved in [BEGZ10] that the current obtained by this trivial extension is closed. In particular, when $k = n$ and $u_1 = \dots = u_n$ this procedure defines the (non-pluripolar) Monge–Ampère measure of u .

For a general θ -psh function u , the approximants $u_j := \max(u, V_\theta - j)$ have minimal singularities. One can show that the sequence of positive Borel measures $\mathbf{1}_{\{u > V_\theta - j\}}(\theta + dd^c u_j)^n$ is increasing in j . Its limit

$$\text{MA}_\theta(u) := \lim \nearrow \mathbf{1}_{\{u > V_\theta - j\}}(\theta + dd^c u_j)^n$$

(in the strong sense of Borel measures) is *the non-pluripolar Monge–Ampère measure* of u . We denote it by $\text{MA}(u)$ if the form θ is fixed.

The *volume* of the class $[\theta]$ is given by the total mass of the Monge–Ampère measure of V_θ (see [Bou02, BEGZ10]).

A set $E \subset X$ is called pluripolar if locally it is contained in the $-\infty$ locus of some plurisubharmonic function. It was shown in [GZ05] that E is pluripolar if and only if $E \subset \{\phi = -\infty\}$ for some $\phi \in \text{PSH}(X, \omega)$. One can replace the Kähler form ω by any big form θ . Indeed if θ is big then there exists $\psi \in \text{PSH}(X, \theta)$ such that $\theta + dd^c\psi \geq \varepsilon\omega$ for some positive constant ε . The function $\psi' := \varepsilon\phi + \psi$ is θ -psh and its $-\infty$ -locus contains E .

1.2. Convergence in capacity. Following [BT82], [Kol03], [GZ05], [BEGZ10] we consider the Monge–Ampère capacity with respect to the form θ ,

$$\text{Cap}_\theta(E) := \sup \left\{ \int_E \text{MA}(u) : u \in \text{PSH}(X, \theta), V_\theta - 1 \leq u \leq V_\theta \right\}.$$

A sequence (u_j) is said to converge in capacity to u if for any $\varepsilon > 0$,

$$\lim_{j \rightarrow +\infty} \text{Cap}_\theta(|u_j - u| > \varepsilon) \rightarrow 0.$$

The definition does not depend on θ because Cap_θ and $\text{Cap}_{\theta'}$ are comparable for any big form θ' . Indeed, it was proved in [DDL18, Theorem 2.3] that there exists a constant $C = C(\theta, \theta') \geq 1$ such that

$$C^{-1}\text{Cap}_\theta^n \leq \text{Cap}_{\theta'} \leq C\text{Cap}_\theta^{1/n}.$$

The following lemma will be used several times in this paper:

Lemma 1.1. *Assume that a sequence (u_j) in $\text{PSH}(X, \theta)$ has uniformly minimal singularities (i.e. there exists $C > 0$ such that $u_j \geq V_\theta - C$ for all j) and converges in capacity to u . Then $\text{MA}(u_j)$ weakly converges to $\text{MA}(u)$. Moreover, if φ_1, φ_2 are quasi-plurisubharmonic functions then*

$$\liminf_{j \rightarrow +\infty} \mathbf{1}_{\{\varphi_1 < \varphi_2\}} \text{MA}(u_j) \geq \mathbf{1}_{\{\varphi_1 < \varphi_2\}} \text{MA}(u),$$

and $\limsup_{j \rightarrow +\infty} \mathbf{1}_{\{\varphi_1 \leq \varphi_2\}} \text{MA}(u_j) \leq \mathbf{1}_{\{\varphi_1 \leq \varphi_2\}} \text{MA}(u).$

Proof. The weak convergence of $\text{MA}(u_j)$ to $\text{MA}(u)$ on each relatively compact open subset U of the ample locus Ω follows from [X96, Theorem 1]. The complement $X \setminus U$ can be chosen to have arbitrarily small capacity, as $X \setminus \Omega$ is pluripolar. This justifies the weak convergence of the Monge–Ampère measures on the whole of X . The second statement follows since $\{\varphi_1 < \varphi_2\}$ is open and $\{\varphi_1 \leq \varphi_2\}$ is closed in the plurifine topology (see [BT87]). q.e.d.

We now provide a simple criterion insuring that a sequence converges in capacity (recall that convergence in L^1 does not imply convergence of the Monge–Ampère measures [Ceg83]):

Lemma 1.2. *Assume that $(u_j) \in \text{PSH}(X, \theta)^\mathbb{N}$ converges to $u \in \text{PSH}(X, \theta)$ in $L^1(X)$. If $u_j \geq u + \varepsilon_j \psi$ for any $j \in \mathbb{N}$, where ψ is quasi-plurisubharmonic and (ε_j) decreases to 0, then (u_j) converges in capacity towards u .*

The result may be well-known to experts. The (sketch of) the proof is given below for the reader's convenience.

Proof. By multiplying ω with some big constant $A > 0$ we can assume that u_j, u, ψ are ω -psh. We can also assume that all functions involved here are negative. By [GZ05, Proposition 3.6] we have

$$\text{Cap}_\omega(\varphi < -t) \leq \frac{C}{t}, \forall t > 0,$$

where $C > 0$ depends on $\int_X \varphi \omega^n$. It thus suffices to prove that $\max(u_j, -t)$ converges to $\max(u, -t)$ in capacity for any $t > 0$. For this reason we can now assume that u_j, u are bounded.

Classical arguments show that the sequence $v_j := \max(u_j, u)$ converges in capacity to u . Thus it remains to show that for a fixed $\varepsilon > 0$,

$$\lim_{j \rightarrow +\infty} \text{Cap}_\omega(u_j \leq u - \varepsilon) = 0.$$

The proof is completed by noting that

$$\{u_j \leq u - \varepsilon\} \subset \left\{ \psi \leq -\frac{\varepsilon}{\varepsilon_j} \right\}.$$

q.e.d.

1.3. Finite energy classes. Finite energy classes have been introduced in [GZ07] and further studied in [BEGZ10, DDL18]. They provide a very convenient frame to study convergence properties of θ -psh potentials.

Definition 1.3. A function $u \in \text{PSH}(X, \theta)$ belongs to $\mathcal{E}(X, \theta)$ if the total mass of $\text{MA}(u)$ is equal to the volume of θ .

It is proved in [DDL18] building on [Dar16] that the singularity type of functions in $\mathcal{E}(X, \theta)$ is the same as that of θ -psh functions with minimal singularities (in particular they have the same Lelong numbers at every point).

Definition 1.4. For $p > 0$, the set $\mathcal{E}^p(X, \theta)$ consists of functions u in $\mathcal{E}(X, \theta)$ such that $\int_X |u - V_\theta|^p \text{MA}(u) < +\infty$.

More generally, a weight is a smooth increasing function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ such that $\chi(-\infty) = -\infty$. The class $\mathcal{E}_\chi(X, \theta)$ consists of functions $u \in \mathcal{E}(X, \theta)$ such that

$$\int_X |\chi(u)| \text{MA}(u) < +\infty.$$

The (unnormalized) Monge–Ampère energy of a θ -psh function u with minimal singularities is

$$E(u) := \frac{1}{n+1} \sum_{k=0}^n \int_X (u - V_\theta)(\theta + dd^c u)^k \wedge (\theta + dd^c V_\theta)^{n-k}.$$

One extends the definition to arbitrary θ -psh functions by

$$E(\varphi) := \inf\{E(u) : u \in \text{PSH}(X, \theta) \text{ with minimal singularities, } u \geq \varphi\}.$$

It is shown in [BEGZ10, BBGZ13] that $\varphi \in \mathcal{E}^1(X, \theta)$ if and only if $E(\varphi)$ is finite. Moreover E is increasing, concave along affine curves in $\mathcal{E}^1(X, \theta)$ and upper semicontinuous with respect to the L^1 -topology.

We need the following generalization of [BEGZ10, Proposition 2.10]:

Lemma 1.5. *Fix $p > 0$. Assume that (u_j) is a decreasing sequence of θ -psh functions with minimal singularities and $\varphi \in \mathcal{E}^p(X, \theta)$ is such that*

$$\sup_{j \in \mathbb{N}^*} \int_X |u_j - \varphi|^p \text{MA}(u_j) < +\infty.$$

Then $u := \lim_j u_j$ belongs to $\mathcal{E}^p(X, \theta)$.

Proof. If $p = 1$ we can use the concavity of the Monge–Ampère energy E . It follows from the assumption and from [BBGZ13, proposition 2.1] that

$$E(u_j) - E(\varphi) \geq \int_X (u_j - \varphi) \text{MA}(u_j) \geq -C,$$

for some $C > 0$. Hence Proposition 2.10 in [BEGZ10] gives the conclusion.

We now deal with the general case $p > 0$. If θ is additionally semi-positive, and $p > 1$, it follows from [DNG18] (see [Dar17, Dar15] for the Kähler case) that the functional

$$I_p(u, v) := \int_X |u - v|^p (\text{MA}(u) + \text{MA}(v))$$

satisfies a quasi-triangle inequality. From this observation and the assumption we get a uniform bound on $I_p(u_j, 0)$, hence $\lim_j u_j$ belongs to $\mathcal{E}^p(X, \theta)$. For an arbitrary θ , and $p > 0$, we use the quasi-triangle inequality from Theorem 1.6 below. q.e.d.

The following quasi-triangle inequality insures that I_p induces a uniform structure on $\mathcal{E}^p(X, \theta)$:

Theorem 1.6. *Let $p > 0$ and $u, v, \varphi \in \text{PSH}(X, \theta)$ with minimal singularities. There exists a uniform constant $C > 0$ depending on p, n such that*

$$I_p(u, v) \leq C(I_p(u, \varphi) + I_p(v, \varphi)).$$

As indicated above, this result was already known when θ is semi-positive. The proof given here covers the general case and also provides a simpler proof of the previous cases.

Proof. Observe that for a positive measure μ and a non-negative measurable function f on X the integral $\int_X f d\mu$ can be expressed as

$$(1.1) \quad \int_X f d\mu = \int_0^{+\infty} \mu(f > t) dt.$$

Using this and a change of variable $t = (2s)^p$ we can write

$$(1.2) \quad \begin{aligned} \int_X |u - v|^p \text{MA}(u) &= \int_0^{+\infty} \text{MA}(u)(|u - v|^p > t) dt \\ &= 2^p p \int_0^{+\infty} s^{p-1} \text{MA}(u)(|u - v| > 2s) ds. \end{aligned}$$

For $s > 0$ we observe that the following inclusion holds

$$(\varphi - s \leq u < v - 2s) \subset \left(\varphi < \frac{u + 2v}{3} - \frac{s}{3} \right).$$

Hence, using this and the trivial inclusion

$$(u < v - 2s) \subset (u < \varphi - s) \cup (\varphi - s \leq u < v - 2s)$$

we can write

$$\begin{aligned} \text{MA}(u)(u < v - 2s) &\leq \text{MA}(u)(u < \varphi - s) \\ &\quad + \text{MA}(u) \left(\varphi < \frac{u + 2v}{3} - \frac{s}{3} \right). \end{aligned}$$

Using this, the inequality $\text{MA}(u) \leq 3^n \text{MA}((u + 2v)/3)$ and the comparison principle, we obtain

$$\text{MA}(u) \left(\varphi < \frac{u + 2v}{3} - \frac{s}{3} \right) \leq 3^n \text{MA}(\varphi) \left(\varphi < \frac{u + 2v}{3} - \frac{s}{3} \right).$$

The comparison principle also yields

$$\text{MA}(u)(v < u - 2s) \leq \text{MA}(v)(v < u - 2s).$$

We then use the same argument as above to get

$$\begin{aligned} \text{MA}(u)(|u - v| > 2s) &\leq \text{MA}(u)(u < \varphi - s) + \text{MA}(v)(v < \varphi - s) \\ &\quad + 3^n \text{MA}(\varphi) \left(\varphi < \frac{u + 2v}{3} - \frac{s}{3} \right) + 3^n \text{MA}(\varphi) \left(\varphi < \frac{v + 2u}{3} - \frac{s}{3} \right). \end{aligned}$$

From this and (1.1) we thus obtain

$$\begin{aligned}
 \int_0^{+\infty} ps^{p-1} \text{MA}(u)(|u-v| > 2s) &\leq \int_0^{+\infty} ps^{p-1} \text{MA}(u)(u < \varphi - s) ds \\
 &+ \int_0^{+\infty} ps^{p-1} \text{MA}(v)(v < \varphi - s) ds \\
 &+ 3^n \int_0^{+\infty} ps^{p-1} \text{MA}(\varphi) \left(\varphi < \frac{u+2v}{3} - \frac{s}{3} \right) ds \\
 &+ 3^n \int_0^{+\infty} ps^{p-1} \text{MA}(\varphi) \left(\varphi < \frac{2u+v}{3} - \frac{s}{3} \right) ds \\
 &\leq \int_X |u-\varphi|^p \text{MA}(u) + \int_X |v-\varphi|^p \text{MA}(v) \\
 &+ 3^{n+p} \int_X \left| \varphi - \frac{u+2v}{3} \right|^p \text{MA}(\varphi) \\
 &+ 3^{n+p} \int_X \left| \varphi - \frac{2u+v}{3} \right|^p \text{MA}(\varphi).
 \end{aligned}$$

Using this, (1.2) and the elementary inequality

$$(a+b)^p \leq \max(2^{p-1}, 1)(a^p + b^p)$$

for $a, b > 0, p > 0$, we arrive at

$$\begin{aligned}
 \int_X |u-v|^p \text{MA}(u) &\leq 2^p \int_X |u-\varphi|^p \text{MA}(u) + 2^p \int_X |v-\varphi|^p \text{MA}(v) \\
 &+ 3^n 2^{2p+1} \int_X (|\varphi-u|^p + |\varphi-v|^p) \text{MA}(\varphi).
 \end{aligned}$$

We then proceed similarly to treat the term $\int_X |u-v|^p \text{MA}(v)$ and arrive at the conclusion. q.e.d.

1.4. Convergence in energy. Monotone convergence implies convergence in capacity, which insures convergence of the Monge–Ampère operator, as indicated above. A stronger notion of convergence has been introduced in [BBGZ13, BBEGZ19]:

Definition 1.7. A sequence $(u_j) \in \mathcal{E}^1(X, \theta)^\mathbb{N}$ converges in energy to $u \in \mathcal{E}^1(X, \theta)$ if

$$0 \leq I(u_j, u) := \int_X (u_j - u)(\text{MA}(u) - \text{MA}(u_j)) \rightarrow 0.$$

The functional I is well adapted to *normalized* potentials. For un-normalized ones, one should use

$$I_1(u_j, u) = \int_X |u_j - u| (\text{MA}(u_j) + \text{MA}(u)).$$

We let the reader check that if a sequence $(u_j) \in \mathcal{E}^1(X, \theta)^\mathbb{N}$ is normalized by $\sup_X u_j = 0$, it converges in energy to u if and only if $I_1(u_j, u) \rightarrow 0$.

It is shown in [BBGZ13] that convergence in energy implies convergence in capacity. The converse is however not true as the following example shows:

Example 1.8. Assume ω is a Kähler form and φ is an ω -psh function which locally near a point z_0 (identified with zero in a local coordinate chart) is defined by $\varphi(z) = a \log |z|^2$, for some $a > 0$ small enough. Then the ω -psh function φ has a Dirac Monge–Ampère mass at 0.

Define

$$u_j := \frac{1}{j} \varphi_j ; \quad \varphi_j := \max(\varphi, -j^{n+2}).$$

We let the reader check that u_j converges to $u = 0$ in capacity. For any $j \in \mathbb{N}$ we have $\text{MA}(\varphi_j)(\varphi \leq -j^{n+2}) \geq c > 0$. In fact, we only need to know that $\varphi \notin \mathcal{E}(X, \omega)$ and c is the loss of the total mass of the non-pluripolar Monge–Ampère measure of φ . Then the energy of u_j is computed by

$$\begin{aligned} \int_X |u_j| \text{MA}(u_j) &\geq \int_{\{\varphi \leq -j^{n+2}\}} |u_j| \text{MA}(u_j) \\ &\geq j^{n+1} \int_{\{\varphi \leq -j^{n+2}\}} j^{-n} \text{MA}(\varphi_j) \geq jc. \end{aligned}$$

This shows that (u_j) does not converge to 0 in energy.

The next result says that these convergences are equivalent if the sequence is bounded from below by a finite energy function.

Proposition 1.9. *Let $u_j, u \in \mathcal{E}^p(X, \theta)$, $p > 0$, and assume that u_j converges in capacity to u . If there exists $\varphi \in \mathcal{E}^p(X, \theta)$ such that $u_j \geq \varphi, \forall j$, then u_j converges to u with respect to the quasi-distance I_p , i.e.*

$$\int_X |u_j - u|^p (\text{MA}(u_j) + \text{MA}(u)) \rightarrow 0.$$

Conversely one can show that if u_j converges in energy to u , then up to extracting and relabeling, there exists $\varphi \in \mathcal{E}^p(X, \theta)$ such that $u_j \geq \varphi, \forall j$.

Proof. The idea of the proof is essentially contained in [GZ07], [BDL15]. It costs no generality to assume that $u_j \leq 0$ which also implies that $u \leq 0$. By dominated convergence it suffices to prove that

$$\lim_{j \rightarrow +\infty} \int_X |u_j - u|^p \text{MA}(u_j) = 0.$$

By using a truncation argument as in [GZ07, Theorem 2.6] one can show that $\text{MA}(u_j)$ converges weakly to $\text{MA}(u)$. We claim that

$$\lim_{s \rightarrow +\infty} \int_{\{u_j \leq u - s\}} |u_j - u|^p \text{MA}(u_j) = 0$$

uniformly in j . Indeed the comparison principle in $\mathcal{E}(X, \theta)$ shows that

$$\begin{aligned} \int_{\{u_j \leq u-s\}} |u_j - u|^p \text{MA}(u_j) &= p \int_s^{+\infty} t^{p-1} \text{MA}(u_j)(u_j < u - t) dt \\ &\leq 2^n p \int_s^{+\infty} t^{p-1} \text{MA} \left(\frac{u_j + u}{2} \right) \left(\varphi < \frac{u_j + u}{2} - \frac{t}{2} \right) dt \\ &\leq 2^n p \int_s^{+\infty} t^{p-1} \text{MA}(\varphi)(\varphi < V_\theta - t/2) dt. \end{aligned}$$

The last term converges to zero as $s \rightarrow +\infty$ because $\varphi \in \mathcal{E}^p(X, \theta)$. Thus the claim is proved. One can also show that

$$\lim_{s \rightarrow +\infty} \int_{\{u \leq u_j - s\}} |u_j - u|^p \text{MA}(u_j) = 0$$

uniformly in j .

Therefore, it remains to show that

$$\lim_{j \rightarrow +\infty} \int_{\{|u_j - u| \leq s\}} |u_j - u|^p \text{MA}(u_j) = 0$$

for any fixed $s > 0$. But the latter follows from Theorem 1.11 below. q.e.d.

1.5. Generalized capacities. Let (X, ω) be a compact Kähler manifold of dimension n and fix θ a smooth closed $(1, 1)$ -form such that $[\theta]$ is big.

Definition 1.10. For $\varphi, \psi \in \text{PSH}(X, \theta)$ such that $\varphi < \psi$ the (φ, ψ) -capacity is defined by

$$\text{Cap}_{\varphi, \psi}(E) := \sup \left\{ \int_E \text{MA}(u) : u \in \text{PSH}(X, \theta), \varphi \leq u \leq \psi \right\}, \quad E \subset X.$$

When θ is Kähler these generalized Monge–Ampère capacities were introduced in [DNL17, DNL15]. When $\varphi = \psi - 1 = V_\theta - 1$ we recover the Monge–Ampère capacity Cap_θ .

It was observed by Di Nezza [DN15] that for each $t \geq 1$, setting $\varphi = V_\theta - t, \psi = V_\theta$, one has

$$\text{Cap}_\theta(E) \leq \text{Cap}_{\varphi, \psi}(E) \leq t^n \text{Cap}_\theta(E), \quad \forall E \subset X.$$

When θ is Kähler all (φ, ψ) -capacities with $\varphi \in \mathcal{E}(X, \theta)$ are comparable [DNL15]. We generalize this result in the context of big classes.

Theorem 1.11. *Assume that $0 \geq \varphi \in \mathcal{E}_\chi(X, \theta)$ for some convex weight $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$. Then there exists a continuous function $F_\chi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ such that $F_\chi(0) = 0$ and for all $E \subset X$,*

$$\text{Cap}_{\varphi, \psi}(E) \leq F_\chi(\text{Cap}_\theta(E)).$$

Proof. We can assume without loss of generality that $\sup_X \psi = 0$. Then $\psi \leq V_\theta$, hence

$$\text{Cap}_{\varphi,\psi}(E) \leq \text{Cap}_{\varphi,V_\theta}(E).$$

It thus suffices to treat the case $\psi = V_\theta$.

Fix $E \subset X$ a non-pluripolar Borel subset. We can assume $\text{Cap}_\theta(E) < 1$. Fix a constant $t > 1$, a θ -psh function u such that $\varphi \leq u \leq V_\theta$ and set $u_t := \max(u, V_\theta - t)$. Observe that $V_\theta - t \leq u_t \leq V_\theta$ hence

$$V_\theta - 1 \leq t^{-1}u_t + (1 - t^{-1})V_\theta \leq V_\theta.$$

It follows that for all $E \subset X$,

$$\int_E \text{MA}(u_t) \leq t^n \int_E \text{MA}(t^{-1}u_t + (1 - t^{-1})V_\theta) \leq t^n \text{Cap}_\theta(E).$$

On the other hand by the comparison principle we also have

$$\begin{aligned} \int_{\{u \leq V_\theta - t\}} \text{MA}(u) &\leq 2^n \int_{\{\varphi \leq (u + V_\theta - t)/2\}} \text{MA}((u + V_\theta - t)/2) \\ &\leq 2^n \int_{\{\varphi \leq V_\theta - t/2\}} \text{MA}(\varphi) \\ &\leq \frac{2^n}{|\chi(-t/2)|} \int_X |\chi(\varphi - V_\theta)| \text{MA}(\varphi). \end{aligned}$$

Thus there is a constant $C > 0$ depending on $E_\chi(\varphi)$ such that

$$\int_E \text{MA}(u) \leq t^n \text{Cap}_\theta(E) + \frac{C}{|\chi(-t/2)|}.$$

We can choose $t = (\text{Cap}_\theta(E))^{-1/(n+1)}$ and get the conclusion. q.e.d.

1.6. Degenerate Monge–Ampère equations. Given a non pluripolar positive Radon measure μ on X , it is useful to consider the equations

$$(1.3) \quad \text{MA}(\varphi) = e^{\lambda\varphi} \mu,$$

where $\lambda \in \mathbb{R}$ is a constant. When $\lambda = 0$ a normalization condition $\mu(X) = \text{Vol}(\theta)$ should be imposed. We will use the following result obtained by a variational method in [BBGZ13]:

Theorem 1.12. *Assume that $\lambda > 0$. Then there exists a unique $\varphi \in \mathcal{E}^1(X, \theta)$ solving (1.3).*

The main idea of the proof in [BBGZ13] is to maximize the functional

$$\mathcal{F}_\lambda(\psi) = E(\psi) + \frac{1}{\lambda} \log \left(\int_X e^{\lambda\psi} d\mu \right).$$

The continuity and the coercivity of the functional \mathcal{F}_λ is automatic when $\lambda > 0$. When $\lambda = 0$ the problem is more subtle and the resolution so far relies on a regularity result of V_θ obtained in [BD12]: one uses the fact that $\text{MA}(V_\theta)$ has bounded density with respect to Lebesgue

measure. The latter is a direct consequence of our [Theorem A](#) (see [Proposition 5.2](#)). An alternative proof of [Proposition 5.2](#) has been given recently in [\[Ber19\]](#), [\[DDL18\]](#) using ideas from the viscosity theory in [\[EGZ11\]](#).

We will also need the *domination principle* which, in the context of big classes, was first established in [\[BEGZ10\]](#) for θ -psh functions with minimal singularities. The result still holds for functions in $\mathcal{E}(X, \theta)$ as follows from an argument due to Dinew (see [\[BL12, DDL18\]](#)).

Proposition 1.13 ([\[BEGZ10, DDL18\]](#)). *Let φ, ψ be θ -psh functions on X and assume that $\varphi \in \mathcal{E}(X, \theta)$. If $\text{MA}(\varphi)(\varphi < \psi) = 0$ then $\varphi \geq \psi$ on X .*

We will occasionally use the following version of the comparison principle, which follows from the domination principle (see [\[DDL18\]](#)):

Proposition 1.14. *Let $\varphi, \psi \in \mathcal{E}(X, \theta)$, μ be a non-pluripolar positive measure and f be a Borel measurable function on X such that*

$$\text{MA}(\varphi) \geq e^{\beta\varphi} e^{-f} \mu ; \text{MA}(\psi) \leq e^{\beta\psi} e^{-f} \mu,$$

where $\beta > 0$ is a constant. Then $\varphi \leq \psi$.

2. Envelopes

Upper envelopes are classical objects in Potential Theory. They were used in the Perron method for solving the Dirichlet problem for the Poisson equation as well as the free boundary problems for the Laplace operator.

Upper envelopes of psh functions were considered by Bremermann, Walsh, Siciak, and Bedford and Taylor to solve the Dirichlet problem for the complex Monge–Ampère equation in strictly pseudo-convex domains (see [\[BT76\]](#)).

We consider here envelopes of quasi-psh functions on compact Kähler manifolds, following [\[GZ05, Ber19\]](#).

2.1. Usual envelopes. Let X be a compact Kähler manifold of complex dimension n and let θ be a closed smooth real $(1, 1)$ -form on X whose cohomology class is big.

Definition 2.1. Given a Lebesgue measurable function $h : X \rightarrow \mathbb{R}$ which is bounded from below, we define the θ -psh envelope of h as follows

$$P_\theta(h) := (\sup\{u \in \text{PSH}(X, \theta) : u \leq h \text{ in } X\})^*,$$

where the star means that we take the upper semi-continuous regularization.

We will also denote by $P(h)$ the envelope $P_\theta(h)$ if the form θ is fixed. In this section we start a systematic study of these envelopes.

When $h = -\mathbf{1}_E$ is the negative characteristic function of a subset E then $P(h) = h_E^*$ is the so called relative extremal function of E [GZ05].

When $h = 0$ then $P(0) = V_\theta$ was introduced in [DPS01] as an example of a θ -psh function with minimal singularities. When h is smooth Berman and Demailly have shown in [BD12] that $P(h)$ has locally bounded laplacian in $\text{Amp}(\theta)$. In particular $V_\theta = P(0)$ has locally bounded laplacian and the Monge–Ampère measure of V_θ can be described as

$$\text{MA}(V_\theta) = \mathbf{1}_{\{V_\theta=0\}}\theta^n.$$

In the case when the class $[\theta]$ is big and nef, a PDE proof of this result was given by Berman in [Ber19]. The fundamental observation of Berman is that the envelope can be obtained as the limit of solutions to a one-parameter family of complex Monge–Ampère equations (this idea has been recently used in [LN15], [KN16], [BL18]). By establishing a uniform laplacian estimate for this family of solutions Berman showed that $P_\theta(h)$ has locally bounded laplacian in the ample locus.

In particular, when the cohomology class of θ is Kähler then $P_\theta(h)$ has bounded laplacian on X . The optimal $\mathcal{C}^{1,1}$ regularity of $P_\theta(h)$, conjectured by Berman, has recently been confirmed by Tosatti [Tos18] and independently by J. Chu and B. Zhou [CZ17] using the $\mathcal{C}^{1,1}$ estimate in [CTW17] and the convergence method of Berman [Ber19].

We need here to study these envelopes for functions h that are less regular. When $h = 0$ on E and $+\infty$ on $X \setminus E$ then $P(h)$ is the global extremal θ -psh function of E that was considered in [GZ05, BEGZ10]. It follows from [GZ05, BEGZ10] that if h is finite on a non-pluripolar set then $P(h)$ is a well-defined θ -psh function on X .

Proposition 2.2.

1. *If $h : X \rightarrow \mathbb{R}$ is a bounded measurable function then $P(h)$ is a θ -psh function with minimal singularities which satisfies $P(h) \leq h$ quasi everywhere in X . Moreover*

$$P(h) := \sup\{u \in \text{PSH}(X, \theta) : u \leq h \text{ quasi everywhere in } X\}.$$

2. *If (h_j) is a decreasing sequence of bounded measurable functions which converge pointwise to h in X , then $P(h_j)$ decreases to $P(h)$ in X .*

3. *If (h_j) is an increasing sequence of bounded quasi-lsc functions converging pointwise to h , then $P(h_j)$ increases to $P(h)$ quasi everywhere.*

4. *If h is continuous and θ is Kähler then $P_\theta(h)$ is continuous in X .*

In these statements quasi everywhere means outside a pluripolar set.

Proof. Assume that h is a bounded Lebesgue measurable function on X . The fact that $P(h)$ is a θ -psh function with minimal singularities follows directly from the definition. The set $\{x \in X : P(h)(x) > h(x)\}$

is negligible. It follows from a classical result in pluripotential theory [BT82] that negligible sets are pluripolar, thus $P(h) \leq h$ quasi every where on X .

We now prove the identity in the first statement. Let φ denote the function on the right-hand side. It is obvious that $P(h) \leq \varphi$. As a countable union of pluripolar sets is also pluripolar, by Choquet’s lemma and the same argument as above we see that $\varphi \leq h$ quasi everywhere on X . The equality follows if we can show that $P(h_1) = P(h_2)$ for two bounded functions such that $\{h_1 \neq h_2\}$ is pluripolar. Indeed, the set

$$E := \{P(h_1) > h_1\} \cup \{h_1 \neq h_2\}$$

is also pluripolar. Hence there exists $\phi \in \text{PSH}(X, \theta)$ such that $\phi = -\infty$ on E . Now for any $\lambda \in (0, 1)$ the function $\lambda\phi + (1 - \lambda)P(h_1)$ is θ -psh on X and bounded from above by h_2 . Letting $\lambda \rightarrow 0$ one sees that $P(h_1) \leq P(h_2)$ off a pluripolar set, hence the inequality holds everywhere. Conversely one can show that $P(h_2) \leq P(h_1)$, completing the proof of the claim, hence the first statement is proved.

The second statement is straightforward. We now prove the third one. Assume that (h_j) is a sequence of bounded quasi-lsc functions that increase pointwise to a bounded function h . Then $P(h_j)$ also increase quasi everywhere to some $\varphi \in \text{PSH}(X, \theta)$ with minimal singularities. One observes immediately that $\varphi \leq P(h)$. It follows from Lemma 2.3 that $\text{MA}(P(h_j))$ vanishes in $\{P(h_j) < h_j\}$. As $P(h_j) \leq \varphi$ and $h_j \nearrow h$ it follows that $\text{MA}(\varphi)$ also vanishes in $\{\varphi < P(h)\}$. The domination principle (Proposition 1.13) insures that $\varphi = P(h)$.

One can prove the last statement by approximation. Let (h_j) be a sequence of smooth functions on X converging uniformly to h . It follows from [Ber19] that $P(h_j)$ has bounded laplacian, in particular it is continuous. As $P(h_j)$ uniformly converges to $P(h)$ the conclusion follows. q.e.d.

The following result is an analogue of the corresponding result of Bedford and Taylor [BT82, Corollary 9.2]:

Lemma 2.3. *Let $h : X \rightarrow \mathbb{R}$ be a bounded Lebesgue measurable function on X and let $L(h)$ be the lower semi-continuity set of h . Then $\text{MA}(P(h))$ puts no mass on the set $L(h) \cap \{P(h) < h\}$. In particular if $\text{Cap}_\omega^*(X \setminus L(h)) = 0$, then*

$$\int_X (P(h) - h)\text{MA}(P(h)) = 0.$$

Proof. The proof proceeds as in the classical case using a balayage argument (see [BT82, Corollary 9.2]). We repeat it here for the convenience of the reader. For notational convenience we set $\hat{h} := P(h)$. Fix a point $x_0 \in L(h) \cap \{\hat{h} < h\}$. Observe that by lower semi-continuity at x_0 , the point x_0 lies in the interior of the set $\{\hat{h} < h\}$. Indeed, fix

$\delta > 0$ such that $\hat{h}(x_0) - h(x_0) \leq -2\delta$. By upper semi-continuity of \hat{h} and lower semi-continuity of h at x_0 there exists a small ball B of center x_0 such that $\max_{\bar{B}} \hat{h} < \min_{\bar{B}} h - \delta$.

Let ρ be a smooth local potential of θ in a neighborhood D of \bar{B} . Shrinking the ball if necessary we can assume that $\text{osc}_{\bar{B}} \rho < \delta$. Then $u := \hat{h} + \rho$ is psh in D . By Bedford and Taylor (see [BT82, Proposition 9.1]) there exists a psh function v in D such that $v = u$ in $D \setminus B$, $v \geq u$ in D and $(dd^c v)^n = 0$ in B .

Since $v = u$ in ∂B , the comparison principle insures $v \geq u$ in \bar{B} . On the other hand on ∂B , we have

$$v = \hat{h} + \rho \leq \max_{\bar{B}} \hat{h} + \max_{\bar{B}} \rho.$$

By the classical maximum principle we get $v \leq \max_{\bar{B}} \hat{h} + \max_{\bar{B}} \rho$ in \bar{B} , hence $v - \rho \leq h - \delta + \text{osc}_{\bar{B}} \rho \leq h$ in B .

Therefore since $v - \rho = \hat{h}$ in ∂B , the function w defined by $w := v - \rho$ in B and $w = \hat{h}$ in $X \setminus B$ is θ -psh in X and satisfies $w \leq h$ in B and $w = \hat{h} \leq h$ quasi everywhere in $X \setminus B$. This yields $w \leq \hat{h}$ in X .

By construction we have $w := v - \rho \geq u - \rho = \hat{h}$ in B and then $w = \hat{h}$ in B and $(\theta + dd^c \hat{h})^n = (dd^c v)^n = 0$ in B . q.e.d.

The result above extends to any function h which is quasi lower semi-continuous in the sense that for any $\delta > 0$ there exists a compact set $K \subset X$ such that $\text{Cap}_\omega(X \setminus K) \leq \delta$ and the restriction $h|_K$ is a lower semi-continuous function.

We need the following fact which follows from the Tietze–Urysohn lemma.

Lemma 2.4. *Let h be a quasi lower semi-continuous function in X . Then there exists a decreasing sequence (h_j) of lower semi-continuous functions in X which converges to h in capacity and quasi everywhere in X .*

Proof. By definition there exists a sequence of compact sets (K_ℓ) such that $\text{Cap}_\omega(X \setminus K_\ell) \leq 2^{-\ell}$ and the restriction $h|_{K_\ell}$ is a lower semi-continuous function in K_ℓ . Take $\tilde{K}_j := \cup_{1 \leq \ell \leq j} K_\ell$ instead of (K_j) , we can assume that the sequence (K_j) is increasing.

Since a lower semi-continuous function on a compact set is the limit of an increasing sequence of continuous functions, it follows from the Tietze–Urysohn Lemma that there exists a lower semi-continuous function H_j in X such that $H_j|_{K_j} = h|_{K_j}$.

Let $h_j := \sup\{H_\ell : \ell \geq j\}$. Then (h_j) is a decreasing sequence of lower semi-continuous functions in X such that $h_j = h$ in K_j , hence it converges to h in $F := \cup_j K_j$. Since $\text{Cap}_\omega^*(X \setminus F) = 0$ it follows that h_j converges to h quasi everywhere in X .

We claim that (h_j) converges to h in capacity. Indeed let $\delta > 0$ be fixed and set for $j \in \mathbb{N}$, $E_j := \{x \in X : h_j \geq h + \delta\}$. Since $h_j = h$ in K_j , it follows that $E_j \subset X \setminus K_j$. Hence $\text{Cap}_\omega^*(E_j) \leq \text{Cap}_\omega^*(X \setminus K_j) \leq 2^{-j}$, and then

$$\lim_{j \rightarrow +\infty} \text{Cap}_\omega^*(\{x \in X : h_j \geq h + \delta\}) = 0,$$

which proves our claim. q.e.d.

Proposition 2.5. *Let h be a bounded quasi lower semi-continuous function in X and set $\hat{h} = P_\theta(h)$ the θ -psh envelope of h . Then $\text{MA}_\theta(\hat{h})$ puts no mass on the set $\{\hat{h} < h\}$ i.e.*

$$\int_X (\hat{h} - h)\text{MA}(\hat{h}) = 0.$$

Proof. By the previous lemma there exists a decreasing sequence (h_j) of lsc functions in X such that (h_j) converges to h in capacity.

From Lemma 2.3 we know that $\int_X (h_j - \hat{h}_j)\text{MA}(\hat{h}_j) = 0$ for any j . We also know by Lemma 2.2 that (\hat{h}_j) decreases to \hat{h} . In particular the convergence holds in energy, hence

$$\lim_{j \rightarrow +\infty} \int_X \hat{h}_j \text{MA}(\hat{h}_j) = \int_X \hat{h} \text{MA}(\hat{h}).$$

On the other hand the functions h_j are lower semi-continuous, uniformly bounded and converge to h in capacity, hence (see [GZ, DDL18])

$$\int_X h \text{MA}(\hat{h}) \leq \liminf_{j \rightarrow +\infty} \int_X h \text{MA}(\hat{h}_j) = \liminf_{j \rightarrow +\infty} \int_X h_j \text{MA}(\hat{h}_j).$$

This implies that $\text{MA}(\hat{h})$ puts no mass on the set $\{\hat{h} < h\}$. q.e.d.

The orthogonal relation $\int_X (P(u) - u)\text{MA}(P(u)) = 0$ does not hold in general, as the following example shows:

Example 2.6. Assume $X = \mathbb{C}\mathbb{P}^n$ is the complex projective space equipped with the Fubini–Study metric $\theta = \omega_{FS}$. Let B denote the unit ball in $\mathbb{C}^n \subset \mathbb{C}\mathbb{P}^n$ and consider u to be -1 on B and 0 elsewhere. Then $P(u)$ is the relative extremal function (see [GZ05]) of B which takes values -1 on the boundary ∂B : we let the reader check that for $z \in \mathbb{C}^n$,

$$P(u)(z) + \log \sqrt{1 + |z|^2} = \max \left\{ \log \sqrt{1 + |z|^2} - 1; \log |z| + \frac{\log 2}{2} - 1 \right\},$$

thus $\text{MA}(P(u))$ does not vanish on $\partial B \subset \{P(u) < u\}$.

Proposition 2.5 generalizes to any upper bounded Borel function h which admits a subextension $\psi \in \mathcal{E}(X, \theta)$ i.e. $\psi \leq h$ in X .

Theorem 2.7. *Let h be a quasi lower semi-continuous function bounded from above in X . Assume there exists $\psi \in \mathcal{E}(X, \theta)$ s.t. $\psi \leq h$ in X . Then*

1. $\hat{h} = P_\theta(h) \in \mathcal{E}(X, \theta)$ and $\hat{h} \leq h$ quasi everywhere in X .
2. $\text{MA}_\theta(\hat{h})$ puts no mass on the set $\{\hat{h} < h\}$.

Proof. We may assume that $h \leq 0$ in X . Since $\psi \in \mathcal{E}(X, \theta)$ there exists a convex increasing weight $\chi : \mathbb{R}^- \rightarrow \mathbb{R}^-$ such that $\psi \in \mathcal{E}_\chi(X, \theta)$ (see [GZ07, BEGZ10]). Since $\psi \leq h$, we conclude from the definition that $\psi \leq \hat{h}$ in X , hence $\hat{h} \in \mathcal{E}(X, \theta)$ by [GZ07, BBEGZ19].

Set $h_j := \max(h, -j)$, by the previous theorem $\text{MA}(\hat{h}_j)$ is carried by the set $\{\hat{h}_j = h_j\}$. Hence for any j ,

$$\int_X \min(h_j - \hat{h}_j, 1) \text{MA}(\hat{h}_j) = 0.$$

Since $h_j = h$ off the set $\{h < -j\} \subset \{\psi < -j\}$ and $\text{Cap}_\omega(\{\psi < -j\}) \rightarrow 0$ as $j \rightarrow +\infty$, it follows that $h_j \rightarrow h$ in capacity in X . Hence $\phi_j := \min(h_j - \hat{h}_j, 1)$ converges to $\phi := \min(h - \hat{h}, 1)$ in capacity. Lemma 2.8 insures

$$\lim_{j \rightarrow +\infty} \int_X |\phi_j - \phi| \text{MA}(\hat{h}_j) = 0.$$

On the other hand since ϕ is bounded and lower semi-continuous on X it follows from convergence property of the complex Monge–Ampère operator that

$$\int_X \phi \text{MA}(\hat{h}) \leq \liminf_{j \rightarrow +\infty} \int_X \phi \text{MA}(\hat{h}_j).$$

As $\phi \geq 0$ we thus get $\int_X \phi \text{MA}(\hat{h}) = 0$, finishing the proof. q.e.d.

Lemma 2.8. *Fix $0 \geq \varphi \in \mathcal{E}(X, \theta)$ and let (f_j) be a sequence of positive uniformly bounded measurable functions on X which converges in capacity to 0. Then*

$$\lim_{j \rightarrow +\infty} \sup \left\{ \int_X f_j \text{MA}(\psi) : \psi \in \text{PSH}(X, \theta), \varphi \leq \psi \leq 0 \right\} = 0.$$

Proof. Fix $\psi \in \mathcal{E}(X, \theta)$ such that $\varphi \leq \psi \leq 0$. Since f_j is uniformly bounded, we have $f_j \leq C$ for any j . Now for fixed $\varepsilon > 0$ we have

$$\int_X f_j \text{MA}(\psi) \leq C \int_{\{f_j \geq \varepsilon\}} \text{MA}(\psi) + \varepsilon \text{Vol}(\theta) \leq C \text{Cap}_{\varphi, V_\theta}(f_j \geq \varepsilon) + \varepsilon \text{Vol}(\theta).$$

It follows from Theorem 1.11 that $\text{Cap}_{\varphi, V_\theta}(f_j \geq \varepsilon)$ converges to 0 as $j \rightarrow +\infty$. The conclusion follows by letting $\varepsilon \rightarrow 0$. q.e.d.

2.2. Envelopes with respect to a measure. Let μ be a positive measure on X which does not charge pluripolar sets.

Definition 2.9. The (θ, μ) -envelope of a measurable function $u : X \rightarrow \mathbb{R}$ is defined by

$$P_{\theta, \mu}(u) := (\sup\{\varphi \in \text{PSH}(X, \theta) : \varphi \leq u \text{ } \mu\text{-a.e.}\})^*.$$

This notion generalizes the one introduced in Definition 2.1.

Proposition 2.10. *Assume that u is bounded from below and there is $b > 0$ such that $\mu(u < b) > 0$. Then $P_{\theta, \mu}(u)$ is a well-defined θ -psh function with minimal singularities. Moreover, $P_{\theta, \mu}(u) \leq u$ holds μ -almost everywhere.*

If (h_j) is a decreasing sequence of bounded measurable functions which converge pointwise to h in X , then $P_{\theta, \mu}(h_j)$ decreases to $P_{\theta, \mu}(h)$ in X .

Proof. We first prove that $P_{\theta, \mu}(u)$ is bounded from above. Indeed, fix an arbitrary θ -psh function φ such that $\varphi \leq u$ μ -a.e. Set

$$K := \{x \in X : \varphi(x) < b\},$$

so that $\mu(K) > 0$. Then $\varphi - b \leq V_{\theta, K}^*$, where $V_{\theta, K}^*$ is the global extremal θ -psh function of K . It follows from Theorem 1.11 that $\text{Cap}_\theta(\varphi < b) \geq c > 0$, where c does not depend on φ . Indeed, since μ is non pluripolar we can find $\psi \in \mathcal{E}(X, \theta)$ with $\sup_X \psi = -1$ such that $\text{MA}(\psi) = \mu$. Hence by definition of the capacity,

$$\text{Cap}_{\psi, V_\theta}(\varphi < b) \geq \mu(\varphi < b) \geq \mu(u < b) =: c_1 > 0.$$

By Theorem 1.11 we know that $F(\text{Cap}_\theta(E)) \geq \text{Cap}_{\psi, V_\theta}(E)$ for every Borel subset E . The function F is continuous and increasing, hence $\text{Cap}_\theta(\varphi < b) \geq F^{-1}(c_1) =: c_2 > 0$.

Since the set K has capacity $\geq c$, it follows from [GZ05, BEGZ10] that $\sup_X \varphi \leq C$, where C depends only on c .

Now, since the family defining $P_{\theta, \mu}(u)$ is uniformly bounded from above, the sup envelope is well defined as a θ -psh function with minimal singularities. It follows from Choquet’s lemma that $P_{\theta, \mu}(u) \leq u$ holds μ -almost everywhere on X .

The proof of the last assertion is straightforward. q.e.d.

Proposition 2.11. *If μ is a volume form and u is bounded and quasi upper semi-continuous on X , then $P_{\theta, \mu}(u) = P_\theta(u)$.*

Proof. Fix an arbitrary function $\varphi \in \text{PSH}(X, \theta)$ such that $\varphi \leq u$ almost everywhere with respect to Lebesgue measure. Assume that u is upper semi-continuous. Fix $x_0 \in X$ and consider a local chart around x_0 . Let ρ be a smooth local potential of θ in this chart. The sub-mean value inequality yields

$$\varphi(x_0) + \rho(x_0) \leq \limsup_{\varepsilon \rightarrow 0} \int_{B(x_0, \varepsilon)} (u(x) + \rho(x)) dV(x) \leq u(x_0) + \rho(x_0),$$

where the last inequality holds because u is upper semicontinuous and ρ is continuous on X .

Assume now that u is quasi upper semi-continuous on X . For each fixed $j \in \mathbb{N}$ there exists a compact set $K_j \subset X$ such that $\text{Cap}_\omega(X \setminus K_j) \leq 2^{-j}$ and the restriction of u on K_j is upper semi-continuous. We can also assume that K_j is increasing in j . Let u_j be a bounded function on X which is upper semi-continuous and $u_j = u$ on K_j . We can impose that (u_j) is increasing in j . It follows from the sub-mean value inequality that

$$\begin{aligned} \varphi(x_0) &\leq \limsup_{\varepsilon \rightarrow 0} \int_{B(x_0, \varepsilon)} u(x) dV(x) \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{B(x_0, \varepsilon)} (u(x) - u_j(x)) dV(x) + \limsup_{\varepsilon \rightarrow 0} \int_{B(x_0, \varepsilon)} u_j(x) dV(x) \\ &\leq 2^{-j} \sup_X (u - u_j) + u_j(x_0). \end{aligned}$$

Thus for $x \in \cup_j K_j$ we have that $\varphi(x) \leq u(x)$. Since $X \setminus \cup_j K_j$ is pluripolar, it follows that $\varphi \leq u$ quasi everywhere on X , thus $\varphi \leq P_\theta(u)$.
q.e.d.

In general the (θ, μ) -envelope is different from the usual one as the following example shows:

Example 2.12. Take a non-pluripolar set $E \subset X$ which has zero Lebesgue measure. Let u be the function that takes value -1 on E and 0 on $X \setminus E$. Take $\mu = \omega^n$ and $\theta = \omega$. Then the (θ, μ) -envelope of u is identically 0 while its usual envelope $P_\omega(u)$ is the relative extremal function of E which is not identically zero because E is non-pluripolar.

We will see later on that for any constant $C > 0$, the function u defined in the example above cannot be a viscosity supersolution of the equation

$$-(\omega + dd^c u)^n + C\omega^n = 0.$$

2.3. Approximation of envelopes and proof of Theorem B. Fix a probability measure μ on X which does not charge pluripolar sets. For each $j \in \mathbb{N}^*$ it follows from [BEGZ10, BBGZ13] that there exists a unique θ -psh function φ_j with minimal singularities such that

$$(2.1) \quad \text{MA}(\varphi_j) = e^{j(\varphi_j - u)} \mu.$$

If μ is a smooth volume form and u is smooth on X , Berman proved in [Ber19] that (φ_j) converges in energy toward the Monge–Ampère envelope $P_\theta(u)$ ¹. The purpose of this section is to relax the regularity assumption on μ and u . We first observe the following

¹The convergence result in [Ber19] has been recently generalized to measures satisfying the Bernstein–Markov condition.

Lemma 2.13. *Let ϕ be the unique function in $\mathcal{E}^1(X, \theta)$ such that $MA_\theta(\phi) = e^\phi \mu$. Then*

$$(2.2) \quad \varphi_j \geq \left(1 - \frac{1}{j}\right) P_{\theta, \mu}(u) + \frac{\phi}{j} + \frac{1}{j}(-n \log j + \inf_X u).$$

Proof. Denote by ψ_j the right-hand side of (2.2) and note that $\psi_j \in \mathcal{E}^1(X, \theta)$. Using the fact that $P_{\theta, \mu}(u) \leq u$ holds μ -a.e. on X , one can check that ψ_j is a pluripotential subsolution of (2.1). It thus follows from the pluripotential comparison principle that $\varphi_j \geq \psi_j$. q.e.d.

Theorem 2.14. *Assume that μ is a non-pluripolar positive measure and u is a bounded Borel measurable function. Then the sequence (φ_j) converges in energy to the envelope $P_{\theta, \mu}(u)$.*

Proof. In view of Lemma 1.9, Lemma 2.13 and Lemma 1.2 it suffices to prove that (φ_j) converges to $P_{\theta, \mu}(u)$ in L^1 .

We claim that the sequence $(\sup_X \varphi_j)$ is bounded. Indeed, assume this is not the case. After extracting and relabeling we can assume that $\sup_X \varphi_j \nearrow +\infty$. The sequence $\psi_j := \varphi_j - \sup_X \varphi_j$ is contained in a compact set of $L^1(X, \omega^n)$. We can thus extract a subsequence, still denoted by (ψ_j) that converges to $\psi \in \text{PSH}(X, \theta)$ in $L^1(X, \omega^n)$. The set

$$P := \{x \in X : \sup_j \varphi_j(x) < +\infty\}$$

is contained in $\{\psi = -\infty\}$, hence it is pluripolar.

By assumption on u there exists $s > 0$ such that $u \leq s$ on X . Consider

$$A_j := \{x \in X : \varphi_j > 2s\}.$$

As $u \leq s$ in X , using (2.1) we obtain $\mu(A_j) \leq e^{-js} \text{Vol}(\{\theta\})$. Thus for j large enough,

$$\mu \left(\bigcup_{k \geq j} A_k \right) \leq \frac{e^{-sj} \text{Vol}(\{\theta\})}{1 - e^{-s}} < \mu(X).$$

Now the complement of $\bigcup_{k \geq j} A_k$ in X is contained in P , a pluripolar set which is negligible with respect to μ , a contradiction. Thus the claim is proved, i.e. $\sup_X \varphi_j$ is bounded.

It follows now from compactness properties of θ -psh functions (see [GZ05]) that the sequence (φ_j) is relatively compact in L^1 . It suffices to prove that any cluster point of this sequence coincides with $P_{\theta, \mu}(u)$. Let φ be such a cluster point. Extracting and relabeling we can assume that φ_j converges in L^1 to φ . It follows from Lemma 2.13 that $\varphi \geq P_{\theta, \mu}(u)$. Consider

$$\tilde{\varphi}_j := \left(\sup_{k \geq j} \varphi_k \right)^*.$$

Then $\tilde{\varphi}_j$ decreases pointwise to φ . Fix $\varepsilon > 0$ and set

$$U_j := \{x \in X : \varphi_j(x) > u(x) + \varepsilon\} ; \tilde{U}_j := \{x \in X : \tilde{\varphi}_j(x) > u(x) + \varepsilon\}.$$

As negligible sets are also pluripolar and μ does not charge these sets we get

$$\mu(\tilde{U}_j) \leq \sum_{k \geq j} \mu(U_k) \leq \frac{\text{Vol}(\{\theta\})e^{-j\varepsilon}}{1 - e^{-\varepsilon}}.$$

Thus $\mu(U) = 0$ where $U := \bigcap_{j \in \mathbb{N}^*} \tilde{U}_j \supset \{\varphi > u + \varepsilon\}$. Letting $\varepsilon \rightarrow 0$ we obtain $\mu(\varphi > u) = 0$ hence $\varphi \leq P_{\theta, \mu}(u)$, finishing the proof. q.e.d.

3. Viscosity vs pluripotential supersolutions

Let (X, ω) be a compact Kähler manifold of dimension n and fix θ a smooth closed $(1, 1)$ -form on X which represents a big cohomology class.

3.1. Background on viscosity solutions. Let $f : X \rightarrow \mathbb{R}^+$ be a continuous function.

Definition 3.1. Let $u : X \rightarrow \mathbb{R} \cup \{-\infty\}$ be an upper semicontinuous function. An upper test function for u at x_0 is a function $q : V_{x_0} \rightarrow \mathbb{R}$ defined in a neighborhood V_{x_0} of x_0 such that $u \leq q$ in V_{x_0} with $u(x_0) = q(x_0)$.

One can define similarly lower tests. Upper test functions are used to define the notion of viscosity subsolutions:

Definition 3.2. Let u be a real-valued upper semi-continuous function in an open set $U \subset X$. We say that the inequality

$$(\theta + dd^c u)^n \geq e^u f dV$$

holds in the viscosity sense in U if u is finite on U and if for any $x_0 \in U$ and any \mathcal{C}^2 upper test function q of u at x_0 the inequality $(\theta + dd^c q)^n \geq e^q f dV$ holds at x_0 .

We shall equivalently say that u is a viscosity subsolution of the equation $(\theta + dd^c u)^n = e^u f dV$.

The notion of viscosity supersolution is defined similarly with a subtle twist that we emphasize now:

Definition 3.3. Let u be a real-valued lower semi-continuous function in an open set $U \subset X$. We say that the inequality

$$(\theta + dd^c u)^n \leq e^u f dV$$

holds in the viscosity sense in U if u is finite in U and if for any $x_0 \in U$ and any \mathcal{C}^2 lower test function q of u at x_0 the inequality

$$(\theta + dd^c q)_+^n \leq e^q f dV$$

holds at x_0 .

Here $\alpha_+ = \alpha$ if the $(1, 1)$ -form α is semipositive and $\alpha_+ = 0$ otherwise.

We shall equivalently say that u is a viscosity supersolution of the equation $(\theta + dd^c\varphi)^n = e^\varphi f dV$.

Definition 3.4. A *viscosity solution* is a function that is both a viscosity subsolution and a viscosity supersolution in the ample locus of $[\theta]$.

We refer the reader to [EGZ11, W12, Ze13] for basic properties of viscosity sub/super-solutions to degenerate complex Monge–Ampère equations. We stress that viscosity subsolutions are θ -psh [EGZ11] and admit upper tests at almost every point by [DD19]. The analogous properties for viscosity supersolutions are far less obvious.

3.2. The global context. Consider the following Monge–Ampère equation

$$(3.1) \quad (\theta + dd^c u)^n = e^u f \omega^n,$$

where f is a non-negative continuous function on X .

3.2.1. Envelope of viscosity supersolutions.

Theorem 3.5. *If u is a viscosity super-solution of (3.1) then $P_\theta(u)$ is a pluripotential super-solution of (3.1).*

This connection has been observed in [EGZ11, Lemma 4.7.3] when u is \mathcal{C}^2 -smooth. The key idea of the proof given here is to approximate the Monge–Ampère envelope $P_\theta(u)$ as in Theorem 2.14.

Proof. For each $\beta > 0$ let φ_β be the unique θ -psh function with minimal singularities such that

$$(3.2) \quad (\theta + dd^c \varphi_\beta)^n = e^{\beta(\varphi_\beta - u)} e^{\varphi_\beta} f \omega^n$$

holds in the pluripotential sense. The existence and uniqueness of the solution φ_β with minimal singularities follows from the main result of [BEGZ10, BBGZ13]. As shown in [EGZ11, EGZ17] equation (3.2) holds in the viscosity sense in $\Omega := \text{Amp}(\theta)$ (the ample locus of $\{\theta\}$) as well.

Step 1. We claim that $\varphi_\beta \leq u$ in Ω , for all $\beta > 0$. Recall that u is a viscosity super-solution of (3.1). Thus the claim would follow if we could apply the viscosity comparison principle [EGZ11, EGZ17]. However, the density in the second term of (3.2), $e^{-\beta u} f$, is not continuous in Ω , hence one cannot directly apply the results from [EGZ11, EGZ17].

To prove the claim we proceed by approximation. Fix $\beta > 0$ and let (u_j) be an increasing sequence of continuous functions converging to u . Such a sequence exists because u is lower semicontinuous. For each j let $\varphi_{\beta,j}$ be the unique θ -psh function with minimal singularities such that

$$(3.3) \quad (\theta + dd^c \varphi_{\beta,j})^n = e^{\beta(\varphi_{\beta,j} - u_j)} e^{\varphi_{\beta,j}} f \omega^n.$$

As $u_j \leq u$ one can check that u is a viscosity super-solution of (3.3). Indeed

$$(\theta + dd^c u)_+^n \leq e^u f \omega^n \leq e^{\beta(u-u_j)} e^u f \omega^n.$$

Moreover, the density function $e^{-\beta u_j} f$ is continuous on X . Hence by the viscosity comparison principle [EGZ17] it follows that $\varphi_{\beta,j} \leq u$ in Ω . For $j > k$, as $u_j \geq u_k$ we have

$$(\theta + dd^c \varphi_{\beta,k})^n = e^{\beta(\varphi_{\beta,k}-u_k)} e^{\varphi_{\beta,k}} f \omega^n \geq e^{\beta(\varphi_{\beta,k}-u_j)} e^{\varphi_{\beta,k}} f \omega^n$$

in the pluripotential sense. In other words, $\varphi_{\beta,k}$ is a pluripotential sub-solution of (3.3). It follows therefore from the pluripotential comparison principle (Proposition 1.14) that $j \mapsto \varphi_{\beta,j}$ is increasing. The increasing limit is a θ -psh function with minimal singularities which solves equation (3.2) (recall that the Monge–Ampère operator is continuous along monotonous sequences). By uniqueness it follows that this limit is φ_β . This proves the claim since $\varphi_{\beta,j} \leq u$ in Ω .

Step 2. We now claim that φ_β increases towards $P(u)$, as $\beta \rightarrow +\infty$. Since $\varphi_\beta \leq u$, we observe that φ_γ is a subsolution to (3.2) $_\beta$ if $\gamma \leq \beta$, hence $\beta \mapsto \varphi_\beta$ is increasing. It converges almost everywhere to some function $\varphi \in \text{PSH}(X, \theta)$ with minimal singularities such that $\varphi \leq u$ in Ω . Since $X \setminus \Omega$ is pluripolar we infer $\varphi \leq P_\theta(u)$ on X .

We now show that $\varphi \geq P_\theta(u)$. Using the domination principle it suffices to prove that $\text{MA}(\varphi)$ vanishes in $\{\varphi < P_\theta(u)\}$. Fix $\delta > 0$. Using (3.2) and observing that

$$\{\varphi < P(u) - \delta\} \subset \{\varphi_\beta < u - \delta\},$$

we obtain

$$\int_{\{\varphi < P(u) - \delta\}} \text{MA}(\varphi_\beta) \leq e^{-\beta\delta} \int_X e^{P(u)} f dV, \quad \forall \beta > 1.$$

Since $\text{MA}(\varphi_\beta)$ weakly converges to $\text{MA}(\varphi)$ as $\beta \rightarrow +\infty$, it follows from Lemma 1.1 that

$$\int_{\{\varphi < P(u) - \delta\}} \text{MA}(\varphi) \leq \liminf_{\beta \rightarrow +\infty} \int_{\{\varphi < P(u) - \delta\}} \text{MA}(\varphi_\beta) = 0.$$

Letting $\delta \rightarrow 0$ insures that $\text{MA}(\varphi)$ vanishes in $\{\varphi < P(u)\}$.

Conclusion. Recall that

$$(\theta + dd^c \varphi_\beta)^n \leq e^{\varphi_\beta} f \omega^n$$

in the pluripotential sense. Since φ_β increases to $P_\theta(u)$, the continuity of the Monge–Ampère operator along monotonous sequences insures that

$$(\theta + dd^c P(u))^n \leq e^{P(u)} f \omega^n$$

in the pluripotential sense, as desired.

q.e.d.

Remark 3.6. Recall that a partial converse to this implication has been given in [EGZ11, Lemma 4.7]: if ψ is a bounded pluripotential supersolution, then its lower semi-continuous regularization ψ_* is a viscosity super-solution.

3.2.2. More general RHS. We consider the following generalization.

Theorem 3.7. *Assume that $F : X \times \mathbb{R} \rightarrow \mathbb{R}^+$ is a continuous function which is non-decreasing in $t \in \mathbb{R}$. Let u be a viscosity supersolution of the equation*

$$(3.4) \quad (\theta + dd^c u)^n = F(x, u)\omega^n.$$

Then the envelope $P_\theta(u)$ is a pluripotential supersolution of (3.4).

Proof. The proof, similar to that of Theorem 3.5, is left to the reader. q.e.d.

3.2.3. Continuity of envelopes. Let θ be a semi-positive and big form on X . We want to investigate conditions under which the envelope $P_\theta(u)$ is continuous in X when h is continuous in X . We say that (X, θ) satisfies the approximation property (AP) if any $\varphi \in \text{PSH}(X, \theta)$ can be approximated by a decreasing sequence of continuous θ -psh functions.

Theorem 3.8. *The following properties are equivalent*

- (i) *For any $h \in C^0(X)$, its envelope $P_\theta(h)$ is continuous on X ;*
- (ii) *(X, θ) satisfies the approximation property (AP);*
- (iii) *For any density $0 \leq f \in L^\infty(X, \mathbb{R})$ with $\int_X f\theta^n = \int_X \theta^n$, the unique solution to the complex Monge–Ampère equation*

$$(\theta + dd^c \varphi)^n = f dV, \quad \max_X \varphi = 0,$$

is continuous on X .

The approximation property has been introduced in [EGZ09] where it is proved that (ii) \implies (iii) holds. (AP) is known to hold when $\{\theta\}$ is a Kähler class [Dem92, BK07, Ber19], or when it is a Hodge class on a singular variety [CGZ13].

Proof. (i) \implies (ii). Assume that $\varphi \in \text{PSH}(X, \theta)$. Let (h_j) be a sequence of smooth functions decreasing to φ on X . Then using (i) we conclude that $(P(h_j))$ is a decreasing sequence of continuous θ -psh functions in X that converges to φ .

(ii) \implies (iii). This property follows from [EGZ09].

(iii) \implies (i). Assume first that h is smooth on X . Then $(\theta + dd^c h)_+^n = f dV$, where f is a continuous function on X . Then h is a viscosity supersolution to the complex Monge–Ampère equation

$$(\theta + dd^c \phi)^n = e^{\phi-h}(\theta + dd^c h)_+^n = e^\phi e^{-h} f dV,$$

where $f e^{-h}$ is a continuous density on X .

By [Theorem A](#), $P(h)$ is then a pluripotential supersolution of the same equation. Therefore $(\theta + dd^c P(h))^n \leq f dV$, in the weak sense, hence there exists a function $g \in L^\infty(X)$ such that $(\theta + dd^c P(h)) = g dV$ weakly on X . Hence by (iii), $P(h)$ is continuous on X .

The general case follows by approximation. Let $h \in C^0(X, \mathbb{R})$ then approximate h by a decreasing sequence of smooth functions h_j in X . Since $|P(h_j) - P(h)| \leq \sup_X |h_j - h|$, it follows that $P(h_j)$ converges to $P(h)$ uniformly on X , hence $P(h)$ is continuous on X . q.e.d.

3.3. The local context. Let $D \Subset \mathbb{C}^n$ be a bounded hyperconvex domain. By definition D admits a continuous negative plurisubharmonic exhaustion ρ . The domain D is said to be strictly pseudoconvex if the exhaustion function ρ can be chosen strictly plurisubharmonic in D . Let $0 \leq f$ be a continuous function in D and dV be the Euclidean volume form on D .

For a bounded function u in D , the upper envelope $P_D(u)$ of u in D is defined by

$$P_D(u) := (\sup\{\varphi \in \text{PSH}(D) : \varphi \leq u\})^*.$$

We will also need to consider the following envelope which takes care of the boundary values:

$$P_{\bar{D}}(u) := (\sup\{\varphi \in \text{PSH}(D) : \varphi^* \leq u \text{ on } \bar{D}\})^*,$$

where u is defined on \bar{D} and φ^* is the upper semicontinuous extension of φ to \bar{D} defined by

$$\varphi^*(\xi) := \lim_{r \rightarrow 0^+} \sup_{B(\xi, r) \cap D} \varphi, \quad \xi \in \partial D.$$

Note that the extension of φ to \bar{D} is upper semicontinuous on \bar{D} .

If u is continuous on \bar{D} then $P_D(u) = P_{\bar{D}}(u)$ on D . This does not hold in general as the example in [Remark 3.12](#) below shows.

We now state and prove the local version of [Theorem A](#):

Theorem 3.9. *Let $D \Subset \mathbb{C}^n$ be a bounded pseudoconvex domain. Assume that a bounded lower semi-continuous function u is a viscosity super-solution of the equation*

$$(3.5) \quad (dd^c u)^n = f dV,$$

in D . Then $P_D(u)$ is a pluripotential super-solution of (3.5) in D .

Proof. We first assume that D is strictly pseudoconvex and u is continuous in \bar{D} and prove that $P_D(u)$ is a pluripotential super-solution of (3.5). For each $\beta > 0$, let φ_β be the unique function in $\text{PSH}(D) \cap C^0(\bar{D})$ such that

$$(dd^c \varphi_\beta)^n = e^{\beta(\varphi_\beta - u)} f dV \quad \text{in } D$$

with boundary values u i.e. $\varphi_\beta = u$ in ∂D [[BT76](#)].

Using the local viscosity comparison principle [EGZ11], we deduce that $\varphi_\beta \leq u$ in \bar{D} for any $\beta > 0$. Arguing as in Step 2 of the proof of Theorem 3.5, we can prove that φ_β increases to $P_D(u)$ a.e. in D as β increases to $+\infty$. Therefore by letting $\beta \rightarrow +\infty$ in the above equation, we conclude that $(dd^c P_D(u))^n \leq f dV$ in D .

Now assume that $D \Subset \mathbb{C}^n$ is a pseudoconvex domain and u is continuous in D . Then the result follows by taking an exhaustive sequence of strictly pseudoconvex domains $D_j \Subset D$ (in view of the lemma below) and applying what have been done above.

For the general case when u is merely lower-semi continuous, we approximate u by inf-convolution i.e. we consider

$$u^j(z) := \inf\{u(\zeta) + j|z - \zeta|^2\}, \quad z \in D^j, j \in \mathbb{N}^*,$$

where (D^j) is an exhaustive sequence of pseudoconvex domains converging to D . Then we know that (u^j) is an increasing sequence of continuous functions converging to u in D and for each j , the function u^j is a supersolution of

$$(dd^c v)^n = f_j dV, \quad \text{in } D^j,$$

where f_j is continuous in D^j and the sequence (f_j) decreases to f in D (see [CIL92, CC95]).

Fix any pseudoconvex domain $B \Subset D$. The previous result insures that for $j > 1$ large enough so that $B \subset D^j$, the function $P_B(u_j)$ satisfies $(dd^c P_B(u^j))^n \leq f_j$ in the pluripotential sense in B . Applying Lemma 3.10 below we obtain at the limit that the differential inequality $(dd^c P_{\bar{B}}(u))^n \leq f$ holds in the pluripotential sense on B .

Again taking an exhaustive sequence (B_j) of pseudoconvex domains converging to D and applying Lemma 3.11 below we obtain the required result. q.e.d.

Lemma 3.10. *Assume that (u_j) is an increasing sequence of lower semicontinuous functions on \bar{D} which converges pointwise to u . Then $P_{\bar{D}}(u_j)$ increases almost everywhere to $P_{\bar{D}}(u)$. As a consequence, if u_j is continuous on \bar{D} for all j , then $P_D(u_j)$ increases almost everywhere to $P_{\bar{D}}(u)$.*

Proof. Let φ be an arbitrary psh function in D such that $\varphi \leq u$ on \bar{D} .

Fix $\varepsilon > 0$. We will show that $u_j \geq \varphi - \varepsilon$ on \bar{D} , for j large enough. Assume that this was not the case. Then we can find a sequence $(x_j) \subset \bar{D}$ such that $u_j(x_j) < \varphi(x_j) - \varepsilon$. For $k \in \mathbb{N}$ fixed and $j > k$ we have

$$u_k(x_j) \leq u_j(x_j) \leq \varphi(x_j) - \varepsilon.$$

We can assume that $x_j \rightarrow x \in \bar{D}$. Since u_k is lsc and φ is usc on \bar{D} it follows that $u_k(x) \leq \varphi(x) - \varepsilon$. Since this is true for any k we deduce that $u(x) \leq \varphi(x) - \varepsilon \leq u(x) - \varepsilon$, a contradiction.

We thus have that $P_{\bar{D}}(u_j) \geq \varphi - \varepsilon$, for j large enough. By letting $j \rightarrow +\infty$ and then $\varepsilon \rightarrow 0$ we conclude that $(\lim P_{\bar{D}}(u_j))^* \geq \varphi$, ultimately giving $(\lim P_{\bar{D}}(u_j))^* \geq P_{\bar{D}}(u)$. The reverse inequality is obvious. q.e.d.

Lemma 3.11. *Let (D_j) be an increasing sequence of relatively compact bounded domains in D such that $\cup D_j = D$. Let u be a lower semi-continuous function in D . Then $P_{\bar{D}_j}(u)$ decreases pointwise to $P_D(u)$.*

Proof. Set $\varphi_j = P_{\bar{D}_j}(u)$ and note that this is a psh function in D_j . Clearly, φ_j is decreasing in j . The decreasing limit $\lim_j \varphi_j$ is psh in any D_k . As (D_k) is an exhaustive sequence of D , these limits define a psh function φ in D .

We need to prove that $\varphi = P_D(u)$. Indeed, if v is a psh function in D such that $v \leq u$ in D then v is also a candidate defining $P_{\bar{D}_j}(u)$, thus $\varphi_j \geq v$. We then get $\varphi \geq P_D(u)$. On the other hand, φ is psh in D and $\varphi \leq u$ quasi everywhere in D , i.e. there exists $E \subset D$ a pluripolar set such that $\varphi \leq u$ in $D \setminus E$. By Josefson theorem [GZ, Corollary 4.41] there exists a negative psh function ψ in D such that $\psi \not\equiv -\infty$ and $\psi = -\infty$ on E . For any $\delta > 0$ observe that $\varphi + \delta\psi \leq u$ in D . Therefore $\varphi + \delta\psi \leq P_D(u)$ for all $\delta > 0$. Letting $\delta \rightarrow 0^+$ we get $\varphi \leq P_D(u)$ quasi everywhere hence everywhere in D . q.e.d.

Remark 3.12. We stress that the envelopes $P_D(u)$ and $P_{\bar{D}}(u)$ are in general different if u is not continuous near the boundary. Indeed, take the function u defined on \bar{D} by $u = 0$ in D and -1 in ∂D . Then u is lower semi-continuous in \bar{D} , $P_D(u) = 0$ in D while $P_{\bar{D}}(u) = -1$ in D .

4. The minimum principle

4.1. The minimum principle. The following property is inspired by the fact that the minimum of two viscosity supersolutions is again a viscosity supersolution (see Section 3).

Lemma 4.1. *Let $u, v \in \mathcal{E}(X, \theta)$ and set $\varphi := P_\theta(\min(u, v))$. Then $\varphi \in \mathcal{E}(X, \theta)$ and*

$$\text{MA}(\varphi) \leq \mathbf{1}_{\{\varphi=u\}}\text{MA}(u) + \mathbf{1}_{\{\varphi=v\}}\text{MA}(v).$$

When θ is semipositive and u, v have bounded laplacian the result follows from Darvas' work [Dar17, Dar15] which uses a strong regularity result on the Monge–Ampère envelope. Our proof is inspired by the convergence method of Berman [Ber19].

Proof. The fact that φ belongs to $\mathcal{E}(X, \theta)$ follows from [DDL18]. Without loss of generality we can assume that u, v have minimal singularities.

For each $j \in \mathbb{N}^*$ it follows from Lemma 4.2 below that there exists a unique $\varphi_j \in \text{PSH}(X, \theta)$ with minimal singularities such that

$$\text{MA}(\varphi_j) = e^{j(\varphi_j - u)}\text{MA}(u) + e^{j(\varphi_j - v)}\text{MA}(v).$$

As both u and v are pluripotential supersolutions of the above equation it follows that $\varphi_j \leq \varphi$. By the pluripotential comparison principle we also have that φ_j increases almost everywhere to $\varphi_\infty \in \mathcal{E}(X, \theta)$.

We first prove that $\varphi_\infty = \varphi$. For each fixed $\varepsilon > 0$ one has

$$\int_{\{\varphi_\infty < \varphi - \varepsilon\}} \text{MA}(\varphi_j) \leq \int_{\{\varphi_j < \varphi - \varepsilon\}} \text{MA}(\varphi_j) \leq 2\text{Vol}(\theta)e^{-j\varepsilon} \rightarrow 0.$$

Letting $j \rightarrow +\infty$ we obtain that $\text{MA}(\varphi_\infty)$ vanishes in $\{\varphi_\infty < \varphi\}$. Applying the domination principle, Proposition 1.13, gives $\varphi_\infty = \varphi$.

Now we prove the inequality in the statement of the lemma. For each fixed $A > 0$ and $j > A$, since $\varphi_j \leq \min(u, v)$ we have

$$\text{MA}(\varphi_j) \leq e^{A(\varphi_j - u)}\text{MA}(u) + e^{A(\varphi_j - v)}\text{MA}(v).$$

Since $\text{MA}(\varphi_j)$ converges to $\text{MA}(\varphi)$, by dominated convergence theorem we obtain

$$\text{MA}(\varphi) \leq e^{A(\varphi - u)}\text{MA}(u) + e^{A(\varphi - v)}\text{MA}(v).$$

Now, letting $A \rightarrow +\infty$ we obtain the result. q.e.d.

Lemma 4.2. *Assume that u, v are θ -psh functions with minimal singularities and fix $\beta > 0$. Then there exists a unique θ -psh function φ with minimal singularities such that*

$$\text{MA}(\varphi) = e^{\beta(\varphi - u)}\text{MA}(u) + e^{\beta(\varphi - v)}\text{MA}(v).$$

Proof. The uniqueness follows from the comparison principle.

To prove existence, without loss of generality we can assume that $\beta = 1$. For each $j \in \mathbb{N}$ set $u_j = \max(u, -j), v_j = \max(v, -j)$. As $e^{-u_j} + e^{-v_j}$ is bounded on X it follows from [BBGZ13] that there exists a unique $\varphi_j \in \mathcal{E}^1(X, \theta)$ such that

$$\text{MA}(\varphi_j) = e^{\varphi_j - u_j}\text{MA}(u) + e^{\varphi_j - v_j}\text{MA}(v).$$

By the comparison principle we know that φ_j is decreasing in j and $\varphi_j \geq \frac{u+v}{2} - C$, for some constant $C > 0$ independent of j . We can thus pass to the limit $j \rightarrow +\infty$ and obtain the result. q.e.d.

Lemma 4.3. *Let $u_j, u \in \mathcal{E}^p(X, \theta)$, $p > 0$, be such that (u_j) converges to u in energy I_p . Then there exists a subsequence still denoted by (u_j) such that*

$$\varphi_j := P_\theta \left(\inf_{k \geq j} u_k \right) \nearrow u.$$

The proof is an adaptation of an argument due to Darvas [Dar17, Dar15].

Proof. After extracting a subsequence we can assume that

$$\int_X |u_j - u|^p \text{MA}_\theta(u_j) \leq 2^{-j}, \quad \forall j.$$

For each $j < k$ set $\varphi_j^k := P_\theta(\inf_{j \leq \ell \leq k} u_\ell)$. It follows from [DDL18] that φ_j^k belongs to $\mathcal{E}^p(X, \theta)$. It follows from Lemma 4.1 that

$$\int_X |u - \varphi_j^k|^p \text{MA}(\varphi_j^k) \leq \sum_{\ell=j}^k \int_X |u - u_\ell|^p \text{MA}(u_\ell) \leq 2^{-j+1}.$$

It then follows from Lemma 1.5 that the decreasing limit $\varphi_j := \lim_{k \rightarrow +\infty}(\varphi_j^k)$ belongs to $\mathcal{E}^p(X, \theta)$. Moreover the continuity of the Monge–Ampère operator (see [BEGZ10]) gives

$$\int_X |u - \varphi_j|^p \text{MA}(\varphi_j) \leq 2^{-j+1}.$$

Hence the increasing limit $\varphi := \lim_{j \rightarrow +\infty} \varphi_j \leq u$ satisfies $\int_{\{\varphi < u\}} \text{MA}(\varphi) = 0$. The domination principle then reveals that $u = \varphi$.
q.e.d.

4.2. Solving complex Monge–Ampère equations. Let (X, ω) be a compact Kähler manifold and μ a positive non-pluripolar Radon measure in some open subset Ω of X . Here we allow μ to have infinite total mass (i.e. $\mu(\Omega)$ may be $+\infty$).

Let θ be a smooth closed $(1, 1)$ -form on X which represents a big class. We assume that there exists $u_0 \in \mathcal{E}(X, \theta)$, such that

$$(\theta + dd^c u_0)^n \geq e^{u_0} \mu, \quad \text{in } \Omega.$$

In particular $e^{u_0} \mu$ extends as a Radon measure in all of X .

We would like to solve the complex Monge–Ampère equation

$$(4.1) \quad (\theta + dd^c \varphi)^n = \mathbf{1}_\Omega e^\varphi \mu,$$

by considering the lower envelope of supersolutions. We first note the following simple consequence of Lemma 4.1 which is basic to what follows:

Proposition 4.4. *Assume $u, v \in \mathcal{E}(X, \theta)$ are both supersolutions of (4.1), i.e. $\text{MA}(u) \leq \mathbf{1}_\Omega e^u \mu$, $\text{MA}(v) \leq \mathbf{1}_\Omega e^v \mu$. Then $P(\min(u, v))$ is also a supersolution,*

$$\text{MA}(P(\min(u, v))) \leq e^{P(\min(u, v))} \mu.$$

Proof. Observe first that since μ is a Radon measure in Ω , $\mu(u = v + r) = 0$ for almost every $r \in \mathbb{R}$. Fix such an r and set $\varphi_r = P_\theta(\min(u, v + r))$. It follows from Lemma 4.1 that

$$\begin{aligned} \text{MA}(\varphi_r) &\leq \mathbf{1}_{\{\varphi_r = u\}} \text{MA}(u) + \mathbf{1}_{\{\varphi_r = v+r\}} \text{MA}(v) \\ &\leq \mathbf{1}_{\{\varphi_r = u\}} e^u \mu + \mathbf{1}_{\{\varphi_r = v+r\}} e^v \mu \\ &\leq e^{\varphi_r + |r|} \mu. \end{aligned}$$

Proposition 2.2 insures that φ_r decreases pointwise to $P_\theta(\min(u, v))$. The latter belongs to $\mathcal{E}(X, \theta)$ as follows from [DDL18]. We conclude by letting $r \rightarrow 0$.
q.e.d.

This result guarantees that the families of pluripotential super-solutions is stable under the operation $P(\min(\cdot, \cdot))$. One can thus hope and solve the equation by taking the infimum of supersolutions; this is the contents of [Theorem C](#) from the introduction which we now establish:

Theorem 4.5. *Assume there exists a subsolution $u_0 \in \mathcal{E}(X, \theta)$, i.e.*

$$(\theta + dd^c u_0)^n \geq e^{u_0} \mu, \quad \text{in } \Omega.$$

Then the envelope of supersolutions

$$\varphi := P\left(\inf\{\psi : \psi \in \mathcal{E}(X, \theta) \text{ and } (\theta + dd^c \psi)^n \leq \mathbf{1}_\Omega e^\psi \mu\}\right)$$

is the unique pluripotential solution of $(\theta + dd^c \varphi)^n = \mathbf{1}_\Omega e^\varphi \mu$.

This result largely generalizes the main result of Berman–Guenancia [[BG14](#), Theorem A]: a projective complex algebraic variety V with semi-log canonical singularities and ample canonical bundle admits a unique Kähler–Einstein metric. Constructing the latter indeed boils down to solving a complex Monge–Ampère equation as above, where

- $\pi : X \rightarrow V$ is a resolution of singularities,
- $\theta = \pi^* \omega_V$ is the pull-back of a Kähler form representing $c_1(V)$,
- $\mu = f dV$ is absolutely continuous with respect to Lebesgue measure with a density $0 \leq f$ which is smooth in $X \setminus D = \pi^{-1} V^{reg}$, and blows up near $D = (s_D = 0)$ like $|s_D|^{-2}$.

One easily constructs a subsolution in this case (take e.g. $-(-\log |s_D|^{-2})^a$ with $0 < a < 1$ and s_D appropriately normalized).

Remark 4.6. The inequality $(\omega + dd^c \psi)^n \leq \mathbf{1}_\Omega e^\psi \mu$ is understood in the sense that the measure $(\omega + dd^c \psi)^n$ is concentrated in Ω where this inequality holds in the sense of positive Borel measures.

Proof. Let K be a compact subset of Ω such that $0 < \mu(K) < +\infty$ and denote by μ_K the restriction of μ on K , which is a positive non-pluripolar measure on X . It follows from [[BBGZ13](#)] that there exists $\varphi_K \in \mathcal{E}(X, \theta)$ such that $\text{MA}(\varphi_K) = e^{\varphi_K} \mu_K$. Hence φ_K is a supersolution of (4.1). The family \mathcal{F} of supersolutions is thus non-empty, and it is uniformly bounded from below by the subsolution u_0 , as follows from the comparison principle and the domination principle.

It follows from Proposition 4.4 that \mathcal{F} is stable by $P \circ \min(\cdot, \cdot)$. Coupled with an analogue of a classical lemma due to Choquet (see Lemma 4.7), this insures that

$$\varphi := P\left(\inf\{\psi : \psi \in \mathcal{E}(X, \theta) \text{ and } (\theta + dd^c \psi)^n \leq \mathbf{1}_\Omega e^\psi \mu\}\right)$$

is again a supersolution to the equation: this is the minimal supersolution. Indeed, in view of Proposition 4.4 and Lemma 4.7 one only needs to check that if $\varphi_j \in \mathcal{E}(X, \theta)$ decreases to $\varphi \in \mathcal{E}(X, \theta)$ and $\text{MA}(\varphi_j) \leq \mathbf{1}_\Omega e^{\varphi_j} \mu$ then $\text{MA}(\varphi) \leq \mathbf{1}_\Omega e^\varphi \mu$. By convergence of the non-pluripolar

Monge–Ampère measures one has that $\mathbf{1}_\Omega \text{MA}(\varphi) \leq \mathbf{1}_\Omega e^\varphi \mu$. It remains to show that $\text{MA}(\varphi)$ is concentrated in Ω , and to do this we use the subsolution u_0 . For each $A > 0$ fixed we set $\Omega_A := \{z \in \Omega : u_0(z) > -A\}$ and define μ_A to be the extension of $\mathbf{1}_{\Omega_A} \mu$ to X . Since $\mu(\Omega_A) < +\infty$, μ_A is a positive Borel measure on X . Moreover,

$$(4.2) \quad \mathbf{1}_{\{u_0 > -A\}} \text{MA}(\varphi_j) \leq e^{\varphi_j} \mu_A,$$

in the sense of Borel measures on X . Observe also that the measures $\text{MA}(\varphi_j)$ are uniformly dominated by the Monge–Ampère capacity Cap_θ as follows from Theorem 1.11. Since the set $\{u_0 > -A\}$ is quasi-open this together with [DDL18, Theorem 2.8] allow us to take the limit as $j \rightarrow +\infty$ in (4.2) to get

$$\mathbf{1}_{\{u_0 > -A\}} \text{MA}(\varphi) \leq e^\varphi \mu_A,$$

in the sense of positive Borel measures on X . In particular

$$\int_{\{u_0 > -A\} \cap (X \setminus \Omega)} \text{MA}(\varphi) = 0.$$

Finally, it suffices to let $A \rightarrow +\infty$.

In order to prove that φ is actually the solution it suffices to show that $\text{MA}(\varphi) = e^\varphi \mu$ in any small ball $B \subset \Omega \cap \text{Amp}(\theta)$. Fix such a ball B . We construct a supersolution ψ which is smaller than φ and which solves the equation in B . The classical method to produce such a supersolution is to solve a local Dirichlet problem in B and glue the local function with φ on the boundary ∂B . This requires a subtle analysis near the boundary as the functions at hand are not continuous (they may be even unbounded). We provide rather a global method which is simpler.

For each $j \in \mathbb{N}$ we let $\psi_j \in \mathcal{E}(X, \theta)$ be the unique solution to

$$\text{MA}(\psi_j) = \mathbf{1}_{\Omega \setminus B} e^{\psi_j - \max(\varphi, -j)} \text{MA}(\varphi) + \mathbf{1}_B e^{\psi_j} \mu.$$

The existence of such a solution follows from [BBGZ13], observing that

$$\nu_j = \mathbf{1}_{\Omega \setminus B} e^{-\max(\varphi, -j)} \text{MA}(\varphi) + \mathbf{1}_B \mu$$

is a non pluripolar Radon measure on X . Since $\text{MA}(\varphi) \leq \mathbf{1}_\Omega e^\varphi \mu$, one can check that u_0 is a subsolution of the above equation. It thus follows from the comparison principle that $\psi_j \geq u_0$ decreases to $\psi \in \mathcal{E}(X, \theta)$ which solves

$$\text{MA}(\psi) = \mathbf{1}_{\Omega \setminus B} e^{\psi - \varphi} \text{MA}(\varphi) + \mathbf{1}_B e^\psi \mu.$$

In order to prove this equality one can first use Fatou’s lemma to show the inequality \geq . For the reverse inequality it suffices to prove that the masses coincide. To do this one can use a truncation argument with the help of the subsolution u_0 as above.

Now, one can check that φ is a supersolution of the above equation while ψ is a supersolution of equation (4.1). By the comparison principle and by minimality of φ we have that $\varphi = \psi$, finishing the proof. q.e.d.

We have used the following analogue of Choquet’s lemma:

Lemma 4.7. *Let \mathcal{U} be a family of upper semicontinuous functions on X . Then there exists a countable subfamily $\mathcal{N} \subset \mathcal{U}$ such that $\inf_{u \in \mathcal{U}} u = \inf_{u \in \mathcal{N}} u$.*

Proof. Replacing each function $u \in \mathcal{U}$ by $u/(1 + |u|)$, we are reduced to the case when the family \mathcal{U} is uniformly bounded.

Fix a distance d on X (e.g. induced by a Riemann metric on X). For an upper semicontinuous function v on X we consider the sup-convolution $\Phi(v, j)$ defined by

$$\Phi(v, j)(x) := \sup\{v(y) - jd(x, y) : y \in X\}, j \in \mathbb{N}.$$

Then for each $j \in \mathbb{N}$, $\Phi(v, j)$ is continuous on X and as $j \rightarrow +\infty$ the sequence $\Phi(v, j)$ decreases pointwise to v .

Set $u := \inf_{v \in \mathcal{U}} v$. By Choquet’s lemma (see e.g. [GZ, Lemma 4.31]), for each $j \in \mathbb{N}$ there exists a sequence $(\varphi_k^j)_{k \in \mathbb{N}} \subset \mathcal{U}$ such that

$$\left(\inf_k \Phi(\varphi_k^j, j)\right)_* = \left(\inf_{v \in \mathcal{U}} \Phi(v, j)\right)_*.$$

Set $\varphi := \inf_{j,k} \varphi_k^j$. The lemma is now reduced to showing that $\varphi = u$. Indeed, it is obvious that $\varphi \geq u$. For each $j \in \mathbb{N}$ and any $v \in \mathcal{U}$, we have

$$\left(\Phi(\varphi, j)\right)_* \leq \left(\inf_k \Phi(\varphi_k^j, j)\right)_* \leq \Phi(v, j).$$

We observe also that $\left(\Phi(\varphi, j)\right)_* = \Phi(\varphi, j)$ for any j . The function $\Phi(\varphi, j)$ is continuous on X . We thus have $\Phi(\varphi, j) \leq \Phi(v, j)$ for all $v \in \mathcal{U}$. Note also that φ, u are upper semicontinuous on X . Letting $j \rightarrow +\infty$ we get $\varphi \leq u$ completing the proof. q.e.d.

At the end of the proof of Theorem C we could also have used a local Dirichlet problem and do a gluing process (balayage technique). We prove in the following that this process works well for measures with finite masses:

Lemma 4.8. *Let u be a θ -psh function with minimal singularities such that $\text{MA}(u) \leq e^u \nu$, where ν is a non-pluripolar positive Radon measure on X . Let B be a small ball in the ample locus of θ . Let v be a bounded θ -psh function in B such that $v \leq u$ in B , $\lim_{z \rightarrow \partial B} (v - u) \geq 0$, and $\text{MA}(v) \leq e^v \nu$ in B . Set*

$$\psi(x) = \begin{cases} u(x) & \text{if } x \in X \setminus B \\ v(x) & \text{if } x \in B. \end{cases}$$

Then $P_\theta(\psi)$ is a (pluripotential) supersolution of the equation $\text{MA}(\varphi) = e^\varphi \nu$ on X .

Proof. The proof uses the machinery we have developed so far. It consists in showing that $P_{\theta,\nu}(\psi)$ is a pluripotential supersolution, which moreover coincides with $P_{\theta}(\psi)$, by using the Berman approximation process.

For each $\beta > 1$ let φ_{β} be the unique function in $\mathcal{E}^1(X, \theta)$ such that (see [BBGZ13])

$$\text{MA}(\varphi_{\beta}) = e^{\beta(\varphi_{\beta}-\psi)} e^{\varphi_{\beta}} \nu = e^{(\beta+1)(\varphi_{\beta}-\psi)} e^{\psi} \nu.$$

Since $\psi \leq u$ and $\text{MA}(u) \leq e^u \mu$ one can check that u is a supersolution of the above equation. It follows from the comparison principle (see Proposition 1.14) that $\varphi_{\beta} \leq u$, for all $\beta > 0$.

We claim that $\varphi_{\beta} \leq v$ in B , for all $\beta > 0$. This could follow from the local comparison principle, if we knew that φ_{β} belongs by restriction to a local finite energy class in B . Since the definition and properties of φ_{β} are global in nature, we need to make a technical detour.

Fix $\beta > 0$ and let g be a local potential of θ in a neighborhood of B (i.e. $dd^c g = \theta$). Fix ρ a negative strictly psh function in B . Set

$$\phi_{\beta,j} := g + \max(\varphi_{\beta}, V_{\theta} - j), \quad j \in \mathbb{N}$$

and note that $(dd^c \phi_{\beta,j})^n$ converges to $(\theta + dd^c \varphi_{\beta})^n$ as $j \rightarrow +\infty$ in the strong sense of Borel measures on B . Since v is bounded and $v = u \geq \varphi_{\beta}$ on ∂B it follows that, for j big enough, $\phi_{\beta,j} \leq v + g$ on ∂B . Fix $\varepsilon > 0$ and set

$$U_{\beta,\varepsilon,j} := B \cap \{v + g < \varepsilon \rho + \phi_{\beta,j}\}, \quad U_{\beta,\varepsilon} := B \cap \{v < \varepsilon \rho + \varphi_{\beta}\}.$$

Observe that $U_{\beta,\varepsilon} \subset U_{\beta,\varepsilon,j} \Subset B$ and $\{v \leq \varepsilon \rho + \varphi_{\beta}\} \subset \{v < \varphi_{\beta}\}$. The comparison principle for bounded psh functions [BT82, Theorem 4.1] yields

$$\begin{aligned} \int_{U_{\beta,\varepsilon}} [\varepsilon^n (dd^c \rho)^n + (dd^c \phi_{\beta,j})^n] &\leq \int_{U_{\beta,\varepsilon,j}} (dd^c(\varepsilon \rho + \phi_{\beta,j}))^n \\ &\leq \int_{U_{\beta,\varepsilon,j}} (dd^c(g + v))^n. \end{aligned}$$

Letting $j \rightarrow +\infty$ and using that v is a supersolution we obtain

$$\int_{U_{\beta,\varepsilon}} [\varepsilon^n (dd^c \rho)^n + e^{\beta(\varphi_{\beta}-v)} e^{\varphi_{\beta}} d\nu] \leq \int_{\{v < \varphi_{\beta}\}} e^v d\nu.$$

In $U_{\beta,\varepsilon}$ we have $\varphi_{\beta} > v$. Thus letting $\varepsilon \rightarrow 0$ in the above inequality we obtain $\nu(v < \varphi_{\beta}) = 0$. Since ρ is strictly psh we conclude that the set $U_{\beta,\varepsilon}$, and hence also the set $\{v < \varphi_{\beta}\}$, has Lebesgue measure zero. It is thus empty, proving the claim.

Thus $\varphi_{\beta} \leq \psi$ on X for all $\beta > 0$. It follows that $\text{MA}(\varphi_{\beta}) \leq e^{\varphi_{\beta}} \nu$. As $\beta \rightarrow +\infty$ Theorem B shows that φ_{β} converges in energy to $P_{\theta,\nu}(\psi)$ which satisfies $P_{\theta,\nu}(\psi) \leq \psi$ quasi everywhere on X . Thus $P_{\theta,\nu}(\psi) \leq P_{\theta}(\psi)$. Since the inequality $P_{\theta}(\psi) \leq P_{\theta,\nu}(\psi)$ is always satisfied for a

non pluripolar measure, we obtain the equality $P_{\theta,\nu}(\psi) = P_\theta(\psi)$. By continuity of the Monge–Ampère measure along convergence in energy, this eventually shows that $\text{MA}(P(\psi)) \leq e^{P(\psi)}\nu$. q.e.d.

5. Further applications

5.1. Controlling the mass of viscosity super-solutions. Let θ be a smooth closed $(1, 1)$ -form such that $[\theta]$ is big.

Proposition 5.1. *Let $f : X \rightarrow \mathbb{R}^+$ be a continuous function. There exists a viscosity super-solution of*

$$(\theta + dd^c u)^n = f\omega^n$$

if and only if $\int_X f\omega^n \geq \text{Vol}(\theta)$.

Proof. Assume that $\int_X f\omega^n \geq \text{Vol}(\theta)$. Let $\varphi \in \text{PSH}(X, \theta)$ be the unique function with minimal singularities normalized by $\sup_X \varphi = 0$ such that $\text{MA}(\varphi) = cf\omega^n$ where $c > 0$ is a normalization constant. It follows from [EGZ11] that φ is also a viscosity solution. Since $c \leq 1$ the result follows.

Conversely, assume that u is a viscosity supersolution. It follows from Theorem 3.7 that $P_\theta(u)$ is also a pluripotential supersolution, hence the inequality follows. q.e.d.

The connection between pluripotential and viscosity supersolutions of complex Monge–Ampère equations allows us to derive the following interesting inequality:

Proposition 5.2. *Assume that θ is a closed smooth $(1, 1)$ -form such that $[\theta]$ is big. Then the envelope V_θ satisfies*

$$\text{MA}(V_\theta) \leq \mathbf{1}_{\{V_\theta=0\}}\theta^n.$$

This is a particular case of an important result of Berman and Demailly [BD12], which uses strong regularity information on the function V_θ . We provide a proof of independent interest. A slightly different proof has recently been given in [Ber19], [DDL18] using the viscosity theory developed in [EGZ11].

Proof. The function 0 is a viscosity super-solution of the equation

$$(\theta + dd^c u)^n = e^u \theta_+^n,$$

where θ_+ is defined pointwise to be θ if $\theta \geq 0$ and zero otherwise. It follows from Theorem 3.5 that V_θ is a pluripotential super-solution of the same equation, thus $\text{MA}(V_\theta) \leq e^{V_\theta} \theta_+^n$ in the pluripotential sense. As $\text{MA}(V_\theta)$ is concentrated on the contact set $\{V_\theta = 0\}$, the conclusion follows. q.e.d.

5.2. Examples of viscosity supersolutions. As the concept of viscosity super-solutions to complex Monge–Ampère equations is relatively new and still a bit mysterious, it is probably helpful to discuss in some details a few elementary examples.

Proposition 5.3. *Assume that u is a bounded viscosity supersolution of*

$$(\theta + dd^c u)^n = C\omega^n,$$

where $C > 0$ is a constant. Then $P_{\theta, dV}(u) = P_{\theta}(u)$. In other words, if a θ -psh function φ satisfies $\varphi \leq u$ almost everywhere with respect to Lebesgue measure then the inequality holds quasi everywhere.

Proof. We use the convergence method developed in Section 2.3. For each $\beta > 1$ let φ_{β} be the unique θ -psh function with minimal singularities such that

$$(5.1) \quad \text{MA}(\varphi_{\beta}) = e^{\beta(\varphi_{\beta} - u)}\omega^n.$$

We claim that $\varphi_{\beta} \leq u + \frac{\log C}{\beta}$ in the ample locus of θ , for any $\beta > 1$. Indeed, observe that $u + \log(C)/\beta$ is a viscosity supersolution of (5.1). Using an approximation argument and the viscosity comparison principle as in the proof of Theorem 3.5 we can show that $\varphi_{\beta} \leq u + \frac{\log C}{\beta}$ in $\text{Amp}(\theta)$. But by Theorem 2.14 we know that φ_{β} converges in energy to the modified envelope $P_{\theta, dV}(u)$. It thus follows that $P_{\theta, dV}(u) = P_{\theta}(u)$.
q.e.d.

Proposition 5.4. *For $n = 1$, the viscosity supersolutions*

$$(dd^c v)_+ \leq dd^c |z|^2$$

are precisely the functions v such that $v - |z|^2$ is super-harmonic.

Proof. Let q be a \mathcal{C}^2 upper test for $|z|^2 - v$ at $a \in \mathbb{C}$. Then the function $|z|^2 - q$ is a lower test for v at a . It follows that $(dd^c(|z|^2 - q))_+ \leq dd^c |z|^2$ at a , hence $dd^c q \geq 0$ at a . It thus follows that $|z|^2 - v$ is a viscosity subsolution, hence it is subharmonic as follows from [Hor, Prop. 3.2.10, p. 147].
q.e.d.

One could expect a similar property to hold in higher dimension: if

$$(dd^c v)_+^n \leq (dd^c |z|^2)^n$$

in the viscosity sense, one would like to conclude that $v - |z|^2$ is 1-concave. This is however not true in general:

Proposition 5.5. *Let B be the unit ball in \mathbb{C}^2 and consider the function u defined by $u(z_1, z_2) = -1$ if $|z_1| = |z_2|$ and $u(z_1, z_2) = 0$ elsewhere. Then u is a viscosity supersolution of the Monge–Ampère equation*

$$(dd^c u)^n = 0.$$

Proof. We set $D := \{(z_1, z_2) \in B : |z_1| = |z_2|\}$. If $x_0 \in B \setminus D$ then u is smooth near x_0 and the result follows from [EGZ11].

Assume now that $x_0 = (a, a) \in D$ and q is a lower test function for u at x_0 . The function $p(z) := q(z, z)$ is a lower test function for the constant -1 near a . It follows that $dd^c p$ is not positive at a hence $(dd^c q)_+^2(x_0) = 0$. Thus u is a viscosity supersolution of the above equation. q.e.d.

Remark 5.6. We let the reader check that the Monge–Ampère envelope $P_B(u)$ is identically -1 in B . Its Monge–Ampère measure is thus identically 0. This is consistent with Theorem 3.9.

The example in Proposition 5.5 indicates that viscosity supersolutions (in a local context and without boundary constraints) are in general not quasi-continuous. One can not construct a similar example on a compact Kähler manifold. More precisely, we have the following:

Lemma 5.7. *Let (X, ω) be a compact Kähler manifold of dimension n and θ be a closed smooth semipositive $(1, 1)$ -form on X such that $\int_X \theta^n > 0$. Let $E \subset X$ be a closed subset of X that has zero Lebesgue measure. If the function $u = -\mathbf{1}_E$ satisfies*

$$(\theta + dd^c u)^n \leq C\omega^n$$

in the viscosity sense on X , then E is pluripolar.

Proof. Indeed, by Proposition 5.3 we have that $P_{\theta, dV}(u) = P_\theta(u)$. Therefore since $u = 0$ a.e. in X , it follows that $P_\theta(u) = 0$ in X , which implies that E is pluripolar since $P_\theta(u) = h_{E, \theta}^*$ is the relative θ -plurisubharmonic extremal function of E (see [GZ05]). q.e.d.

Our analysis above motivates the following:

Question 5.8. Assume that u is a bounded lower semicontinuous function on X such that

$$(\omega + dd^c u)^n \leq C\omega^n$$

holds in the viscosity sense for some positive constant C . Is u quasi-continuous on X ?

Understanding the regularity properties of viscosity supersolutions is an important problem. We establish a refined semi-continuity property:

Proposition 5.9. *Assume that v is a bounded viscosity supersolution of $-(\theta + dd^c v)^n + e^v C dV = 0$. Then*

$$v(a) = \liminf_{x \rightarrow a, x \neq a} v(x)$$

for all $a \in X$.

Proof. Assume that it were not the case. Then we can find $a \in X$ and $\varepsilon > 0$ such that

$$v(a) + \varepsilon \leq \liminf_{x \rightarrow a, x \neq a} v(x).$$

Thus, we can find a small ball $B(a, r)$ in a local coordinate chart around a such that $v(a) + \varepsilon \leq \inf_{x \in B(a, r), x \neq a} v(x)$. Now, for any $A > 0$ the function $q_A(z) := A|z - a|^2 + v(a)$ is a smooth sub test of v at a . If A is large enough then $(\theta + dd^c q_A)$ is positive definite at a and the inequality $(\theta + dd^c q_A)_+^n \leq e^{q_A} C dV$ does not hold at a , which is a contradiction.

q.e.d.

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