J. DIFFERENTIAL GEOMETRY
113 (2019) 189-226

CRITICAL POINTS OF THE CLASSICAL EISENSTEIN SERIES OF WEIGHT TWO

Zhijie Chen & Chang-Shou Lin

Abstract

In this paper, we completely determine the critical points of the normalized Eisenstein series $E_2(\tau)$ of weight 2. Although $E_2(\tau)$ is not a modular form, our result shows that $E_2(\tau)$ has at most one critical point in every fundamental domain of the form $\gamma(F_0)$ of $\Gamma_0(2)$, where $\gamma(F_0)$ are translates of the basic fundamental domain F_0 via the Möbius transformation of $\gamma \in \Gamma_0(2)$. We also give a criteria for such fundamental domain containing a critical point of $E_2(\tau)$. Furthermore, under the Möbius transformations of $\Gamma_0(2)$ action, all critical points can be mapped into the basic fundamental domain F_0 and their images in F_0 give rise to a dense subset of the union of three connected smooth curves in F_0. A geometric interpretation of these smooth curves is also given. It turns out that these curves coincide with the degeneracy curves of trivial critical points of a multiple Green function related to flat tori.

1. Introduction

The Jacobi theta functions, the Eisenstein series and the Weierstrass functions arise in numerous theories and applications of both mathematics and physics. Since their discovery in the early 19th century, the mathematical foundation of elliptic functions was subsequently developed. It turns out that, besides their applications in science, these special functions in the elliptic function theory are rather deep objects by themselves.

The main goal of this paper is to completely locate all the critical points of the classical function $\eta_1(\tau)$ or equivalently the normalized Eisenstein series $E_2(\tau)$ of weight 2. Throughout the paper, we use the notations $\mathbb{R}^+ = (0, +\infty)$, $\omega_1 = 1$, $\omega_2 = \tau$, $\omega_3 = 1 + \tau$ and $\Lambda_\tau = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau \in \mathbb{H} = \{\tau | \operatorname{Im} \tau > 0\}$. Let $\wp(z) = \wp(z|\tau)$ be the Weierstrass $\wp$-function with periods Λ_τ, defined by

$$\wp(z|\tau) := \frac{1}{z^2} + \sum_{\omega \in \Lambda_\tau \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

Received July 17, 2017.

Let $\zeta(z) = \zeta(z|\tau) := -\int^z \wp(\xi|\tau)d\xi$ be the Weierstrass zeta function, which is odd and has two quasi-periods $\eta_k(\tau) := 2\zeta(\frac{\omega_k}{2}|\tau)$, $k = 1, 2$:

$$\eta_1(\tau) = \zeta(z+1|\tau) - \zeta(z|\tau), \quad \eta_2(\tau) = \zeta(z+\tau|\tau) - \zeta(z|\tau). \tag{1.1}$$

The well-known Legendre relation gives $\eta_2(\tau) = \tau\eta_1(\tau) - 2\pi i$. In the literature, $\eta_1(\tau)$ is known as the Weierstrass eta function (cf. [1]), which is just a multiple of the normalized Eisenstein series $E_2(\tau)$ of weight 2:

$$\begin{aligned} \frac{3}{\pi^2}\eta_1(\tau) = E_2(\tau) :=& \frac{3}{\pi^2}\sum_{m=-\infty}^{\infty}\sideset{}{'}\sum_{n=-\infty}^{\infty}\frac{1}{(m\tau+n)^2} \\ =& 1 - 24\sum_{n=1}^{\infty} b_n e^{2n\pi i\tau}, \quad b_n = \sum_{1\le d|n} d. \end{aligned} \tag{1.2}$$

Conventionally, $\sum'$ means to sum over $(n,m) \in \mathbb{Z}^2 \setminus \{(0,0)\}$. Besides, $\eta_1(\tau)$ is also connected with Dedekind eta function

$$\eta(\tau) := e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1 - e^{2n\pi i\tau})$$

through the following logarithmic differential formula (cf. [1, p. 696]):

$$\frac{1}{\eta(\tau)}\eta'(\tau) = \frac{i}{4\pi}\eta_1(\tau).$$

Unlike the other Eisenstein series of weight $2k$ with $k \ge 2$, $E_2(\tau)$ is not a modular form. Its transformation under the action of $SL(2,\mathbb{Z})$ satisfies

$$E_2\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2E_2(\tau) - \frac{6ic}{\pi}(c\tau+d), \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \tag{1.3}$$

Thus it is surprising that its critical points possess the following property.

Theorem 1.1. *Let $F = \gamma(F_0)$ be a fundamental domain of $\Gamma_0(2)$. Then $E_2(\tau)$ has at most one critical point in F.*

Here $\Gamma_0(2)$ is the congruence subgroup of $SL(2,\mathbb{Z})$ defined by

$$\Gamma_0(2) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{Z}) \middle| c \equiv 0 \mod 2 \right\},$$

and F_0 is the basic fundamental domain of $\Gamma_0(2)$:

$$F_0 := \{\tau \in \mathbb{H} \mid 0 \leqslant \text{ Re } \tau \leqslant 1 \text{ and } |\tau - \tfrac{1}{2}| \geqslant \tfrac{1}{2}\}.$$

Then for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ (i.e. *consider γ and $-\gamma$ to be the same*),

$$\gamma(F_0) := \left\{ \gamma\cdot\tau := \tfrac{a\tau+b}{c\tau+d} \middle| \tau \in F_0 \right\} = (-\gamma)(F_0)$$

is another fundamental domain of $\Gamma_0(2)$.

Recently, there are some works studying the zeros of $E_2(\tau)$; see [**9**, **18**] and references therein. As far as we know, there seems no results concerning the critical points of $E_2(\tau)$ in the literature.

In view of Theorem 1.1, a natural question is: *What are those fundamental domains containing critical points*? This is completely solved in the following result. Note that $\gamma(F_0) = F_0 + m$ for some $m \in \mathbb{Z}$ if and only if $c = 0$.

Theorem 1.2. *Let $F = \gamma(F_0)$ be a fundamental domain of $\Gamma_0(2)$ with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$. Then F contains a critical point of $E_2(\tau)$ if and only if $c \neq 0$.*

By Theorem 1.2, we can transform every critical point of $E_2(\tau)$ via the Möbius transformation of $\Gamma_0(2)$ action to locate it in F_0. Denote *the collection of such corresponding points in F_0 by $\mathcal{D}$*, which consists of infinitely many points. A fundamental question is: *What is the geometry of the set $\mathcal{D}$*?

Surprisingly, it turns out that $\mathcal{D}$ locates on the union of *three connected smooth curves* $\tau(C)$'s in F_0, which are parameterized by $C \in \mathbb{R} \setminus \{0, 1\}$ via the following identity

$$C = \tau - \frac{2\pi i}{\eta_1(\tau) \pm \sqrt{g_2(\tau)/12}}, \quad \tau \in F_0. \tag{1.4}$$

Here $g_2(\tau) = 60G_4(\tau)$ is the well-known invariant coming from

$$\wp'(z|\tau)^2 = 4\wp(z|\tau)^3 - g_2(\tau)\wp(z|\tau) - g_3(\tau),$$

and $G_4(\tau)$ is the Eisenstein series of weight 4. We will prove in Section 2 that for each $C \in \mathbb{R} \setminus \{0, 1\}$, there is *a unique point* $\tau(C) \in F_0$ such that (1.4) holds.[1] Consequently, the parametrization (1.4) will give three connected smooth curves

$$\mathcal{C}_0 := \{\tau(C) | C \in (0, 1)\},$$
$$\mathcal{C}_- := \{\tau(C) | C \in (-\infty, 0)\}, \quad \mathcal{C}_+ := \{\tau(C) | C \in (1, +\infty)\}.$$

The relation between (1.4) and $\eta_1'(\tau)$ comes from the classical formula (see e.g. [**1**, p. 704], or from Ramanujan's formula: $E_2'(\tau) = \frac{\pi i}{6}(E_2^2 - E_4)$)

$$\eta_1'(\tau) = \tfrac{i}{2\pi} \left(\eta_1(\tau)^2 - \tfrac{1}{12} g_2(\tau)\right). \tag{1.5}$$

Theorem 1.3. *Let $\tau(C)$ be defined by (1.4) for $C \in \mathbb{R} \setminus \{0, 1\}$. Then*

$$\mathcal{D} = \left\{ \tau(\tfrac{-d}{c}) \,\middle|\, \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0 \right\} \subset \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+. \tag{1.6}$$

Furthermore, the closure of $\mathcal{D}$ in F_0 is precisely the union of the three connected smooth curves:

$$\overline{\mathcal{D}} \cap F_0 = \overline{\mathcal{D}} \setminus \{0, 1\} = \mathcal{C}_- \cup \mathcal{C}_0 \cup \mathcal{C}_+. \tag{1.7}$$

[1] Note that the right hand side (RHS) of (1.4) is actually a multi-valued function, please see Theorem 3.1 for the precise definition of this unique $\tau(C)$.

Remark 1.4. In fact, we will prove $\tau(C) \in \mathring{F}_0$, where $\mathring{F}_0 = F_0 \backslash \partial F_0$ denotes the set of interior points of F_0. Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$, we will prove in Theorem 4.1 that the unique critical point of $E_2(\tau)$ in $\gamma(F_0)$ is precisely $\frac{a\tau(\frac{-d}{c})+b}{c\tau(\frac{-d}{c})+d} \in \gamma(\mathring{F}_0)$. Given $\gamma_j = \left(\begin{smallmatrix} a_j & b_j \\ c_j & d_j \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c_j \neq 0$ such that $\gamma_1 \neq \pm\gamma_2$, we have $\gamma_1(\mathring{F}_0) \cap \gamma_2(\mathring{F}_0) = \emptyset$ (note that $\gamma_1(\partial F_0) \cap \gamma_2(\partial F_0) \neq \emptyset$ may happen) and so

$$\frac{a_1\tau(\frac{-d_1}{c_1}) + b_1}{c_1\tau(\frac{-d_1}{c_1}) + d_1} \neq \frac{a_2\tau(\frac{-d_2}{c_2}) + b_2}{c_2\tau(\frac{-d_2}{c_2}) + d_2}.$$

Therefore, the map from $\mathcal{D}$ to the set of critical points of $E_2(\tau)$ is one-to-one and onto. The above results completely locate all the critical points of the Eisenstein series $E_2(\tau)$ or equivalently $\eta_1(\tau)$. To the best of our knowledge, such fundamental results have not appeared in the literature and are new. We believe that they will have important applications. For example, we consider $\tau = \frac{1}{2} + ib$ with $b > 0$. Then $\eta_1(\tau) \in \mathbb{R}$. In order to study the behavior of the Green function on rhombus tori, Wang and the second author [**14**] considered the monotone property of $\eta_1(\tau)$ and their numerical computation [**14**, Figure 2] suggests that η_1 should increase from 0 to some b_0 and then decrease after b_0, but they can not prove this assertion in [**14**] because (1.2) implies

$$\frac{3}{\pi^2}\eta_1(\tfrac{1}{2} + ib) = 1 - 24\sum_{n=1}^{\infty}(-1)^n b_n e^{-2n\pi b}, \quad b_n = \sum_{1\leq d|n} d > 0,$$

from which it seems difficult to obtain the monotone property shown in [**14**, Figure 2]. Now this assertion is confirmed by the following corollary.

Corollary 1.5. *There exists $b_0 \in (\frac{5}{24}, \frac{1}{2\sqrt{3}})$ such that $\eta_1(\frac{1}{2} + ib)$ is strictly increasing for $b \in (0, b_0)$ and strictly decreasing for $b \in (b_0, +\infty)$.*

One of our motivations of studying critical points of $\eta_1(\tau)$ comes from the Green function on flat tori. Let $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be a flat torus and $G(z) = G(z;\tau)$ be the Green function on the torus E_τ:

$$-\Delta G(z;\tau) = \delta_0 - \frac{1}{|E_\tau|} \text{ on } E_\tau, \quad \int_{E_\tau} G(z;\tau) = 0,$$

where δ_0 is the Dirac measure at 0 and $|E_\tau|$ is the area of the torus E_τ. See [**14**] for a detailed study of $G(z;\tau)$. In [**2**, **13**, **15**, **16**], Chai, Wang and the second author introduced a multiple Green function G_n, $n \in \mathbb{N}$. Geometrically, any critical point of G_n is closely related to bubbling phenomena of nonlinear partial differential equations with exponential nonlinearities in two dimension; see [**2**, **13**, **16**] for typical examples. Thus, understanding the critical points of G_n is important for applications.

For the case $n = 2$, the multiple Green function G_2 is defined by

$$G_2(z_1, z_2; \tau) := G(z_1 - z_2; \tau) - 2G(z_1; \tau) - 2G(z_2; \tau), \tag{1.8}$$

where $0 \neq z_1 \neq z_2 \neq 0$. A critical point (a_1, a_2) of G_2 satisfies

$$2\nabla G(a_1; \tau) = \nabla G(a_1 - a_2; \tau), \quad 2\nabla G(a_2; \tau) = \nabla G(a_2 - a_1; \tau).$$

Clearly if (a_1, a_2) is a critical point then so does (a_2, a_1), and we consider such two critical points to be *the same one*. A critical point (a_1, a_2) is called a *trivial critical point* if

$$\{a_1, a_2\} = \{-a_1, -a_2\} \text{ in } E_\tau.$$

Recall $\omega_1 = 1, \omega_2 = \tau$ and $\omega_3 = 1 + \tau$. It is known [**16**] that G_2 has only five trivial critical points $\{(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j) | i \neq j\}$ and $\{(q_\pm, -q_\pm) | \wp(q_\pm|\tau) = \pm\sqrt{g_2(\tau)/12}\}$, and the Hessian at $(q_\pm, -q_\pm)$ is given by

$$\begin{aligned} &\det D^2 G_2(q_\pm, -q_\pm; \tau) \\ =&\frac{3|g_2(\tau)|}{4\pi^4 \operatorname{Im}\tau} |\wp(q_\pm|\tau) + \eta_1(\tau)|^2 \operatorname{Im}\left(\tau - \frac{2\pi i}{\eta_1(\tau) \pm \sqrt{g_2(\tau)/12}}\right). \end{aligned} \tag{1.9}$$

From here and (1.4), we will prove in Section 5 that the three curves coincide with the degeneracy curves of G_2 (i.e. *the curves consisting of those τ's such that the Hessian at some trivial critical points of G_2 vanishes*) related to $(q_\pm, -q_\pm)$. The Hessian at $(\frac{1}{2}\omega_i, \frac{1}{2}\omega_j)$ is related to the critical points of the classical function $e_k(\tau) := \wp(\frac{\omega_k}{2}|\tau)$, $\{i, j, k\} = \{1, 2, 3\}$. We will study the critical points of $e_k(\tau)$ in another paper.

Our proof of the existence and uniqueness of $\tau(C)$ relies on a *pre-modular form* $Z^{(2)}_{r,s}(\tau)$ of weight 3 introduced in [**15**, Example 5.8]. See also [**7**]. For each pair $(r, s) \in \mathbb{R}^2 \setminus \frac{1}{2}\mathbb{Z}^2$, $Z^{(2)}_{r,s}(\tau)$ is defined by

$$Z^{(2)}_{r,s}(\tau) := Z_{r,s}(\tau)^3 - 3\wp(r + s\tau|\tau) Z_{r,s}(\tau) - \wp'(r + s\tau|\tau),$$

where $Z_{r,s}(\tau)$ is introduced by Hecke [**11**]:

$$Z_{r,s}(\tau) := \zeta(r + s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau). \tag{1.10}$$

Indeed, if $(r, s) \in \mathbb{Q}^2$, $Z_{r,s}(\tau)$ is the well-known Eisenstein series of weight 1 with characteristic (r, s); see [**8**, p. 139]. It is not difficult to see that $Z_{r,s}(\tau)$ is a modular form of weight 1 with respect to $\Gamma(N)$ if (r, s) is a N-torsion point, so $Z^{(2)}_{r,s}(\tau)$ is a modular form of weight 3. See Section 2. The importance of $Z^{(2)}_{r,s}(\tau)$ lies on the fact that at any zero τ_0 of $Z^{(2)}_{r,s}(\cdot)$, the pair (r, s) contains all the monodromy data of the classical Lamé equation

$$y''(z) = [n(n + 1)\wp(z|\tau_0) + B] y(z), \quad n = 2 \tag{1.11}$$

for some $B \in \mathbb{C}$; see [**15**, Theorem 4.3]. Therefore, it is important to study the zero of $Z^{(2)}_{r,s}(\cdot)$, which has not been settled yet. In this paper, we study the structure of the zeros of $Z_{r,s}^{(2)}(\cdot)$. Define four open triangles (see Figure 1 in Section 2):

$$\begin{aligned}\triangle_0 &:= \{(r,s) \mid 0<r,s<\tfrac{1}{2},\ r+s>\tfrac{1}{2}\},\\ \triangle_1 &:= \{(r,s) \mid \tfrac{1}{2}<r<1,\ 0<s<\tfrac{1}{2},\ r+s>1\},\\ \triangle_2 &:= \{(r,s) \mid \tfrac{1}{2}<r<1,\ 0<s<\tfrac{1}{2},\ r+s<1\},\\ \triangle_3 &:= \{(r,s) \mid r>0,\ s>0,\ r+s<\tfrac{1}{2}\}.\end{aligned} \tag{1.12}$$

Theorem 1.6. *Let $(r,s)\in[0,1]\times[0,\frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$. Then $Z_{r,s}^{(2)}(\tau)=0$ has a solution τ in F_0 if and only if $(r,s)\in\triangle_1\cup\triangle_2\cup\triangle_3$. Furthermore, for any $(r,s)\in\triangle_1\cup\triangle_2\cup\triangle_3$, $Z_{r,s}^{(2)}(\tau)$ has a unique zero τ in F_0 which satisfies $\tau\in\mathring{F}_0$.*

Remark that $Z_{r,s}^{(2)}(\tau)$ is not well-defined for $(r,s)=(0,0)$ since $Z_{0,0}\equiv\infty$ and so do $\wp(0),\wp'(0)$. To prove Theorems 1.2–1.3, we will "blow up" $Z_{r,s}^{(2)}(\tau)$ by considering $\lim_{s\to 0}\frac{1}{s}Z_{-Cs,s}^{(2)}(\tau)$, $C\in\mathbb{R}$, and the existence and uniqueness of $\tau(C)$ will follow from that of the zero of $Z_{-Cs,s}^{(2)}(\tau)$ as $s\to 0$.

The rest of this paper is organized as follows. Theorem 1.6 will be proved in Section 2. In Section 3, we apply Theorem 1.6 to prove the existence and uniqueness of $\tau(C)$. See Theorem 3.1. In Section 4, we give the detailed proofs of our main results Theorems 1.1–1.3 and Corollary 1.5. Some precise characterizations of the three curves (see Theorem 4.2) will also be given. In Section 5, we introduce the relation between the three curves and the degeneracy curve of G_2 and prove the smoothness of the curves. Finally in Appendix A, we give another application of Theorem 1.6.

Acknowledgments. The authors thank the referee for valuable comments and Prof. Chin-Lung Wang for providing the file of Figure 3 to us. The research of the first author was supported by NSFC (No. 11701312).

2. Zeros of pre-modular forms

This section is devoted to the proof of Theorem 1.6. First we recall the modularity of $g_2(\tau)$ and $\wp(z|\tau)$. Given any $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)\in SL(2,\mathbb{Z})$, it is well known that

$$\begin{aligned} g_2\left(\tfrac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^4 g_2(\tau),\\ \wp\left(\tfrac{z}{c\tau+d}\,\middle|\,\tfrac{a\tau+b}{c\tau+d}\right) &= (c\tau+d)^2\,\wp(z|\tau).\end{aligned} \tag{2.1}$$

From here we can obtain

$$\zeta\left(\tfrac{z}{c\tau+d}\,\middle|\,\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)\,\zeta(z|\tau),$$

and so

$$\begin{pmatrix} \eta_2(\frac{a\tau+b}{c\tau+d}) \\ \eta_1(\frac{a\tau+b}{c\tau+d}) \end{pmatrix} = (c\tau+d)\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \eta_2(\tau) \\ \eta_1(\tau) \end{pmatrix}. \tag{2.2}$$

In the rest of this paper, we will freely use the formulas (2.1)–(2.2).

As in [**7**, **15**], for any $(r,s)\in\mathbb{R}^2\setminus\mathbb{Z}^2$, we define pre-modular forms

$$\begin{aligned} Z_{r,s}(\tau) :=& \zeta(r+s\tau|\tau) - r\eta_1(\tau) - s\eta_2(\tau) \\ =& \zeta(r+s\tau|\tau) - (r+s\tau)\eta_1(\tau) + 2\pi i s, \end{aligned} \tag{2.3}$$

$$Z_{r,s}^{(2)}(\tau) := Z_{r,s}(\tau)^3 - 3\wp(r+s\tau|\tau)Z_{r,s}(\tau) - \wp'(r+s\tau|\tau). \tag{2.4}$$

As mentioned before, $Z_{r,s}^{(2)}(\tau)$ is not well-defined for $(r,s)\in\mathbb{Z}^2$. If $(r,s)\in\frac{1}{2}\mathbb{Z}^2\backslash\mathbb{Z}^2$, where

$$\tfrac{1}{2}\mathbb{Z}^2 := \{(\tfrac{m}{2},\tfrac{n}{2})\,|\,m,n\in\mathbb{Z}\},$$

then (1.1) and the oddness of $\zeta(z|\tau)$ imply $Z_{r,s}(\tau)\equiv 0$ and so $Z_{r,s}^{(2)}(\tau)\equiv 0$, where we used $\wp'(\frac{\omega_k}{2})=0$. Therefore, we only consider $(r,s)\in\mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$. Then both $Z_{r,s}(\tau)$ and $Z_{r,s}^{(2)}(\tau)$ are holomorphic in $\mathbb{H}$, and it is easy to see that the following properties hold:

(i) $Z_{r,s}(\tau) = \pm Z_{m\pm r,n\pm s}(\tau)$ and hence $Z_{r,s}^{(2)}(\tau) = \pm Z_{m\pm r,n\pm s}^{(2)}(\tau)$ for any $(m,n)\in\mathbb{Z}^2$.

(ii) $Z_{r',s'}(\tau') = (c\tau+d)Z_{r,s}(\tau)$ and hence $Z_{r',s'}^{(2)}(\tau') = (c\tau+d)^3 Z_{r,s}^{(2)}(\tau)$ for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in SL(2,\mathbb{Z})$, where $\tau' = \gamma\cdot\tau := \frac{a\tau+b}{c\tau+d}$ and $(s',r') = (s,r)\cdot\gamma^{-1}$.

In particular, when $(r,s)\in Q_N$ is a N-torsion point for some $N\in\mathbb{N}_{\geq 3}$, where

$$Q_N := \left\{ \left(\tfrac{k_1}{N},\tfrac{k_2}{N}\right) \middle| \gcd(k_1,k_2,N)=1,\ 0\leq k_1,k_2\leq N-1 \right\}, \tag{2.5}$$

and $\gamma\in\Gamma(N) := \{\gamma\in SL(2,\mathbb{Z})|\gamma\equiv I_2 \bmod N\}$, then $(r',s')\equiv(r,s) \bmod \mathbb{Z}^2$. In other words, if $(r,s)\in Q_N$, then

$$Z_{r,s}\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)Z_{r,s}(\tau), \quad Z_{r,s}^{(2)}\left(\tfrac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^3 Z_{r,s}^{(2)}(\tau)$$

hold for any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\in\Gamma(N)$, namely $Z_{r,s}(\tau)$ and $Z_{r,s}^{(2)}(\tau)$ are *modular forms* of weight 1 and 3, respectively, with respect to the principal congruence subgroup $\Gamma(N)$. Due to this reason, $Z_{r,s}(\tau)$ and $Z_{r,s}^{(2)}(\tau)$ are called *pre-modular forms* in this paper as in [**15**].

We are interested in the structure of the zeros of $Z_{r,s}(\tau)$ and $Z_{r,s}^{(2)}(\tau)$ for $(r,s)\in\mathbb{R}^2\setminus\frac{1}{2}\mathbb{Z}^2$. By property (ii), we can restrict τ in the fundamental domain F_0 of $\Gamma_0(2)$:

$$F_0 := \{\tau\in\mathbb{H}\mid 0\leqslant \text{ Re }\tau\leqslant 1 \text{ and } |\tau-\tfrac{1}{2}|\geqslant\tfrac{1}{2}\},$$

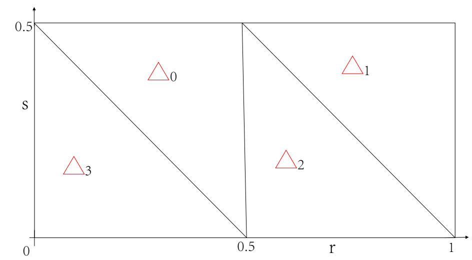

Figure 1. The four open trianlges $\triangle_k$.

and by (i), we only need to consider $(r,s) \in [0,1] \times [0,\frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$. Recall the four open triangles defined in (1.12) (see Figure 1). Clearly $[0,1] \times [0,\frac{1}{2}] = \cup_{k=0}^{3}\overline{\triangle_k}$. The following result was proved in [**6**].

Theorem A ([**6**]). *Let $(r,s) \in [0,1] \times [0,\frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$. Then $Z_{r,s}(\tau) = 0$ has a solution τ in F_0 if and only if $(r,s) \in \triangle_0$. Furthermore, for any $(r,s) \in \triangle_0$, $Z_{r,s}(\tau)$ has a unique zero τ in F_0 which satisfies $\tau \in \mathring{F}_0$.*

In this paper, we will prove an analogous result for $Z_{r,s}^{(2)}(\tau)$.

Theorem 2.1 (=Theorem 1.6). *Let $(r,s) \in [0,1] \times [0,\frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$. Then $Z_{r,s}^{(2)}(\tau) = 0$ has a solution τ in F_0 if and only if $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$. Furthermore, for any $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$, $Z_{r,s}^{(2)}(\tau)$ has a unique zero τ in F_0 which satisfies $\tau \in \mathring{F}_0$.*

Unlike $Z_{r,s}(\tau)$, Theorem 2.1 shows an interesting phenomena for $Z_{r,s}^{(2)}(\tau)$. For example, $Z_{r,s}^{(2)}(\tau)$ has zeros in F_0 for $(r,s) \in \triangle_1 \cup \triangle_2$, but it has no zeros in F_0 for $(r,s) \in \partial\triangle_1 \cap \partial\triangle_2$.

The rest of this section is to prove Theorem 2.1. The reason why we choose the fundamental domain F_0 will be clear from the proof, particularly Lemma 2.3. The basic strategy is similar to that of proving Theorem A in [**6**]. However, the argument is more involved and new techniques are needed. For example, for the same assertion of the pre-modular forms having no zero in F_0 for $(r,s) \in \cup_{k=0}^{3}\partial\triangle_k$, it is a trivial consequence of the same assertion for $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$ in Theorem A; but obviously, this is not the case in Theorem 2.1.

First we need the following important results from the viewpoint of partial differential equations (PDE).

Theorem B ([**15**, **4**]).

(1) [**15**, Theorem 0.4] *The mean field equation*

$$\Delta u + e^u = 16\pi\delta_0 \quad \text{on } E_\tau := \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \tag{2.6}$$

has solutions if and only if there exists $(r,s)\in\mathbb{R}^2\setminus\frac{1}{2}\mathbb{Z}^2$ *such that* τ *is a zero of* $Z_{r,s}^{(2)}(\cdot)$.

(2) [**4**, Theorems 1.1 and 3.1] *If* $\tau\in\{e^{\pi i/3}\}\cup i\mathbb{R}^+$, *then equation (2.6) has no solutions.*

Remark 2.2. In [**2**, **15**], Chai, Wang and the second author studied the following singular Liouville equation

$$\Delta u + e^u = 8n\pi\delta_0 \quad \text{on } E_\tau, \tag{2.7}$$

where $n\in\mathbb{N}$. The solvability of (2.7) depends essentially on the moduli τ of the flat torus E_τ and is intricate from the PDE point of view. To settle this challenging problem, they studied it from the viewpoint of algebraic geometry. They developed a theory to connect this PDE problem with the Lamé equation (1.11) and pre-modular forms. In particular, Wang and the second author [**15**] proved the existence of a pre-modular form $Z_{r,s}^{(n)}(\cdot)$ of weight $\frac{n(n+1)}{2}$ such that (2.7) on E_τ has solutions if and only if $Z_{r,s}^{(n)}(\tau)=0$ for some $(r,s)\in\mathbb{R}^2\setminus\frac{1}{2}\mathbb{Z}^2$. Theorem B-(1) is a special case of this statement for $n=2$. Theorem B-(2) is a purely PDE result. See also [**10**], where the non-existence of even and symmetric solutions (i.e. $u(z)=u(-z)=u(\bar z)$) of (2.7) for $\tau\in i\mathbb{R}^+$ was proved. We will see that Theorem B plays a crucial role in the proof of Lemma 2.3 and hence Theorem 2.1. This is the only place where the PDE results are used.

Lemma 2.3. *Let* $(r,s)\in[0,1]\times[0,\frac{1}{2}]\backslash\frac{1}{2}\mathbb{Z}^2$. *Then* $Z_{r,s}^{(2)}(\tau)\neq 0$ *for any* $\tau\in\{e^{\pi i/3}\}\cup(\partial F_0\cap\mathbb{H})$.

Proof. It does not seem that this assertion could be obtained directly from the expression (2.4) of $Z_{r,s}^{(2)}(\tau)$. Indeed, this lemma is a consequence of the result of *nonlinear PDEs* (i.e. Theorem B).

Given $\tau\in\{e^{\pi i/3}\}\cup(\partial F_0\cap\mathbb{H})$. If $\tau\in\{e^{\pi i/3}\}\cup i\mathbb{R}^+$, then Theorem B implies $Z_{r,s}^{(2)}(\tau)\neq 0$ for any $(r,s)\in\mathbb{R}^2\backslash\frac{1}{2}\mathbb{Z}^2$. If $\tau\in i\mathbb{R}^+ +1$, then by applying $\gamma=\left(\begin{smallmatrix}1&-1\\0&1\end{smallmatrix}\right)$ in property (ii), we have that $\tau-1\in i\mathbb{R}^+$ and

$$Z_{r,s}^{(2)}(\tau)=Z_{r+s,s}^{(2)}(\tau-1)\neq 0 \text{ for any } (r,s)\in\mathbb{R}^2\backslash\tfrac{1}{2}\mathbb{Z}^2.$$

If $|\tau-\frac{1}{2}|=\frac{1}{2}$, then again by applying $\gamma=\left(\begin{smallmatrix}1&0\\-1&1\end{smallmatrix}\right)$ in property (ii) we see that $\frac{\tau}{1-\tau}\in i\mathbb{R}^+$ and

$$(1-\tau)^3 Z_{r,s}^{(2)}(\tau)=Z_{r,r+s}^{(2)}(\tfrac{\tau}{1-\tau})\neq 0 \text{ for any } (r,s)\in\mathbb{R}^2\backslash\tfrac{1}{2}\mathbb{Z}^2.$$

This completes the proof. q.e.d.

Recalling Q_N in (2.5), we define

$$M_N(\tau):=\prod_{(r,s)\in Q_N} Z_{r,s}^{(2)}(\tau). \tag{2.8}$$

By properties (i)–(ii), it is easy to see that $M_N(\tau)$ is a *modular form* with respect to $SL(2,\mathbb{Z})$ of weight $3|Q_N|$ (i.e. for any $\gamma\in SL(2,\mathbb{Z})$, when (r,s) runs over all elements of Q_N, then so does (r',s') after modulo $\mathbb{Z}^2$), where $|Q_N|=\#Q_N$. To apply the theory of modular forms, we recall the following classical formula. See e.g. [**8**, **17**] for the proof.

Theorem C. *Let $f(\tau)$ be a nonzero modular form with respect to $SL(2,\mathbb{Z})$ of weight k. Then*

$$\sum_{\tau\in\mathbb{H}\setminus\{i,\rho\}}\nu_\tau(f)+\nu_\infty(f)+\frac{1}{2}\nu_i(f)+\frac{1}{3}\nu_\rho(f)=\frac{k}{12}, \tag{2.9}$$

where $\rho:=e^{\pi i/3}$, $\nu_\tau(f)$ denotes the zero order of f at τ and the summation over τ is performed modulo $SL(2,\mathbb{Z})$ equivalence.

We note for each $(r,s)\in Q_N$, there exists a unique $(\tilde{r},\tilde{s})\in Q_N$ such that $(\tilde{r},\tilde{s})\equiv(-r,-s)\bmod\mathbb{Z}^2$. Then property (i) of $Z^{(2)}_{r,s}(\tau)$ gives $Z^{(2)}_{\tilde{r},\tilde{s}}(\tau)=-Z^{(2)}_{r,s}(\tau)$, which implies that

$$\nu_\tau(M_N)\in 2\mathbb{N}\cup\{0\}\quad\text{for any}\quad\tau\in\mathbb{H}. \tag{2.10}$$

To apply Theorem C, we need the asymptotics of $Z^{(2)}_{r,s}(\tau)$ as $\operatorname{Im}\tau\to+\infty$.

Lemma 2.4. *Let $(r,s)\in[0,1)\times[0,1)\setminus\frac{1}{2}\mathbb{Z}^2$ and $q=e^{2\pi i\tau}$ with $\tau\in F_0$. Then the asymptotics of $Z^{(2)}_{r,s}(\tau)$ at the three cusps $\tau=0,1,\infty$ are as follows:*

(a) As $F_0\ni\tau\to\infty$,

$$Z^{(2)}_{r,s}(\tau)=4\pi^3 is(1-s)(2s-1)+o(1)\ \text{ if }\ s\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right),$$
$$Z^{(2)}_{r,s}(\tau)=-48\pi^3\sin(2\pi r)q+O(q^2)\ \text{ if }\ s=0,$$
$$Z^{(2)}_{r,s}(\tau)=-12\pi^3\sin(2\pi r)q^{1/2}+O(q)\quad\text{if}\quad s=1/2.$$

(b) As $F_0\ni\tau\to0$,

$$\lim_{\tau\to0}Z^{(2)}_{r,s}(\tau)=\infty\ \text{ if }\ r\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right).$$

(c) As $F_0\ni\tau\to1$,

$$\lim_{\tau\to1}Z^{(2)}_{r,s}(\tau)=\infty\ \text{ if }\ (r+s)\in\left(0,\tfrac{1}{2}\right)\cup\left(\tfrac{1}{2},1\right)\cup\left(1,\tfrac{3}{2}\right).$$

Proof. By using the q-expansions of $\wp(z|\tau)$ and $Z_{r,s}(\tau)$ (see (3.12)–(3.13) in Section 3), the asymptotics of $Z^{(2)}_{r,s}(\tau)$ as $\tau\to\infty$ can be easily calculated. Because the calculation is straightforward and is already done in [**7**, **15**], we omit the details for (a) here.

The asymptotics of $Z_{r,s}^{(2)}(\tau)$ at the cusp 0 can be obtained by using property (ii) and the assertion (a). Letting $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ leads to

$$Z_{r+s,-r}^{(2)}\left(\tfrac{\tau-1}{\tau}\right) = \tau^3 Z_{r,s}^{(2)}(\tau).$$

When $\tau \in F_0$ and $\tau \to 0$, we have $\frac{\tau-1}{\tau} \in F_0$ and $\frac{\tau-1}{\tau} \to \infty$. Then applying (a) we obtain that as $F_0 \ni \tau \to 0$,

$$\begin{aligned} Z_{r,s}^{(2)}(\tau) &= \frac{-1}{\tau^3} Z_{-(r+s),r}^{(2)}\left(\tfrac{\tau-1}{\tau}\right) \\ &= \frac{-1}{\tau^3}\left[4\pi^3 ir(1-r)(2r-1)+o(1)\right] \quad \text{if} \quad r \in (0,\tfrac{1}{2}) \cup (\tfrac{1}{2},1). \end{aligned} \tag{2.11}$$

This proves (b).

Similarly, when $\tau \in F_0$ and $\tau \to 1$, we have $\frac{\tau-1}{\tau} \in F_0$ and $\frac{\tau-1}{\tau} \to 0$. Applying property (ii) and (2.11) we obtain that as $F_0 \ni \tau \to 1$,

$$\begin{aligned} Z_{r,s}^{(2)}(\tau) &= \frac{1}{\tau^3} Z_{r+s,-r}^{(2)}\left(\tfrac{\tau-1}{\tau}\right) \\ &= \frac{-1}{(\tau-1)^3}\left[4\pi^3 i(r+s)(1-r-s)(2r+2s-1)+o(1)\right] \end{aligned}$$

for $r+s \in (0,\frac{1}{2}) \cup (\frac{1}{2},1)$. The remaining case $r+s \in (1,\frac{3}{2})$ follows from $Z_{r,s}^{(2)}(\tau) = Z_{r-1,s}^{(2)}(\tau)$. This proves (c). q.e.d.

Lemma 2.4-(a) implies

$$\text{the vanishing order of } Z_{r,s}^{(2)}(\tau) \text{ at } \infty \text{ is } \begin{cases} 0 & \text{if} \quad s \neq 0, 1/2 \\ 1 & \text{if} \quad s = 0 \\ \frac{1}{2} & \text{if} \quad s = 1/2 \end{cases}. \tag{2.12}$$

Recall $\triangle_k$, $k = 0,1,2,3$, defined in (1.12).

Lemma 2.5. *Fix $k \in \{0,1,2,3\}$. Then the number of zeros of $Z_{r,s}^{(2)}(\tau)$ in F_0 is a constant for $(r,s) \in \triangle_k$.*

Proof. Since $(r,s) \in \triangle_k$, we have $r,s,r+s \notin \{0,\frac{1}{2},1,\frac{3}{2}\}$, so Lemma 2.4 (a)–(c) imply that

$$Z_{r,s}^{(2)}(\tau) \not\to 0 \quad \text{as} \quad F_0 \ni \tau \to \infty, 0, 1$$

respectively. Together with Lemma 2.3 that $Z_{r,s}^{(2)}(\tau) \neq 0$ on $\partial F_0 \cap \mathbb{H}$, it is easy to apply the argument principle to conclude that the number of zeros of $Z_{r,s}^{(2)}(\tau)$ in F_0 is a constant for $(r,s) \in \triangle_k$. q.e.d.

Lemma 2.6. *Let $(r,s) \in Q_3$. Then $Z_{r,s}^{(2)}(\tau) \neq 0$ for any $\tau \in \mathbb{H}$.*

Proof. Note that $3|Q_3| = 24$. Since $s \neq \frac{1}{2}$ and $(\frac{1}{3},0)$, $(\frac{2}{3},0) \in Q_3$, we see from Lemma 2.4-(a) (or (2.12)) that $M_3(\tau) \sim q^2$ as $F_0 \ni \tau \to \infty$, i.e. $\nu_\infty(M_3) = 2$. Therefore, we deduce from (2.9) that $M_3(\tau)$ has no zeros in $\mathbb{H}$. q.e.d.

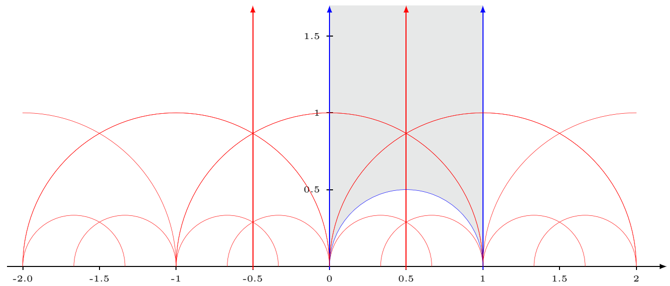

Figure 2. $F_0 = F \cup \gamma_1(F) \cup \gamma_2(F)$.

Let F be a fundamental domain of $SL(2,\mathbb{Z})$ defined by[2]

$$F := \{\tau \in \mathbb{H} \mid 0 \le \operatorname{Re}\tau \le 1, |\tau| \ge 1, |\tau - 1| > 1\} \cup \{e^{\pi i/3}\}. \tag{2.13}$$

Define $\gamma(F) := \{\gamma \cdot \tau | \tau \in F\}$ for any $\gamma \in SL(2,\mathbb{Z})$, then $\gamma(F)$ is also a fundamental domain of $SL(2,\mathbb{Z})$. Let

$$\gamma_1 := \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad \gamma_2 := \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.14}$$

then it is easy to prove that

$$F_0 = F \cup \gamma_1(F) \cup \gamma_2(F). \tag{2.15}$$

See Figure 2, which is copied from [6]. Now we are in the position to prove Theorem 2.1.

Proof of Theorem 2.1. Recall Lemma 2.3 that $Z^{(2)}_{r,s}(\tau) \ne 0$ for $\tau \in \{\rho\} \cup (\partial F_0 \cap \mathbb{H})$. We divide the proof of Theorem 2.1 into several steps.

Step 1. We claim that $Z^{(2)}_{r,s}(\tau)$ has no zeros in F_0 for $(r,s) \in \triangle_0$.

Lemma 2.6 says that $Z^{(2)}_{\frac{1}{3},\frac{1}{3}}(\tau)$ has no zeros in F_0. Since $(\frac{1}{3},\frac{1}{3}) \in \triangle_0$, our claim follows directly from Lemma 2.5.

Step 2. We claim that $Z^{(2)}_{r,s}(\tau)$ has a unique zero in F_0 for $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$.

Since

$$(\tfrac{5}{6},\tfrac{1}{3}) \in \triangle_1, \quad (\tfrac{2}{3},\tfrac{1}{6}) \in \triangle_2, \quad (\tfrac{1}{6},\tfrac{1}{6}) \in \triangle_3, \tag{2.16}$$

by Lemma 2.5 we only need to prove the claim for $(r,s) \in \{(\frac{1}{6},\frac{1}{6}), (\frac{2}{3},\frac{1}{6}), (\frac{5}{6},\frac{1}{3})\}$. Note that $M_6(\tau)$ is a modular form of weight $3|Q_6| = 72$. For

---

[2] Of course, the standard definition of F should be $F := \{\tau \in \mathbb{H} \mid 0 \le \operatorname{Re}\tau < 1, |\tau| \ge 1, |\tau - 1| > 1\} \cup \{e^{\pi i/3}\}$, i.e. $\operatorname{Re}\tau = 1$ is not needed. In this paper, to guarantee the validity of (2.15), it is more convenient for us to use the definition (2.13), which does not effect our following argument because Lemma 2.3 says that $Z^{(2)}_{r,s}(\tau) \ne 0$ if $\tau \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau = 1\} \cup \gamma_1(\{\tau \in \mathbb{H} | \operatorname{Re}\tau = 1\}) \cup \gamma_2(\{\tau \in \mathbb{H} | \operatorname{Re}\tau = 1\}) \subset \partial F_0 \cap \mathbb{H}$.

$(r,s) \in Q_6$, Lemma 2.4-(a) shows that $\lim_{\tau\to\infty} Z^{(2)}_{r,s}(\tau) = 0$ if and only if $(r,s) \in \{(\frac{1}{6},0),(\frac{5}{6},0),(\frac{1}{6},\frac{1}{2}),(\frac{2}{6},\frac{1}{2}),(\frac{4}{6},\frac{1}{2}),(\frac{5}{6},\frac{1}{2})\}$. Thus we see from (2.12) that

$$\nu_\infty(M_6) = 1 \times 2 + \tfrac{1}{2} \times 4 = 4.$$

On the other hand, Lemma 2.3 says $\nu_i(M_6) = \nu_\rho(M_6) = 0$. Therefore, it follows from (2.9) that

$$\sum_{\tau \in \mathbb{H}\setminus\{i,\rho\}} \nu_\tau(M_6) = 2. \tag{2.17}$$

Recall that F defined in (2.13) is a fundamental domain of $SL(2,\mathbb{Z})$. Applying (2.10), there exists a *unique* $\tau_0 \in (F \cap \mathring{F}_0) \setminus \{\rho = e^{\pi i/3}\}$ such that $\nu_{\tau_0}(M_6) = 2$, i.e. there exists a *unique* $(r_1,s_1) \in Q_6'$ such that

$$Z^{(2)}_{r_1,s_1}(\tau_0) = 0, \tag{2.18}$$

where

$$Q_6' := \left\{ \begin{array}{l} (0,\frac{1}{6}),(\frac{1}{6},0),(\frac{1}{6},\frac{1}{6}),(\frac{1}{6},\frac{1}{3}),(\frac{1}{6},\frac{1}{2}),(\frac{1}{3},\frac{1}{6}), \\ (\frac{1}{3},\frac{1}{2}),(\frac{1}{2},\frac{1}{6}),(\frac{1}{2},\frac{1}{3}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{6}),(\frac{5}{6},\frac{1}{3}) \end{array} \right\} \subset [0,1) \times [0,\tfrac{1}{2}].$$

Remark that for any $(r,s) \in Q_6$, either $(r,s) \in Q_6'$ or there exists a unique $(\tilde{r},\tilde{s}) \in Q_6'$ such that $(r,s) \equiv (-\tilde{r},-\tilde{s}) \mod \mathbb{Z}^2$.

Step 2-1. We prove that $(r_1,s_1) \in \{(\frac{1}{6},\frac{1}{6}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})\}$.

Assume by contradiction that $(r_1,s_1) \notin \{(\frac{1}{6},\frac{1}{6}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})\}$. Then by (2.17)–(2.18), we have

$$Z^{(2)}_{r,s}(\tau) \neq 0 \text{ in } F \text{ for } (r,s) \in \{(\tfrac{1}{6},\tfrac{1}{6}),(\tfrac{2}{3},\tfrac{1}{6}),(\tfrac{5}{6},\tfrac{1}{3})\}. \tag{2.19}$$

Recall (2.14)–(2.15). Letting $\gamma = \gamma_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right)$ in property (ii) leads to

$$Z^{(2)}_{-s,r+s}(\gamma_1 \cdot \tau) = (1-\tau)^3 Z^{(2)}_{r,s}(\tau). \tag{2.20}$$

Applying this to $(r,s) \in \{(\frac{1}{6},\frac{1}{6}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})\}$, it follows from property (i) that

$$Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\gamma_1 \cdot \tau) = Z^{(2)}_{\frac{-1}{6},\frac{1}{3}}(\gamma_1 \cdot \tau) = (1-\tau)^3 Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\tau), \tag{2.21}$$

$$-Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\gamma_1 \cdot \tau) = Z^{(2)}_{\frac{-1}{6},\frac{5}{6}}(\gamma_1 \cdot \tau) = (1-\tau)^3 Z^{(2)}_{\frac{2}{3},\frac{1}{6}}(\tau), \tag{2.22}$$

$$Z^{(2)}_{\frac{2}{3},\frac{1}{6}}(\gamma_1 \cdot \tau) = Z^{(2)}_{\frac{-1}{3},\frac{7}{6}}(\gamma_1 \cdot \tau) = (1-\tau)^3 Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\tau). \tag{2.23}$$

Together with (2.19), we obtain

$$Z^{(2)}_{r,s}(\tau) \neq 0 \text{ in } \gamma_1(F) \text{ for } (r,s) \in \{(\tfrac{1}{6},\tfrac{1}{6}),(\tfrac{2}{3},\tfrac{1}{6}),(\tfrac{5}{6},\tfrac{1}{3})\}. \tag{2.24}$$

Similarly, letting $\gamma = \gamma_2 = \left(\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ in property (ii) leads to

$$Z^{(2)}_{r+s,-r}(\gamma_2 \cdot \tau) = \tau^3 Z^{(2)}_{r,s}(\tau) \tag{2.25}$$

and so

$$-Z^{(2)}_{\frac{2}{3},\frac{1}{6}}(\gamma_2\cdot\tau)=Z^{(2)}_{\frac{1}{3},\frac{-1}{6}}(\gamma_2\cdot\tau)=\tau^3 Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\tau),\tag{2.26}$$

$$Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\gamma_2\cdot\tau)=Z^{(2)}_{\frac{5}{6},\frac{-2}{3}}(\gamma_2\cdot\tau)=\tau^3 Z^{(2)}_{\frac{2}{3},\frac{1}{6}}(\tau),\tag{2.27}$$

$$Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\gamma_2\cdot\tau)=Z^{(2)}_{\frac{7}{6},\frac{-5}{6}}(\gamma_2\cdot\tau)=\tau^3 Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\tau).\tag{2.28}$$

Together with (2.19), we obtain

$$Z^{(2)}_{r,s}(\tau)\neq 0 \text{ in } \gamma_2(F) \text{ for } (r,s)\in\{(\tfrac{1}{6},\tfrac{1}{6}),(\tfrac{2}{3},\tfrac{1}{6}),(\tfrac{5}{6},\tfrac{1}{3})\}.$$

Therefore, it follows from (2.15) (i.e. $F_0=F\cup\gamma_1(F)\cup\gamma_2(F)$) that $Z^{(2)}_{r,s}(\tau)\neq 0$ in F_0 for $(r,s)\in\{(\frac{1}{6},\frac{1}{6}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})\}$. By (2.16), we conclude from Lemma 2.5 and Step 1 that

$$Z^{(2)}_{r,s}(\tau)\neq 0 \quad\text{in}\quad F_0 \quad\text{for any}\quad (r,s)\in\cup_{k=0}^3\triangle_k.\tag{2.29}$$

From (2.29), (2.18) and $(r_1,s_1)\in Q_6'\subset[0,1)\times[0,\frac{1}{2}]$, we obtain $(r_1,s_1)\in\cup_{k=0}^3\partial\triangle_k$. This, together with $Z^{(2)}_{r_1,s_1}(\tau_0)=0$ and the argument principle, implies the existence of $(r,s)\in\cup_{k=0}^3\triangle_k$ close to (r_1,s_1) such that $Z^{(2)}_{r,s}(\tau)=0$ for some $\tau\in F_0$ close to τ_0, a contradiction with (2.29). Therefore, $(r_1,s_1)\in\{(\frac{1}{6},\frac{1}{6}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})\}$.

Step 2-2. We prove that $Z^{(2)}_{r,s}(\tau)$ has a unique zero in F_0 for each $(r,s)\in\{(\frac{1}{6},\frac{1}{6}),(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})\}$.

By Step 2-1, without loss of generality, we may assume $(r_1,s_1)=(\frac{1}{6},\frac{1}{6})$ (the other two cases $(r_1,s_1)=(\frac{2}{3},\frac{1}{6}),(\frac{5}{6},\frac{1}{3})$ can be discussed in the same way).

Then τ_0 is the unique zero of $Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\tau)$ in F and (2.17) implies

$$Z^{(2)}_{r,s}(\tau)\neq 0 \text{ in } F \text{ for } (r,s)\in\{(\tfrac{2}{3},\tfrac{1}{6}),(\tfrac{5}{6},\tfrac{1}{3})\}.\tag{2.30}$$

This together with (2.22) and (2.28) implies

$$Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\tau)\neq 0 \text{ in } \gamma_1(F)\cup\gamma_2(F).$$

Therefore, by (2.15) we conclude that $Z^{(2)}_{\frac{1}{6},\frac{1}{6}}(\tau)$ has a unique zero in F_0.

For $(r,s)=\left(\frac{5}{6},\frac{1}{3}\right)$, by applying (2.21), we see that $\gamma_1\cdot\tau_0$ is the unique zero of $Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\tau)$ in $\gamma_1(F)$. Clearly (2.27) and (2.30) give

$$Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\tau)\neq 0 \text{ in } \gamma_2(F).$$

Together with (2.30) and (2.15), we conclude that $Z^{(2)}_{\frac{5}{6},\frac{1}{3}}(\tau)$ has a unique zero in F_0. By a similar discussion, $Z^{(2)}_{\frac{2}{3},\frac{1}{6}}(\tau)$ also has a unique zero in F_0.

Now by (2.16) and Lemma 2.5, we conclude that $Z_{r,s}^{(2)}(\tau)$ has a unique zero in F_0 for $(r,s) \in \triangle_1 \cup \triangle_2 \cup \triangle_3$. This proves Step 2.

Step 3. We prove that $Z_{r,s}^{(2)}(\tau) \neq 0$ in F_0 for any $(r,s) \in \cup_{k=0}^{3} \partial \triangle_k \backslash \frac{1}{2}\mathbb{Z}^2$.

Suppose that there exists $(r_0, s_0) \in \cup_{k=0}^{3} \partial \triangle_k \backslash \frac{1}{2}\mathbb{Z}^2$ such that $Z_{r_0,s_0}^{(2)}(\tau) = 0$ has a zero τ_0 in F_0. Lemma 2.3 implies $\tau_0 \in \mathring{F}_0 \setminus \{\rho\}$. Clearly there exists a sequence of *prime numbers* $N \to \infty$ and $(\tilde{r}_N, \tilde{s}_N) \in Q_{2N}$, $\tilde{s}_N \leq \frac{1}{2}$, such that

$$(\tilde{r}_N, \tilde{s}_N) \in \cup_{k=0}^{3} \partial \triangle_k \backslash \tfrac{1}{2}\mathbb{Z}^2 \text{ and } (\tilde{r}_N, \tilde{s}_N) \to (r_0, s_0) \text{ as } N \to \infty.$$

Again by the argument principle, it follows that

$$Z_{\tilde{r}_N,\tilde{s}_N}^{(2)}(\tau) \text{ has a zero } \tilde{\tau}_N \in \mathring{F}_0 \setminus \{\rho\} \text{ for } N \text{ large.} \tag{2.31}$$

By (2.15), (2.20) and (2.25), we may always assume $\tilde{\tau}_N \in F$ by replacing $\tilde{\tau}_N$ with one of $\{\gamma_1^{-1} \cdot \tilde{\tau}_N, \gamma_2^{-1} \cdot \tilde{\tau}_N\}$ if necessary.

Now fix such a large prime number N. Recalling (2.8) that $M_{2N}(\tau)$ is a modular form with respect to $SL(2,\mathbb{Z})$ of weight $3|Q_{2N}| = 9(N^2-1)$. Since for any $k \in \{1, 3, \cdots, 2N-1\} \backslash \{N\}$, $(\frac{k}{2N}, 0) \in Q_{2N}$, and for any $k \in \{1, 2, \cdots, 2N-1\} \backslash \{N\}$, $(\frac{k}{2N}, \frac{1}{2}) \in Q_{2N}$, it follows from Lemma 2.4-(a) that

$$\nu_\infty(M_{2N}) = 1 \times (N-1) + \tfrac{1}{2} \times 2(N-1) = 2(N-1).$$

Together with Lemma 2.3 that $\nu_i(M_{2N}) = \nu_\rho(M_{2N}) = 0$, we see from (2.9) that

$$\sum_{\tau \in \mathbb{H} \backslash \{i, \rho\}} \nu_\tau(M_{2N}) = \frac{3(N^2-1)}{4} - 2(N-1) = \frac{3N^2 - 8N + 5}{4}, \tag{2.32}$$

where the summation over τ is performed modulo $SL(2,\mathbb{Z})$ equivalence.

On the other hand, recall

$$\triangle_3 = \{(r,s) \mid r > 0,\ s > 0,\ r + s < \tfrac{1}{2}\}.$$

It is easy to compute that in Q_{2N}, there are

$$1 + 2 + \cdots + \frac{N-3}{2} = \frac{(N-1)(N-3)}{8}$$

$(r,s) = (\frac{k_1}{2N}, \frac{k_2}{2N})$'s belonging to $\triangle_3$ such that k_1 is odd and k_2 is even (resp. k_1 is even and k_2 is odd); and there are

$$1 + 2 + \cdots + \frac{N-1}{2} = \frac{(N+1)(N-1)}{8}$$

$(r,s) = (\frac{k_1}{2N}, \frac{k_2}{2N})$'s belonging to $\triangle_3$ such that k_1 and k_2 are both odd. Thus

$$|Q_{2N} \cap \triangle_3| = \frac{(N-1)(N-3)}{4} + \frac{(N+1)(N-1)}{8} = \frac{3N^2-8N+5}{8}. \tag{2.33}$$

Write

$$Q_{2N} \cap \triangle_3 = \left\{ (r_k, s_k) \middle| 1 \le k \le \frac{3N^2-8N+5}{8} \right\}.$$

Applying Step 2, we see that $Z_{r_k,s_k}(\tau)$ has a unique zero $\tau_k \in \mathring{F}_0$ in F_0. If $\tau_k \in \gamma_1(F) \cup \gamma_2(F)$, say $\tau_k \in \gamma_1(F)$ for example, for some k, then by (2.20) and property (i) it is easy to see the existence of $(r_k', s_k') \in \triangle_2 \cap Q_{2N}$ such that $\gamma_1^{-1} \cdot \tau_k \in F$ is the unique zero of $Z^{(2)}_{r_k',s_k'}(\tau)$ in F_0. Therefore, together with (2.31) and (2.33), we conclude that

$$\sum_{\tau \in \mathbb{H}\setminus\{i,\rho\}} \nu_\tau(M_{2N}) \ge 2|Q_{2N} \cap \triangle_3| + 2 = \frac{3N^2-8N+5}{4} + 2,$$

which is a contradiction with (2.32). This proves Step 3 and hence the proof of Theorem 2.1 is complete. q.e.d.

3. Existence and uniqueness of $\tau(C)$

The purpose of this section is to prove the existence and uniqueness of $\tau(C)$ for $C \in \mathbb{R} \setminus \{0,1\}$ by applying Theorem 1.6. Given $C \in \mathbb{R}$, we define the holomorphic function $f_C(\tau)$ on $\mathbb{H}$ by

$$f_C(\tau) := 12(C\eta_1(\tau) - \eta_2(\tau))^2 - g_2(\tau)(C-\tau)^2. \tag{3.1}$$

By $\eta_2 = \tau\eta_1 - 2\pi i$, we see that $f_C(\tau) = 0$ if and only if (1.4) holds. This $f_C(\tau)$ appears in the expression of solutions of certain Painlevé VI equation (cf. [**3**, **5**]). The following result proves the existence and uniqueness of $\tau(C)$ as zeros of $f_C(\tau)$. Recall the fundamental domain F_0 of $\Gamma_0(2)$:

$$F_0 = \{\tau \in \mathbb{H} \mid 0 \le \operatorname{Re}\tau \le 1, \quad |\tau - \tfrac{1}{2}| \ge \tfrac{1}{2}\}.$$

Theorem 3.1 (Zero structure of $f_C(\tau)$ in F_0)**.**

(1) *For any $C \in \mathbb{R}\setminus\{0,1\}$, $f_C(\tau)$ has a unique zero $\tau(C)$ in F_0. Furthermore, $\tau(C) \in \mathring{F}_0$.*
(2) *For $C \in \{0,1\}$, $f_C(\tau)$ has no zeros in F_0.*

Recall the classical result (cf. [**1**, p. 704]) that

$$\eta_1'(\tau) = \frac{i}{2\pi}\left(\eta_1(\tau)^2 - \tfrac{1}{12}g_2(\tau)\right). \tag{3.2}$$

To prove Theorem 3.1, first we need the following lemma.

Lemma 3.2. *If $\tau = ib$ with $b > 0$, then $\eta_1'(\tau) \neq 0$ and*

$$g_2(\tau) - 12\eta_1(\tau)^2 > 0, \tag{3.3}$$

$$12\eta_2(\tau)^2 - \tau^2 g_2(\tau) > 0. \tag{3.4}$$

Proof. Denote $q = e^{2\pi i\tau}$. Recall the q-expansion of $\eta_1(\tau)$ (see (1.2)):

$$\eta_1(\tau) = \frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} b_k q^k, \quad \text{where } b_k = \sum_{1\le d|k} d. \tag{3.5}$$

Let $\tau = ib$ with $b > 0$. Then $q = e^{-2\pi b}$ and hence $\frac{d}{db}\eta_1(ib) > 0$ for $b > 0$. So $\eta_1'(\tau) \neq 0$ and (3.3) follows from (3.2). To prove (3.4), we use the following modular property (see (2.2)):

$$\eta_1(\tfrac{-1}{\tau}) = \tau\eta_2(\tau), \;\; g_2(\tfrac{-1}{\tau}) = \tau^4 g_2(\tau). \tag{3.6}$$

It follows that

$$12\eta_2(\tau)^2 - \tau^2 g_2(\tau) = \frac{1}{\tau^2}\left[12\eta_1(\tfrac{-1}{\tau})^2 - g_2(\tfrac{-1}{\tau})\right] > 0,$$

i.e. (3.4) holds. q.e.d.

Lemma 3.3. *For any $C \in \mathbb{R}\backslash\{0,1\}$, $f_C(\tau) \neq 0$ for $\tau \in \partial F_0 \cap \mathbb{H}$.*

Proof. Suppose $f_C(\tau) = 0$ for some $\tau \in \partial F_0 \cap \mathbb{H}$.

Case 1. $\tau \in i\mathbb{R}^+$.

Then it is known that $g_2(\tau) > 0$, $\eta_1(\tau) \in \mathbb{R}$ (see e.g. Lemma 3.2) and $\eta_2(\tau) = \tau\eta_1(\tau) - 2\pi i \in i\mathbb{R}$. It follows from $f_C(\tau) = 0$ and (3.1) that

$$\frac{2\pi i}{\tau - C} = \eta_1(\tau) \pm \sqrt{g_2(\tau)/12} \in \mathbb{R},$$

a contradiction with our assumption $C \in \mathbb{R} \setminus \{0\}$.

Case 2. $|\tau - \frac{1}{2}| = \frac{1}{2}$.

Then $\tau' = \frac{\tau}{1-\tau} \in i\mathbb{R}^+$. Define $C' := \frac{C}{1-C} \in \mathbb{R} \setminus \{0\}$. By $g_2(\tau') = (1-\tau)^4 g_2(\tau)$ and

$$\eta_2(\tau') = (1-\tau)\eta_2(\tau), \; \eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)), \tag{3.7}$$

a straightforward computation leads to

$$f_{C'}(\tau') = \frac{(1-\tau)^2}{(1-C)^2} f_C(\tau) = 0.$$

Then we obtain a contradiction as Case 1.

Case 3. $\tau \in 1 + i\mathbb{R}^+$.

Then $\tau' = \tau - 1 \in i\mathbb{R}^+$. Define $C' := C - 1 \in \mathbb{R} \setminus \{0\}$. By using $g_2(\tau') = g_2(\tau)$ and

$$\eta_1(\tau') = \eta_1(\tau), \; \eta_2(\tau') = \eta_2(\tau) - \eta_1(\tau), \tag{3.8}$$

we easily obtain $f_{C'}(\tau') = f_C(\tau) = 0$, again a contradiction as Case 1.

The proof is complete. q.e.d.

Recall the pre-modular form $Z_{r,s}^{(2)}(\tau)$ in Section 2. Now we study the precise relation between $Z_{r,s}^{(2)}(\tau)$ and $f_C(\tau)$. This is *the key point* of our whole idea. Fix any $C\in\mathbb{R}$, and for $s\in(0,\frac{1}{4(1+|C|)^2})$ we define

$$F_{C,s}(\tau):=\frac{4(\tau-C)}{s}Z_{-Cs,s}^{(2)}(\tau).$$

Lemma 3.4. *Letting $s\to 0$, $F_{C,s}(\tau)$ converges to $f_C(\tau)$ uniformly in any compact subset of $F_0=\bar{F}_0\cap\mathbb{H}$.*

Proof. Denote $u=-Cs+s\tau=s(\tau-C)$ for convenience. Then $u\to 0$ as $s\to 0$. Let $\tau\in K$ where K is any compact subset of F_0. Then $g_2(\tau)$ and $g_3(\tau)$ are uniformly bounded for $\tau\in K$. So it follows from the Laurent series of $\zeta(\cdot|\tau)$ and $\wp(\cdot|\tau)$ that

$$\begin{aligned}\zeta(-Cs+s\tau|\tau)&=\frac{1}{u}-\frac{g_2(\tau)}{60}u^3+O(|u|^5),\\ \wp(-Cs+s\tau|\tau)&=\frac{1}{u^2}+\frac{g_2(\tau)}{20}u^2+O(|u|^4),\\ \wp'(-Cs+s\tau|\tau)&=\frac{-2}{u^3}+\frac{g_2(\tau)}{10}u+O(|u|^3),\end{aligned}\tag{3.9}$$

hold uniformly for $\tau\in K$ as $s\to 0$. From here and (2.3), we see that

$$Z_{-Cs,s}(\tau)=\frac{1}{u}+2\pi i s-\eta_1 u-\frac{g_2}{60}u^3+O(|u|^5)\tag{3.10}$$

and so

$$\begin{aligned}Z_{-Cs,s}^{(2)}(\tau)&=Z_{-Cs,s}(\tau)^3-3\wp(-Cs+s\tau|\tau)Z_{-Cs,s}(\tau)-\wp'(-Cs+s\tau|\tau)\\ &=\frac{-12\pi^2 s}{\tau-C}-12\pi i\eta_1 s+3\eta_1^2 u-\frac{g_2}{4}u+O(|u|^2)\end{aligned}\tag{3.11}$$

uniformly for $\tau\in K$ as $s\to 0$. Consequently, we derive from $u=s(\tau-C)$ and $\eta_2=\tau\eta_1-2\pi i$ that

$$\begin{aligned}F_{C,s}(\tau)&=\frac{4(\tau-C)}{s}Z_{-Cs,s}^{(2)}(\tau)\\ &=-48\pi^2-48\pi i\eta_1(\tau-C)+12\eta_1^2(\tau-C)^2-g_2(\tau-C)^2+O(s)\\ &=12((\tau-C)\eta_1-2\pi i)^2-g_2(\tau-C)^2+O(s)\\ &=12(C\eta_1-\eta_2)^2-g_2(\tau-C)^2+O(s)\to f_C(\tau)\end{aligned}$$

uniformly for $\tau\in K$ as $s\to 0$. The proof is complete. q.e.d.

Lemma 3.5. *Let $s>0$. Then as $s\to 0$, any zero $\tau(s)\in\{\tau\in\mathbb{H}|\operatorname{Re}\tau\in[-1,1]\}$ of $Z_{-Cs,s}^{(2)}(\tau)$ (if exist) is uniformly bounded.*

Proof. Suppose by contradiction that up to a subsequence of $s\to 0$, $Z_{-Cs,s}^{(2)}(\tau)$ has a zero $\tau(s)\in\{\tau\in\mathbb{H}|\operatorname{Re}\tau\in[-1,1]\}$ such that $\tau(s)\to\infty$ as $s\to 0$. Write $\tau=\tau(s)=a(s)+ib(s)$, then $a(s)\in[-1,1]$ and $b(s)\to+\infty$.

Denote $q = e^{2\pi i\tau}$ as before. We recall the q-expansions (cf. [**12**, p. 46] for $\wp$ and [**6**, (5.3)] for $Z_{r,s}$): for $|q| < |e^{2\pi iz}| < |q|^{-1}$,

$$\frac{\wp(z|\tau)}{-4\pi^2} = \frac{1}{12} + \frac{e^{2\pi iz}}{(1-e^{2\pi iz})^2} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} nq^{nm}(e^{2\pi inz} + e^{-2\pi inz} - 2), \tag{3.12}$$

$$\begin{aligned}\frac{\wp'(z|\tau)}{-4\pi^2} =& \frac{2\pi ie^{2\pi iz}}{(1-e^{2\pi iz})^2} + \frac{4\pi ie^{4\pi iz}}{(1-e^{2\pi iz})^3}\\ &+ 2\pi i\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} n^2q^{nm}(e^{2\pi inz} - e^{-2\pi inz}),\end{aligned}$$

$$\begin{aligned}Z_{r,s}(\tau) =& 2\pi is - \pi i\frac{1+e^{2\pi iz}}{1-e^{2\pi iz}}\\ &- 2\pi i\sum_{n=1}^{\infty}\left(\frac{e^{2\pi iz}q^n}{1-e^{2\pi iz}q^n} - \frac{e^{-2\pi iz}q^n}{1-e^{-2\pi iz}q^n}\right),\end{aligned} \tag{3.13}$$

where $z = r + s\tau$ in (3.13). We will also use the q-expansion of $g_2(\tau)$ (cf. [**12**, p. 44]):

$$g_2(\tau) = \frac{4}{3}\pi^4 + 320\pi^4\sum_{k=1}^{\infty}\sigma_3(k)q^k, \text{ where } \sigma_3(k) = \sum_{1\le d|k} d^3. \tag{3.14}$$

Now we let $z = -Cs + s\tau = s(a(s) - C + ib(s))$ and denote $x = e^{2\pi iz}$ for convenience. Then

$$e^{2\pi b(s)} = |q|^{-1} > |x| = e^{-2\pi sb(s)} > |q| = e^{-2\pi b(s)},$$

so we can apply the above q-expansions. Notice that $|x| \in (0,1)$ and $|x^{-1}q| = e^{-2\pi(1-s)b(s)} \to 0$ as $s \to 0$. There are two cases.

Case 1. Up to a subsequence $|x^{-1}q| = o(s|1-x|^2)$.

Then we derive from (3.12)–(3.13) that

$$\frac{\wp(z|\tau)}{-4\pi^2} = \frac{1}{12} + \frac{x}{(1-x)^2} + o(s|1-x|^2), \tag{3.15}$$

$$\frac{\wp'(z|\tau)}{-4\pi^2} = \frac{2\pi ix}{(1-x)^2} + \frac{4\pi ix^2}{(1-x)^3} + o(s|1-x|^2),$$

$$Z_{-Cs,s}(\tau) = -\pi i\frac{1+x}{1-x} + 2\pi is + o(s|1-x|^2). \tag{3.16}$$

We note that if $x \to 1$, then $sb(s) \to 0$ and

$$\begin{aligned}1 - x &= 1 - e^{2\pi is(a(s)-C+ib(s))}\\ &= -2\pi is(a(s) - C + ib(s)) + o(sb(s)).\end{aligned}$$

Together with $b(s) \to +\infty$ as $s \to 0$, we always have

$$\frac{s^2}{1-x} = o(s). \tag{3.17}$$

Then by (3.15)–(3.16) and (3.17), a straightforward computation gives

$$\begin{aligned}0 &= Z^{(2)}_{-Cs,s}(\tau(s)) = Z_{-Cs,s}(\tau)^3 - 3\wp(z|\tau)Z_{-Cs,s}(\tau) - \wp'(z|\tau)\\ &= -4\pi^3 is + o(s),\end{aligned}$$

which is a contradiction.

Case 2. Up to a subsequence $|x^{-1}q| \geq ds|1-x|^2$ for some constant $d > 0$.

Then we see from (3.17) that

$$e^{-2\pi(1-s)b(s)} = |x^{-1}q| \geq ds|1-x|^2 \geq s^3$$

and so $b(s) \leq \ln\frac{1}{s}$ for $s > 0$ small. Then $u := s(\tau - C) = s(a(s) - C + ib(s)) \to 0$ and

$$s = o(|u|), \quad u^2 = o(s).$$

Recall $q = e^{2\pi i\tau}$. Since $b(s) \to +\infty$, (3.5) and (3.14) show that

$$g_2(\tau) = \frac{4}{3}\pi^4 + O(|q|), \quad \eta_1(\tau) = \frac{1}{3}\pi^2 + O(|q|)$$

are uniformly bounded, so (3.9)–(3.11) still hold, namely

$$0 = Z^{(2)}_{-Cs,s}(\tau(s)) = -12\pi i\eta_1 s - \frac{12\pi^2 s}{\tau - C} + 3\eta_1^2 u - \frac{g_2}{4}u + O(|u|^2).$$

Since

$$3\eta_1^2 - \frac{g_2}{4} = O(|q|) = O(e^{-2\pi b(s)}) = O\left(|\tau - C|^{-2}\right),$$

we have

$$\frac{-12\pi^2 s}{\tau - C} + 3\eta_1^2 u - \frac{g_2}{4}u = O\left(\tfrac{s}{|\tau - C|}\right) = o(s).$$

Therefore, we finally obtain

$$\begin{aligned}0 &= -12\pi i\eta_1 s - \frac{12\pi^2 s}{\tau - C} + 3\eta_1^2 u - \frac{g_2}{4}u + O(|u|^2)\\ &= -4\pi^3 is + o(s),\end{aligned}$$

which is a contradiction. The proof is complete. q.e.d.

Recall $\triangle_k, k = 0,1,2,3$, defined in (1.12). We define

$$\begin{aligned}\tilde{\triangle}_1 :=& \{(r,s) | (r+1,s) \in \triangle_1\}\\ =& \left\{(r,s) | 0 < s < \tfrac{1}{2}, \tfrac{-1}{2} < r < 0, r+s > 0\right\},\end{aligned}$$

$$\begin{aligned}\tilde{\triangle}_2 :=& \{(r,s) | (r+1,s) \in \triangle_2\}\\ =& \left\{(r,s) | 0 < s < \tfrac{1}{2}, \tfrac{-1}{2} < r < 0, r+s < 0\right\}.\end{aligned}$$

Since property (i) in Section 2 gives $Z^{(2)}_{r,s}(\tau) = Z^{(2)}_{r+1,s}(\tau)$, we see from Theorem 1.6 that

(3.18) For $(r,s) \in \tilde{\triangle}_1 \cup \tilde{\triangle}_2 \cup \triangle_3$, $Z^{(2)}_{r,s}(\tau)$ has a unique zero in F_0.

From now on, we fix $C \in (-\infty,0)\cup(0,1)\cup(1,\infty)$. Then $s \in (0,\frac{1}{4(1+|C|)^2})$ implies $(-Cs,s)\in \tilde{\triangle}_1\cup\tilde{\triangle}_2\cup\triangle_3$, so (3.18) implies that $Z^{(2)}_{-Cs,s}(\tau)$ has a unique zero $\tau(s)\in F_0$. By the definition of F_0, we easily see that

$$\frac{-1}{\tau(s)},\frac{\tau(s)}{1-\tau(s)}\in\{\tau\in\mathbb{H}|\operatorname{Re}\tau\in[-1,1]\}.$$

Lemma 3.6. *As $s\to 0$, the unique zero $\tau(s)\in F_0$ of $Z^{(2)}_{-Cs,s}(\tau)$ can not converge to any of $\{0,1,\infty\}$.*

Proof. Lemma 3.5 shows that $\tau(s)\not\to\infty$. To prove $\tau(s)\not\to\{0,1\}$, we use the modular property (ii) of $Z^{(2)}_{r,s}(\tau)$ in Section 2:

$$Z^{(2)}_{r',s'}(\tau')=(c\tau+d)^3Z^{(2)}_{r,s}(\tau),$$

whenever

$$\tau'=\frac{a\tau+b}{c\tau+d}\text{ and }(s',r')=(s,r)\begin{pmatrix}d&-b\\-c&a\end{pmatrix},\ \begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2,\mathbb{Z}).$$

We also use property (i):

$$Z^{(2)}_{m\pm r,n\pm s}(\tau)=\pm Z^{(2)}_{r,s}(\tau),\ \forall m,n\in\mathbb{Z}. \tag{3.19}$$

Letting $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)=\left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)$, we obtain

$$Z^{(2)}_{s,-r}(\tfrac{-1}{\tau})=\tau^3Z^{(2)}_{r,s}(\tau). \tag{3.20}$$

Recall $C\in(-\infty,0)\cup(0,1)\cup(1,+\infty)$ and $s\in(0,\frac{1}{4(1+|C|)^2})$.

Case 1. $C\in(-\infty,0)$.
By defining

$$\tilde{C}:=\frac{-1}{C},\quad \tilde{s}:=-Cs,$$

we have $\tilde{s}\in(0,\frac{1}{4(1+|\tilde{C}|)^2})$ for s small and

$$\tau^3Z^{(2)}_{-Cs,s}(\tau)=Z^{(2)}_{s,Cs}(\tfrac{-1}{\tau})=-Z^{(2)}_{-s,-Cs}(\tfrac{-1}{\tau})=-Z^{(2)}_{-\tilde{C}\tilde{s},\tilde{s}}(\tfrac{-1}{\tau}).$$

Therefore, $Z^{(2)}_{-\tilde{C}\tilde{s},\tilde{s}}(\tau)$ has zero $\frac{-1}{\tau(s)}\in\{\tau\in\mathbb{H}|\operatorname{Re}\tau\in[-1,1]\}$. Since $\tilde{s}\to 0$ as $s\to 0$, Lemma 3.5 implies $\frac{-1}{\tau(s)}\not\to\infty$, i.e. $\tau(s)\not\to 0$ as $s\to 0$.

Case 2. $C\in(0,1)\cup(1,+\infty)$.
By defining

$$\tilde{C}:=\frac{-1}{C},\quad \tilde{s}:=Cs,$$

we have $\tilde{s}\in(0,\frac{1}{4(1+|\tilde{C}|)^2})$ for s small and

$$\tau^3Z^{(2)}_{-Cs,s}(\tau)=Z^{(2)}_{s,Cs}(\tfrac{-1}{\tau})=Z^{(2)}_{-\tilde{C}\tilde{s},\tilde{s}}(\tfrac{-1}{\tau}).$$

Again we obtain $\tau(s)\not\to 0$ as $s\to 0$.

Therefore, we have proved $\tau(s)\not\to 0$ as $s\to 0$.

Finally, to prove $\tau(s)\not\to 1$ as $s\to 0$, we let $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0\\ -1 & 1\end{smallmatrix}\right)$ and obtain

$$Z^{(2)}_{r,r+s}\left(\tfrac{\tau}{1-\tau}\right)=(1-\tau)^3 Z^{(2)}_{r,s}(\tau). \tag{3.21}$$

Case 3. $C\in(-\infty,0)\cup(0,1)$.

By defining

$$\tilde{C}:=\frac{C}{1-C},\quad \tilde{s}:=(1-C)s,$$

we have $\tilde{s}\in(0,\frac{1}{4(1+|\tilde{C}|)^2})$ for s small and

$$(1-\tau)^3 Z^{(2)}_{-Cs,s}(\tau)=Z^{(2)}_{-Cs,(1-C)s}\left(\tfrac{\tau}{1-\tau}\right)=Z^{(2)}_{-\tilde{C}\tilde{s},\tilde{s}}\left(\tfrac{\tau}{1-\tau}\right).$$

So $Z^{(2)}_{-\tilde{C}\tilde{s},\tilde{s}}(\tau)$ has zero $\frac{\tau(s)}{1-\tau(s)}\in\{\tau\in\mathbb{H}|\operatorname{Re}\tau\in[-1,1]\}$, and Lemma 3.5 implies $\frac{\tau(s)}{1-\tau(s)}\not\to\infty$, i.e. $\tau(s)\not\to 1$ as $s\to 0$.

Case 4. $C\in(1,+\infty)$.

By defining

$$\tilde{C}:=\frac{C}{1-C},\quad \tilde{s}:=-(1-C)s,$$

we have $\tilde{s}\in(0,\frac{1}{4(1+|\tilde{C}|)^2})$ for s small and

$$\begin{aligned}(1-\tau)^3 Z^{(2)}_{-Cs,s}(\tau)&=Z^{(2)}_{-Cs,(1-C)s}\left(\tfrac{\tau}{1-\tau}\right)\\ &=-Z^{(2)}_{Cs,-(1-C)s}\left(\tfrac{\tau}{1-\tau}\right)=-Z^{(2)}_{-\tilde{C}\tilde{s},\tilde{s}}\left(\tfrac{\tau}{1-\tau}\right).\end{aligned}$$

Again we obtain $\tau(s)\not\to 1$ as $s\to 0$.

The proof is complete. q.e.d.

Now we are in a position to prove Theorem 3.1-(1).

Proof of Theorem 3.1-(1). Fix $C\in\mathbb{R}\backslash\{0,1\}$ and let $s\in(0,\frac{1}{4(1+|C|)^2})$. Recall that $\tau(s)$ is the unique zero of $Z^{(2)}_{-Cs,s}(\tau)$ in F_0. By Lemma 3.6, up to a subsequence of $s\to 0$, we have

$$\tau(C):=\lim_{s\to 0}\tau(s)\in\bar{F}_0\cap\mathbb{H}=F_0. \tag{3.22}$$

Recalling

$$F_{C,s}(\tau)=\frac{4(\tau-C)}{s}Z^{(2)}_{-Cs,s}(\tau),$$

we have $F_{C,s}(\tau(s))=0$. Then Lemma 3.4 implies $f_C(\tau(C))=0$, namely $f_C(\tau)$ has a zero $\tau(C)\in F_0$. Applying Lemma 3.3, we have $\tau(C)\in\mathring{F}_0$. Suppose $f_C(\tau)$ has another zero $\tau_1\neq\tau(C)$ in $\mathring{F}_0$. Since $F_{C,s}(\tau)$ and $f_C(\tau)$ are all holomorphic functions, it follows from Lemma 3.4 and Rouché's theorem that $F_{C,s}(\tau)$ has a zero $\tau_1(s)$ satisfying $\tau_1(s)\to\tau_1$ as $s\to 0$, namely $Z^{(2)}_{-Cs,s}(\tau)$ has two different zeros $\tau(s)$ and $\tau_1(s)$ in F_0 when $s>0$ small, a contradiction with (3.18). Therefore, $\tau(C)$ is the unique zero of $f_C(\tau)$ in F_0. This also implies that (3.22) actually holds for $s\to 0$ (i.e. not only for a subsequence). The proof is complete.

q.e.d.

The proof of Theorem 3.1-(2) will be postponed in the next section. As in Theorem 3.1-(1), we always denote by $\tau(C)$ the unique zero of $f_C(\tau)$ in F_0. Since we proved in [**3**, Theorem 1.1] that for fixed C,

$$f_C(\tau) \textit{ has at most simple zeros on } \mathbb{H}, \tag{3.23}$$

the implicit function theorem infers that $\tau(C)$ is a *smooth* function of $C\in\mathbb{R}\backslash\{0,1\}$. We conclude this section by proving some basic properties of $\tau(C)$.

Lemma 3.7. *The smooth function $\tau(C)$ satisfies*

$$\tau\left(\frac{1}{1-C}\right)=\frac{1}{1-\tau(C)},\quad \forall C\in\mathbb{R}\backslash\{0,1\}. \tag{3.24}$$

Proof. Let $\tau'=\frac{1}{1-\tau}$ and $C'=\frac{1}{1-C}$, then it easy to prove that

$$\tau'\in F_0 \Longleftrightarrow \tau\in F_0 \text{ and } C'\in\mathbb{R}\backslash\{0,1\}\Longleftrightarrow C\in\mathbb{R}\backslash\{0,1\}. \tag{3.25}$$

By using $g_2(\tau')=(1-\tau)^4 g_2(\tau)$ and

$$\eta_2(\tau')=(1-\tau)\eta_1(\tau),\ \eta_1(\tau')=(1-\tau)(\eta_1(\tau)-\eta_2(\tau)), \tag{3.26}$$

a straightforward computation leads to

$$f_{C'}(\tau')=\frac{(1-\tau)^2}{(1-C)^2}f_C(\tau).$$

So $f_C(\tau(C))=0$ gives $f_{C'}(\frac{1}{1-\tau(C)})=0$. Applying Theorem 3.1-(1), we obtain (3.24). This completes the proof. q.e.d.

Lemma 3.8. *Write $\tau(C)=a(C)+b(C)i$ with $a(C),b(C)\in\mathbb{R}$. Then*

$$b(C)\to+\infty,\ a(C)\to\begin{cases}1/4 & \textit{if } C\to+\infty,\\ 3/4 & \textit{if } C\to-\infty,\end{cases} \tag{3.27}$$

$$\tau(C)\to 0 \textit{ as } C\to 0 \quad \textit{and} \quad \tau(C)\to 1 \textit{ as } C\to 1. \tag{3.28}$$

Proof. Recalling (1.4), we define

$$\phi_\pm(\tau):=\tau-\frac{2\pi i}{\eta_1(\tau)\pm\sqrt{g_2(\tau)/12}},\quad \tau\in F_0. \tag{3.29}$$

Write $\tau=a+bi$ and $q=e^{2\pi i\tau}$ as before. Recall from the q-expansions (3.5) and (3.14) that

$$\eta_1(\tau)=\frac{1}{3}\pi^2-8\pi^2(q+3q^2)+O(|q|^3), \tag{3.30}$$

$$g_2(\tau)=\frac{4}{3}\pi^4+320\pi^4(q+9q^2)+O(|q|^3). \tag{3.31}$$

For $\tau \in F_0$, we fix the branch of $\sqrt{g_2(\tau)/12}$ near $b = +\infty$ such that $\sqrt{g_2(\tau)/12} = \frac{1}{3}\pi^2 + O(|q|)$ near $b = +\infty$. Then we easily obtain

$$\eta_1(\tau) - \sqrt{g_2(\tau)/12} = -48\pi^2 q \left(1 - 42q + O(|q|^2)\right),$$

and so

$$\begin{aligned}\phi_-(\tau) &= \tau + \frac{i}{24\pi} q^{-1} + \frac{7i}{4\pi} + O(|q|) \\ &= a + \frac{\sin 2\pi a}{24\pi} e^{2\pi b} + i\left(b + \frac{\cos 2\pi a}{24\pi} e^{2\pi b} + \frac{7}{4\pi}\right) + O(|q|).\end{aligned}$$

Therefore, when $C \in \mathbb{R}$ and $|C| \to +\infty$, it is easy to prove the existence of $\tau_1(C) = a_1(C) + ib_1(C) \in \mathring{F}_0$ such that $C = \phi_-(\tau_1(C))$ and

$$b_1(C) \to +\infty, \ a_1(C) \to \begin{cases} 1/4 & \text{if } C \to +\infty, \\ 3/4 & \text{if } C \to -\infty, \end{cases}$$

i.e. $\tau_1(C) \to \infty$ as $C \to \pm\infty$. By $\eta_2 = \tau\eta_1 - 2\pi i$, (3.29) and (3.1), it follows that $C = \phi_-(\tau_1(C))$ implies $f_C(\tau_1(C)) = 0$. Since $\tau(C)$ is the unique zero of f_C in F_0, we conclude $\tau(C) = \tau_1(C)$. This proves (3.27). Finally, (3.28) follows from (3.27) and (3.24). q.e.d.

4. Critical points of $\eta_1(\tau)$ or equivalently $E_2(\tau)$

4.1. Location of critical points of $\eta_1(\tau)$. This section is devoted to the proof of our main results. Note that $e^{\pi i/3} = \frac{1}{1-e^{\pi i/3}}$. Then by

$$\eta_1(\tfrac{1}{1-\tau}) = (1-\tau)(\eta_1(\tau) - \eta_2(\tau))$$

and the Legendre relation $\eta_2(\tau) = \tau\eta_1(\tau) - 2\pi i$, we easily obtain

$$\eta_1(e^{\pi i/3}) = 2\pi/\sqrt{3}. \tag{4.1}$$

First we prove the following result, which implies Theorems 1.1–1.2 as consequences.

Theorem 4.1. *Let $\tau(C)$ be the unique zero of $f_C(\tau)$ for $C \in \mathbb{R}\backslash\{0,1\}$ in Theorem 3.1-(1). Then the followings hold:*

(1) *For any $m \in \mathbb{Z}$, there holds $\eta_1'(\tau) \neq 0$ in $F_0 + m$. Consequently, $\eta_1'(\tau) \neq 0$ whenever $\operatorname{Im}\tau \geq \frac{1}{2}$.*

(2) *Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$. Then $\frac{a\tau(-d/c)+b}{c\tau(-d/c)+d}$ is the unique zero of $\eta_1'(\tau)$ in the fundamental domain $\gamma(F_0)$ of $\Gamma_0(2)$. In particular,*

$$\Theta := \left\{ \frac{a\tau(\frac{-d}{c})+b}{c\tau(\frac{-d}{c})+d} \,\middle|\, \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma_0(2)/\{\pm I_2\} \text{ with } c \neq 0 \right\} \tag{4.2}$$

gives rise to all the zeros of $\eta_1'(\tau)$ in $\mathbb{H}$.

Proof. (1). First we claim that

$$12\eta_1(\tau)^2 - g_2(\tau) \neq 0 \text{ for } \tau \in \partial F_0 \cap \mathbb{H}. \tag{4.3}$$

If $\tau \in i\mathbb{R}^+$, $12\eta_1(\tau)^2 - g_2(\tau) \neq 0$ follows from Lemma 3.2. If $\tau \in i\mathbb{R}^+ + 1$, then

$$12\eta_1(\tau)^2 - g_2(\tau) = 12\eta_1(\tau-1)^2 - g_2(\tau-1) \neq 0.$$

If $|\tau - \frac{1}{2}| = \frac{1}{2}$, then $\tau' = \frac{\tau}{1-\tau} \in i\mathbb{R}^+$. By using (3.7), we see from (3.1) with $C = -1$ that

$$\begin{aligned} f_{-1}(\tau') &= 12(\eta_1(\tau') + \eta_2(\tau'))^2 - g_2(\tau')(1+\tau')^2 \\ &= (1-\tau)^2[12\eta_1(\tau)^2 - g_2(\tau)]. \end{aligned}$$

Since Lemma 3.3 shows $f_{-1}(\tau') \neq 0$, we obtain $12\eta_1(\tau)^2 - g_2(\tau) \neq 0$. This proves (4.3).

Suppose by contradiction that $12\eta_1(\tau)^2 - g_2(\tau)$ has a zero τ_0 in $\mathring{F}_0$. Then

$$\text{either } \eta_1(\tau_0) - \sqrt{g_2(\tau_0)/12} = 0 \text{ or } \eta_1(\tau_0) + \sqrt{g_2(\tau_0)/12} = 0.$$

Without loss of generality, we may assume $\eta_1(\tau_0) + \sqrt{g_2(\tau_0)/12} = 0$. Recall (4.1) and the fact that $g_2(\tau) = 0$ in F_0 if and only if $\tau = e^{\pi i/3}$. So $\tau_0 \neq e^{\pi i/3}$ and $g_2(\tau_0) \neq 0$, i.e. $\eta_1(\tau) + \sqrt{g_2(\tau)/12}$ is holomorphic at τ_0 with τ_0 being a zero. Recalling $\phi_\pm(\tau)$ in (3.29), it follows that $\phi_+(\tau)$ is meromorphic at τ_0 with τ_0 being a pole and so maps a small neighborhood $U \subset \mathring{F}_0$ of τ_0 onto a neighborhood of ∞. Then for $C > 0$ large enough, there exists $\tau_1(C) \in U$ such that $C = \phi_+(\tau_1(C))$, which implies $f_C(\tau_1(C)) = 0$. Applying Theorem 3.1-(1) and Lemma 3.8, we obtain $\tau_1(C) = \tau(C) \to \infty$ as $C \to +\infty$, which contradicts with $\tau_1(C) \in U$.

Therefore, we have proved that

$$12\eta_1(\tau)^2 - g_2(\tau) \neq 0 \quad \text{for any } \tau \in F_0. \tag{4.4}$$

Since

$$\eta_1'(\tau) = \frac{i}{2\pi}\left(\eta_1(\tau)^2 - \tfrac{1}{12}g_2(\tau)\right)$$

and $\eta_1(\tau+1) = \eta_1(\tau)$, we conclude that $\eta_1'(\tau) \neq 0$ for any $\tau \in F_0 + m$ and $m \in \mathbb{Z}$. This proves (1).

(2). Given $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)/\{\pm I_2\}$ with $c \neq 0$. Write $\tau' = \gamma \cdot \tau = \frac{a\tau+b}{c\tau+d}$ with $\tau \in F_0$. By using

$$\eta_1(\tau') = (c\tau+d)(c\eta_2(\tau) + d\eta_1(\tau)), \quad g_2(\tau') = (c\tau+d)^4 g_2(\tau), \tag{4.5}$$

we have

$$\begin{aligned} 12\eta_1(\tau')^2 - g_2(\tau') &= c^2(c\tau+d)^2\left[12(\tfrac{d}{c}\eta_1 + \eta_2)^2 - g_2(\tau)(\tfrac{d}{c}+\tau)^2\right] \\ &= c^2(c\tau+d)^2 f_{\frac{-d}{c}}(\tau). \end{aligned}$$

Clearly $\frac{-d}{c} \in \mathbb{Q}\backslash\mathbb{Z}$, so Theorem 3.1-(1) shows that $\tau(\frac{-d}{c}) \in \mathring{F}_0$ is the unique zero of $f_{\frac{-d}{c}}(\tau)$ in F_0. Consequently,

$$\gamma \cdot \tau(\tfrac{-d}{c}) = \frac{a\tau(\frac{-d}{c})+b}{c\tau(\frac{-d}{c})+d} \in \gamma(\mathring{F}_0)$$

is the unique zero of $12\eta_1^2 - g_2$ in $\gamma(F_0)$. Since

$$\mathbb{H} = \bigcup_{\gamma\in\Gamma_0(2)/\{\pm I_2\}} \gamma(F_0),$$

we conclude that the set Θ defined in (4.2) gives all the zeros of $12\eta_1^2 - g_2$ and so η_1'. This proves (2). The proof is complete. q.e.d.

Recall the curves defined in Section 1:

$$\mathcal{C}_- = \{\tau(C)|C \in (-\infty, 0)\}, \quad \mathcal{C}_+ = \{\tau(C)|C \in (1, +\infty)\},$$
$$\mathcal{C}_0 = \{\tau(C)|C \in (0, 1)\}.$$

Proof of Theorem 1.3. The smoothness of the three curves will be given in Section 5. The assertion that under the Möbius transformation of $\Gamma_0(2)$ action, the collection of all critical points of $\eta_1(\tau)$ is precisely the set $\mathcal{D}$ given by (1.6), is a direct consequence of the expression (4.2) of the critical point set Θ. Recall that $\tau(C)$ is smooth as a function of $C \in \mathbb{R}\setminus\{0, 1\}$. To prove the denseness, i.e. the identity (1.7), it suffices to prove that

$$Q_0 := \{\tfrac{-d}{c} \mid d \in \mathbb{Z},\, c \in 2\mathbb{Z}\setminus\{0\},\, (c, d) = 1\}$$

is dense in $\mathbb{Q}$ and hence dense in $\mathbb{R}$. Take any $\frac{m}{n} \in \mathbb{Q}\setminus\{0\}$ such that $m, n \in \mathbb{Z}$ and $(m, n) = 1$.

Case 1. n is even. Then $\frac{m}{n} \in Q_0$.

Case 2. n is odd and m is odd. Then

$$Q_0 \ni \frac{m(2^k+n)}{2^k n} \to \frac{m}{n} \quad \text{as } k \to +\infty.$$

Case 3. n is odd and m is even. Then

$$Q_0 \ni \frac{2^k m+n}{2^k n} \to \frac{m}{n} \quad \text{as } k \to +\infty.$$

This proves $\overline{Q_0} = \mathbb{R}$ and so completes the proof. q.e.d.

Proof of Corollary 1.5. Let $\gamma = \left(\begin{smallmatrix} 1 & -1 \\ 2 & -1 \end{smallmatrix}\right) \in \Gamma_0(2)$. Then it is easy to prove that

$$\gamma(F_0) = \left\{\tau \in \mathbb{H} \,|\, |\tau - \tfrac{1}{2}| \le \tfrac{1}{2},\ |\tau - \tfrac{1}{4}| \ge \tfrac{1}{4},\ |\tau - \tfrac{3}{4}| \ge \tfrac{1}{4}\right\},$$

and so

$$\left\{\tau \in \mathbb{H} | \operatorname{Re}\tau = \tfrac{1}{2}\right\} \subset F_0 \cup \gamma(F_0).$$

Applying Theorem 4.1-(2), we see that $\tilde{\tau} := \frac{\tau(1/2)-1}{2\tau(1/2)-1}$ is the unique zero of η_1' in $\gamma(F_0)$. We will prove in Theorem 4.2 (iii) that $\tau(1/2) = \frac{1}{2} + i\hat{b}$ for some $\hat{b} \in (\frac{\sqrt{3}}{2}, \frac{6}{5})$. Then $\tilde{\tau} = \frac{1}{2} + \frac{i}{4\hat{b}} \in \{\tau \in \mathbb{H} | \operatorname{Re}\tau = \frac{1}{2}\}$, namely $\tilde{\tau}$ is the unique zero of η_1' on the line $\{\tau \in \mathbb{H} | \operatorname{Re}\tau = \frac{1}{2}\}$ with $b_0 := \operatorname{Im}\tilde{\tau} \in (\frac{5}{24}, \frac{1}{2\sqrt{3}})$. Recall (3.5) and (4.1) that $\eta_1(\frac{1}{2} + i\frac{\sqrt{3}}{2}) > \frac{\pi^2}{3} = \lim_{b\to+\infty} \eta_1(\frac{1}{2} + ib)$. Moreover, it follows from (1.2) and (1.3) that for $\tau = \frac{1}{2} + ib$,

$$\eta_1\left(\tfrac{1}{2} + \tfrac{i}{4b}\right) = \eta_1(\tfrac{\tau-1}{2\tau-1}) = -4b^2\eta_1(\tau) + 8\pi b \to -\infty \text{ as } b \to +\infty.$$

Thus $\eta_1(\frac{1}{2}+ib)$ is strictly increasing for $b \in (0, b_0)$ and strictly decreasing for $b \in (b_0, +\infty)$. The proof is complete. q.e.d.

Now we can finish the proof of Theorem 3.1.

Proof of Theorem 3.1-(2). First we consider $C = 0$, i.e.

$$f_0(\tau) = 12\eta_2(\tau)^2 - g_2(\tau)\tau^2.$$

Suppose $f_0(\tau) = 0$ for some $\tau \in F_0$. Then it is easy to see that $\tau' := \frac{\tau-1}{\tau} \in F_0$. By

$$\eta_1(\tau') = \tau\eta_2(\tau), \quad g_2(\tau') = \tau^4 g_2(\tau),$$

we obtain

$$12\eta_1(\tau')^2 - g_2(\tau') = \tau^2 f_0(\tau) = 0,$$

a contradiction with (4.4).

Now we consider $C = 1$, i.e.

$$f_1(\tau) := 12(\eta_1(\tau) - \eta_2(\tau))^2 - g_2(\tau)(1-\tau)^2.$$

Suppose $f_1(\tau) = 0$ for some $\tau \in F_0$. Then it is easy to see that $\tau' := \frac{1}{1-\tau} \in F_0$. By

$$\eta_1(\tau') = (1-\tau)(\eta_1(\tau) - \eta_2(\tau)), \quad g_2(\tau') = (1-\tau)^4 g_2(\tau),$$

we obtain

$$12\eta_1(\tau')^2 - g_2(\tau') = (1-\tau)^2 f_1(\tau) = 0,$$

again a contradiction with (4.4). The proof is complete. q.e.d.

4.2. Geometry of curves $\mathcal{C}_-$, $\mathcal{C}_0$ and $\mathcal{C}_+$. In this section, we want to describe some geometry about these three curves, including their intersection with the line $\operatorname{Re}\tau = \frac{1}{2}$.

Theorem 4.2.

(i) *The function $C \mapsto \tau(C)$ is one-to-one whenever C is restricted in one of $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$, i.e. any one of curves $\mathcal{C}_-$, $\mathcal{C}_0$, $\mathcal{C}_+$ has no self-intersection. Furthermore,*

$$\partial\mathcal{C}_0 = \{0, 1\}, \quad \partial\mathcal{C}_- = \{0, \tfrac{3}{4} + i\infty\}, \quad \partial\mathcal{C}_+ = \{1, \tfrac{1}{4} + i\infty\}.$$

(ii) *The curve $\mathcal{C}_0$ is symmetric with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$; $\mathcal{C}_-$ and $\mathcal{C}_+$ are symmetric with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$.*

(iii) $\tau(\frac{1}{2})$ *is the unique intersection point of the curve* $\mathcal{C}_0$ *with the line* $\operatorname{Re}\tau = \frac{1}{2}$*. Furthermore,* $\operatorname{Im}\tau(\frac{1}{2}) \in (\frac{\sqrt{3}}{2}, \frac{6}{5})$*.*

(iv) $\mathcal{C}_-$ *(resp.* $\mathcal{C}_+$*) has a unique intersection point* τ_- *with the line* $\operatorname{Re}\tau = \frac{1}{2}$*. Furthermore,* $\operatorname{Im}\tau_- \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$*.*

(v) τ_- *is the unique intersection point of* $\mathcal{C}_-$ *with* $\mathcal{C}_+$*.*

(vi) $\frac{1}{1-\tau_-}$ *(resp.* $\frac{\tau_-}{\tau_- - 1}$*) is the unique intersection point of* $\mathcal{C}_0$ *with* $\mathcal{C}_-$ *(resp.* $\mathcal{C}_+$*).*

First we prove that any one of these curves has no self-intersection.

Lemma 4.3. *The function* $C \mapsto \tau(C)$ *is one-to-one whenever* C *is restricted in one of* $(-\infty, 0)$*,* $(0,1)$ *and* $(1, +\infty)$*.*

Proof. We recall the following results (cf. [**14**, Section 6]): when $\tau = \frac{1}{2} + ib$ with $b > 0$,

$$\eta_1(\tau), g_2(\tau) \in \mathbb{R} \quad \text{and} \quad g_2(\tau) \begin{cases} > 0 & \text{if} \;\; b > \frac{\sqrt{3}}{2}, \\ = 0 & \text{if} \;\; b = \frac{\sqrt{3}}{2}, \\ < 0 & \text{if} \;\; b \in [\frac{1}{2}, \frac{\sqrt{3}}{2}). \end{cases} \tag{4.6}$$

As pointed out before, $f_C(\tau(C)) = 0$ is equivalent to

$$\text{either} \;\; C = \phi_+(\tau(C)) = \tau(C) - \frac{2\pi i}{\eta_1(\tau(C)) + \sqrt{g_2(\tau(C))/12}} \tag{4.7}$$

$$\text{or} \;\; C = \phi_-(\tau(C)) = \tau(C) - \frac{2\pi i}{\eta_1(\tau(C)) - \sqrt{g_2(\tau(C))/12}}.$$

If $\tau(C) = \frac{1}{2} + ib(C)$ with $b(C) \geq \frac{\sqrt{3}}{2}$ for some $C \in \mathbb{R}\backslash\{0,1\}$, then it follows from (4.6) and (4.4) that $C = \phi_\pm(\tau(C)) = \operatorname{Re}\tau(C) = \frac{1}{2}$. Therefore,

$$\mathcal{C}_-, \mathcal{C}_+ \subset F_0 \backslash \left\{\tau = \tfrac{1}{2} + ib \mid b \geq \tfrac{\sqrt{3}}{2}\right\}. \tag{4.8}$$

Remark that $g_2(\tau) = 0$ for $\tau \in F_0$ if and only if $\tau = \frac{1}{2} + \frac{\sqrt{3}}{2}i$. We restrict $\tau \in F_0\backslash\{\tau = \frac{1}{2} + ib | b \geq \frac{\sqrt{3}}{2}\}$ and fix a branch of $\sqrt{g_2(\tau)/12}$. Then it follows from (4.4) that both $\phi_+(\tau)$ and $\phi_-(\tau)$ are *single-valued holomorphic* functions in $F_0\backslash\{\tau = \frac{1}{2} + ib \mid b \geq \frac{\sqrt{3}}{2}\}$. Define

$$D_1 := \{C \in (-\infty, 0) | C = \phi_-(\tau(C))\},$$
$$D_2 := \{C \in (-\infty, 0) | C = \phi_+(\tau(C))\}.$$

Since $\tau(C)$ is continuous as a function of C, we see that D_1, D_2 are both closed subsets of $(-\infty, 0)$, so

$$\text{either } D_1 = (-\infty, 0), D_2 = \emptyset \;\; \text{or} \;\; D_1 = \emptyset, D_2 = (-\infty, 0).$$

Without loss of generality we may assume $D_1 = (-\infty, 0)$. Then $C = \phi_-(\tau(C))$ for any $C \in (-\infty, 0)$. This proves that $(-\infty, 0) \ni C \mapsto$

$\tau(C) \in \mathcal{C}_-$ is one-to-one. Similarly, $(1,+\infty) \ni C \mapsto \tau(C) \in \mathcal{C}_+$ is one-to-one. Finally, since

$$C \in (0,1) \Longleftrightarrow \frac{1}{1-C} \in (1,+\infty),$$

it follows from (3.24) that $(0,1) \ni C \mapsto \tau(C) \in \mathcal{C}_0$ is also one-to-one. The proof is complete. q.e.d.

Lemma 4.4. *Let $\tau = \frac{1}{2} + ib$ with $b \geq \frac{\sqrt{3}}{2}$ and recall (4.6) that $g_2(\tau) \geq 0$. Then*

$$\eta_1(\tau) - \sqrt{g_2(\tau)/12} < \frac{2\pi}{b}. \tag{4.9}$$

Proof. Clearly $q = e^{2\pi i\tau} = -e^{-2\pi b}$. It follows from the q-expansions (3.5) and (3.14) that

$$\eta_1(\tau) = \frac{\pi^2}{3} - 8\pi^2 \sum_{k=1}^{\infty} (-1)^k b_k e^{-2k\pi b}, \quad b_k = \sum_{1 \leq d | k} d, \tag{4.10}$$

$$g_2(\tau) = \frac{4}{3}\pi^4 + 320\pi^4 \sum_{k=1}^{\infty} (-1)^k \sigma_3(k) e^{-2k\pi b}, \;\; \sigma_3(k) = \sum_{1 \leq d | k} d^3. \tag{4.11}$$

Since $b \geq \frac{\sqrt{3}}{2}$, we have $e^{\pi b} > 15$. It is easy to prove that

$$b_k < 15^{\frac{k}{4}} < e^{\frac{k\pi b}{4}}, \quad \sigma_3(k) \leq b_k^3 < e^{\frac{3k\pi b}{4}}, \quad \forall k \geq 1.$$

Consequently,

$$\sum_{k=3}^{\infty} b_k e^{-2k\pi b} \leq \sum_{k=3}^{\infty} e^{-\frac{7}{4}k\pi b} = \frac{e^{-\frac{21}{4}\pi b}}{1 - e^{-\frac{7}{4}\pi b}} < e^{-4\pi b},$$

i.e.

$$-\sum_{k=1}^{\infty} (-1)^k b_k e^{-2k\pi b} = e^{-2\pi b} - 3e^{-4\pi b} - \sum_{k=3}^{\infty} (-1)^k b_k e^{-2k\pi b} \in (0, e^{-2\pi b}).$$

From here and (4.10) we obtain

$$\frac{\pi^2}{3} < \eta_1(\tau) < \frac{\pi^2}{3} + 8\pi^2 e^{-2\pi b}, \quad \forall\, b \geq \frac{\sqrt{3}}{2}. \tag{4.12}$$

Together with Corollary 1.5 that $\frac{d}{db}\eta_1(\frac{1}{2} + ib) \neq 0$ for $b \geq \frac{1}{2\sqrt{3}}$, we conclude that

$$\frac{d}{db}\eta_1(\tfrac{1}{2} + ib) < 0 \quad \text{for} \quad b \geq \frac{1}{2\sqrt{3}}, \tag{4.13}$$

so (4.1) gives

$$\eta_1(\tfrac{1}{2} + ib) \leq \eta_1(e^{i\pi/3}) = \frac{2\pi}{\sqrt{3}} \;\text{ for any } b \geq \frac{\sqrt{3}}{2}.$$

Since $g(\tau) \geq 0$ for $b \geq \frac{\sqrt{3}}{2}$, we see that (4.9) holds for any $b \in [\frac{\sqrt{3}}{2}, \sqrt{3})$.

Now we assume $b \geq \frac{6}{5}$ (indeed $b \geq \sqrt{3}$ is enough, here we assume $b \geq \frac{6}{5}$ for later use). Then $e^{\pi b} > 40$. Similarly we have

$$\sum_{k=2}^{\infty} \sigma_3(k) e^{-2k\pi b} \leq \sum_{k=2}^{\infty} e^{-\frac{5}{4}k\pi b} = \frac{e^{-\frac{5}{2}\pi b}}{1-e^{-\frac{5}{4}\pi b}} < \frac{1}{2}e^{-2\pi b},$$

i.e.

$$\sum_{k=1}^{\infty}(-1)^k \sigma_3(k) e^{-2k\pi b} = -e^{-2\pi b} + \sum_{k=2}^{\infty}(-1)^k \sigma_3(k) e^{-2k\pi b} > -\frac{3}{2}e^{-2\pi b}.$$

Thus (4.11) and $e^{\pi b} > 40$ give

$$g_2(\tau) > \frac{4}{3}\pi^4\left(1-360e^{-2\pi b}\right) > \frac{4}{3}\pi^4(1-210e^{-2\pi b})^2,$$

i.e.

$$\sqrt{g_2(\tau)/12} > \frac{\pi^2}{3}(1-210e^{-2\pi b}), \quad \forall b \geq \frac{6}{5}. \tag{4.14}$$

Together with (4.12), we obtain

$$\eta_1(\tau) - \sqrt{g_2(\tau)/12} < 78\pi^2 e^{-2\pi b}, \quad \forall b \geq \frac{6}{5}.$$

Since it is trivial to see that $78\pi^2 e^{-2\pi b} < \frac{2\pi}{b}$ for $b \geq \sqrt{3}$, we conclude that (4.9) holds for any $b \geq \sqrt{3}$. This completes the proof. q.e.d.

Proof of Theorem 4.2. (i) is just Lemmas 3.8 and 4.3.

(ii). We will prove in Theorem A.1 below that $\eta_1(1-\bar{\tau}) = \overline{\eta_1(\tau)}$ and $\eta_2(1-\bar{\tau}) = \overline{\eta_1(\tau)} - \overline{\eta_2(\tau)}$. By the q-expansion (3.14) we also have $g_2(1-\bar{\tau}) = \overline{g_2(\tau)}$. Since $C \in \mathbb{R}\backslash\{0,1\}$, we easily obtain

$$\begin{aligned} f_{1-C}(1-\bar{\tau}) =& 12\left((1-C)\eta_1(1-\bar{\tau}) - \eta_2(1-\bar{\tau})\right)^2 - g_2(1-\bar{\tau})(C-\bar{\tau})^2 \\ =& 12(C\overline{\eta_1(\tau)} - \overline{\eta_2(\tau)})^2 - \overline{g_2(\tau)}(C-\bar{\tau})^2 = \overline{f_C(\tau)}. \end{aligned}$$

Therefore, it follows from Theorem 3.1-(1) that

$$\tau(1-C) = 1 - \overline{\tau(C)}. \tag{4.15}$$

Since τ and $1-\bar{\tau}$ is symmetric with respect to the line $\operatorname{Re}\tau = \frac{1}{2}$, we see that assertion (ii) holds.

(iii). By (ii), $\mathcal{C}_0$ has intersections with the line $\operatorname{Re}\tau = \frac{1}{2}$. Let $\tau_0 = \frac{1}{2} + ib$ be such an intersection point. Then $\tau_0 = \tau(C)$ for a unique $C \in (0,1)$. Applying (4.15), we have $\tau(1-C) = \tau_0 = \tau(C)$, so Lemma 4.3 gives $1-C = C$, i.e. $C = \frac{1}{2}$. This proves that $\tau_0 = \tau(\frac{1}{2})$ is the unique intersection point of the curve $\mathcal{C}_0$ with the line $\operatorname{Re}\tau = \frac{1}{2}$. By (4.7),

$$\begin{aligned} \frac{1}{2} = \phi_{\pm}(\tau_0) &= \tau_0 - \frac{2\pi i}{\eta_1(\tau_0) \pm \sqrt{g_2(\tau_0)/12}} \\ &= \frac{1}{2} + ib - \frac{2\pi i}{\eta_1(\tau_0) \pm \sqrt{g_2(\tau_0)/12}}, \end{aligned}$$

we see that $\eta_1(\tau_0) \pm \sqrt{g_2(\tau_0)/12} = \frac{2\pi}{b}$ is real, so $g_2(\tau_0) \geq 0$, i.e. $b = \operatorname{Im}\tau_0 \geq \frac{\sqrt{3}}{2}$. Since (4.1) gives $\eta_1(e^{\pi i/3}) = \frac{2\pi}{\sqrt{3}} \neq \frac{4\pi}{\sqrt{3}}$, we obtain $b = \operatorname{Im}\tau_0 > \frac{\sqrt{3}}{2}$. Then $\operatorname{Im}\frac{\tau(1/2)-1}{2\tau(1/2)-1} < \frac{1}{2\sqrt{3}}$ and so Lemma 4.4 applies. In particular, (4.9) infers

$$\eta_1(\tau_0) + \sqrt{g_2(\tau_0)/12} = \frac{2\pi}{b}. \tag{4.16}$$

Suppose $b = \operatorname{Im}\tau_0 \geq \frac{6}{5}$. Then (4.12), (4.14) and (4.16) imply

$$\frac{2\pi^2}{3} - 70\pi^2 e^{-2\pi b} < \frac{2\pi}{b} \leq \frac{5\pi}{3},$$

which is equivalent to $\frac{210\pi}{2\pi-5} > e^{2\pi b}$, clearly a contradiction with $e^{\pi b} \geq e^{6\pi/5} > 40$. Thus $b = \operatorname{Im}\tau_0 < \frac{6}{5}$. This proves (iii).

(iv)–(v). Note from (4.8) that if $\mathcal{C}_-$ (resp. $\mathcal{C}_+$) has a intersection point τ with the line $\operatorname{Re}\tau = \frac{1}{2}$, then $\operatorname{Im}\tau \in [\frac{1}{2}, \frac{\sqrt{3}}{2})$. Assume $\tau = \frac{1}{2} + ib$ with $b \in [\frac{1}{2}, \frac{\sqrt{3}}{2})$ is a intersection point of $\mathcal{C}_-$ with the line $\operatorname{Re}\tau = \frac{1}{2}$. Then (4.6) gives $\sqrt{g_2(\tau)/12} = \pm i\sqrt{|g_2(\tau)|/12}$. Clearly there exists $C \in (-\infty, 0)$ such that $\tau = \tau(C)$, which implies $C = \phi_\pm(\tau)$, i.e.

$$\begin{aligned} C &= \tau - \frac{2\pi i}{\eta_1(\tau) \pm i\sqrt{|g_2(\tau)|/12}} \\ &= \frac{1}{2} + bi - \frac{2\pi i(\eta_1(\tau) \mp i\sqrt{|g_2(\tau)|/12})}{\eta_1(\tau)^2 - \frac{g_2(\tau)}{12}}. \end{aligned}$$

Thus

$$b - \frac{2\pi\eta_1(\tau)}{\eta_1(\tau)^2 - \frac{g_2(\tau)}{12}} = 0. \tag{4.17}$$

For $\tau = \frac{1}{2} + ib$ with $b \in [\frac{1}{2}, \frac{\sqrt{3}}{2}]$, we define

$$\theta(b) := \frac{b\eta_1(\frac{1}{2}+ib)}{2\pi} \quad \text{and} \quad \theta_1(b) := \frac{\eta_1(\frac{1}{2}+ib)^2}{\eta_1(\frac{1}{2}+ib)^2 - \frac{g_2(\frac{1}{2}+ib)}{12}}.$$

Then (4.17) shows that if $\tau = \frac{1}{2} + ib$ with $b \in [\frac{1}{2}, \frac{\sqrt{3}}{2})$ is a intersection point of $\mathcal{C}_-$ with the line $\operatorname{Re}\tau = \frac{1}{2}$, then

$$\theta(b) - \theta_1(b) = 0. \tag{4.18}$$

Now we want to prove that (4.18) has a unique solution in $[\frac{1}{2}, \frac{\sqrt{3}}{2}]$. For this, first we note from (4.13) that

$$\eta_1(\tfrac{1}{2} + bi) > \pi^2/3 \quad \text{for} \quad b \geq \tfrac{1}{2\sqrt{3}}. \tag{4.19}$$

We also recall the following fundamental results (cf. [14]):

$$e_1(\tfrac{1}{2}+\tfrac{1}{2}i)=0 \text{ and } e_1(\tfrac{1}{2}+bi)>0 \text{ for } b>\tfrac{1}{2}, \tag{4.20}$$

$$g_3(\tfrac{1}{2}+bi)=4e_1(\tfrac{1}{2}+bi)|e_2(\tfrac{1}{2}+bi)|^2>0 \text{ for } b\in(\tfrac{1}{2},\tfrac{\sqrt{3}}{2}], \tag{4.21}$$

$$g_2(\tfrac{1}{2}+bi)^3-27g_3(\tfrac{1}{2}+bi)^2<0 \text{ for } b\geq\tfrac{1}{2}. \tag{4.22}$$

By using (see e.g. [1, p. 704] or [3, Appendix B])

$$\frac{d}{db}\eta_1(\tfrac{1}{2}+bi)=i\eta_1'(\tau)=\frac{-1}{4\pi}(2\eta_1^2-\tfrac{1}{6}g_2), \tag{4.23}$$

$$\frac{d}{db}g_2(\tfrac{1}{2}+bi)=ig_2'(\tau)=\frac{1}{\pi}(3g_3-2\eta_1g_2),$$

$$\frac{d}{db}g_3(\tfrac{1}{2}+bi)=ig_3'(\tau)=\frac{1}{\pi}(-3g_3\eta_1+\tfrac{1}{6}g_2^2), \tag{4.24}$$

we easily obtain

$$\theta'(b)=\frac{\eta_1}{2\pi}-\frac{b}{8\pi^2}(2\eta_1^2-\tfrac{1}{6}g_2)=\frac{\eta_1(\frac{1}{2}+ib)}{2\pi\theta_1(b)}(\theta_1(b)-\theta(b)), \tag{4.25}$$

$$\begin{aligned}\theta_1'(b)&=\frac{1}{12}\frac{d}{db}\left(\frac{g_2(\frac{1}{2}+ib)}{\eta_1(\frac{1}{2}+ib)^2-\frac{g_2(\frac{1}{2}+ib)}{12}}\right)\\&=\frac{\eta_1}{12\pi(\eta_1^2-\frac{g_2}{12})^2}\left(3g_3\eta_1-\eta_1^2g_2-\tfrac{1}{12}g_2^2\right).\end{aligned} \tag{4.26}$$

Step 1. We claim that $\theta'(\frac{1}{2})<0$.
Indeed, by (3.6) and $\eta_2(\tau)=\tau\eta_1(\tau)-2\pi i$ we obtain

$$\eta_1(i)=\pi \quad\text{and}\quad \eta_2(i)=-\pi i,$$

which imply

$$\eta_1(\tfrac{1}{2}+\tfrac{1}{2}i)=\eta_1(\tfrac{1}{1-i})=(1-i)(\eta_1(i)-\eta_2(i))=2\pi.$$

Recalling Lemma 3.2 that $g_2(i)>12\eta_1(i)^2=12\pi^2$, we have

$$g_2(\tfrac{1}{2}+\tfrac{1}{2}i)=g_2(\tfrac{1}{1-i})=(1-i)^4g_2(i)=-4g_2(i)<-48\pi^2,$$

so

$$0<\theta_1(\tfrac{1}{2})=\frac{\eta_1(\frac{1}{2}+\frac{1}{2}i)^2}{\eta_1(\frac{1}{2}+\frac{1}{2}i)^2-\frac{g_2(\frac{1}{2}+\frac{1}{2}i)}{12}}<\tfrac{1}{2}=\theta(\tfrac{1}{2}).$$

This, together with (4.25), proves $\theta'(\frac{1}{2})<0$.

Step 2. We claim that for any $b\in(\frac{1}{2},\frac{\sqrt{3}}{2})$ satisfying $\theta_1'(b)=0$, there holds $\psi'(b)>0$, where

$$\psi(b):=(3g_3\eta_1-\eta_1^2g_2-\tfrac{1}{12}g_2^2)(\tfrac{1}{2}+bi).$$

Applying (4.23)–(4.24), a straightforward computation leads to

$$\psi'(b) = \frac{1}{\pi}\left(3g_2\eta_1^3 - \tfrac{27}{2}g_3\eta_1^2 + \tfrac{3}{4}g_2^2\eta_1 - \tfrac{3}{8}g_2g_3\right).$$

By (4.19), (4.26) and $\theta_1'(b) = 0$, we have $\psi(b) = 0$, i.e.

$$\eta_1^2 g_2 = 3g_3\eta_1 - \tfrac{1}{12}g_2^2.$$

Inserting this into the term $3g_2\eta_1^3$ of $\psi'(b)$, we easily obtain

$$\psi'(b) = \frac{1}{\pi}\left(-\tfrac{9}{2}g_3\eta_1^2 + \tfrac{1}{2}g_2^2\eta_1 - \tfrac{3}{8}g_2g_3\right),$$

and so

$$\begin{aligned} g_2\psi'(b) &= \frac{1}{\pi}\left(-\tfrac{9}{2}g_3(3g_3\eta_1 - \tfrac{1}{12}g_2^2) + \tfrac{1}{2}g_2^3\eta_1 - \tfrac{3}{8}g_2^2g_3\right) \\ &= \frac{\eta_1}{2\pi}(g_2^3 - 27g_3^2). \end{aligned}$$

Applying (4.6), (4.19) and (4.22) we conclude $\psi'(b) > 0$.

Step 3. As a direct consequence of Step 2 and (4.26), we have that if $\theta_1'(b_0) \geq 0$ for some $b_0 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$, then $\theta_1'(b) > 0$ for any $b \in (b_0, \frac{\sqrt{3}}{2})$.

Step 4. We claim that there exists $b_1 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ such that $\theta'(b) < 0$ for $b \in [\frac{1}{2}, b_1)$, $\theta'(b_1) = 0$ and $\theta'(b) > 0$ for $b - b_1 > 0$ small. Consequently,

$$\theta_1'(b_1) \geq 0 \text{ and } \theta_1'(b) > 0 \text{ for any } b \in (b_1, \tfrac{\sqrt{3}}{2}). \tag{4.27}$$

Since $\eta_1(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = \frac{2\pi}{\sqrt{3}}$ gives $\theta(\frac{\sqrt{3}}{2}) = \frac{1}{2} = \theta(\frac{1}{2})$, Step 1 implies that there exists $b_1 \in (\frac{1}{2}, \frac{\sqrt{3}}{2})$ such that $\theta'(b) < 0$ for $b \in [\frac{1}{2}, b_1)$ and $\theta'(b_1) = 0$. Suppose $\theta'(b) \leq 0$ for $b - b_1 > 0$ sufficiently small, then (4.25) says that $\theta_1(b) - \theta(b)$ has a local maximum 0 at b_1, so $\theta_1'(b_1) = 0$ and then Step 3 gives $\theta_1'(b) > 0$ for $b - b_1 > 0$ small. However, this implies $\theta_1'(b) - \theta'(b) > 0$ for $b - b_1 > 0$ sufficiently small, which contradicts with that $\theta_1(b) - \theta(b)$ has a local maximum 0 at b_1. Therefore, $\theta'(b) > 0$ for $b - b_1 > 0$ small. This also gives $\theta_1(b) - \theta(b) > 0$ for $b - b_1 > 0$ small. Since $\theta_1(b_1) - \theta(b_1) = 0$, we obtain $\theta_1'(b_1) = \theta_1'(b_1) - \theta'(b_1) \geq 0$ and so (4.27) holds.

Step 5. We claim that $\theta'(b) > 0$ for $b \in (b_1, \frac{\sqrt{3}}{2}]$.

Since $g_2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 0$ gives $\theta_1(\frac{\sqrt{3}}{2}) = 1 > \theta(\frac{\sqrt{3}}{2})$, we have $\theta'(\frac{\sqrt{3}}{2}) > 0$. Suppose there exists $b_2 \in (b_1, \frac{\sqrt{3}}{2})$ such that $\theta'(b_2) = 0$ and $\theta'(b) > 0$ for $b \in (b_1, b_2)$. Then (4.25) gives $\theta_1(b) - \theta(b) > 0$ for $b \in (b_1, b_2)$ and $\theta_1(b_2) - \theta(b_2) = 0$, i.e. $\theta_1'(b_2) - \theta'(b_2) \leq 0$. However, by (4.27) we have $\theta_1'(b_2) - \theta'(b_2) = \theta_1'(b_2) > 0$, a contradiction.

Step 6. We finish the proof of assertions (iv)–(v).

By Steps 4–5 and (4.25), $\theta_1(b)-\theta(b)=0$ has a unique solution b_1 in $[\frac{1}{2},\frac{\sqrt{3}}{2}]$ and $b_1\in(\frac{1}{2},\frac{\sqrt{3}}{2})$. Together with (4.18), we see that $\mathcal{C}_-$ has at most one intersection point $\frac{1}{2}+ib_1$ with the line $\operatorname{Re}\tau=\frac{1}{2}$. Since Lemma 3.8 implies that $\mathcal{C}_-$ must intersect with the line $\operatorname{Re}\tau=\frac{1}{2}$, we conclude that $\tau_-:=\frac{1}{2}+ib_1$ is the unique intersection point of $\mathcal{C}_-$ with the line $\operatorname{Re}\tau=\frac{1}{2}$. Finally, it follows from assertion (ii) that τ_- is also the unique intersection point of $\mathcal{C}_+$ with the line $\operatorname{Re}\tau=\frac{1}{2}$ (resp. $\mathcal{C}_-$).

(vi). Clearly this follows readily from the assertion (v) and (3.24). The proof is complete. q.e.d.

5. Geometric interpretation and smoothness of the curves

The purpose of this section is to give the geometric meaning of the three curves from the multiple Green function G_2. As pointed out in Section 1, $\{(q_\pm,-q_\pm)|\wp(q_\pm)=\pm\sqrt{g_2/12}\}$ are trivial critical points of G_2, and it was calculated in [**16**, Example 4.2] that the Hessian is given by

$$\det D^2G_2(q_\pm,-q_\pm;\tau)=\frac{3|g_2(\tau)|}{4\pi^4\operatorname{Im}\tau}|\wp(q_\pm|\tau)+\eta_1(\tau)|^2\operatorname{Im}\phi_\pm(\tau),\tag{5.1}$$

where $\phi_\pm(\tau)$ is defined in (3.29). Define the following *degeneracy curve* of G_2 on F_0 related to $(q_\pm,-q_\pm)$:[3]

$$L_{+,-}:=\left\{\tau\in F_0\setminus\{e^{\pi i/3}\}\;\middle|\;\begin{array}{c}\det D^2G_2(q_+,-q_+;\tau)=0\\ \text{or }\det D^2G_2(q_-,-q_-;\tau)=0\end{array}\right\}.$$

Theorem 5.1. *The degeneracy curve $L_{+,-}=\mathcal{C}_-\cup\mathcal{C}_0\cup\mathcal{C}_+$. In particular, $\mathcal{C}_-$, $\mathcal{C}_0$, $\mathcal{C}_+$ are all smooth curves.*

Proof. Remark that $\wp(q_\pm|\tau)+\eta_1(\tau)=\eta_1(\tau)\pm\sqrt{g_2(\tau)/12}=0$ implies $12\eta_1(\tau)^2-g_2(\tau)=0$. So it follows from (4.4) that $\eta_1(\tau)\pm\sqrt{g_2(\tau)/12}\neq 0$ for any $\tau\in F_0$. Consequently, we deduce from (5.1) that

$$\begin{aligned}L_{+,-}&=\left\{\tau\in F_0\setminus\{e^{\pi i/3}\}\,|\operatorname{Im}\phi_+(\tau)=0\text{ or }\operatorname{Im}\phi_-(\tau)=0\right\}\\&=\left\{\tau\in F_0\setminus\{e^{\pi i/3}\}\;\middle|\;\begin{array}{r}\phi_+(\tau)=C\text{ or }\phi_-(\tau)=C\\ \text{for some }C\in\mathbb{R}\end{array}\right\}\\&=\left\{\tau\in F_0\setminus\{e^{\pi i/3}\}\,|f_C(\tau)=0\text{ for some }C\in\mathbb{R}\right\}\\&=\{\tau(C)|C\in\mathbb{R}\setminus\{0,1\}\}=\mathcal{C}_-\cup\mathcal{C}_0\cup\mathcal{C}_+,\end{aligned}\tag{5.2}$$

where we have used Theorem 3.1 and Theorem 4.2 (iii)–(iv), which show that $e^{\pi i/3}\notin\mathcal{C}_-\cup\mathcal{C}_0\cup\mathcal{C}_+$.

Now we prove that the three curves are all smooth curves in F_0. Recalling (4.8) in Lemma 4.3, we restrict $\tau\in F_0\backslash\{\tau=\frac{1}{2}+ib|b\geq\frac{\sqrt{3}}{2}\}$ and

[3] $g_2(e^{\pi i/3})=0$ implies that the two trivial critical points $(q_\pm,-q_\pm)$ degenerate to one trivial critical point at $\tau=e^{\pi i/3}$, so we exclude the trivial case in the definition of $L_{+,-}$.

fix a branch of $\sqrt{g_2(\tau)/12}$. Then both $\phi_+(\tau)$ and $\phi_-(\tau)$ are *single-valued* in $F_0\backslash\{\tau=\frac{1}{2}+ib|b\geq\frac{\sqrt{3}}{2}\}$ and it follows from [**3**, Theorem 3.1] that

$$\phi_+'(\tau)\neq 0,\quad \phi_-'(\tau)\neq 0,\quad \forall\tau\in F_0\backslash\{\tau=\tfrac{1}{2}+ib|b\geq\tfrac{\sqrt{3}}{2}\}. \tag{5.3}$$

By (5.2), the same argument as Lemma 4.3 implies that either

$$\mathcal{C}_-\subset\{\tau\in F_0\backslash\{\tau=\tfrac{1}{2}+ib|b\geq\tfrac{\sqrt{3}}{2}\}|\operatorname{Im}\phi_+(\tau)=0\} \tag{5.4}$$

or

$$\mathcal{C}_-\subset\{\tau\in F_0\backslash\{\tau=\tfrac{1}{2}+ib|b\geq\tfrac{\sqrt{3}}{2}\}|\operatorname{Im}\phi_-(\tau)=0\}.$$

Say (5.4) holds for example. Write $\tau=a+bi$ with $a,b\in\mathbb{R}$. By (5.4), (5.3) and

$$\frac{\partial\operatorname{Im}\phi_+}{\partial a}=\operatorname{Im}\phi_+',\quad \frac{\partial\operatorname{Im}\phi_+}{\partial b}=\operatorname{Re}\phi_+',$$

we see that $\mathcal{C}_-$ is smooth at any $\tau\in\mathcal{C}_-$. Therefore, in both cases, we can apply (5.3) to conclude that $\mathcal{C}_-$ is a smooth curve. Then Theorem 4.2-(ii) shows that $\mathcal{C}_+$ is also a smooth curve. For $\mathcal{C}_0$, we note from Theorem 4.2-(iii) that $\mathcal{C}_0\subset F_0\backslash\{\tau=\frac{1}{2}+ib|b\leq\frac{\sqrt{3}}{2}\}$. Restrict $\tau\in F_0\backslash\{\tau=\frac{1}{2}+ib|b\leq\frac{\sqrt{3}}{2}\}$ and fix a branch of $\sqrt{g_2(\tau)/12}$. Again both $\phi_+(\tau)$ and $\phi_-(\tau)$ are *single-valued* in $F_0\backslash\{\tau=\frac{1}{2}+ib|b\leq\frac{\sqrt{3}}{2}\}$, so the same argument shows that $\mathcal{C}_0$ is a smooth curve. q.e.d.

The numerical simulation for the degeneracy curves of G_2 and hence the three smooth curves is shown in Figure 3, which is copied from C. L. Wang [**15**]. The other six curves appearing in Figure 3 are those degeneracy curves of G_2 at other trivial critical points $\{(\frac{1}{2}\omega_i,\frac{1}{2}\omega_j)|i\neq j\}$. In another paper, we will prove that under the $\Gamma_0(2)$ action, all critical points of $e_k(\tau)=\wp(\frac{\omega_k}{2}|\tau)$ will be mapped to locate on these six smooth curves. Figure 3 indicates that critical points of $E_2(\tau)$ and $e_k(\tau)$'s could be approximately computed via mathematical softwares such as Mathematica.

Appendix A. Application

In this appendix, we apply Theorem 1.6 back to the mean field equation (2.6). Define

$$L:=\{(r,s)\in\triangle_1\mid 2r+s=2\}.$$

By Theorem 1.6, for any $(r,s)\in L$, $Z_{r,s}^{(2)}(\tau)$ has a unique zero, denoted by τ_s, in F_0.

Theorem A.1. *For any $(r,s)\in L$, the unique zero τ_s of $Z_{r,s}^{(2)}(\tau)$ in F_0 satisfies $\tau_s=\frac{1}{2}+ib_s$ for some $b_s>\frac{1}{2}$. Furthermore, $\lim_{s\to\frac{1}{2}}b_s=+\infty$ and*

$$b^*:=\lim_{s\to 0}b_s\ \textit{exists and}\ \ b^*\in(\sqrt{3}/2,6/5). \tag{A.1}$$

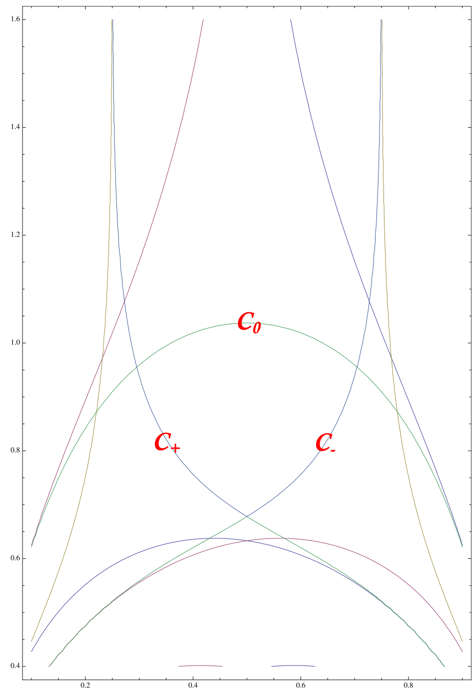

Figure 3. The smooth curves.

In particular, for any $\tau = \frac{1}{2}+ib$ with $b \in (b^*, +\infty)$, there exists $(r,s) \in L$ such that this τ is the unique zero of $Z_{r,s}^{(2)}$ in F_0.

Proof. Given any $(r,s) \in L$. By the definition of $\zeta(z|\tau)$ and $\wp(z|\tau)$, it is easy to prove

$$\overline{\zeta(z|\tau)} = \zeta(\bar{z}|1-\bar{\tau}),\ \overline{\wp(z|\tau)} = \wp(\bar{z}|1-\bar{\tau}),$$
$$\overline{\wp'(z|\tau)} = \wp'(\bar{z}|1-\bar{\tau}).$$

Thus

$$\begin{aligned}\overline{\eta_1(\tau)} &= 2\overline{\zeta(1/2|\tau)} = 2\zeta(1/2|1-\bar{\tau}) = \eta_1(1-\bar{\tau}),\\ \overline{\eta_2(\tau)} &= 2\overline{\zeta(\tau/2|\tau)} = 2\zeta(\bar{\tau}/2|1-\bar{\tau})\\ &= 2\zeta(1/2|1-\bar{\tau}) - 2\zeta((1-\bar{\tau})/2|1-\bar{\tau})\\ &= \eta_1(1-\bar{\tau}) - \eta_2(1-\bar{\tau}),\end{aligned}$$

i.e.

$$\begin{aligned}\overline{Z_{r,s}(\tau)} &= \zeta(r+s-s(1-\bar{\tau})|1-\bar{\tau}) - (r+s)\eta_1(1-\bar{\tau}) + s\eta_2(1-\bar{\tau})\\ &= Z_{r+s,-s}(1-\bar{\tau}).\end{aligned}$$

From here and

$$\overline{\wp(r+s\tau|\tau)} = \wp(r+s-s(1-\bar{\tau})|1-\bar{\tau}),$$

$$\overline{\wp'(r+s\tau|\tau)} = \wp'(r+s-s(1-\bar\tau)|1-\bar\tau),$$

we obtain

$$\begin{aligned}\overline{Z^{(2)}_{r,s}(\tau)} &= Z^{(2)}_{r+s,-s}(1-\bar\tau) = -Z^{(2)}_{-(r+s),s}(1-\bar\tau)\\ &= -Z^{(2)}_{2-r-s,s}(1-\bar\tau) = -Z^{(2)}_{r,s}(1-\bar\tau).\end{aligned}$$

Then $1-\overline{\tau_s}$ is also a zero of $Z^{(2)}_{r,s}(\tau)$ in F_0, so $1-\overline{\tau_s}=\tau_s$, i.e. $\tau_s=\frac{1}{2}+ib_s$ for some $b_s\in(\frac{1}{2},+\infty)$. Suppose by contradiction that, up to a sequence,

$$\lim_{s\to 1/2} b_s = b \in [1/2,+\infty).$$

Then $\frac{1}{2}+ib$ is a zero of $Z^{(2)}_{\frac{3}{4},\frac{1}{2}}(\tau)$ in F_0, which is a contradiction with Theorem 1.6 because $(\frac{3}{4},\frac{1}{2})\in\partial\triangle_1$. This proves $\lim_{s\to\frac{1}{2}} b_s=+\infty$. Note $Z^{(2)}_{r,s}(\tau)=Z^{(2)}_{1-\frac{1}{2}s,s}(\tau)=Z^{(2)}_{-\frac{1}{2}s,s}(\tau)$. To prove (A.1), we recall that the proof of Theorem 3.1-(1) shows that $\lim_{s\to 0}\tau_s$ exists and

$$\lim_{s\to 0}\tau_s = \tau(\tfrac{1}{2}),$$

where $\tau(\frac{1}{2})$ is the unique zero of $f_{\frac{1}{2}}(\tau)$ in F_0. Together with Theorem 4.2-(iii), we obtain $\lim_{s\to 0} b_s = \operatorname{Im}\tau(\frac{1}{2})\in(\frac{\sqrt{3}}{2},\frac{6}{5})$. This proves (A.1) and hence completes the proof. q.e.d.

The following result, which was announced in [**4**], give new existence results for the mean field equation (2.6) when E_τ is a rhombus torus.

Theorem A.2. *Let $\tau=\frac{1}{2}+ib$ with $b>b^*$, where $b^*\in(\frac{\sqrt{3}}{2},\frac{6}{5})$ is in Theorem A.1. Then equation (2.6) on E_τ has a solution.*

Proof. This theorem is an immediate consequence of Theorem A.1 and Theorem B-(1). q.e.d.

Remark that Theorem A.2 is almost optimal in the sense of Theorem B-(2), which says that if $\tau=\frac{1}{2}+\frac{\sqrt{3}}{2}i$, then equation (2.6) on E_τ has *no* solutions.

References

[1] Y. V. Brezhnev; *Non-canonical extension of ϑ-functions and modular integrability of ϑ-constants.* Proc. Roy. Soc. Edinburgh Sect. A **143** (2013), no. 4, 689–738, MR3082297, Zbl 1318.33034.

[2] C.L. Chai, C.S. Lin and C.L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: I.* Cambridge Journal of Mathematics **3** (2015), 127–274, MR3356357, Zbl 1327.35116.

[3] Z. Chen, T.J. Kuo and C.S. Lin; *Simple zero property of some holomorphic functions on the moduli space of tori.* Science China Mathematics, published online. arXiv:1703.05521v1 [math. CV].

[4] Z. Chen, T.J. Kuo and C.S. Lin; *Non-existence of solutions for a mean field equation on flat tori at critical parameter* 16π. Comm. Anal. Geom. to appear. arXiv:1610.01787v2 [math. AP].

[5] Z. Chen, T.J. Kuo, C.S. Lin and K. Takemura; *On reducible monodromy representations of some generalized Lamé equation*. Math. Z. **288** (2018), 679–688, MR3778971, Zbl 1395.34087.

[6] Z. Chen, T.J. Kuo, C.S. Lin and C.L. Wang; *Green function, Painlevé VI equation and Eisenstein series of weight one*. J. Differ. Geom. **108** (2018), 185–241, MR3763067, Zbl 1390.34242.

[7] S. Dahmen; *Counting integral Lamé equations with finite monodromy by means of modular forms*. Master Thesis, Utrecht University, 2003.

[8] F. Diamond and J. Shurman; *A First Course in Modular Forms*. Springer-Verlag UTM vol. **228**, New York, 2005, MR2112196, Zbl 1062.11022.

[9] A. El Basraoui and A. Sebbar; *Zeros of the Eisenstein series E_2*. Proc. Amer. Math. Soc. **138** (2010), 2289–2299, MR2607858, Zbl 1206.11050.

[10] A. Eremenko and A. Gabrielov; *Spherical Rectangles*. Arnold Math. J. **2** (2016), 463–486, MR3564884, Zbl 1365.30006.

[11] E. Hecke; *Zur Theorie der elliptischen Modulfunctionen*. Math. Ann. **97** (1926), 210–242.

[12] S. Lang; *Elliptic Functions*. Graduate Text in Mathematics **112**, Springer-Verlag 1987.

[13] C.S. Lin; *Green function, mean field equation and Painlevé VI equation*. Current Developments in Mathematics, 2015, 137–188, MR3642545, Zbl 06751632.

[14] C.S. Lin and C.L. Wang; *Elliptic functions, Green functions and the mean field equations on tori*. Annals of Math. **172** (2010), no. 2, 911–954, MR2680484, Zbl 1207.35011.

[15] C.S. Lin and C.L. Wang; *Mean field equations, Hyperelliptic curves, and Modular forms: II*. J. Éc. polytech. Math. **4** (2017), 557–593, MR3665608, Zbl 06754336.

[16] C.S. Lin and C.L. Wang; *Geometric quantities arising from bubbling analysis of mean field equations*. arXiv:1609.07204v1 [math. AP] 2016.

[17] J.P. Serre; *A course in Arithmetic*. Springer-Verlag, 1973.

[18] R. Wood and M. Young; *Zeros of the weight two Eisenstein series*. J. Number Theory **143** (2014), 320–333, MR3227351, Zbl 1296.11026.

Department of Mathematical Sciences
Yau Mathematical Sciences Center
Tsinghua University
Beijing, 100084
China

E-mail address: zjchen2016@tsinghua.edu.cn

Taida Institute for Mathematical Sciences (TIMS)
Center for Advanced Study in Theoretical Sciences (CASTS)
National Taiwan University
Taipei 10617
Taiwan

E-mail address: cslin@math.ntu.edu.tw