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STABLE BLOWUP FOR THE SUPERCRITICAL YANG-MILLS HEAT FLOW

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Abstract

In this paper, we consider the heat flow for Yang-Mills connections on $\mathbb{R}^5 \times SO(5)$. In the $SO(5)$-equivariant setting, the Yang-Mills heat equation reduces to a single semilinear reaction-diffusion equation for which an explicit self-similar blowup solution was found by Weinkove [**31**]. We prove the nonlinear asymptotic stability of this solution under small perturbations. In particular, we show that there exists an open set of initial conditions in a suitable topology such that the corresponding solutions blow up in finite time and converge to a non-trivial self-similar blowup profile on an unbounded domain. Convergence is obtained in suitable Sobolev norms and in L^∞.

1. Introduction

1.1. Equivariant Yang-Mills connections on $\mathbb{R}^d \times SO(d)$. For $\mu = 1, \dots, d$, we consider mappings $A_\mu : \mathbb{R}^d \to \mathfrak{so}(d)$, where $\mathfrak{so}(d)$ denotes the Lie algebra of the Lie group $SO(d)$, i.e., $\mathfrak{so}(d)$ can be considered as the set of skew-symmetric $(d \times d)$-matrices endowed with the commutator bracket. In the following, Einstein's summation convention is in force. For

$$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu],$$

the Yang-Mills functional is defined as

$$\mathcal{F}_A = \int_{\mathbb{R}^d} \operatorname{tr}(F_{\mu\nu} F^{\mu\nu}). \tag{1.1}$$

The Euler-Lagrange equations associated to this functional are given by

$$\partial_\mu F^{\mu\nu}(x) + [A_\mu(x), F^{\mu\nu}(x)] = 0 \tag{1.2}$$

and solutions are referred to as Yang-Mills connections. We note that there is still a gauge freedom in this equation which, however, is of no relevance here since we will only consider the equivariant case. A standard approach to find solutions to Eq. (1.2) is to add an artificial time

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dependence to the model which implies that Yang-Mills connections are static solutions to the corresponding heat flow equation

$$\partial_t A_\mu(t,x) + \partial^\nu F_{\mu\nu}(t,x) + [A^\nu(t,x), F_{\mu\nu}(t,x)] = 0, \quad t > 0, \tag{1.3}$$

for some initial condition $A_\mu(0,x) = A_{0\mu}(x)$. This model is usually referred to as the *Yang-Mills heat flow* for connections on the trivial bundle $\mathbb{R}^d \times SO(d)$. The equation enjoys scale invariance in the sense that if A_μ is a solution, then, for any $\lambda > 0$,

$$A_\mu^\lambda(t,x) := \lambda A_\mu(\lambda^2 t, \lambda x),$$

solves Eq. (1.3) with initial condition $A_\mu^\lambda(0,x) = \lambda A_{0\mu}(\lambda x)$. Since the Yang-Mills functional in four space dimensions is invariant under this scaling transformation, the model is referred to as *critical* for $d = 4$. Consequently, the Yang-Mills equation is *supercritical* for $d \geq 5$. To simplify matters, we consider $SO(d)$-equivariant connections of the form

$$A_\mu^{ij}(t,x) = u(t,|x|)\ \sigma_\mu^{ij}(x),$$

where $\sigma_\mu^{ij}(x) = \delta_\mu^i x^j - \delta_\mu^j x^i$, for $i,j = 1,\dots,d$, cf. for example [**16**], [**14**] and the references therein. This ansatz reduces Eq. (1.3) to a single equation for a radial function $u : [0,\infty) \times [0,\infty) \to \mathbb{R}$, given by

$$\begin{aligned} \partial_t u(t,r) - \partial_r^2 u(t,r) - \frac{d+1}{r}\partial_r u(t,r) \\ + 3(d-2)u^2(t,r) + (d-2)r^2 u^3(t,r) = 0, \end{aligned} \tag{1.4}$$

see e.g. [**14**] for a detailed derivation. It was shown by Grotowski [**16**] that this symmetry is preserved by the flow. Hence, for equivariant initial data $A_{0\mu}(x) = u_0(|x|)\ \sigma_\mu^{ij}(x)$ it suffices to consider Eq. (1.4) with $u(0,r) = u_0(r)$. The scale invariance of Eq. (1.3) implies that Eq. (1.4) is invariant under the transformation $u \mapsto u_\lambda$, where

$$u_\lambda(t,r) = \lambda^2 u(\lambda^2 t, \lambda r)$$

and $\lambda > 0$. It was shown by Gastel [**14**] that self-similar blowup solutions to Eq. (1.4) exist in dimensions $5 \leq d \leq 9$. Explicit examples were found by Weinkove [**31**] and they are of the form

$$\mathbf{w}_T(t,r) = \tfrac{1}{T-t}\mathbf{W}\left(\tfrac{r}{\sqrt{T-t}}\right), \quad \mathbf{W}(\rho) = -\frac{1}{a_1(d)\rho^2 + a_2(d)}, \tag{1.5}$$

for some $T > 0$, with constants

$$a_1(d) = \tfrac{\sqrt{d-2}}{2\sqrt{2}}, \quad a_2(d) = \tfrac{1}{2}(6d - 12 - (d+2)\sqrt{2d-4}),$$

for $5 \leq d \leq 9$. By setting $\mathbf{A}_{\mu,T}(t,x) = \mathbf{w}_T(t,|x|)\ \sigma_\mu(x)$, a one-parameter family of blowup solutions for Eq. (1.3) is obtained. These solutions are invariant under the natural scaling of the equation (up to a change of the blowup time) and obviously, they blow up in L^∞ as $t \to T^-$. In this paper, we address the stability of these solutions under equivariant perturbations in five space dimensions. The main results are summarized below.

1.2. Stable self-similar blowup for $d = 5$. We consider the equation

$$\partial_t u(t,r) = \partial_r^2 u(t,r) + \frac{6}{r}\partial_r u(t,r) - 9u^2(t,r) - 3r^2u^3(t,r), \tag{1.6}$$

for $r \in [0,\infty)$ and $t > 0$, subject to the initial condition $u(0,\cdot) = u_0$. We set $\mathcal{E} = C^\infty_{\mathrm{e},0}(\mathbb{R})$, where

$$C^\infty_{\mathrm{e},0}(\mathbb{R}) := \{u \in C^\infty_0(\mathbb{R}) : u(x) = u(-x)\}. \tag{1.7}$$

On $\mathcal{E}$ a norm is defined by

$$\|u\|^2_{\mathcal{E}} := \|u\|^2_1 + \|u\|^2_2,$$

where

$$\|u\|^2_1 := \int_0^\infty |r^3u''(r) + 6r^2u'(r)|^2 dr,$$

$$\|u\|^2_2 := \int_0^\infty |r^3u^{(4)}(r) + 12r^2u^{(3)}(r) + 24ru''(r) - 24u'(r)|^2 dr.$$

The following result shows that the blowup described by $\mathbf{A}_{\mu,T}$ is stable under equivariant perturbations.

Theorem 1.1. *Fix $T_0 > 0$. There are constants $\delta, K > 0$ such that for all real-valued functions $u_0 \in \mathcal{E}$ satisfying $\|u_0 - \mathbf{w}_{T_0}(0,\cdot)\|_{\mathcal{E}} \le \frac{\delta}{K}$, the following holds: There is a $T = T(u_0) \in [T_0 - \delta, T_0 + \delta]$ such that a unique classical solution $u(t,\cdot)$ to Eq. (1.6) exists for all $t \in (0,T)$ with $u(0,\cdot) = u_0$. At $t = T$, the solution blows up at the origin and converges to $\mathbf{w}_T$ according to*

$$\frac{\|u(t,\cdot) - \mathbf{w}_T(t,\cdot)\|_1}{\|\mathbf{w}_T(t,\cdot)\|_1} \lesssim (T-t)^{\frac{1}{150}}, \quad \frac{\|u(t,\cdot) - \mathbf{w}_T(t,\cdot)\|_2}{\|\mathbf{w}_T(t,\cdot)\|_2} \lesssim (T-t)^{\frac{1}{150}}. \tag{1.8}$$

Furthermore, we have convergence in $L^\infty(\mathbb{R}^+)$, i.e.,

$$\frac{\|u(t,\cdot) - \mathbf{w}_T(t,\cdot)\|_{L^\infty(\mathbb{R}^+)}}{\|\mathbf{w}_T(t,\cdot)\|_{L^\infty(\mathbb{R}^+)}} \lesssim (T-t)^{\frac{1}{150}}.$$

Remark 1.2. (i) Although the problem is posed on $\mathbb{R}^d$, $d = 5$, the effective dimension of the Laplacian in Eq. (1.4) is $d + 2$. From a mathematical point of view it is therefore reasonable to consider u as a radial function on $\mathbb{R}^7$. Obviously,

$$\|u\|_1 \simeq \|\Delta u(|\cdot|)\|_{L^2(\mathbb{R}^7)}, \quad \|u\|_2 \simeq \|\Delta^2 u(|\cdot|)\|_{L^2(\mathbb{R}^7)}.$$

(ii) In all of the above bounds the left hand side is normalized to the behavior of $\mathbf{w}_T$ in the respective norm. However, the given rate of convergence is not sharp.

(iii) The smoothness assumptions on the initial data seem to be quite restrictive. In fact, the result holds for a much larger class of initial conditions, see Section 5.6. A more general version for solutions that satisfy the equation in a suitable weak sense is given in Theorem 1.3 below.

(iv) The restriction to $d = 5$ is by no means crucial and our techniques easily extend to the cases $d = 7$ and $d = 9$. For the sake of simplicity, however, we only consider the case $d = 5$.

We note that our approach is very robust since we do not make use of any Lyapunov functionals or monotonicity formulas. Instead, we rewrite Eq. (1.4) in similarity coordinates and study perturbations around $\mathbf{w}_T$ by means of strongly continuous semigroups, operator theory and spectral analysis.

1.3. Known results. The study of Yang-Mills (YM) functionals in mathematics was initiated in the 1980's and triggered profound developments in differential geometry, see [**6**]. The corresponding heat flow equation has been studied extensively in various geometric settings, see for example the recent monograph by Feehan [**13**]. For an $SU(2)$-bundle over the unit ball in $\mathbb{R}^4$, global existence of smooth solutions was established by Schlatter, Struwe and Tahvildar-Zadeh [**30**] in the equivariant setting. In supercritical dimensions, blow up in finite time is due to Naito [**24**], see also Grotowski [**16**] and Gastel [**14**] for the case of a trivial $SO(d)$–bundle over $\mathbb{R}^d$. Weinkove [**31**] investigated the nature of singularities under certain assumptions on the blowup rate and it was found that locally around the blowup point solutions converge in a suitable sense to homothetically shrinking solitons, also referred to as *YM-solitons*. These objects correspond to solutions of the YM heat flow on the trivial bundle over $\mathbb{R}^d$, which is our main motivation to study the problem in this geometrical setting. Moreover, Weinkove gave explicit examples of such solitons on $\mathbb{R}^d \times SO(d)$, $5 \leq d \leq 9$, see Eq. (1.5). Very recently, a description of general blowup solutions for the YM heat flow over closed Riemannian manifolds was obtained by Kelleher and Streets [**22**] for $d \geq 4$ and it was shown that singularities can be described either by suitably rescaled Yang-Mills connections, i.e., by static solutions, or by YM-solitons. The results of Weinkove have raised interest in the stability of YM-solitons in recent years, see e.g. Kelleher and Streets [**21**], Chen and Zhang [**4**]. However, to the best of our knowledge no rigorous proof on the stability of the Weinkove solution given in Eq. (1.5) has been obtained so far and Theorem 1.1 is the first result in this direction. We note that in higher dimensions, $d \geq 10$, the existence of self-similar solutions to Eq. (1.4) was excluded by Bizoń and Wasserman [**2**].

1.4. Related problems. The above model bears many similarities with the heat flow for co-rotational harmonic maps from $\mathbb{R}^d \to \mathbb{S}^d$, which is supercritical for $d \geq 3$. For $3 \leq d \leq 6$, Fan [**12**] has constructed a family of self-similar blowup solutions. In analogy to the YM heat flow, self-similar blowup is excluded for $d \geq 7$, see [**2**]. In this regime, Biernat and Seki [**1**] have recently constructed explicit examples for type II blowup solutions. It is most likely that a similar result for the YM heat flow can be obtained in dimensions $d \geq 10$. The problem of non-uniqueness of weak solutions in the supercritical case has recently been addressed by Germain, Ghoul and Miura [**15**]. In the critical case $d = 2$, finite-time blowup of solutions has been proved by Chang and Ding [**3**] which contrasts the result of [**30**] for the YM heat flow, see also [**17**]. Stable type II blowup for $d = 2$ is due to Raphaël and Schweyer [**27**], [**28**].

1.5. Formulation of the problem in similarity coordinates. We fix $T_0 > 0$ and write the initial condition as

$$u(0, r) = \mathbf{w}_{T_0}(0, r) + v_0(r), \tag{1.9}$$

for $r \in [0, \infty)$, where $\mathbf{w}_{T_0}$ denotes the self-similar blowup solution in Eq. (1.5) with fixed blowup time $t = T_0$. We introduce similarity coordinates $(\tau, \rho) \in [0, \infty) \times [0, \infty)$ defined by

$$\tau = -\log(T - t) + \log T, \quad \rho = \frac{r}{\sqrt{T - t}},$$

for $T > 0$, which enters the analysis as a free parameter that will be fixed only at the very end of the argument. By setting

$$\psi(-\log(T - t) + \log T, \tfrac{r}{\sqrt{T-t}}) := (T - t)u(t, r),$$

the initial value problem given by Eq. (1.6) and Eq. (1.9) transforms into

$$\begin{aligned} \partial_\tau \psi(\tau, \rho) = \partial_\rho^2 \psi(\tau, \rho) + \frac{6}{\rho} \partial_\rho \psi(\tau, \rho) - \frac{1}{2} \rho \partial_\rho \psi(\tau, \rho) \\ - \psi(\tau, \rho) - 9\psi^2(\tau, \rho) - 3\rho^2 \psi^3(\tau, \rho) \end{aligned} \tag{1.10}$$

for $\tau > 0$ with initial condition

$$\psi(0, \rho) = Tu(0, \sqrt{T}\rho) = \tfrac{T}{T_0} \mathbf{W}\left(\tfrac{\sqrt{T}}{\sqrt{T_0}}\rho\right) + Tv_0(\sqrt{T}\rho).$$

The Weinkove solution

$$\mathbf{W}(\rho) = -\frac{1}{a_1 \rho^2 + a_2}, \qquad a_1 = \frac{1}{2}\sqrt{\frac{3}{2}}, \quad a_2 = \frac{1}{2}(18 - 7\sqrt{6}), \tag{1.11}$$

is a static solution to Eq. (1.10). The differential operator on the right hand side of Eq. (1.10) has a natural extension to $\mathbb{R}^7$. In fact, Eq. (1.10) can be written as

$$\begin{aligned} \partial_\tau \Psi(\tau, \xi) =& \Delta \Psi(\tau, \xi) - \tfrac{1}{2} \xi \cdot \nabla \Psi(\tau, \xi) \\ & - \Psi(\tau, \xi) - 9\Psi^2(\tau, \xi) - 3|\xi|^2 \Psi^3(\tau, \xi) \end{aligned} \tag{1.12}$$

for $\tau > 0$, $\xi \in \mathbb{R}^7$ and a radial function $\Psi(\tau,\xi) = \psi(\tau,|\xi|)$. Hence, we study Eq. (1.12) with initial data of the form

$$\Psi(0,\xi) = \tfrac{T}{T_0}\mathbf{W}\left(\tfrac{\sqrt{T}}{\sqrt{T_0}}|\xi|\right) + Tv_0(\sqrt{T}|\xi|).$$

In a more abstract way, the problem can be formulated as

$$\frac{d}{d\tau}\Psi(\tau) = L_0\Psi(\tau) + F(\Psi(\tau)), \quad \tau > 0, \tag{1.13}$$

with $\Psi(0) = \Psi_0^T$, where L_0 represents the linear differential operator on the right hand side of Eq. (1.12) and F denotes the nonlinearity. In the following, we study Eq. (1.13) on a Hilbert space $(\mathcal{H}, \|\cdot\|)$ defined as the completion of the set of radial, compactly supported functions on $\mathbb{R}^7$ with respect to the norm

$$\|u\|^2 = \|\Delta u\|^2_{L^2(\mathbb{R}^7)} + \|\Delta^2 u\|^2_{L^2(\mathbb{R}^7)},$$

see Section 2 for details. In this setting, L_0 has a realization as an unbounded, closed operator on a suitable domain $\mathcal{D}(L_0) \subset \mathcal{H}$. In order to study small perturbations of $\mathbf{W}$, we insert the ansatz $\Psi(\tau) = \mathbf{W} + \Phi(\tau)$, which yields

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= (L_0 + L')\Phi(\tau) + N(\Phi(\tau)), \quad \tau > 0, \\ \Phi(0) &= U(v_0, T). \end{aligned} \tag{1.14}$$

Here, $L'u = V(|\cdot|)u$ is a linear perturbation with

$$V(\rho) = -18\mathbf{W}(\rho) - 9\rho^2\mathbf{W}(\rho)^2 = \frac{72(36 - 14\sqrt{6} + (\sqrt{6}-2)\rho^2)}{(36 - 14\sqrt{6} + \sqrt{6}\rho^2)^2} \tag{1.15}$$

for $\rho \in [0,\infty)$ and N denotes the remaining nonlinearity. A short calculation shows that

$$N(\Phi(\tau)) = -9\big[1 + |\cdot|^2\mathbf{W}(|\cdot|)\big]\Phi(\tau)^2 - 3|\cdot|^2\Phi(\tau)^3.$$

Furthermore,

$$U(v_0, T) := Tv_0(\sqrt{T}|\cdot|) + \tfrac{T}{T_0}\mathbf{W}\left(\tfrac{\sqrt{T}}{\sqrt{T_0}}|\cdot|\right) - \mathbf{W}(|\cdot|)$$

denotes the transformed initial condition. In Section 3 we show that the operator $L := L_0 + L'$, equipped with a suitable domain, generates a strongly continuous one-parameter semigroup $\{S(\tau) : \tau \geq 0\}$ of bounded linear operators on $\mathcal{H}$. For general initial conditions in $\mathcal{H}$ we do not expect to obtain classical solutions to Eq. (1.14). Thus, we look for mild solutions $\Phi \in C([0,\infty), \mathcal{H})$ that satisfy the integral equation

$$\Phi(\tau) = S(\tau)U(v_0, T) + \int_0^\tau S(\tau - \tau')N(\Phi(\tau'))d\tau' \tag{1.16}$$

for $\tau \geq 0$. With these preliminaries we can state the following theorem.

Theorem 1.3. *Fix $T_0 > 0$. Let $M > 0$ be sufficiently large and $\delta > 0$ sufficiently small. For every $v_0 \in \mathcal{H}$ with $\|v_0\| \leq \frac{\delta}{M^2}$, there exists a $T = T_{v_0} \in [T_0 - \frac{\delta}{M}, T_0 + \frac{\delta}{M}]$ and a function $\Phi \in C([0,\infty),\mathcal{H})$ that satisfies Eq. (1.16) for all $\tau \geq 0$. Furthermore,*

$$\|\Phi(\tau)\| \lesssim e^{-\frac{1}{150}\tau}, \quad \forall \tau \geq 0.$$

Sections 2–5 are mainly devoted to the proof of Theorem 1.3. In Section 5.6 we show that Theorem 1.3 implies Theorem 1.1.

1.6. Notation and conventions. We write $\mathbb{N}$ for the natural numbers $\{1,2,3,\dots\}$, $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Furthermore, $\mathbb{R}^+ := \{x \in \mathbb{R} : x > 0\}$. The notation $a \lesssim b$ means $a \leq Cb$ for an absolute constant $C > 0$ and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$. If $a \leq C_\varepsilon b$ for a constant $C_\varepsilon > 0$ depending on some parameter ε, we write $a \lesssim_\varepsilon b$. We use the common notation $\langle x \rangle := \sqrt{1+|x|^2}$ also known as the *Japanese bracket*. For a function $x \mapsto g(x)$, we denote by $g^{(n)}(x) = \frac{d^n g(x)}{dx^n}$ the derivatives of order $n \in \mathbb{N}$. For $n = 1, 2$, we also write $g'(x)$ and $g''(x)$, respectively. For a function $(x,y) \mapsto f(x,y)$, partial derivatives of order n will be denoted by $\partial_x^n f(x,y) = \frac{\partial^n}{\partial x^n} f(x,y)$. Throughout the paper, $W(f,g)$ denotes the Wronskian of two functions $x \mapsto f(x)$ and $x \mapsto g(x)$, where we use the convention $W(f,g) = fg' - f'g$. By $C_0^\infty(\mathbb{R}^d)$ we denote the set of compactly supported functions on $\mathbb{R}^d$, $d \geq 1$. The spaces $L^2(\mathbb{R}^d)$ and $H^k(\mathbb{R}^d)$, $k \in \mathbb{N}_0$, denote the standard Lebesgue and Sobolev spaces with the usual norm

$$\|u\|^2_{H^k(\mathbb{R}^d)} := \sum_{\alpha : |\alpha| \leq k} \|\partial^\alpha u\|^2_{L^2(\mathbb{R}^d)}.$$

The set of bounded linear operators on a Hilbert space $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. For a closed linear operator $(L, \mathcal{D}(L))$, we write $\sigma(L)$ for the spectrum. The resolvent set is defined as $\rho(L) := \mathbb{C} \setminus \sigma(L)$ and we write $R_L(\lambda) := (\lambda - L)^{-1}$ for $\lambda \in \rho(L)$.

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2. Function spaces

On $\mathbb{R}^7$, we introduce the set of radial test functions,

$$C^\infty_{\mathrm{rad},0}(\mathbb{R}^7) := \{\tilde{u} \in C^\infty_0(\mathbb{R}^7) : \tilde{u} \text{ is radial}\},$$

and define $C^\infty_{\mathrm{e},0}(\mathbb{R})$, the set of even test functions on $\mathbb{R}$, as in Eq. (1.7). Note that if $u \in C^\infty_{\mathrm{e},0}(\mathbb{R})$, then $u^{(2k+1)}(0) = 0$ for all $k \in \mathbb{N}_0$. Furthermore, every $u \in C^\infty_{\mathrm{e},0}(\mathbb{R})$ defines a function $\tilde{u} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ by $\tilde{u}(\xi) := u(|\xi|)$. Conversely, if $\tilde{u} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$, then $\tilde{u}(\xi) = u(|\xi|)$ for some $u \in C^\infty_{\mathrm{e},0}(\mathbb{R})$. In the following, we set

$$L^2_{\mathrm{rad}}(\mathbb{R}^7) := \{\tilde{u} \in L^2(\mathbb{R}^7) : \tilde{u} \text{ is radial}\}.$$

For $\tilde{u} \in L^2_{\mathrm{rad}}(\mathbb{R}^7)$, we have $\tilde{u}(\xi) = u(|\xi|)$ a.e. on $\mathbb{R}^7$, for some $u \in L^2(\mathbb{R})$, satisfying $u(x) = u(-x)$ a.e. on $\mathbb{R}$. By using polar coordinates (ρ, ω), $\rho = |\xi|, \omega = \frac{\xi}{|\xi|}$, the inner product on $L^2_{\mathrm{rad}}(\mathbb{R}^7)$ can be written as

$$(\tilde{u}|\tilde{v})_{L^2(\mathbb{R}^7)} = \int_{\mathbb{R}^7} u(|\xi|)\overline{v(|\xi|)}d\xi = C\int_0^\infty u(\rho)\overline{v(\rho)}\rho^6 d\rho,$$

where the constant comes from the integration over $\mathbb{S}^6$. Furthermore, we consider Sobolev spaces

$$H^k_{\mathrm{rad}}(\mathbb{R}^7) := \{\tilde{u} \in H^k(\mathbb{R}^7) : \tilde{u} \text{ is radial}\},$$

for $k \in \mathbb{N}$, where the norm on H^k is defined in the usual manner. For $\tilde{u} \in H^{2j}_{\mathrm{rad}}(\mathbb{R}^7)$, $j = 1, 2$, the operators Δ^j exist in the weak sense, i.e., there are functions $\tilde{f}_j \in L^2_{\mathrm{rad}}(\mathbb{R}^7)$ such that

$$\int_{\mathbb{R}^7} \tilde{u}(\xi)\Delta^j\tilde{\phi}(\xi)d\xi = \int_{\mathbb{R}^7} \tilde{f}_j(\xi)\tilde{\phi}(\xi)d\xi \tag{2.1}$$

for all $\tilde{\phi} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ (this is not a restriction since for non-radial test functions, Eq. (2.1) can be recast by introducing polar coordinates and writing $\tilde{\phi}$ as a spherical mean). For the radial Laplace operator, we use the notation

$$\Delta_{\mathrm{rad}}u(\rho) = u''(\rho) + \tfrac{6}{\rho}u'(\rho).$$

2.1. Definition of the Hilbert space $\mathcal{H}$.

On $C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ we introduce the inner product

$$(\tilde{u}|\tilde{v}) := (\Delta\tilde{u}|\Delta\tilde{v})_{L^2(\mathbb{R}^7)} + (\Delta^2\tilde{u}|\Delta^2\tilde{v})_{L^2(\mathbb{R}^7)},$$

and a norm $\|\tilde{u}\| := \sqrt{(\tilde{u}|\tilde{u})}$.

Lemma 2.1. *Let $\tilde{u} = u(|\cdot|) \in C^4_{\mathrm{rad}}(\mathbb{R}^7)$, such that $\|\tilde{u}\| < \infty$ and*

$$\lim_{\rho\to\infty} \rho^3|\Delta_{\mathrm{rad}}u(\rho)| = 0, \quad \lim_{\rho\to\infty} \rho^3|(\Delta_{\mathrm{rad}}u)'(\rho)| = 0.$$

Then we have the bound

$$\|\nabla\Delta\tilde{u}\|_{L^2(\mathbb{R}^7)} \lesssim \|\tilde{u}\|.$$

If in addition $\lim_{\rho\to\infty}\rho^{\frac{3}{2}}|u(\rho)| = 0$ *and* $\lim_{\rho\to\infty}\rho^{\frac{5}{2}}|u'(\rho)| = 0$, *then*

$$\|(\cdot)^\alpha u\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [0,1],$$
$$\|(\cdot)^\alpha u'\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [0,2],$$
$$\|(\cdot)^\alpha u''\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [1,3],$$
$$\|(\cdot)^\alpha u^{(3)}\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [2,3],$$

as well as $\|(\cdot)^3 u^{(4)}\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\|$. *Furthermore,*

$$\|(\cdot)^\alpha u\|_{L^\infty(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [0,\tfrac{3}{2}],$$
$$\|(\cdot)^\alpha u'\|_{L^\infty(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [1,\tfrac{5}{2}],$$
$$\|(\cdot)^\alpha \Delta_{\text{rad}} u\|_{L^\infty(\mathbb{R}^+)} \lesssim \|\tilde{u}\| \text{ for } \alpha \in [2,3],$$
$$\|(\cdot)^3 (\Delta_{\text{rad}} u)'\|_{L^\infty(\mathbb{R}^+)} \lesssim \|\tilde{u}\|.$$

Proof. By scaling, it is natural to expect the bounds

$$\sum_{j=1}^{4}\|(\cdot)^{j-1}u^{(j)}\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\|_{\dot{H}^4(\mathbb{R}^7)}, \quad \sum_{j=0}^{2}\|(\cdot)^{j+1}u^{(j)}\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{u}\|_{\dot{H}^2(\mathbb{R}^7)}$$

and Hardy's inequality shows that these estimates are indeed correct. Based on this, the stated assertions follow easily by interpolation and Sobolev embedding. A self-contained and elementary proof including all details can be found in the arXiv preprint version [**10**] of this article. The same applies to the next Lemma, which is a result of the standard construction of the completion. q.e.d.

Lemma 2.2. *Let* $\mathcal{H}$ *denote the completion of* $(C^\infty_{\text{rad},0}(\mathbb{R}^7), \|\cdot\|)$. *Then* $\mathcal{H}$ *is a Hilbert space and its elements can be identified with functions* $\tilde{u} = u(|\cdot|) \in C_{\text{rad}}(\mathbb{R}^7) \cap C^3_{\text{rad}}(\mathbb{R}^7 \setminus \{0\})$, *that satisfy*

$$\lim_{\rho\to\infty}\rho^{\frac{3}{2}}|u(\rho)| = 0, \quad \lim_{\rho\to\infty}\rho^{\frac{5}{2}}|u'(\rho)| = 0.$$

The norm induced by the inner product on $\mathcal{H}$ *is given by*

$$\|\tilde{u}\|^2 = \|\Delta\tilde{u}\|^2_{L^2(\mathbb{R}^7)} + \|\Delta^2\tilde{u}\|^2_{L^2(\mathbb{R}^7)},$$

where $\Delta\tilde{u}$ *can be interpreted as a classical differential operator and* $\Delta^2\tilde{u}$ *has to be understood in a weak sense, cf. Eq.* (2.1). *Finally, for all* $\tilde{u} \in \mathcal{H}$,

$$\|\nabla\Delta\tilde{u}\|_{L^2(\mathbb{R}^7)} \lesssim \|\tilde{u}\|, \quad \|u\|_{L^\infty(\mathbb{R}^+)} \lesssim \|\tilde{u}\|.$$

3. Semigroup theory and spectral analysis

3.1. The Ornstein-Uhlenbeck operator on $\mathcal{H}$. We set

$$\Lambda\tilde u(\xi) := \frac{1}{2}\xi\cdot\nabla\tilde u(\xi),$$

for $\xi \in \mathbb{R}^7$ and define the formal differential expression

$$\mathcal{L}_0\tilde u(\xi) := \Delta\tilde u(\xi) - \Lambda\tilde u(\xi) - \tilde u(\xi).$$

In polar coordinates, $\mathcal{L}_0$ decouples into a radial and an angular part. In particular, for $\tilde u = u(|\cdot|) \in C^\infty_{\mathrm{rad}}(\mathbb{R}^7)$, $\mathcal{L}_0\tilde u(\cdot) = \tilde\ell_0 u(|\cdot|)$, where

$$\tilde\ell_0 u(\rho) = \Delta_{\mathrm{rad}}u(\rho) - \tfrac{1}{2}\rho u'(\rho) - u(\rho).$$

We define $\tilde L_0\tilde u := \mathcal{L}_0\tilde u$ on the domain

$$\mathcal{D}(\tilde L_0) := \Big\{\tilde u = u(|\cdot|) \in \mathcal{H}\cap C^6_{\mathrm{rad}}(\mathbb{R}^7) : \Delta^3\tilde u \in L^2(\mathbb{R}^7), \mathcal{L}_0\tilde u \in \mathcal{H},$$
$$\lim_{\rho\to\infty}\rho^3|\Delta^j_{\mathrm{rad}}u(\rho)| = 0,\ \lim_{\rho\to\infty}\rho^3|(\Delta^j_{\mathrm{rad}}u)'(\rho)| = 0,\ \text{for } j = 1,2\Big\}.$$

The operator $\tilde L_0$ is densely defined since $C^\infty_{\mathrm{rad},0}(\mathbb{R}^7) \subset \mathcal{D}(\tilde L_0)$. Note that functions in $\mathcal{D}(\tilde L_0)$ satisfy the assumptions of Lemma 2.1 by Lemma 2.2. We need the following result, which is based on [**23**].

Lemma 3.1. *Consider the operator* $(\Lambda, \mathcal{D}(\Lambda))$*, with*

$$\mathcal{D}(\Lambda) = \{\tilde u \in L^2(\mathbb{R}^7) : \Lambda\tilde u \in L^2(\mathbb{R}^7)\},$$

where $\Lambda\tilde u$ *is understood in the sense of distributions. Then, for all* $\tilde u \in \mathcal{D}(\Lambda)$*,*

$$\mathrm{Re}(-\Lambda\tilde u|\tilde u)_{L^2(\mathbb{R}^7)} \leq \frac{7}{4}\|\tilde u\|^2_{L^2(\mathbb{R}^7)}. \tag{3.1}$$

Proof. As defined above, the operator is closed by Lemma 2.1 in [**23**]. Furthermore, $C^\infty_0(\mathbb{R}^7)$ is a core by [**23**], Proposition 2.2. Integration by parts shows that Eq. (3.1) holds for all $\tilde u \in C^\infty_0(\mathbb{R}^7)$. This implies the claim. q.e.d.

Lemma 3.2. *For all* $\tilde u \in \mathcal{D}(\tilde L_0)$*, we have*

$$\mathrm{Re}(\tilde L_0\tilde u|\tilde u) \leq -\frac{1}{4}\|\tilde u\|^2. \tag{3.2}$$

Proof. First, one can easily check that for all $\tilde u \in C^6(\mathbb{R}^7)$,

$$\Delta\Lambda\tilde u = \Lambda\Delta\tilde u + \Delta\tilde u, \quad \Delta^2\Lambda\tilde u = \Lambda\Delta^2\tilde u + 2\Delta^2\tilde u.$$

Hence,

$$\Delta\tilde L_0\tilde u = \Delta^2\tilde u - \Lambda\Delta\tilde u - 2\Delta\tilde u, \quad \Delta^2\tilde L_0\tilde u = \Delta^3\tilde u - \Lambda\Delta^2\tilde u - 3\Delta^2\tilde u. \tag{3.3}$$

Let $\tilde u \in \mathcal{D}(\tilde L_0)$, then $\Delta^j\tilde u \in L^2_{\mathrm{rad}}(\mathbb{R}^7)$, $j = 1,2,3$. In view of Eq. (3.3) and the fact that $\tilde L_0 u \in \mathcal{H}$, we get that $\Lambda\Delta\tilde u, \Lambda\Delta^2\tilde u \in L^2_{\mathrm{rad}}(\mathbb{R}^7)$. In particular, $\Delta\tilde{u}, \Delta^2\tilde{u} \in \mathcal{D}(\Lambda)$ and $\operatorname{Re}(-\Lambda\Delta\tilde{u}|\Delta\tilde{u})_{L^2(\mathbb{R}^7)} \leq \frac{7}{4}\|\Delta\tilde{u}\|^2_{L^2(\mathbb{R}^7)}$, $\operatorname{Re}(-\Lambda\Delta^2\tilde{u}|\Delta^2\tilde{u})_{L^2(\mathbb{R}^7)} \leq \frac{7}{4}\|\Delta^2\tilde{u}\|^2_{L^2(\mathbb{R}^7)}$, by Lemma 3.1. Integration by parts implies that

$$\operatorname{Re}(\Delta^2\tilde{u}|\Delta\tilde{u})_{L^2(\mathbb{R}^7)} \leq 0, \quad \operatorname{Re}(\Delta^3\tilde{u}|\Delta^2\tilde{u})_{L^2(\mathbb{R}^7)} \leq 0.$$

In view of Eq. (3.3) we infer that Eq. (3.2) holds. q.e.d.

Lemma 3.3. *The range of* $(\mu - \tilde{L}_0)$ *is dense in* $\mathcal{H}$ *for* $\mu = \frac{5}{2}$.

Proof. Let $\tilde{f} \in C^\infty_{\text{rad},0}(\mathbb{R}^7)$ and let $\hat{f} = \mathcal{F}(\tilde{f})$ denote its Fourier transform. The properties of $\tilde{f}$ imply that $\hat{f}$ is a radial Schwartz function, i.e., $\hat{f}(\zeta) = \hat{g}(|\zeta|)$ for some even function $g \in \mathcal{S}(\mathbb{R})$. For $\eta = |\zeta|$, we set

$$\hat{h}(\eta) = \int_\eta^\infty \frac{2e^{-(s^2-\eta^2)}}{s}\hat{g}(s)ds$$

and define $\hat{u}(\cdot) := \hat{h}(|\cdot|)$. It is easy to check that $\hat{u}$ satisfies the equation

$$|\zeta|^2\hat{u}(\zeta) - \tfrac{1}{2}\zeta \cdot \nabla\hat{u}(\zeta) = \hat{f}(\zeta), \tag{3.4}$$

which reduces to

$$\eta^2\hat{h}(\eta) - \tfrac{1}{2}\eta\hat{h}'(\eta) = \hat{g}(\eta)$$

in polar coordinates. We consider the integral

$$\eta^3\hat{h}(\eta) = \int_0^\infty K(\eta, s)\hat{g}(s)s^3 ds$$

with $K(\eta, s) := 2s^{-4}\eta^3 e^{-(s^2-\eta^2)}1_{[0,\infty)}(s - \eta)$. It is easy to check that $|K(\eta, s)| \lesssim \min\{\eta^{-1}, s^{-1}\}$, for all $\eta, s \in [0, \infty)$. By [7], Lemma 5.5, the kernel induces a bounded integral operator on $L^2(\mathbb{R}^+)$. Hence,

$$\|\hat{u}\|_{L^2(\mathbb{R}^7)} \simeq \|(\cdot)^3\hat{h}\|_{L^2(\mathbb{R}^+)} \lesssim \|(\cdot)^3\hat{g}\|_{L^2(\mathbb{R}^+)} \simeq \|\hat{f}\|_{L^2(\mathbb{R}^7)}.$$

In a similar manner one can show that

$$\|\langle\cdot\rangle^k\hat{u}\|_{L^2(\mathbb{R}^7)} \lesssim_k \|\langle\cdot\rangle^k\hat{f}\|_{L^2(\mathbb{R}^7)}$$

for all $k \in \mathbb{N}_0$. The right hand side is finite since $\hat{f} \in \mathcal{S}(\mathbb{R}^7)$. We define $\tilde{u} := \mathcal{F}^{-1}(\hat{u})$. Then $\tilde{u} \in H^k(\mathbb{R}^7)$ for all $k \in \mathbb{N}_0$ by Plancherel's theorem. In particular, $\tilde{u}$ can be approximated with respect to $\|\cdot\|$ by functions in $C^\infty_{0,\text{rad}}(\mathbb{R}^7)$ and thus, $\tilde{u} \in C^6_{\text{rad}}(\mathbb{R}^7) \cap \mathcal{H}$. Applying $\mathcal{F}^{-1}$ to Eq. (3.4) shows that $\tilde{u}$ is a solution to the equation

$$\tfrac{7}{2}\tilde{u}(\xi) - \Delta\tilde{u}(\xi) + \tfrac{1}{2}\xi \cdot \nabla\tilde{u}(\xi) = \tilde{f}(\xi),$$

and we infer that $(\frac{5}{2} - \tilde{L}_0)\tilde{u} = \tilde{f}$. Elementary calculations show that $\|(\cdot)^3(\Delta^j_{\text{rad}}v)'\|_{L^\infty(1,\infty)} \lesssim \|\tilde{v}\|_{H^6(\mathbb{R}^7)}$, $\|(\cdot)^3\Delta^j_{\text{rad}}v\|_{L^\infty(1,\infty)} \lesssim \|\tilde{v}\|_{H^6(\mathbb{R}^7)}$ for all $\tilde{v} = v(|\cdot|) \in C^\infty_{\text{rad},0}(\mathbb{R}^7)$ and $j = 1, 2$. By density of $C^\infty_0(\mathbb{R}^7)$ in $H^6(\mathbb{R}^7)$, we can always find a radial sequence $(\tilde{u}_n)_{n\in\mathbb{N}}$, $\tilde{u}_n = u_n(|\cdot|) \in C^\infty_{\text{rad},0}(\mathbb{R}^7)$ that approximates $\tilde{u}$ in $H^6(\mathbb{R}^7)$. We have $(\cdot)^3(\Delta^j_{\text{rad}}u_n)' \to$

$(\cdot)^3(\Delta^j_{\mathrm{rad}}u)'$ in $L^\infty(1,\infty)$, hence $\lim_{\rho\to\infty}\rho^3|(\Delta^j_{\mathrm{rad}}u)'(\rho)|=0$ for $j=1,2$. Similarly, $\lim_{\rho\to\infty}\rho^3|\Delta^j_{\mathrm{rad}}u(\rho)|=0$. Hence, $\tilde u\in\mathcal{D}(\tilde L_0)$. The density of $C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ in $\mathcal{H}$ finally implies the claim. q.e.d.

Lemma 3.4. *The operator $(\tilde L_0,\mathcal{D}(\tilde L_0))$ is closable and the closure $(L_0,\mathcal{D}(L_0))$ generates a strongly continuous one-parameter semigroup $\{S_0(\tau):\tau\geq 0\}$ of bounded operators on $\mathcal{H}$. The semigroup satisfies*

$$\|S_0(\tau)\tilde u\|\leq e^{-\frac{1}{4}\tau}\|\tilde u\|$$

for all $\tilde u\in\mathcal{H}$ and $\tau\geq 0$. Furthermore, $L_0\tilde u(\xi)=\ell_0u(|\xi|)$, where

$$\ell_0u(\rho)=\Delta_{\mathrm{rad}}u(\rho)-\tfrac{1}{2}\rho u'(\rho)-u(\rho),\quad \rho>0,$$

in a classical sense and $\lim_{\rho\to 0}\ell_0u(\rho)$ exists.

Proof. The first part of the statement follows from Lemma 3.2 and 3.3 and an application of the Lumer-Phillips Theorem [**11**], p. 83. The closure is constructed in the usual way: Let $(\tilde u_n)_{n\in\mathbb{N}}\subset\mathcal{D}(\tilde L_0)$, $\tilde u_n(\cdot)=u_n(|\cdot|)$, be such that $\tilde u_n\to\tilde u$ and $\tilde L_0\tilde u_n\to\tilde f$ in $\mathcal{H}$. Then we say $\tilde u\in\mathcal{D}(L_0)$ and define $L_0\tilde u:=\tilde f$. Next, we describe $L_0\tilde u$ in more detail. Convergence in $\mathcal{H}$ implies that

$$\Delta\tilde u_n\to\Delta\tilde u\in L^2_{\mathrm{rad}}(\mathbb{R}^7)\tag{3.5}$$

and $\tilde u\in C_{\mathrm{rad}}(\mathbb{R}^7)\cap C^3_{\mathrm{rad}}(\mathbb{R}^7\setminus\{0\})$. Furthermore, $\tilde u(\cdot)=u(|\cdot|)$ and we have $u_n\to u$ in $L^\infty(\mathbb{R}^+)$ (Lemma 2.1). Analogously, $L_0\tilde u\in C_{\mathrm{rad}}(\mathbb{R}^7)$, $L_0u(\cdot)=\ell_0u(|\cdot|)$ and

$$\Delta_{\mathrm{rad}}u_n-\tfrac{1}{2}(\cdot)u_n'-u_n\to\ell_0u \text{ in } L^\infty(\mathbb{R}^+).$$

By Lemma 2.1, $(\cdot)u_n'\to(\cdot)u'$ in $L^\infty(\mathbb{R}^+)$. This and Eq. (3.5) imply that $\Delta_{\mathrm{rad}}u_n\to\Delta_{\mathrm{rad}}u$ in $L^\infty(\mathbb{R}^+)$. By uniqueness of the limit function we get that

$$\ell_0u(\rho)=\Delta_{\mathrm{rad}}u(\rho)-\tfrac{1}{2}\rho u'(\rho)-u(\rho),\quad \rho>0.$$

Finally, $\lim_{\rho\to 0}\ell_0u(\rho)$ exists by continuity of $L_0\tilde u$ at the origin. q.e.d.

3.2. The perturbed problem.

Lemma 3.5. *Let V be defined as in Eq. (1.15). Then*

$$L'\tilde u:=V(|\cdot|)\tilde u$$

defines a bounded operator on $\mathcal{H}$. Moreover, L' is compact relative to $(L_0,\mathcal{D}(L_0))$.

Proof. First, we observe that the potential satisfies $|V^{(2k)}(\rho)|\lesssim_k\langle\rho\rangle^{-2-2k}$, $|V^{(2k+1)}(\rho)|\lesssim_k\rho\langle\rho\rangle^{-4-2k}$, for all $k\in\mathbb{N}_0$ and $\rho\in[0,\infty)$. We show below that for all $\tilde u\in\mathcal{D}(\tilde L_0)$ and $R\geq 1$,

$$\|\Delta L'\tilde u\|_{H^1(\mathbb{B}^7)}\lesssim\|\tilde u\|,\quad \|\Delta L'\tilde u\|_{H^1(\mathbb{R}^7\setminus\mathbb{B}^7_R)}\lesssim R^{-1}\|\tilde u\|,\tag{3.6}$$

as well as

$$\|\Delta^2 L'\tilde{u}\|_{L^2(\mathbb{B}^7)} \lesssim \|\tilde{u}\|, \quad \|\Delta^2 L'\tilde{u}\|_{L^2(\mathbb{R}^7\setminus\mathbb{B}^7_R)} \lesssim R^{-1}\|\tilde{u}\|. \tag{3.7}$$

These bounds imply that $\|L'\tilde{u}\| \lesssim \|\tilde{u}\|$ for all $\tilde{u} \in \mathcal{D}(\tilde{L}_0)$. By density of $\mathcal{D}(\tilde{L}_0)$ in $\mathcal{H}$, L' extends to a bounded operator on $\mathcal{H}$. Next, we set $\mathcal{G} := (\mathcal{D}(L_0), \|\cdot\|_{\mathcal{G}})$, where

$$\|\tilde{u}\|_{\mathcal{G}} := \|\tilde{u}\| + \|L_0\tilde{u}\|,$$

denotes the graph norm. We show below that for all $\tilde{u} \in \mathcal{D}(\tilde{L}_0)$ and $R \geq 1$,

$$\|\Delta^2 L'\tilde{u}\|_{\dot{H}^1(\mathbb{B}^7)} \lesssim \|\tilde{u}\|_{\mathcal{G}}, \quad \|\Delta^2 L'\tilde{u}\|_{\dot{H}^1(\mathbb{R}^7\setminus\mathbb{B}^7_R)} \lesssim R^{-1}\|\tilde{u}\|_{\mathcal{G}}. \tag{3.8}$$

By definition of the closure, these bounds extend to all of $\mathcal{D}(L_0)$. In view of Eq. (3.6)–(3.8) we get that for $j = 1, 2$, and all $\tilde{u} \in \mathcal{D}(L_0)$,

$$\|\Delta^j L'\tilde{u}\|_{H^1(\mathbb{R}^7)} \lesssim \|\tilde{u}\|_{\mathcal{G}}, \tag{3.9}$$

$$\|\Delta^j L'\tilde{u}\|_{H^1(\mathbb{R}^7\setminus\mathbb{B}^7_R)} \lesssim R^{-1}\|\tilde{u}\|_{\mathcal{G}}. \tag{3.10}$$

Let $\mathcal{B}_{\mathcal{G}} := \{\tilde{u} \in \mathcal{D}(L_0) : \|\tilde{u}\|_{\mathcal{G}} \leq 1\}$. To see that the perturbation is compact as an operator $L' : \mathcal{G} \to \mathcal{H}$, we convince ourselves that the sets

$$K_1 := \Delta L'(\mathcal{B}_{\mathcal{G}}), \quad K_2 := \Delta^2 L'(\mathcal{B}_{\mathcal{G}})$$

are totally bounded in $L^2(\mathbb{R}^7)$. By Eq. (3.9), $K_1, K_2 \subset H^1(\mathbb{R}^7)$. By equation (3.10), there exists a constant $C > 0$ such that for all $\tilde{u} \in \mathcal{B}_{\mathcal{G}}$ and $R \geq 1$,

$$\|\Delta^j L'\tilde{u}\|_{H^1(\mathbb{R}^7\setminus\mathbb{B}^7_R)} \leq CR^{-1}.$$

The right hand side becomes arbitrarily small (uniformly in $\tilde{u}$) by choosing R large enough. Hence, we can apply the result of [**18**], Theorem 10, which implies that $K_1, K_2 \subset L^2(\mathbb{R}^7)$ are totally bounded. Now let $(\tilde{u}_n)_{n\in\mathbb{N}} \subset \mathcal{B}_{\mathcal{G}}$ and consider the sequence $(L'\tilde{u}_n)_{n\in\mathbb{N}}$. Then $(\Delta L'\tilde{u}_n)_{n\in\mathbb{N}} \subset K_1$. Since K_1 is totally bounded in $L^2(\mathbb{R}^7)$, there is a subsequence, still denoted by $(\Delta L'\tilde{u}_n)_{n\in\mathbb{N}}$, that is a Cauchy sequence in $L^2(\mathbb{R}^7)$. By applying the Laplace operator we obtain a sequence $(\Delta^2 L'\tilde{u}_n)_{n\in\mathbb{N}} \subset K_2$. Again, we find a subsequence that converges in $L^2(\mathbb{R}^7)$ and thus, we have identified a subsequence of $(L'\tilde{u}_n)_{n\in\mathbb{N}}$ that converges in $\mathcal{H}$. This implies the claim. Eq. (3.6) and (3.7) follow easily from Lemma 2.1 and the decay of the potential. It is left to prove Eq. (3.8). First, we observe that

$$\begin{aligned}\|\nabla\Delta^2[V(|\cdot|)\tilde{u}]\|_{L^2(\mathbb{R}^7)} &\lesssim \|V(|\cdot|)\nabla\Delta^2\tilde{u}\|_{L^2(\mathbb{R}^7)} + \|\langle\cdot\rangle^{-4}u\|_{L^2(\mathbb{R}^+)} \\ &+ |(\cdot)\langle\cdot\rangle^{-4}u'\|_{L^2(\mathbb{R}^+)} + \|(\cdot)^2\langle\cdot\rangle^{-4}u''\|_{L^2(\mathbb{R}^+)} + \|(\cdot)^3\langle\cdot\rangle^{-4}u'''\|_{L^2(\mathbb{R}^+)} \\ &+ \|(\cdot)^4\langle\cdot\rangle^{-4}u^{(4)}\|_{L^2(\mathbb{R}^+)} \lesssim \|V(|\cdot|)\nabla\Delta^2\tilde{u}\|_{L^2(\mathbb{R}^7)} + \|\tilde{u}\|,\end{aligned}$$

and

$$\|\nabla\Delta^2[V(|\cdot|)\tilde{u}]\|_{L^2(\mathbb{R}^7\setminus\mathbb{B}^7_R)} \lesssim \|V(|\cdot|)\nabla\Delta^2\tilde{u}\|_{L^2(\mathbb{R}^7\setminus\mathbb{B}^7_R)} + R^{-2}\|\tilde{u}\|.$$

For $\tilde{u} \in \mathcal{D}(\tilde{L}_0)$, we have $\|\nabla\Delta\tilde{L}_0\tilde{u}\|_{L^2(\mathbb{R}^7)} \lesssim \|\tilde{L}_0\tilde{u}\|$, by Lemma 2.2. Hence,

$$\begin{aligned}\|V(|\cdot|)\nabla\Delta^2\tilde{u}\|_{L^2(\mathbb{R}^7)} \lesssim &\|V(|\cdot|)\nabla\Delta\tilde{L}_0\tilde{u}\|_{L^2(\mathbb{R}^7)} \\ &+ \|V(|\cdot|)\nabla\Delta\Lambda\tilde{u}\|_{L^2(\mathbb{R}^7)} + \|V(|\cdot|)\nabla\Delta\tilde{u}\|_{L^2(\mathbb{R}^7)}.\end{aligned}$$

With Lemma 2.1 one easily gets that $\|\langle\cdot\rangle^{-1}\nabla\Delta\Lambda\tilde{u}\|_{L^2(\mathbb{R}^7)} \lesssim \|\tilde{u}\|$, for all $\tilde{u} \in \mathcal{D}(\tilde{L}_0)$. This and the decay of the potential at infinity yield Eq. (3.8). q.e.d.

The next statement is a consequence of Lemma 3.5 and the Bounded Perturbation Theorem, [**11**], p. 158.

Corollary 3.6. *The operator $L := L_0 + L'$, $\mathcal{D}(L) = \mathcal{D}(L_0)$ generates a strongly continuous one-parameter semigroup $\{S(\tau) : \tau \geq 0\}$ of bounded operators on $\mathcal{H}$ satisfying*

$$\|S(\tau)\tilde{u}\| \leq e^{(\|L'\|-\frac{1}{4})\tau}\|\tilde{u}\|$$

for all $\tilde{u} \in \mathcal{H}$ and all $\tau \geq 0$.

In the following, we need some information on the spectrum of the supersymmetric partner of the perturbed Ornstein-Uhlenbeck operator, cf. [**5**], in the self-adjoint setting.

3.3. Spectral analysis for a self-adjoint operator. We introduce the formal differential expression

$$\mathcal{A}v(\rho) = -v''(\rho) + \left(\frac{\rho^2}{16} + \frac{12}{\rho^2} + \frac{3}{4} + Q(\rho)\right)v(\rho), \tag{3.11}$$

with

$$Q(\rho) = \frac{384\sqrt{6} - \rho^2\left(\rho^2 + 24\sqrt{6} - 44\right) - 956}{\left(\rho^2 + 6\sqrt{6} - 14\right)^2}.$$

Lemma 3.7. *Let*

$$\mathcal{D}(A) = \{v \in L^2(\mathbb{R}^+) : v, v' \in AC_{\text{loc}}(\mathbb{R}^+), \mathcal{A}v \in L^2(\mathbb{R}^+)\}$$

and set $Av = \mathcal{A}v$ for $v \in \mathcal{D}(A)$. Then, $(A, \mathcal{D}(A))$ is a self-adjoint operator on $L^2(\mathbb{R}^+)$ and $\sigma(A) \subseteq [\omega_A, \infty)$ for $\omega_A = \frac{1}{75}$.

Proof. Frobenius' method and standard arguments show that $\mathcal{A}$ is limit-point at both endpoints of the interval $(0, \infty)$ and we infer that the (maximal) operator defined as above is self-adjoint. Furthermore, a core is given by $(\tilde{A}, \mathcal{D}(\tilde{A}))$, where $\tilde{A}v := \mathcal{A}u$ and

$$\mathcal{D}(\tilde{A}) = \{v \in \mathcal{D}(A) : v \text{ has compact support}\}.$$

We show that $(\tilde{A}v, v)_{L^2(\mathbb{R}^+)} \geq \omega_A\|v\|^2_{L^2(\mathbb{R}^+)}$ for $\omega_A = \frac{1}{75}$ and all $v \in \mathcal{D}(\tilde{A})$ by using the properties of $q(\rho) := \frac{\rho^2}{16} + \frac{12}{\rho^2} + \frac{3}{4} + Q(\rho)$. Since

$v \in \mathcal{D}(\tilde{A})$ vanishes at the origin the Cauchy-Schwarz inequality implies

$$\int_0^\gamma |v(\rho)|^2 d\rho \leq \gamma^2 \int_0^\gamma |v'(\rho)|^2 d\rho$$

for all $\gamma > 0$. One can easily check that the function q attains its global minimum $q(\rho_{min}) = q_{min}$ at some $\rho_{min} \in (0,\gamma)$, $\gamma := \frac{5}{2}$, and that $\gamma^{-2} + q_{min} > q(\gamma) > \omega_A$. On $[\gamma,\infty)$, q is strictly positive and monotonically increasing. Using integration by parts we estimate

$$\begin{aligned}(\tilde{A}v|v)_{L^2(\mathbb{R}^+)} &\geq \left(\gamma^{-2} + q_{min}\right) \int_0^\gamma |v(\rho)|^2 d\rho + q(\gamma) \int_\gamma^\infty |v(\rho)|^2 d\rho \\ &\geq q(\gamma) \int_0^\infty |v(\rho)|^2 d\rho.\end{aligned}$$

q.e.d.

3.4. Characterization of the spectrum of L.

Lemma 3.8. *Let $\lambda \in \mathbb{C}$ be a spectral point of the operator $(L, \mathcal{D}(L))$. Then either* $\mathrm{Re}\lambda \leq -\frac{1}{75}$ *or $\lambda = 1$ is an eigenvalue.*

Proof. First, we note that in polar coordinates the equation $(\lambda - L)\tilde{u} = 0$ reduces to

$$\lambda u(\rho) - u''(\rho) - \frac{6}{\rho} u'(\rho) + \frac{1}{2}\rho u'(\rho) + (1 - V(\rho))u(\rho) = 0. \tag{3.12}$$

Smoothness of the coefficients on $(0,\infty)$ and an application of Frobenius' method imply that u is smooth on $[0,\infty)$. For $\lambda = 1$, a direct calculation shows that

$$\mathbf{g}(\rho) = (a_1\rho^2 + a_2)^{-2} \tag{3.13}$$

solves Eq. (3.12), where the constants a_1, a_2 are given in Eq. (1.11). By exploiting the decay of $\mathbf{g}$ it follows that $\tilde{\mathbf{g}}$ can be approximated by functions in $C^\infty_{0,\mathrm{rad}}(\mathbb{R}^7)$, with respect to the norm $\|\cdot\|$. Consequently, $\tilde{\mathbf{g}} := \mathbf{g}(|\cdot|) \in \mathcal{H}$ and it is easy to check that $\tilde{\mathbf{g}} \in \mathcal{D}(\tilde{L}_0) \subset \mathcal{D}(L_0)$. Hence, $\tilde{\mathbf{g}}$ is an eigenfunction. To prove the rest of Lemma 3.8, we assume that $\lambda \in \sigma(L)$. If $\mathrm{Re}\lambda \leq -\frac{1}{75}$, then the statement is true. If $\mathrm{Re}\lambda > -\frac{1}{75}$, then compactness of L' implies that λ must be an eigenvalue. The case $\lambda = 1$ has already been discussed, so assume that $\lambda \neq 1$ and let $\tilde{u}_\lambda = u_\lambda(|\cdot|)$ denote the corresponding eigenfunction. We set $v_\lambda(\rho) = \rho^3 e^{-\frac{\rho^2}{8}} u_\lambda(\rho)$. Since $u_\lambda \in C^\infty[0,\infty)$ is a solution to Eq. (3.12), v_λ is smooth and solves the equation

$$\lambda v_\lambda(\rho) - v_\lambda''(\rho) + \left(\frac{\rho^2}{16} + \frac{6}{\rho^2} - \frac{3}{4} - V(\rho)\right) v_\lambda(\rho) = 0. \tag{3.14}$$

We define the differential expressions

$$Bv(\rho) := -v'(\rho) + \beta(\rho)v(\rho), \quad B^+v(\rho) := v'(\rho) + \beta(\rho)v(\rho),$$

for $\beta(\rho) := \frac{3}{\rho} - \frac{\rho}{4} - \frac{4\rho}{6\sqrt{6}-14+\rho^2}$. Using this, Eq. (3.14) can be written as

$$(\lambda + B^+ B - 1)v_\lambda(\rho) = 0. \tag{3.15}$$

The kernel of B is spanned by the transformed symmetry mode $\rho \mapsto \rho^3 e^{-\frac{\rho^2}{8}} \mathbf{g}(\rho)$. We set $w_\lambda := Bv_\lambda$. By applying $-B$ to Eq. (3.15) we infer that w_λ satisfies the equation

$$(-\lambda - BB^+ + 1)w_\lambda(\rho) = 0. \tag{3.16}$$

A straightforward calculation shows that Eq. (3.16) can be written as

$$-\lambda w_\lambda(\rho) - \mathcal{A}(\rho) w_\lambda(\rho) = 0,$$

where $\mathcal{A}$ is given by Eq. (3.11). Recall that $\mathcal{A}$ is the defining differential expression for the self-adjoint operator A described in Lemma 3.7. By inserting the definition we get that

$$w_\lambda(\rho) = -\rho^3 e^{-\frac{\rho^2}{8}} u_\lambda'(\rho) + \rho^2 e^{-\frac{\rho^2}{8}} h(\rho) u_\lambda(\rho),$$

where $h \in C^\infty[0,\infty)$, $\lim_{\rho\to\infty} h(\rho) = c$. Obviously, $w_\lambda \in C^\infty[0,\infty)$ and with Lemma 2.2 it is easy to see that $w_\lambda \in L^2(\mathbb{R}^+)$ and thus $w_\lambda \in \mathcal{D}(A)$. Hence, $\tilde{\lambda} := -\lambda$ is an eigenvalue of the operator $(A, \mathcal{D}(A))$ with eigenfunction w_λ. Our assumption on λ implies that $\mathrm{Re}\tilde{\lambda} < \frac{1}{75}$. However, this contradicts Lemma 3.7. q.e.d.

Lemma 3.9. *We have that* $\ker(1-L) = \mathrm{span}(\tilde{\mathbf{g}})$*, where* $\tilde{\mathbf{g}} = \mathbf{g}(|\cdot|)$*, see Eq.* (3.13)*.*

Proof. It was already shown above that $\tilde{\mathbf{g}}$ is an eigenfunction corresponding to the eigenvalue $\lambda = 1$. Assume that there is another eigenfunction $\tilde{u} = u(|\cdot|) \in \ker(1-L)$. Then u solves

$$u''(\rho) + \tfrac{6}{\rho} u'(\rho) - \tfrac{1}{2}\rho u'(\rho) + V(\rho)u(\rho) - 2u(\rho) = 0.$$

For this equation a fundamental system is given by $\{\mathbf{g}, \mathbf{h}\}$, where

$$\mathbf{h}(\rho) = e^{\frac{\rho^2}{4}} \left(h_1(\rho) + h_2(\rho) e^{-\frac{\rho^2}{4}} \int_0^\rho e^{\frac{s^2}{4}} ds \right), \tag{3.17}$$

$$h_1(\rho) = \frac{\sum_{j=0}^3 \alpha_j \rho^{2j}}{20\rho^5(6\sqrt{6}-14+\rho^2)^2}, \quad h_2(\rho) = \frac{2(61-24\sqrt{6})}{5(6\sqrt{6}-14+\rho^2)^2},$$

for constants $\alpha_0 = 24(8652\sqrt{6} - 21193)$, $\alpha_1 = 4(8347 - 3408\sqrt{6})$, $\alpha_2 = 2(372\sqrt{6} - 923)$, $\alpha_3 = 15$. Their Wronskian is given by $W(\mathbf{g},\mathbf{h})(\rho) = \rho^{-6} e^{\frac{\rho^2}{4}}$. The function $\mathbf{h}$ can also be written as $\mathbf{h}(\rho) = \rho^{-5}\langle\rho\rangle^2 e^{\frac{\rho^2}{4}} H(\rho)$, where H is regular around zero, $H(0) \neq 0$ and $\lim_{\rho\to\infty} H(\rho) = c$ for some $c \in \mathbb{R}\setminus\{0\}$. Hence, $u(\rho) = c_1\mathbf{g}(\rho) + c_2\mathbf{h}(\rho)$, for some constants $c_1, c_2 \in \mathbb{C}$. Since $\mathbf{h}$ diverges at the origin as well as for $\rho \to \infty$, we must have $c_2 = 0$ for $u(|\cdot|) \in \mathcal{D}(L) \subset \mathcal{H}$. Thus, u is a multiple of $\mathbf{g}$. q.e.d.

4. Bounds on the resolvent and growth estimates for $S(\tau)$

In order to translate the spectral information into growth bounds for the semigroup, we use the following result on the resolvent.

Proposition 4.1. *Fix $\alpha > -\frac{1}{75}$ and choose $M_\alpha > 0$ sufficiently large. Define*

$$\Omega_\alpha := \{\lambda \in \mathbb{C} : \lambda = \alpha + \mathrm{i}\,\omega, \text{ where } \omega \in \mathbb{R}, |\omega| \geq M_\alpha\}.$$

There exists a constant $C_\alpha > 0$ such that

$$\|R_L(\lambda)\tilde{f}\| \leq C_\alpha \|\tilde{f}\| \tag{4.1}$$

for all $\lambda \in \Omega_\alpha$ and $\tilde{f} \in \mathcal{H}$.

To prove Proposition 4.1, we follow the strategy developed by the authors in [**9**]. To keep formulas within margin, we introduce the following useful notation.

Definition 4.2. For a function $f : I \subset \mathbb{R} \to \mathbb{C}$ and $\gamma \in \mathbb{R}$, we write $f(x) = \mathcal{O}(x^\gamma)$ if

$$|f^{(k)}(x)| \leq C_k |x|^{\gamma - k}$$

for all $x \in I$ and $k \in \mathbb{N}_0$. Similarly, $f(x) = \mathcal{O}(\langle x \rangle^\gamma)$, if

$$|f^{(k)}(x)| \leq C_k \langle x \rangle^{\gamma - k},$$

for all $x \in I$ and $k \in \mathbb{N}_0$. Functions with this property are said to be of *symbol type* or to have *symbol behavior*. We note that symbol behavior is stable under algebraic operations, e.g., $\mathcal{O}(x^\beta)\mathcal{O}(x^\gamma) = \mathcal{O}(x^{\beta+\gamma})$ for $\beta, \gamma \in \mathbb{R}$. An analogous definition holds for functions depending on more than one variable.

4.1. Explicit representation of the resolvent for large imaginary parts. If $\lambda \in \mathbb{C}$, $\lambda \neq 1$ and $\mathrm{Re}\lambda > -\frac{1}{75}$, then we know from Lemma 3.8 that $\lambda \in \rho(L)$ and that the resolvent $R_L(\lambda) : \mathcal{H} \to \mathcal{D}(L)$ exists as a bounded operator. For $\tilde{f} = f(|\cdot|) \in \mathcal{H}$ we set $\tilde{w} := R_L(\lambda)\tilde{f}$, $\tilde{w} = w(|\cdot|)$. By definition, $(\lambda - L)\tilde{w} = \tilde{f}$, which reduces to

$$\lambda w(\rho) - w''(\rho) - \frac{6}{\rho} w'(\rho) + \frac{1}{2}\rho w'(\rho) + w(\rho) - V(\rho)w(\rho) = f(\rho),$$

in polar coordinates. By setting $u(r) := w(2\rho)$ and changing the sign we obtain

$$u''(r) + \frac{6}{r}u'(r) - 2ru'(r) + \tilde{V}(r)u(r) - \tilde{\lambda}u(r) = -4f(2r) \tag{4.2}$$

for $\tilde{\lambda} = 4(\lambda + 1)$, $\tilde{V}(r) = 4V(2\rho)$. First, we consider the homogeneous version of Eq. (4.2) which corresponds to Eq. (4.1) in [**9**] for $d = 7$ and

$\ell = 0$. Concerning the potential, we only need that $\tilde{V}(r) = \mathcal{O}(\langle r \rangle^{-2})$. By setting $v(r) = r^3 e^{-\frac{r^2}{2}} u(r)$, Eq. (4.2) with $f = 0$ transforms into

$$v''(r) - r^2 v(r) - \frac{4\nu^2 - 1}{4r^2} v(r) - \mu v(r) + \tilde{V}(r) v(r) = 0 \tag{4.3}$$

for $\nu = \frac{5}{2}$ and $\mu := \tilde{\lambda} - 7$. For $r \gg 1$, Eq. (4.3) resembles a Weber equation, whereas for r small we have a perturbed Bessel equation. For the rest of this section, we assume that

$$\mu = b + \mathrm{i}\,\omega$$

with $b > -4$ fixed, corresponding to $\mathrm{Re}\lambda > -\frac{1}{4}$. Finally, we note that most implicit constants depend on b in the following. However, we do not indicate this dependence in order to improve readability. All other dependencies are tracked during the calculations.

4.1.1. A fundamental system away from the center. We write Eq. (4.3) as

$$v''(r) - (r^2 + \mu) v(r) = \mathcal{O}(r^{-2}) v(r) \tag{4.4}$$

for $r \geq 1$. The homogeneous equation can be solved in terms of parabolic cylinder functions. However, this is not very useful for our purpose since we need precise information on the asymptotics for large imaginary parts. Instead, we use a Liouville-Green transform in combination with perturbation theory. First, we assume that $\mu \in \mathbb{R}$, $\mu > 0$ and define

$$\zeta(y) := \int_{10\mu^{-\frac{1}{2}}}^{y} \sqrt{1 + z^2}\, dz = F(y) - F(10\mu^{-\frac{1}{2}}),$$

where $F(z) = \frac{1}{2}\log(z + \sqrt{1+z^2}) + \frac{1}{2} z \sqrt{1+z^2}$. Obviously, $\zeta'(y) = \sqrt{1+y^2}$. Furthermore,

$$q(y) := \frac{2 - 3y^2}{4(1+y^2)^2}.$$

A fundamental system for the equation

$$v''(r) - (r^2 + \mu) v(r) + \mu^{-1} q(\mu^{-\frac{1}{2}} r) v(r) = 0,$$

is given by $\{v^+, v^-\}$, where

$$v^{\pm}(r) = \zeta'(\mu^{-\frac{1}{2}} r)^{-\frac{1}{2}} e^{\pm \mu \zeta(\mu^{-\frac{1}{2}} r)}.$$

This suggests to add $\mu^{-1} q(\mu^{-\frac{1}{2}} r) v(r)$ to both sides of Eq. (4.4) and to treat the right hand side perturbatively. Since we are interested in complex values of μ, we have to extend the above quantities to the complex plane. We define $\sqrt{\cdot} = (\cdot)^{\frac{1}{2}}$ to be the principal branch of the square root, which is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$. More explicitly,

$$\sqrt{z} = \frac{1}{\sqrt{2}} \sqrt{|z| + \mathrm{Re}\, z} + \frac{\mathrm{i}\, \mathrm{sgn}\,(\mathrm{Im}\, z)}{\sqrt{2}} \sqrt{|z| - \mathrm{Re}\, z}, \tag{4.5}$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$. We note that $\sqrt{z}^2 = z$ and $|\sqrt{z}| = \sqrt{|z|}$ for all $z \in \mathbb{C}\setminus(-\infty, 0]$. Furthermore, $\sqrt{z}\sqrt{w} = \sqrt{zw}$ holds at if $-\pi < \arg z + \arg w < \pi$. The function F is defined and holomorphic on $\mathbb{C} \setminus ([\mathrm{i}, \mathrm{i}\infty) \cup [-\mathrm{i}, -\mathrm{i}\infty))$. Since $\mathrm{Re}(\mu^{-\frac{1}{2}} r) > 0$ for all $\mu \in \mathbb{C} \setminus (-\infty, 0]$, the function $\mu \mapsto \zeta(\mu^{-\frac{1}{2}} r)$ is holomorphic on $\mathbb{C} \setminus (-\infty, 0]$.

From now on we assume $\mu = b + \mathrm{i}\,\omega$, where $b > -4$ is fixed and $\omega > 0$.

Lemma 4.3. *For $r \in [3, \infty)$, we define $Q(r, \mu) := \mu^{-1} q(\mu^{-\frac{1}{2}} r)$. The function $Q(r, \cdot)$ is holomorphic and $Q(r, \mu) = \mathcal{O}(r^{-2}\omega^0)$ for $\omega \gg 1$. Furthermore, we define $\xi(r, \mu) := \zeta(\mu^{-\frac{1}{2}} r)$, such that $\xi(r, \cdot)$ is holomorphic and*

$$\partial_r \xi(r, \mu) = \mu^{-\frac{1}{2}} \sqrt{1 + \tfrac{r^2}{\mu}}.$$

For $\omega \gg 1$, we have the representations

$$\mathrm{Re}[\mu \xi(r, \mu)] = \mathrm{Re}\mu^{\frac{1}{2}} (r - 10) + \tilde{\varphi}(r, \omega), \tag{4.6}$$

$$\mathrm{Re}[\mu \xi(r, \mu)] = \tfrac{1}{2} r^2 + \tfrac{b}{2} \log\langle \omega^{-1/2} r \rangle + \varphi(r, \omega), \tag{4.7}$$

where both $\tilde{\varphi}(\cdot, \omega)$ and $\varphi(\cdot, \omega)$ are monotonically increasing functions. Moreover, $|\varphi(10, \omega)| \lesssim 1$, $\tilde{\varphi}(10, \omega) = 0$ and $\tilde{\varphi}(r, \omega) = \mathcal{O}(r^3 \omega^{-\frac{1}{2}})$ provided $r\omega^{-\frac{1}{2}} \lesssim 1$.

Proof. For $\omega \gg 1$, we have $|\mu| \simeq \omega$. Using that

$$|1 + \mu^{-1} r^2|^2 \gtrsim 1 + |\mu|^{-2} r^4 \gtrsim 1 + \omega^{-2} r^4 \gtrsim \langle \omega^{-\frac{1}{2}} r \rangle^4 \tag{4.8}$$

for $\omega \gg 1$, we obtain

$$|Q(r, \mu)| \lesssim |\mu|^{-1} \frac{1 + |\mu|^{-1} r^2}{|1 + \mu^{-1} r^2|^2} \lesssim |\mu|^{-1} \langle \omega^{-\frac{1}{2}} r \rangle^{-2} \lesssim r^{-2}$$

and thus $Q(r, \mu) = \mathcal{O}(r^{-2}\omega^0)$. To see that Eq. (4.6) holds, we use that $\mathrm{Re}[\mu \partial_r \xi(r, \mu)] = \mathrm{Re}\sqrt{\mu + r^2}$. With Eq. (4.5) we get

$$\mathrm{Re}\sqrt{\mu + r^2} = \frac{1}{\sqrt{2}} \sqrt{\sqrt{\omega^2 + (r^2 + b)^2} + r^2 + b}. \tag{4.9}$$

Eq. (4.6) is a consequence of the fact that $(r^2 + b)^2 > b^2$ for $b > -4$ and all $r \geq 3$, which implies

$$\begin{aligned} \partial_r \tilde{\varphi}(r, \omega) &= \mathrm{Re}[\mu \partial_r \xi(r, \mu)] - \mathrm{Re}\mu^{\frac{1}{2}} \\ &= \mathrm{Re}[\mu \partial_r \xi(r, \mu)] - \frac{1}{\sqrt{2}} \sqrt{\sqrt{\omega^2 + b^2} + b} > 0. \end{aligned}$$

Since $\xi(10, \mu) = 0$, $\tilde{\varphi}(10, \omega) = 0$ by definition. If $|r\mu^{-\frac{1}{2}}| \lesssim 1$, a Taylor expansion of $F(z)$ around zero yields $\mu \xi(r, \mu) = \mu^{1/2}(r - 10) + \psi(r, \mu) - \psi(10, \mu)$, where $\psi(r, \mu) = \mathcal{O}(r^3 \omega^{-\frac{1}{2}})$. By definition of $\tilde{\varphi}$, we have $\tilde{\varphi}(r,\omega) = \mathrm{Re}[\psi(R,\mu) - \psi(10,\mu)]$. We now turn to Eq. (4.7). The right hand side of Eq. (4.9) implies that

$$\mathrm{Re}[\mu\partial_r\xi(r,\mu)] \geq \sqrt{r^2+b}$$

and for $b \geq 0$, the proof of Eq. (4.7) is along the lines of [**9**], p. 2497. For $-4 < b < 0$ we distinguish two cases. First, we assume that $3 \leq r \leq \sqrt{2\omega}$. Then, $4|b|r^2 \leq 8|b|\omega < \omega^2$ for $\omega \gg 1$. This implies

$$\omega^2 + (r^2 - |b|)^2 = \omega^2 + r^4 - 2r^2|b| + |b|^2 > (r^2 + |b|)^2.$$

With this, we infer $\mathrm{Re}[\mu\partial_r\xi(r,\mu)] > r$. In particular,

$$\partial_r\varphi(r,\omega) = \mathrm{Re}[\mu\partial_r\xi(r,\mu)] - r + \frac{|b|}{2}\frac{r\omega^{-1}}{1+\omega^{-1}r^2} > 0.$$

For $r > \sqrt{2\omega}$ we have $\frac{\omega}{r^2-|b|} \leq 1$, and since $\sqrt{1+x^2} \geq 1 + \frac{x^2}{4}$ for $0 \leq x \leq 1$ we get

$$\mathrm{Re}[\mu\partial_r\xi(r,\mu)] \geq \sqrt{r^2 - |b| + \tfrac{\omega^2}{8(r^2-|b|)}} > r - \frac{|b|r\omega^{-1}}{2(1+r^2\omega^{-1})},$$

for $\omega \gg 1$, where the last step can be verified by an elementary computation. This implies Eq. (4.7). q.e.d.

By adding $Q(r,\mu)v(r)$ to both sides we rewrite Eq. (4.3) as

$$v''(r) - r^2v(r) - \mu v(r) + Q(r,\mu)v(r) = \mathcal{O}(r^{-2}\omega^0)v(r),$$

and apply perturbation theory to obtain a fundamental system for $r \geq 4$. The next result is partially along the lines of [**9**], Proposition 4.3.

Lemma 4.4. *Define*

$$v_0^\pm(r,\omega) := \tfrac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}(1+\tfrac{r^2}{\mu})^{-\frac{1}{4}}e^{\pm\mu\xi(r,\mu)}$$

with Wronskian $W(v_0^-(\cdot,\omega), v_0^+(\cdot,\omega)) = 1$. *For* $\omega \gg 1$, *Eq.* (4.3) *has a fundamental system* $\{v_-, v_+\}$ *of the form*

$$v_\pm(r,\omega) = v_0^\pm(r,\omega)[1 + \mathcal{O}(r^{-1}\omega^{-\frac{1}{2}})], \quad \textit{for all } r \geq 4. \tag{4.10}$$

Proof. By the variation of constants formula we obtain an integral equation for v_- given by

$$v_-(r,\omega) = v_0^-(r,\omega) + v_0^+(r,\omega)\int_r^\infty v_0^-(s,\omega)\mathcal{O}(s^{-2}\omega^0)v_-(s,\omega)ds$$
$$- v_0^-(r,\omega)\int_r^\infty v_0^+(s,\omega)\mathcal{O}(s^{-2}\omega^0)v_-(s,\omega)ds.$$

First, we assume that $r \geq 3$. Setting $h_- := \frac{v_-}{v_0^-}$ yields

$$h_-(r,\omega) = 1 + \int_r^\infty K(r,s,\omega)h_-(s,\omega)ds, \tag{4.11}$$

where

$$K(r,s,\omega) := \left[\frac{v_0^+(r,\omega)}{v_0^-(r,\omega)} v_0^-(s,\omega)^2 - v_0^- v_0^+(s,\omega)\right] \mathcal{O}(s^{-2}\omega^0)$$
$$= g(s,\omega)\left(e^{-2(\mu\xi(s,\mu)-\mu\xi(r,\mu))} - 1\right),$$

and $g(s,\omega) = \frac{1}{2}\mu^{-\frac{1}{2}}(1+\frac{s^2}{\mu})^{-\frac{1}{2}}\mathcal{O}(s^{-2}\omega^0) = \mathcal{O}(s^{-2}\omega^{-\frac{1}{2}})$. Eq. (4.6) implies that

$$|K(r,s,\omega)| \lesssim s^{-2}\omega^{-\frac{1}{2}}\left[1 + e^{-2\mathrm{Re}\mu^{1/2}(s-r)} e^{-2(\tilde{\varphi}(s,\omega)-\tilde{\varphi}(r,\omega))}\right] \lesssim s^{-2}\omega^{-\frac{1}{2}},$$

since $s > r$ and $\tilde{\varphi}(\cdot,\omega)$ is monotonically increasing on $[3,\infty)$. This yields the bound

$$\int_r^\infty \sup_{r\in[3,s]} |K(r,s,\omega)| ds \lesssim 1.$$

An application of the standard result on Volterra equations, see e.g. Lemma 2.4 in [**29**], yields a solution h_- to Eq. (4.11) with $|h_-(r,\omega)-1| \lesssim r^{-1}\omega^{-\frac{1}{2}}$, for all $r \geq 3$. Furthermore, $|\partial_r^k[h_-(r,\omega)-1]| \lesssim_k r^{-1-k}\omega^{-\frac{1}{2}}$ for all $k \in \mathbb{N}_0$, cf. [**9**], Remark 4.4. Thus, $h_-(r,\omega) = 1 + \mathcal{O}(r^{-1}\omega^{-\frac{1}{2}})$, which yields Eq. (4.10). The second solution v_+ is obtained by setting

$$v_+(r,\omega) := v_-(r,\omega)\left[\frac{v_0^+(3,\omega)}{v_0^-(3,\omega)} + \int_3^r v_-(s,\omega)^{-2} ds\right].$$

Following the lines of [**9**], p. 2499–2500, we use the identity

$$v_0^-(r,\omega)\int_3^r v_0^-(s,\omega)^{-2} ds = v_0^+(r,\omega) - \frac{v_0^+(3,\omega)}{v_0^-(3,\omega)} v_0^-(r,\omega)$$

to obtain an expression for $h_+ := \frac{v_+}{v_0^+}$ given by

$$h_+(r,\omega) = 1 + \mathcal{O}(r^{-1}\omega^{-\frac{1}{2}}) + \frac{v_-(r,\omega)}{v_0^+(r,\omega)}\int_3^r v_0^-(s,\omega)^{-2}\mathcal{O}(s^{-1}\omega^{-\frac{1}{2}}) ds.$$

We show by induction that for all $k \in \mathbb{N}_0$,

$$\partial_r^k[h_+(r,\omega)-1] = \mathcal{O}(r^{-1-k}\omega^{-\frac{1}{2}}) + \int_3^r H_k(r,s,\omega) ds$$
$$+ e^{-2[\mu\xi(r,\mu)-\mu\xi(3,\mu)]}\mathcal{O}(r^0\omega^{-\frac{1}{2}+\frac{k}{2}})\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^k),$$

$H_k(r,s,\omega) = e^{-2[\mu\xi(r,\mu)-\mu\xi(s,\mu)]}\mathcal{O}(s^{-2-k}\omega^{-\frac{1}{2}})\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^k)\mathcal{O}(\langle\omega^{-\frac{1}{2}}s\rangle^{-k})$. For $k = 0$, we use that $v_0^-(s,\omega)^{-2} = \partial_s e^{2\mu\xi(s,\mu)}$ and integrate by parts to get

$$h_+(r,\omega) = 1 + \mathcal{O}(r^{-1}\omega^{-\frac{1}{2}}) + e^{-2[\mu\xi(r,\mu)-\mu\xi(3,\mu)]}\mathcal{O}(r^0\omega^{-\frac{1}{2}})$$
$$+ \int_3^r e^{-2[\mu\xi(r,\mu)-\mu\xi(s,\mu)]}\mathcal{O}(r^0 s^{-2}\omega^{-\frac{1}{2}}) ds.$$

With $\mu\partial_r\xi(r,\mu) = \mu^{\frac{1}{2}}(1+\mu^{-1}r^2)^{\frac{1}{2}}$, we get

$$\begin{aligned}&\partial_r^{k+1}[h_+(r,\omega)-1] = \\ &\qquad \mathcal{O}(r^{-2-k}\omega^{-\frac{1}{2}}) + e^{-2[\mu\xi(r,\mu)-\mu\xi(3,\mu)]}\mathcal{O}(r^0\omega^{\frac{k}{2}})\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{k+1}) \\ &\qquad + \int_3^r \partial_r H_k(r,s,\omega)ds,\end{aligned}$$

where

$$\begin{aligned}\partial_r H_k(r,s,\omega) =& e^{-2[\mu\xi(r,\mu)-\mu\xi(s,\mu)]}\mathcal{O}(r^0 s^{-2-k}\omega^0) \\ & \mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{k+1})\mathcal{O}(\langle\omega^{-\frac{1}{2}}s\rangle^{-k}).\end{aligned}$$

Note that $[\mu\partial_s\xi(s,\mu)]^{-1} = \mu^{-\frac{1}{2}}\mathcal{O}(\langle\omega^{-\frac{1}{2}}s\rangle^{-1})$, see Eq. (4.8). Thus, an integration by parts yields

$$\begin{aligned}&e^{-2\mu\xi(r,\mu)}\mathcal{O}(r^0\omega^0)\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{k+1}) \\ &\int_3^r \left(\partial_s e^{2\mu\xi(s,\mu)}\right)\mathcal{O}(s^{-2-k}\omega^{-\frac{1}{2}})\mathcal{O}(\langle\omega^{-\frac{1}{2}}s\rangle^{-k-1})ds \\ &= \mathcal{O}(r^{-2-k}\omega^{-\frac{1}{2}}) + e^{-2[\mu\xi(r,\mu)-\mu\xi(3,\mu)]}\mathcal{O}(r^0\omega^{-\frac{1}{2}})\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{k+1}) \\ &+ \int_3^r H_{k+1}(r,s,\omega)ds,\end{aligned}$$

where

$$\begin{aligned}H_{k+1}(r,s,\omega) = e^{-2[\mu\xi(r,\mu)-\mu\xi(s,\mu)]}&\mathcal{O}(r^0\omega^{-\frac{1}{2}})\mathcal{O}(s^{-3-k}\omega^0) \\ &\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{k+1})\mathcal{O}(\langle\omega^{-\frac{1}{2}}s\rangle^{-(k+1)}).\end{aligned}$$

For $4 \le r \le \omega^{\frac{1}{2}}$, $\omega \gg 1$, we use Lemma 4.3 to estimate

$$\omega^{-\frac{1}{2}+\frac{k}{2}}e^{-2\mathrm{Re}[\mu\xi(r,\mu)-\mu\xi(3,\mu)]} \lesssim_k r^{-1-k}\omega^{-\frac{1}{2}},$$

and for $r > \omega^{\frac{1}{2}}$ we get,

$$r^k\omega^{-\frac{1}{2}}e^{-2\mathrm{Re}[\mu\xi(r,\mu)-\mu\xi(3,\mu)]} \lesssim_k \omega^{-\frac{1}{2}}r^{4+k}e^{-r^2} \lesssim r^{-1-k}\omega^{-\frac{1}{2}}.$$

The integral kernels satisfy

$$\begin{aligned}&|H_k(r,s,\omega)| \lesssim_k \omega^{-\frac{1}{2}}s^{-2-k}\langle\omega^{-\frac{1}{2}}r\rangle^{k-b}\langle\omega^{-\frac{1}{2}}s\rangle^{-k+b} \\ &e^{-(r^2-s^2)}e^{-2[\varphi(r,\omega)-\varphi(s,\omega)]} \lesssim_k \omega^{-\frac{1}{2}}r^{-2-k}r^{2+k}s^{-2-k}r^{k+4}s^{-k-4}e^{-(r^2-s^2)} \\ &\lesssim_k r^{-2-k}\omega^{-\frac{1}{2}}e^{-(r^2-2(k+3)\log r)}e^{s^2-2(k+3)\log s} \lesssim_k r^{-2-k}\omega^{-\frac{1}{2}}\end{aligned}$$

for all $3 \le s \le r$, $\omega \gg 1$ and $k \in \mathbb{N}_0$. With these estimates, we infer that

$$|\partial_r^k[h_+(r,\omega)-1]| \lesssim_k r^{-1-k}\omega^{-\frac{1}{2}}$$

for all $r \ge 4$, $\omega \gg 1$ and $k \in \mathbb{N}_0$. This justifies the notation $h_+(r,\omega) = 1 + \mathcal{O}(r^{-1}\omega^{-\frac{1}{2}})$ and implies Eq. (4.10) for v_+. q.e.d.

4.1.2. A fundamental system near the origin. For small radii, Eq. (4.3) is written as

$$v''(r) - \tfrac{4\nu^2-1}{4r^2}v(r) - \mu v(r) = \mathcal{O}(\langle r\rangle^2)v(r).$$

The right hand side is again treated perturbatively.

Lemma 4.5. *Choose $c > 1$. Define*

$$\psi_0(r,\omega) := \sqrt{r}J_{5/2}(\mathrm{i}\,\mu^{\frac{1}{2}}r), \quad \tilde{\psi}_1(r,\omega) := \sqrt{r}Y_{5/2}(\mathrm{i}\,\mu^{\frac{1}{2}}r).$$

Eq. (4.3) has a fundamental system $\{v_0, v_1\}$ of the form

$$\begin{aligned} v_0(r,\omega) &= \psi_0(r,\omega)[1+\mathcal{O}(r^2\omega^0)],\\ v_1(r,\omega) &= [\tilde{\psi}_1(r,\omega) + \mathcal{O}(\omega^0)\psi_0(r,\omega)][1+\mathcal{O}(r^0\omega^{-\frac{1}{2}})], \end{aligned}$$

for all $\omega \gg c^2$ and $r \in (0, c\omega^{-\frac{1}{2}}]$. Furthermore, $v_0(r,\omega) = \mathcal{O}(r^3\omega^{\frac{5}{4}})$, $v_1(r,\omega) = \mathcal{O}(r^{-2}\omega^{-\frac{5}{4}})$.

We note that we have the explicit expressions,

$$\begin{aligned} \sqrt{z}J_{5/2}(\mathrm{i}\,z) &= \alpha_0 z^{-2}[(3+z^2)\sinh(z) - 3z\cosh(z)],\\ \sqrt{z}Y_{5/2}(\mathrm{i}\,z) &= \alpha_1 z^{-2}[(3+z^2)\cosh(z) - 3z\sinh(z)], \end{aligned}$$

for some $\alpha_0, \alpha_1 \in \mathbb{C}$. The construction of $\{v_0, v_1\}$ is along the lines of [**9**], Lemma 4.5, for fixed $\nu = \frac{5}{2}$, and is omitted for the sake of brevity. A sketch of the proof can be found in the arXiv preprint version [**10**] of this paper. Finally, we construct a fundamental system for intermediate values of r.

Lemma 4.6. *Choose $c > 1$ sufficiently large and define*

$$\psi_\pm(r,\omega) := \sqrt{r}H^{\mp}_{5/2}(\mathrm{i}\,\mu^{\frac{1}{2}}r),$$

where $H^{\mp}_{5/2} = J_{5/2} \mp \mathrm{i}\,Y_{5/2}$. Provided $\omega \gg c^2$, there exists a fundamental system $\{\tilde{v}_-, \tilde{v}_+\}$ of Eq. (4.3) given by

$$\tilde{v}_\pm(r,\omega) = \psi_\pm(r,\omega)[1+\mathcal{O}(r^0\omega^{-\frac{1}{2}})], \tag{4.12}$$

for $r \in [\frac{1}{2}c\omega^{-\frac{1}{2}}, 40]$. Furthermore, $\tilde{v}_\pm(r,\omega) = \mathcal{O}(r^0\omega^{-\frac{1}{4}})e^{\pm\mu^{1/2}r}$.

Proof. As in the proof of Lemma 4.6 in [**9**] we define $\tilde{\psi}_\pm(r,\omega) := \alpha_\pm\psi_\pm(r,\omega)$, for $\alpha_\pm \in \mathbb{C}\setminus\{0\}$ such that $W(\tilde{\psi}_-(\cdot,\omega), \tilde{\psi}_+(\cdot,\omega)) = 1$. Note that $\tilde{\psi}_\pm$ can be given in closed form by

$$\tilde{\psi}_\pm(r,\omega) = \tfrac{1}{\sqrt{2}}\mu^{-\frac{1}{4}}e^{\pm\mu^{1/2}r}(1 \mp 3r^{-1}\mu^{-\frac{1}{2}} + 3r^{-2}\mu^{-1}).$$

We choose $c > 1$ large enough such that $|\tilde{\psi}_\pm(r,\omega)| > 0$ for $r \geq \frac{1}{4}c\omega^{-\frac{1}{2}}$. The variation of constants formula, see [**9**], Lemma 4.6, which yields an equation for $h_- = \frac{\tilde{v}_-}{\tilde{\psi}_-}$ given by

$$h_-(r,\omega) = 1 + \int_r^R K(r,s,\omega)h_-(s,\omega)ds, \tag{4.13}$$

for some $R > 0$ and

$$K(r,s,\omega) = \left(\tilde{\psi}_-(s,\omega)\tilde{\psi}_+(s,\omega) - \frac{\tilde{\psi}_+(r,\omega)}{\tilde{\psi}_-(r,\omega)}\tilde{\psi}_-(s,\omega)^2\right)\mathcal{O}(\langle s\rangle^2).$$

We set $R = 50$. By using the above explicit formulas we get that $|K(r,s,\omega)| \lesssim \omega^{-\frac{1}{2}} + \omega^{-\frac{1}{2}}e^{-2\mathrm{Re}\mu^{1/2}(s-r)} \lesssim \omega^{-\frac{1}{2}}$, for all $\frac{1}{4}c\omega^{-\frac{1}{2}} \le r \le s \le 50$. The standard result on Volterra equations yields the existence of a solution h_- satisfying $|h_-(r,\omega) - 1| \lesssim \omega^{-\frac{1}{2}}$, for all $r \in [\frac{1}{4}c\omega^{-\frac{1}{2}}, 50]$ and $\omega \gg 1$. In the following, we restrict ourselves to $\frac{1}{4}c\omega^{-\frac{1}{2}} \le r \le 40$. With this, we obtain

$$|\partial_r^k h_-(r,\omega)| \lesssim_k r^{-k}\omega^{-\frac{1}{2}} \tag{4.14}$$

for all $k \in \mathbb{N}$. For the first derivative, we differentiate Eq. (4.13) and integrate by parts to get

$$\begin{aligned}\partial_r h_-(r,\omega) &= \int_r^{50} e^{-2\mu^{1/2}(s-r)}\mathcal{O}(r^0\omega^0 s^0)h_-(s,\omega)ds \\ &= g_1(r,\omega) + \int_r^{50} H_1(r,s,\omega)\partial_s h_-(s,\omega)ds,\end{aligned}$$

where

$$\begin{aligned}g_1(r,\omega) =& \mathcal{O}(r^0\omega^{-\frac{1}{2}}) + e^{-2\mu^{1/2}(50-r)}\mathcal{O}(r^0\omega^{-\frac{1}{2}}) \\ &+ \int_r^{50} e^{-2\mu^{1/2}(s-r)}\mathcal{O}(r^0\omega^{-\frac{1}{2}}s^{-1})h_-(s,\omega)ds,\end{aligned}$$

and $H_1(r,s,\omega) = e^{-2\mu^{1/2}(s-r)}\mathcal{O}(r^0\omega^{-\frac{1}{2}}s^0)$. It is easy to see that $|g_1(r,\omega)| \lesssim r^{-1}\omega^{-\frac{1}{2}}$ and also $|H_1(r,s,\omega)| \lesssim \omega^{-\frac{1}{2}}$ such that the standard result on Volterra equations implies that $|\partial_r h_-(r,\omega)| \lesssim r^{-1}\omega^{-\frac{1}{2}}$. Furthermore, we have the bound $|\partial_r g_1(r,\omega)| \lesssim r^{-2}\omega^{-\frac{1}{2}}$, since $\omega^{\frac{1}{2}}e^{-2\mathrm{Re}\mu^{1/2}(50-r)} \lesssim 1$ for $r \le 40$. For higher derivatives we get terms with a similar structure, which yields Eq. (4.14) as well as Eq. (4.12) for $\tilde{v}_-$ (up to a constant). The second solution is obtained by setting

$$\tilde{v}_+(r,\omega) := \tilde{v}_-(r,\omega)\left[\frac{\tilde{\psi}_+(a,\omega)}{\tilde{\psi}_-(a,\omega)} + \int_a^r \tilde{v}_-(s,\omega)^{-2}ds\right],$$

where $a := \frac{1}{4}c\omega^{-\frac{1}{2}}$. As in the proof of Lemma 4.4 this yields an equation for $h_+ = \frac{\tilde{v}_+}{\tilde{\psi}_+}$ given by

$$h_+(r,\omega) = 1 + \mathcal{O}(r^0\omega^{-\frac{1}{2}}) + \int_a^r e^{-2\mu^{1/2}(r-s)}\mathcal{O}(r^0 s^0\omega^0)ds.$$

Integration by parts yields

$$\begin{aligned}h_+(r,\omega) =& 1 + \mathcal{O}(r^0\omega^{-\frac{1}{2}}) + e^{-2\mu^{1/2}(r-a)}\mathcal{O}(r^0\omega^{-\frac{1}{2}}) \\ &+ \int_a^r e^{-2\mu^{1/2}(r-s)}\mathcal{O}(s^{-1}\omega^{-\frac{1}{2}})ds.\end{aligned}$$

From this it is obvious that $|h_+(r,\omega)-1| \lesssim \omega^{-\frac{1}{2}}$. Restricting ourselves to $\frac{1}{2}c\omega^{-\frac{1}{2}} \leq r \leq 40$, we get that $|\partial_r^k h_+(r,\omega)| \lesssim_k r^{-k}\omega^{-\frac{1}{2}}$ for all $k \in \mathbb{N}$ and $\omega \gg 1$ provided $c > 1$ is chosen sufficiently large. Dividing both $\tilde{v}_\pm(r,\omega)$ by $\alpha_\pm$ yields Eq. (4.12). The explicit formula above shows that $\tilde{v}_\pm(r,\omega) = \mathcal{O}(r^0\omega^{-\frac{1}{4}})e^{\pm\mu^{1/2}r}$. q.e.d.

4.1.3. A global fundamental system. The proof of the following result is along the lines of Corollary 4.7 and Lemma 4.8 in [**9**]. From now on we fix $c > 1$ such that Lemma 4.6 holds.

Lemma 4.7. *Provided $\omega \gg c^2$ we have the representation*

$$\begin{aligned}\tilde{v}_\pm(r,\omega) &= \mathcal{O}(\omega^0)v_0(r,\omega) \mp [\mathrm{i} + \mathcal{O}(\omega^{-\frac{1}{2}})]v_1(r,\omega), \\ v_0(r,\omega) &= [\alpha_- + \mathcal{O}(\omega^{-\frac{1}{2}})]\tilde{v}_-(r,\omega) + [\alpha_+ + \mathcal{O}(\omega^{-\frac{1}{2}})]\tilde{v}_+(r,\omega),\end{aligned}$$

for all $r \in (0,40]$. Under the same assumptions,

$$v_-(r,\omega) = \alpha e^{10\mu^{1/2}}[1 + \mathcal{O}(\omega^{-\frac{1}{2}})]\tilde{v}_-(r,\omega) + e^{-10\mu^{1/2}}\mathcal{O}(\omega^{-\frac{1}{2}})\tilde{v}_+(r,\omega),$$

for all $r \in [\frac{1}{2}c\omega^{-\frac{1}{2}},\infty)$, where $\alpha_\pm, \alpha \in \mathbb{C}\backslash\{0\}$.

Lemma 4.8. *The functions $\{v_0, v_-\}$ provide a fundamental system for Eq. (4.3) and we have the representations*

$$\begin{aligned}v_0(r,\omega) &= e^{10\mu^{1/2}}\mathcal{O}(\omega^{-\frac{1}{2}})v_-(r,\omega) + \alpha_1 e^{10\mu^{1/2}}[1 + \mathcal{O}(\omega^{-\frac{1}{2}})]v_+(r,\omega), \\ v_-(r,\omega) &= e^{10\mu^{1/2}}\mathcal{O}(\omega^0)v_0(r,\omega) + \alpha_2 e^{10\mu^{1/2}}[1 + \mathcal{O}(\omega^{-\frac{1}{2}})]v_1(r,\omega),\end{aligned}$$

for all $r > 0$, all $\omega \gg c^2$ and some constants $\alpha_j \in \mathbb{C}\backslash\{0\}$, $j = 1,2$. The Wronskian is given by

$$W(\omega) := W(v_-(\cdot,\omega), v_0(\cdot,\omega)) = \alpha_1 e^{10\mu^{1/2}}[1 + \mathcal{O}(\omega^{-\frac{1}{2}})].$$

Proof. The fundamental systems $\{v_-, v_+\}$ and $\{\tilde{v}_-, \tilde{v}_+\}$ are valid in the intervals $r \in [4,\infty)$ and $r \in [\frac{1}{2}c\omega^{-\frac{1}{2}}, 40]$, respectively. The connection formula yields

$$\tilde{v}_\pm(r,\omega) = \frac{W_\pm(\cdot,\omega)}{W(v_-(\cdot,\omega),v_+(\cdot,\omega))}v_-(r,\omega) + \frac{\tilde{W}_\pm(\cdot,\omega)}{W(v_+(\cdot,\omega),v_-(\cdot,\omega))}v_+(r,\omega), \tag{4.15}$$

where $W_\pm(\omega) := W(\tilde{v}_\pm(\cdot,\omega), v_+(\cdot,\omega))$, $\tilde{W}_\pm(\omega) := W(\tilde{v}_\pm(\cdot,\omega), v_-(\cdot,\omega))$. By Lemma 4.7, v_0 can be expressed in terms of Hankel functions. In combination with Eq. (4.15) this yields

$$\begin{aligned}v_0(r,\omega) &= [\alpha_- + \mathcal{O}(\omega^{-\frac{1}{2}})]W_-(\omega)v_-(r,\omega)\\ &+ [\alpha_+ + \mathcal{O}(\omega^{-\frac{1}{2}})]W_+(\omega)v_-(r,\omega)\\ &+ [\alpha_- + \mathcal{O}(\omega^{-\frac{1}{2}})]\tilde{W}_-(\omega)v_+(r,\omega) + [\alpha_+ + \mathcal{O}(\omega^{-\frac{1}{2}})]\tilde{W}_+(\omega)v_+(r,\omega),\end{aligned}$$

for some constants $\alpha_\pm \in \mathbb{C}\setminus\{0\}$ and all $r > 0$. Evaluation at $r = 10$ shows that

$$\begin{aligned}\tilde{W}_+(\omega) &= \beta_+ e^{10\mu^{1/2}}[1+\mathcal{O}(\omega^{-\frac{1}{2}})], & \tilde{W}_-(\omega) &= e^{-10\mu^{1/2}}\mathcal{O}(\omega^{-\frac{1}{2}}),\\ W_+(\omega) &= \mathcal{O}(\omega^{-\frac{1}{2}})e^{10\mu^{1/2}}, & W_-(\omega) &= \beta_- e^{-10\mu^{1/2}}[1+\mathcal{O}(\omega^{-\frac{1}{2}})],\end{aligned}$$

for some $\beta_\pm \in \mathbb{C}\backslash\{0\}$. This implies the representation of v_0. The representation for v_- is a consequence of Lemma 4.7. The expression for the Wronskian can be verified easily. q.e.d.

Having a global fundamental system for Eq. (4.3), we can consider the inhomogenous equation, which yields an explicit formula for the resolvent for $\omega \gg 1$.

Lemma 4.9. *Fix $\alpha > -\frac{1}{75}$. Set $b = 4\alpha - 3$ and choose M_α sufficiently large. Let $\lambda = \alpha + \mathrm{i}\,\omega$, $\omega > M_\alpha$, and for $\tilde{g} \in \mathcal{H}$, $\tilde{g} = g(|\cdot|)$, set $f := 4g(2\cdot)$. Then the resolvent $R_L(\lambda) : \mathcal{H} \to \mathcal{D}(L) \subset \mathcal{H}$ exists and*

$$[R_L(\lambda)\tilde{g}](\xi) = [\mathcal{R}(4\omega)f]\,(|\xi|/2)\,,$$

where

$$[\mathcal{R}(\omega)f](r) := \int_r^\infty g_1(r,s,\omega)f(s)ds + \int_0^r g_2(r,s,\omega)f(s)ds, \tag{4.16}$$

$$\begin{aligned}g_1(r,s,\omega) &= \tfrac{r^{-3}e^{\frac{1}{2}(r^2-s^2)}s^3}{W(\omega)}v_0(r,\omega)v_-(s,\omega),\\ g_2(r,s,\omega) &= \tfrac{r^{-3}e^{\frac{1}{2}(r^2-s^2)}s^3}{W(\omega)}v_-(r,\omega)v_0(s,\omega).\end{aligned} \tag{4.17}$$

Proof. First, we use the above results and the variation of constants formula to infer that solutions to the equation

$$v''(r) - r^2v(r) - \frac{4\nu^2-1}{4r^2}v(r) + \tilde{V}(r)v(r) - (b+\mathrm{i}\,\tilde{\omega})v(r) = -r^3e^{-\frac{r^2}{2}}f(r) \tag{4.18}$$

are of the general form

$$\begin{aligned}v(r) = c_0v_0(r,\tilde{\omega}) + c_-v_-(r,\tilde{\omega}) - v_0(r,\tilde{\omega})\int_{r_0}^r \frac{v_-(s,\tilde{\omega})}{W(\tilde{\omega})}s^3e^{-\frac{s^2}{2}}f(s)ds\\ + v_-(r,\tilde{\omega})\int_{r_1}^r \frac{v_0(s,\tilde{\omega})}{W(\tilde{\omega})}s^3e^{-\frac{s^2}{2}}f(s)ds\end{aligned}$$

for constants $c_0, c_- \in \mathbb{C}$, $r_0, r_1 \in \mathbb{R}$, provided $b > -4$ and $\tilde{\omega} \geq M_b$ for some $M_b > 0$ large enough. For fixed $\alpha > -\frac{1}{75}$, we define $b := 4\alpha - 3$, $M_\alpha := \frac{1}{4}M_b$. For $\lambda = \alpha + \mathrm{i}\,\omega \in \Omega_\alpha$, we set $\tilde{\omega} = 4\omega$. Since $\lambda \in \rho(L)$,

$(\lambda - L)R_L(\lambda)\tilde{g} = \tilde{g}$. With $\tilde{w}_\lambda := R_L(\lambda)\tilde{g}$, $\tilde{w}_\lambda = w_\lambda(|\cdot|)$, we infer that w_λ satisfies the equation

$$\lambda w_\lambda(\rho) - w''_\lambda(\rho) - \frac{6}{\rho}w'_\lambda(\rho) + \frac{1}{2}\rho w'_\lambda(\rho) + w_\lambda(\rho) - V(\rho)w_\lambda(\rho) = g(\rho).$$

Thus, $v_\lambda(r) := r^3 e^{-\frac{1}{2}r^2} w_\lambda(2r)$ solves Eq. (4.18) for $f := 4g(2\cdot)$. By definition, $b > -4$ and $\tilde{\omega} \geq M_b$ such that v_λ is of the form stated above. The fact that $R_L(\lambda)\tilde{g} \in \mathcal{H}$ implies that w_λ is continuous and decays at infinity according to Lemma 2.2. The resulting conditions on v_λ can only be satisfied if

$$c_0 = \int_{r_0}^{\infty} \frac{v_-(s,\tilde{\omega})}{W(\tilde{\omega})} s^3 e^{-\frac{s^2}{2}} f(s) ds, \quad c_- = \int_0^{r_1} \frac{v_0(s,\tilde{\omega})}{W(\tilde{\omega})} s^3 e^{-\frac{s^2}{2}} f(s) ds.$$

This shows that

$$\begin{aligned} v_\lambda(r) = & \frac{v_0(r,\tilde{\omega})}{W(\tilde{\omega})} \int_r^\infty v_-(s,\tilde{\omega}) s^3 e^{-\frac{1}{2}s^2} f(s) ds \\ & + \frac{v_-(r,\tilde{\omega})}{W(\tilde{\omega})} \int_0^r v_0(s,\tilde{\omega}) s^3 e^{-\frac{1}{2}s^2} f(s) ds, \end{aligned}$$

which yields Eq. (4.16) and Eq. (4.17). q.e.d.

4.2. Uniform bounds for large imaginary parts. It is sufficient to prove Eq. (4.1) on a dense subset, hence we restrict ourselves to $f \in C^\infty_{\mathrm{e},0}(\mathbb{R})$. We assume again that $\omega \gg 1$. Using Eq. (4.17) we define an integral operator $\mathcal{T}(\omega)$ by

$$[\mathcal{T}(\omega)f](r) := \int_r^\infty \partial_r g_1(r,s,\omega) f(s) ds + \int_0^r \partial_r g_2(r,s,\omega) f(s) ds,$$

for $r > 0$.

Lemma 4.10. *Define* $\delta \in [0, \frac{1}{2})$ *by*

$$\delta := \begin{cases} -\frac{3}{2} - \frac{b}{2} & \text{for } -4 < b \leq -3, \\ 0 & \text{for } b > -3. \end{cases}$$

For $m = 0, 1$,

$$\begin{aligned} & r^{m+2}[\mathcal{T}(\omega)f]^{(m)}(r) = O(r\omega^0)f(r) + O(r^2\omega^0)f'(r) \\ & + \sum_{k_1=0}^{2} \left[\mathcal{J}^m_{1k_1}(\omega)(\cdot)^{k_1+1} f^{(k_1)}\right](r) + \sum_{k_2=0}^{3} \left[\mathcal{J}^m_{2k_2}(\omega)(\cdot)^{k_2} f^{(k_2)}\right](r), \end{aligned}$$

where $[\mathcal{J}^m_{ik_i}(\omega)f](r) = \int_0^\infty J^m_{ik_i}(r,s,\omega)f(s)ds$, *for* $i = 1, 2$. *The integral kernels satisfy*

$$|J^m_{ik_i}(r,s,\omega)| \lesssim \min\{r^{-1+\delta}s^{-\delta}, r^{-\delta}s^{-1+\delta}\},$$

for all $r, s > 0$ *and* $\omega \gg 1$. *For* $n = 0, \dots, 3$,

$$r^n[T(\omega)f]^{(n)}(r) = \sum_{j=0}^{2} O(r^{j+1}\omega^0)f^{(j)}(r) + \sum_{k=0}^{3} O(r^k\omega^0)f^{(k)}(r)$$

$$+ \sum_{j_1=0}^{2} \left[\mathcal{K}^n_{1j_1}(\omega)(\cdot)^{j_1+1}f^{(j_1)}\right](r) + \sum_{j_2=0}^{3} \left[\mathcal{K}^n_{2j_2}(\omega)(\cdot)^{j_2}f^{(j_2)}\right](r)$$

$$+ \sum_{j_3=1}^{4} \left[\mathcal{K}^n_{3j_3}(\omega)(\cdot)^{j_3-1}f^{(j_3)}\right](r),$$

where $[\mathcal{K}^n_{ij_i}(\omega)f](r) = \int_0^\infty K^n_{ij_i}(r,s,\omega)f(s)ds$ *for* $i = 1, 2, 3$, *and*

$$|K^n_{ij_i}(r,s,\omega)| \lesssim \min\{r^{-1+\delta}s^{-\delta}, r^{-\delta}s^{-1+\delta}\},$$

for all $r, s > 0$ *and* $\omega \gg 1$.

Proof. The proof is based on the explicit representations of the kernel functions g_1, g_2 in different regimes. We only sketch the argument here and refer the interested reader to the arXiv preprint version [**10**] of this article for full details. Consider the operator $\int_0^r \partial_r g_2(r,s,\omega)f(s)ds$, i.e., the case $s \leq r$. From the representation of v_0 in terms of Weber functions, see Lemma 4.8, we see that the most significant contribution to the kernel g_2 comes from the term

$$\frac{1}{W(\omega)} r^{-3}s^3 e^{\frac{1}{2}(r^2-s^2)} v_-(r,\omega)\mathcal{O}(\omega^0)e^{10\mu^{1/2}}v_+(s,\omega)$$
$$= e^{\frac{1}{2}r^2-\mu\xi(r,\mu)}e^{-[\frac{1}{2}s^2-\mu\xi(s,\mu)]}\mathcal{O}(r^{-3}\langle\omega^{-\frac{1}{2}}r\rangle^{-\frac{1}{2}}s^3\langle\omega^{-\frac{1}{2}}s\rangle^{-\frac{1}{2}}\omega^{-\frac{1}{2}}),$$

see Lemma 4.4, where we restrict ourselves to the regime $10 \leq s \leq r$. With $\mu\partial_r\xi(r,\mu) = \mu^{\frac{1}{2}}\sqrt{1+\frac{r^2}{\mu}}$ one checks that

$$\partial_r e^{\frac{1}{2}r^2-\mu\xi(r,\mu)} = \mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{-1}\omega^{\frac{1}{2}})e^{\frac{1}{2}r^2-\mu\xi(r,\mu)} \tag{4.19}$$

and also

$$e^{\frac{1}{2}r^2-\mu\xi(r,\mu)} = \mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle\omega^{-\frac{1}{2}})\partial_r e^{\frac{1}{2}r^2-\mu\xi(r,\mu)}. \tag{4.20}$$

Consequently, the dominant contribution to $\partial_r g_2(r,s,\omega)$ is of the form

$$e^{\frac{1}{2}r^2-\mu\xi(r,\mu)}e^{-[\frac{1}{2}s^2-\mu\xi(s,\mu)]}\mathcal{O}(r^{-3}\langle\omega^{-\frac{1}{2}}r\rangle^{-\frac{3}{2}}s^3\langle\omega^{-\frac{1}{2}}s\rangle^{-\frac{1}{2}}\omega^0)$$

and it suffices to consider the operator

$$[T_W(\omega)f](r) :=$$
$$\int_0^r \chi(s)e^{\frac{1}{2}r^2-\mu\xi(r,\mu)} \quad e^{-[\frac{1}{2}s^2-\mu\xi(s,\mu)]}r^{-3}\langle\omega^{-\frac{1}{2}}r\rangle^{-\frac{3}{2}}s^3\langle\omega^{-\frac{1}{2}}s\rangle^{-\frac{1}{2}}f(s)ds,$$

where χ is a smooth cut-off that localizes to $[10,\infty)$. By Eq. (4.20), an integration by parts yields

$$[T_W(\omega)f](r) = \mathcal{O}(r^{-1}\omega^0)f(r) + \int_0^r \Big(\chi(s)e^{\frac{1}{2}r^2-\mu\xi(r,\mu)}e^{-[\frac{1}{2}s^2-\mu\xi(s,\mu)]} \mathcal{O}(r^{-3}\langle\omega^{-\frac{1}{2}}r\rangle^{-\frac{3}{2}}s^0\langle\omega^{-\frac{1}{2}}s\rangle^{\frac{1}{2}}\omega^{-\frac{1}{2}})\partial_s[s^3f(s)]\Big)ds,$$

where we have used $\mathcal{O}(\langle\omega^{-\frac{1}{2}}r\rangle^{-1}\omega^{-\frac{1}{2}}) = \mathcal{O}(r^{-1}\omega^0)$ for the boundary term. From Eqs. (4.19) and (4.20) it follows that one may trade derivatives in r for derivatives in s at the expense of additional weights. More precisely, repeated integration by parts yields

$$\partial_r^n[T_W(\omega)f](r) = \sum_{k=0}^n \mathcal{O}(r^{-1-n+k}\omega^0)f^{(k)}(r) + \int_0^r \Big(\chi(s)e^{\frac{1}{2}r^2-\mu\xi(r,\mu)}e^{-[\frac{1}{2}s^2-\mu\xi(s,\mu)]} \mathcal{O}(r^{-3}\langle\omega^{-\frac{1}{2}}r\rangle^{-\frac{3}{2}-n}s^0\langle\omega^{-\frac{1}{2}}s\rangle^{\frac{1}{2}+n}\omega^{-\frac{1}{2}})\partial_s^{1+n}[s^3f(s)]\Big)ds$$

for any $n\in\mathbb{N}_0$. From Eq. (4.7) we infer the bound

$$\Big|e^{\frac{1}{2}r^2-\mu\xi(r,\mu)}e^{-[\frac{1}{2}s^2-\mu\xi(s,\mu)]}\Big| \lesssim \langle\omega^{-\frac{1}{2}}r\rangle^{-\frac{b}{2}}\langle\omega^{-\frac{1}{2}}s\rangle^{\frac{b}{2}},$$

and it is straightforward to prove the stated estimate for the integral kernel.

In the case $r\leq s$, the most singular contribution to the kernel g_1 comes from the regime $0<r\leq s\lesssim\omega^{-\frac{1}{2}}$ and the term

$$\tfrac{1}{W(\omega)}r^{-3}s^3e^{\frac{1}{2}(r^2-s^2)}v_0(r,\omega)\mathcal{O}(\omega^0)e^{10\mu^{1/2}}v_1(s,\omega) = \mathcal{O}(r^{-3}s^3\omega^0)v_0(r,\omega)v_1(s,\omega),$$

see Lemma 4.8. From Lemma 4.5 we infer $\partial_r[r^{-3}v_0(r,\omega)] = \mathcal{O}(r\omega^{\frac{9}{4}})$ and this shows that the most important contribution to $\partial_r g_1(r,s,\omega)$ is of the form $\mathcal{O}(rs\omega)$. With this it is straightforward to prove the stated bounds. The other cases are in some sense interpolates which can be treated similarly. q.e.d.

Lemma 4.11. *For $m=0,1$, $n=0,\dots,3$, all $f\in C^\infty_{\mathrm{e},0}(\mathbb{R})$ and all $\omega\gg1$, we have the bounds*

$$\|(\cdot)^{m+2}[\mathcal{T}(\omega)f]^{(m)}\|_{L^2(\mathbb{R}^+)} \lesssim \|f(|\cdot|)\|,$$
$$\|(\cdot)^{n}[\mathcal{T}(\omega)f]^{(n)}\|_{L^2(\mathbb{R}^+)} \lesssim \|f(|\cdot|)\|.$$

Proof. Choose $C=C_b>0$ sufficiently large such that the results of Lemma 4.10 hold for all $\omega\geq C$. The integral operators $\mathcal{J}^m_{ik_i}(\omega)$ and

$\mathcal{K}^n_{ij_i}(\omega)$ extend to bounded operators on $L^2(\mathbb{R}^+)$ by Lemma 5.5 in [7]. Since all bounds are uniform in ω, we infer

$$\|\mathcal{J}^m_{ik_i}(\omega)f\|_{L^2(\mathbb{R}^+)} \lesssim \|f\|_{L^2(\mathbb{R}^+)}, \quad \|\mathcal{K}^n_{ij_i}(\omega)f\|_{L^2(\mathbb{R}^+)} \lesssim \|f\|_{L^2(\mathbb{R}^+)},$$

for all $f \in C^\infty_{\mathrm{e},0}(\mathbb{R})$. This yields

$$\begin{aligned}
&\|(\cdot)^{m+2}[\mathcal{T}(\omega)f]^{(m)}\|_{L^2(\mathbb{R}^+)} \lesssim \|(\cdot)f\|_{L^2(\mathbb{R}^+)} + \|(\cdot)^2 f'\|_{L^2(\mathbb{R}^+)}\\
&+ \sum_{k_1=0}^{2} \|\mathcal{J}^m_{1k_1}(\omega)(\cdot)^{k_1+1}f^{(k_1)}\|_{L^2(\mathbb{R}^+)} + \sum_{k_2=0}^{3} \|\mathcal{J}^m_{2k_2}(\omega)(\cdot)^{k_2}f^{(k_2)}\|_{L^2(\mathbb{R}^+)}\\
&\lesssim \sum_{k_1=0}^{2} \|(\cdot)^{k_1+1}f^{(k_1)}\|_{L^2(\mathbb{R}^+)} + \sum_{k_2=0}^{3} \|(\cdot)^{k_2}f^{(k_2)}\|_{L^2(\mathbb{R}^+)} \lesssim \|f(|\cdot|)\|,
\end{aligned}$$

for $m = 0, 1$, where the last step follows from Lemma 2.1. The second estimate can be derived analogously. q.e.d.

4.2.1. Proof of Proposition 4.1. Fix $\alpha > -\frac{1}{75}$. Set $b = 4\alpha - 3$ and choose $C_\alpha > 0$ sufficiently large such that the above results hold for all $\omega \geq C_\alpha$. Set $M_\alpha := \frac{1}{4}C_\alpha$. Let $\lambda = \alpha + \mathrm{i}\omega$, $\omega \geq M_\alpha$. For $\tilde{g} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$, $\tilde{g} = g(|\cdot|)$, we have that $g \in C^\infty_{\mathrm{e},0}(\mathbb{R})$ and we set $f := 4g(2\cdot)$. Furthermore, we define $\tilde{\omega} := 4\omega$. According to Lemma 4.9 we have an explicit expression for the resolvent. By definition, rescaling and the fact that $[\mathcal{R}(\tilde{\omega})f]' = \mathcal{T}(\tilde{\omega})f$, we can apply Lemma 4.11 to infer that

$$\begin{aligned}
\|R_L(\lambda)\tilde{g}\| &\lesssim \|(\cdot)^2[\mathcal{T}(\tilde{\omega})f]\|_{L^2(\mathbb{R}^+)} + \|(\cdot)^3[\mathcal{T}(\tilde{\omega})f]'\|_{L^2(\mathbb{R}^+)}\\
&+ \sum_{k=0}^{3} \|(\cdot)^k[\mathcal{T}(\tilde{\omega})f]^{(k)}\|_{L^2(\mathbb{R}^+)} \lesssim \|\tilde{g}\|.
\end{aligned}$$

The density of $C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ in $\mathcal{H}$ implies that Eq. (4.1) holds for all $\tilde{f} \in \mathcal{H}$. For negative imaginary parts we write $\lambda = \alpha - \mathrm{i}\omega$, $\omega \geq M_\alpha$. Since L has real coefficients, the equation $(\lambda - L)R_L(\lambda)\tilde{g} = \tilde{g}$ yields

$$(\overline{\lambda} - L)\overline{R_L(\lambda)\tilde{g}} = \overline{\tilde{g}}.$$

Hence, $\overline{R_L(\lambda)\tilde{g}} = R(\overline{\lambda})\overline{\tilde{g}}$. By applying the above result we infer that Proposition 4.1 holds.

4.3. Growth estimates for $S(\tau)$.

Lemma 4.12. *There exists a projection $P \in \mathcal{B}(\mathcal{H})$ with* $\mathrm{rg}P = \mathrm{span}(\tilde{\mathbf{g}})$*, such that P commutes with $S(\tau)$ for all $\tau \geq 0$ and*

$$\|PS(\tau)\tilde{u}\| = e^\tau \|P\tilde{u}\|$$

for all $\tilde{u} \in \mathcal{H}$. Moreover,

$$\|(1-P)S(\tau)\tilde{u}\| \leq Ce^{-a\tau}\|(1-P)\tilde{u}\| \tag{4.21}$$

for $a = \frac{1}{150}$, all $\tilde{u} \in \mathcal{H}$, all $\tau \geq 0$ and some constant $C \geq 1$.

Proof. By Lemma 3.8, the eigenvalue $\lambda = 1$ is isolated in the spectrum of L. Hence, we can define a spectral projection $P \in \mathcal{B}(\mathcal{H})$ by

$$P = \frac{1}{2\pi\,\mathrm{i}} \int_\gamma R_L(\lambda) d\lambda,$$

where γ is a positively oriented circle around 1 in the complex plane with radius $r_\gamma = \frac{1}{2}$, cf. [**20**], p. 178. Note that $[P, S(\tau)] = 0$ for all $\tau \geq 0$. Furthermore, $\mathcal{H} = \ker P \oplus \operatorname{rg} P$ and L is decomposed into the parts $L_\mathcal{M}$ and $L_\mathcal{N}$ in $\mathcal{M} := \operatorname{rg} P$ and $\mathcal{N} := \ker P$, respectively. The respective spectra are given by

$$\sigma(L_\mathcal{M}) = \{1\}, \quad \sigma(L_\mathcal{N}) = \sigma(L) \setminus \{1\} \subseteq \{z \in \mathbb{C} : \operatorname{Re} z \leq -\tfrac{1}{75}\},$$

see Lemma 3.8. One always has that $\ker(1 - L) \subseteq \operatorname{rg} P$, see for example [**19**]. We show that in our case also the reverse inclusion holds. First, we observe that P has finite rank. This is a consequence of the invariance of the essential spectrum under relative compact perturbations, [**20**], p. 239, Theorem 5.28, and the fact that $1 \notin \sigma(L_0)$. We infer that the operator $1 - L_\mathcal{M} : \mathcal{M} \to \mathcal{M}$ is finite-dimensional with zero as its only spectral point. Consequently, $1 - L_\mathcal{M}$ is nilpotent and there exists a minimal $k \geq 1$ such that $(1 - L_\mathcal{M})^k \tilde{u} = 0$ for all $\tilde{u} \in \mathcal{M} = \operatorname{rg} P$. If $k = 1$, the desired inclusion $\operatorname{rg} P \subseteq \ker(1 - L)$ follows immediately. Assume that $k \geq 2$. Then there exists a nontrivial $\tilde{v} \in \operatorname{rg}(1 - L_\mathcal{M}) \cap \ker(1 - L_\mathcal{M})$ and thus, a $\tilde{u} = u(|\cdot|) \in \mathcal{M} \subset \mathcal{D}(L)$ satisfying $(1 - L_\mathcal{M})\tilde{u} = (1 - L)\tilde{u} = c\tilde{\mathbf{g}}$ for some $c \in \mathbb{C} \setminus \{0\}$, see Lemma 3.9. Without loss of generality, we set $c = -1$. By introducing polar coordinates we see that u solves

$$u''(\rho) + \tfrac{6}{\rho} u'(\rho) - \tfrac{1}{2}\rho u'(\rho) + V(\rho) u(\rho) - 2u(\rho) = \mathbf{g}(\rho). \tag{4.22}$$

Since $\tilde{u} \in \mathcal{H}$, we have $u \in C[0, \infty)$ and $\lim_{\rho \to \infty} \rho^{\frac{3}{2}} |u(\rho)| = 0$, see Lemma 2.2. We will see that there is no solution of Eq. (4.22) having these properties. For the homogeneous version of Eq. (4.22) we have the fundamental system $\{\mathbf{g}, \mathbf{h}\}$, see the proof of Lemma 3.9, where $\mathbf{h}$ is given by Eq. (3.17). By the variation of constants formula, the general solution to Eq. (4.22) is of the form

$$\begin{aligned} u(\rho) =& c_0 \mathbf{g}(\rho) + c_1 \mathbf{h}(\rho) \\ & - \mathbf{g}(\rho) \int_{\rho_0}^{\rho} \mathbf{h}(s) \mathbf{g}(s) s^6 e^{-\frac{s^2}{4}} ds + \mathbf{h}(\rho) \int_{\rho_1}^{\rho} \mathbf{g}(s)^2 s^6 e^{-\frac{s^2}{4}} ds. \end{aligned}$$

Recall the representation $\mathbf{h}(\rho) = \rho^{-5} \langle \rho \rangle^2 e^{\frac{\rho^2}{4}} H(\rho)$ from the proof of Lemma 3.9. To guarantee continuity of u at zero, we are forced to choose $c_1 = \int_0^{\rho_1} \mathbf{g}(s)^2 s^6 e^{-\frac{s^2}{4}} ds$ such that

$$u(\rho) = c_0 \mathbf{g}(\rho) - \mathbf{g}(\rho) \int_{\rho_0}^{\rho} s\langle s\rangle^2 H(s)\mathbf{g}(s) ds + \rho^{-5}\langle\rho\rangle^2 e^{\frac{\rho^2}{4}} H(\rho) \int_0^{\rho} \mathbf{g}(s)^2 s^6 e^{-\frac{s^2}{4}} ds.$$

To obtain decay at infinity we must have

$$\lim_{\rho\to\infty} \int_0^{\rho} \mathbf{g}(s)^2 s^6 e^{-\frac{s^2}{4}} ds = 0.$$

This, however, is impossible since the integrand is strictly positive on $\mathbb{R}^+$. By contradiction, we infer that $k = 1$ and $\mathcal{M} = \mathrm{rg} P = \ker(1 - L_{\mathcal{M}}) \subseteq \ker(1 - L) = \mathrm{span}(\tilde{\mathbf{g}})$.

Since $\tilde{\mathbf{g}}$ is an eigenfunction of L we have $\|PS(\tau)\tilde{u}\| = \|S(\tau)P\tilde{u}\| = e^{\tau}\|P\tilde{u}\|$ for all $\tilde{u} \in \mathcal{H}$. Standard arguments show that $(L_{\mathcal{N}}, \mathcal{D}(L) \cap \mathcal{N})$ generates a C_0-semigroup $\{S_{\mathcal{N}}(\tau) : \tau \geq 0\}$ with $S(\tau)|_{\mathcal{N}} = S_{\mathcal{N}}(\tau)$. It is well-known that for all $\tau > 0$, $r(S_{\mathcal{N}}(\tau)) = e^{\omega_{\mathcal{N}}\tau}$, where $\omega_{\mathcal{N}}$ denotes the growth bound of the semigroup on $\mathcal{N}$ and $r(S_{\mathcal{N}}(\tau))$ is the spectral radius of the bounded operator $S_{\mathcal{N}}(\tau) : \mathcal{N} \to \mathcal{N}$, see [**11**], p. 251. To obtain Eq. (4.21), it suffices to show that for each $\tau > 0$,

$$\Lambda_\tau := \{z \in \mathbb{C} : |z| > e^{-\frac{1}{75}\tau}\} \subseteq \rho(S_{\mathcal{N}}(\tau)).$$

Let $z \in \Lambda_\tau$ for some fixed $\tau > 0$ and assume that $z = e^{\lambda\tau}$ for some $\lambda \in \mathbb{C}$. Then $\mathrm{Re}\lambda = \frac{1}{\tau}\log|z| > -\frac{1}{75}$ and therefore, $\lambda \in \rho(L_{\mathcal{N}})$. Hence,

$$\{\lambda \in \mathbb{C} : z = e^{\lambda\tau}\} \subseteq \rho(L_{\mathcal{N}}).$$

By Proposition 4.1 and the fact that $R_{L_{\mathcal{N}}}(\lambda)\tilde{f} = R_L(\lambda)|_{\mathcal{N}}\tilde{f}$ for $\tilde{f} \in \mathcal{N}$, we infer

$$\|R_{L_{\mathcal{N}}}(\tfrac{1}{\tau}\log|z| + \tfrac{\mathrm{i}}{\tau}\arg z + \tfrac{2\pi\,\mathrm{i}\,k}{\tau})\tilde{f}\| \leq C\|\tilde{f}\|$$

for all $\tilde{f} \in \mathcal{N}$, all $k \in \mathbb{Z}$, and an absolute constant $C > 0$. In particular,

$$\sup\{\|R_{L_{\mathcal{N}}}(\lambda)\| : z = e^{\lambda\tau}\} < \infty$$

and by [**26**], Theorem 3, $z \in \rho(S_{\mathcal{N}}(\tau))$. This implies $\omega_{\mathcal{N}} \leq -\frac{1}{75}$. Since the growth bound is defined as an infimum (that may not be attained), we see that for each $\varepsilon > 0$ there exists a constant $C_\varepsilon \geq 1$ such that

$$\|(1-P)S(\tau)\tilde{u}\| = \|S_{\mathcal{N}}(\tau)(1-P)\tilde{u}\| \leq C_\varepsilon e^{-(\frac{1}{75}-\varepsilon)\tau}\|(1-P)\tilde{u}\|.$$

Choosing $\varepsilon = \frac{1}{150}$ implies the claim. q.e.d.

5. Nonlinear Stability

5.1. Estimates for the nonlinearity. In the sequel, we denote by $\mathcal{B} \subset \mathcal{H}$ the closed unit ball in $\mathcal{H}$. For $\tilde{u} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ we define

$$N(\tilde{u}) := f_1(|\cdot|)\tilde{u}^2 + f_2(|\cdot|)\tilde{u}^3,$$

where

$$f_1(\rho) := -9(1+\rho^2\mathbf{W}(\rho)), \quad f_2(\rho) := -3\rho^2.$$

Lemma 5.1. *The nonlinearity extends to a continuous mapping $N : \mathcal{H} \to \mathcal{H}$ that satisfies*

$$\|N(\tilde{u}_1) - N(\tilde{u}_2)\| \lesssim (\|\tilde{u}_1\| + \|\tilde{u}_2\|)\|\tilde{u}_1 - \tilde{u}_2\| \tag{5.1}$$

for all $\tilde{u}_1, \tilde{u}_2 \in \mathcal{B} \subset \mathcal{H}$. Furthermore, N is differentiable at every $\tilde{u} \in \mathcal{H}$ with Fréchet-derivative $DN(\tilde{u}) \in \mathcal{B}(\mathcal{H})$ and the mapping $DN : \mathcal{H} \to \mathcal{B}(\mathcal{H})$ is continuous.

Proof. Using Lemma 2.1 and the fact that

$$|f_1^{(2k)}(\rho)| \lesssim_k \langle\rho\rangle^{-2k}, \quad |f_1^{(2k+1)}(\rho)| \lesssim_k \rho\langle\rho\rangle^{-2k-2}$$

for all $\rho \geq 0$ and $k \in \mathbb{N}_0$, it is easy to verify that

$$\|f_1(|\cdot|)\tilde{u}\tilde{v}\| \lesssim \|\tilde{u}\|\|\tilde{v}\|, \quad \|f_2(|\cdot|)\tilde{u}\tilde{v}\tilde{w}\| \lesssim \|\tilde{u}\|\|\tilde{v}\|\|\tilde{w}\| \tag{5.2}$$

for all $\tilde{u}, \tilde{v}, \tilde{w} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$. However, for the sake of brevity, we omit the details of the calculation and refer the reader to the arXiv preprint version [**10**] of this article. Eq. (5.2) implies Eq. (5.1) for all $\tilde{u}_1, \tilde{u}_2 \in \mathcal{B} \cap C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$. By density of $C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ in $\mathcal{H}$, N extends to a continuous mapping $N : \mathcal{H} \to \mathcal{H}$ such that Eq. (5.1) holds for all $\tilde{u}_1, \tilde{u}_2 \in \mathcal{H}$. To see that the nonlinearity is Fréchet-differentiable, we again assume that $\tilde{u}, \tilde{v} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ and set

$$DN(\tilde{u})\tilde{v} := 2f_1(|\cdot|)\tilde{u}\tilde{v} + 3f_2(|\cdot|)\tilde{u}^2\tilde{v}.$$

The above estimates show that $\|DN(\tilde{u})\tilde{v}\| \lesssim_{\tilde{u}} \|\tilde{v}\|$. By density, $DN(\tilde{u})$ extends to a bounded, linear operator on $\mathcal{H}$. Moreover, for all $\tilde{u}_1, \tilde{u}_2 \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$,

$$\|DN(\tilde{u}_1) - DN(\tilde{u}_2)\|_{\mathcal{B}(\mathcal{H})} \leq \gamma_2(\|\tilde{u}_1\|, \|\tilde{u}_2\|)\|\tilde{u}_1 - \tilde{u}_2\|$$

with $\gamma_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ continuous, and thus DN can be extended to a continuous mapping $DN : \mathcal{H} \to \mathcal{B}(\mathcal{H})$. Finally, it is easy to see that $DN(\tilde{u})$ is indeed the Fréchet-derivative of N at $\tilde{u}$. q.e.d.

5.2. Initial data operator. We fix $T_0 > 0$, set

$$\mathbf{W}(T,\rho) := \tfrac{T}{T_0}\mathbf{W}\left(\tfrac{\sqrt{T}}{\sqrt{T_0}}\rho\right)$$

with $\mathbf{W}$ given by Eq. (1.11) and define for $\tilde{v} \in \mathcal{H}$, $T \in [T_0 - \delta, T_0 + \delta]$ and $0 < \delta \leq \frac{T_0}{2}$,

$$U(\tilde{v}, T) := T\tilde{v}(T^{\frac{1}{2}}\cdot) + \mathbf{W}(T, |\cdot|) - \mathbf{W}(|\cdot|).$$

Lemma 5.2. *For fixed $\tilde{v} \in \mathcal{H}$, the mapping $T \mapsto U(\tilde{v}, T)$: $[T_0 - \delta, T_0 + \delta] \to \mathcal{H}$ is continuous. Furthermore, if $\|\tilde{v}\| \leq \delta$ then*

$$\|U(\tilde{v}, T)\| \lesssim \delta$$

for all $T \in [T_0 - \delta, T_0 + \delta]$.

Proof. We prove the statement for $T_0 = 1$ (the general case is analogous). Fix $\tilde{v} \in \mathcal{H}$ and let $\delta \in (0, \frac{1}{2}]$. For $T, \tilde{T} \in [1-\delta, 1+\delta]$, we have

$$\begin{aligned}\|U(\tilde{v}, T) - U(\tilde{v}, \tilde{T})\| &\lesssim |T - \tilde{T}|\|\tilde{v}\| + |T - \tilde{T}| \\ &+ \|\mathbf{W}(T^{\frac{1}{2}}|\cdot|) - \mathbf{W}(\tilde{T}^{\frac{1}{2}}|\cdot|)\| + \|\tilde{v}(T^{\frac{1}{2}}\cdot) - \tilde{v}(\tilde{T}^{\frac{1}{2}}\cdot)\|\end{aligned}$$

by rescaling. The first three terms tend to zero in the limit $T \to \tilde{T}$. Let $\varepsilon > 0$ be arbitrary. By density, there is a $\tilde{u} \in C^\infty_{\mathrm{rad},0}(\mathbb{R}^7)$ such that $\|\tilde{v} - \tilde{u}\| < \varepsilon$, then

$$\|\tilde{v}(T^{\frac{1}{2}}\cdot) - \tilde{v}(\tilde{T}^{\frac{1}{2}}\cdot)\| \lesssim \|\tilde{v} - \tilde{u}\| + \|\tilde{u}(T^{\frac{1}{2}}\cdot) - \tilde{u}(\tilde{T}^{\frac{1}{2}}\cdot)\|,$$

where the last term vanishes as $T \to \tilde{T}$. This implies the continuity of $U(\tilde{v}, \cdot) : [1-\delta, 1+\delta] \to \mathcal{H}$. For $\tilde{v} \in \mathcal{H}$, $\|\tilde{v}\| \leq \delta$ we get

$$\begin{aligned}\|U(\tilde{v}, T)\| &\leq T\|\tilde{v}(T^{\frac{1}{2}}\cdot)\| + \|T\mathbf{W}(T^{\frac{1}{2}}|\cdot|) - \mathbf{W}(|\cdot|)\| \\ &\lesssim \|\tilde{v}\| + \|T\mathbf{W}(T^{\frac{1}{2}}|\cdot|) - \mathbf{W}(|\cdot|)\| \lesssim \delta\end{aligned}$$

for all $T \in [1-\delta, 1+\delta]$. q.e.d.

5.3. Operator formulation of Eq. (1.14). With the above definitions, Eq. (1.14) can now be considered as an abstract initial value problem on $\mathcal{H}$. The corresponding integral equation reads

$$\Phi(\tau) = S(\tau)U(\tilde{v}, T) + \int_0^\tau S(\tau - \tau')N(\Phi(\tau'))d\tau', \tag{5.3}$$

where $\{S(\tau) : \tau \geq 0\}$ is the semigroup generated by $(L, \mathcal{D}(L))$, see Corollary 3.6. We set $a = \frac{1}{150}$ and introduce the Banach space

$$\mathcal{X} := \{\Phi \in C([0,\infty), \mathcal{H}) : \|\Phi\|_{\mathcal{X}} := \sup_{\tau \geq 0} e^{a\tau}\|\Phi(\tau)\| < \infty\}.$$

By $\mathcal{X}_\delta$ we denote the closed subspace

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \leq \delta\}.$$

5.4. Correction of the unstable behavior. We define

$$C(\Phi, \tilde{u}) := P\tilde{u} + \int_0^\infty e^{-\tau'} PN(\Phi(\tau'))d\tau',$$

and set

$$K(\Phi, \tilde{u})(\tau) := S(\tau)\tilde{u} + \int_0^\tau S(\tau - \tau')N(\Phi(\tau'))d\tau' - e^\tau C(\Phi, \tilde{u}).$$

Lemma 5.3. *Choose $\delta > 0$ sufficiently small and $c > 0$ sufficiently large. For every $\tilde{u} \in \mathcal{H}$ with $\|\tilde{u}\| \leq \frac{\delta}{c}$, there exists a unique $\Phi(\tilde{u}) \in \mathcal{X}_\delta$ that satisfies*

$$\Phi(\tilde{u}) = K(\Phi(\tilde{u}), \tilde{u}).$$

Furthermore, the mapping $\tilde{u} \mapsto \Phi(\tilde{u})$ is continuous.

Proof. We first convince ourselves that for fixed $\tilde{u} \in \mathcal{H}$, we have $K(\cdot, \tilde{u}) : \mathcal{X}_\delta \to \mathcal{X}_\delta$. For $\Phi \in \mathcal{X}_\delta$, the continuity of $K(\Phi, \tilde{u})(\tau)$ in τ is a consequence of the strong continuity of the semigroup. We write $K(\Phi, \tilde{u})(\tau) = PK(\Phi, \tilde{u})(\tau) + (1 - P)K(\Phi, \tilde{u})(\tau)$ and use Lemma 5.1 together with $N(0) = 0$ to estimate

$$\begin{aligned}
\|PK(\Phi, \tilde{u})(\tau)\| &\lesssim \int_\tau^\infty e^{-(\tau'-\tau)} \|PN(\Phi(\tau'))\| d\tau' \\
&\lesssim e^{-2a\tau} \|\Phi\|_{\mathcal{X}}^2 \lesssim e^{-2a\tau} \delta^2, \\
\|(1 - P)K(\Phi, \tilde{u})(\tau)\| &\lesssim e^{-a\tau} \|\tilde{u}\| + \int_0^\tau e^{-a(\tau-\tau')} \|N(\Phi(\tau'))\| d\tau' \\
&\lesssim e^{-a\tau} (\tfrac{\delta}{c} + \delta^2).
\end{aligned}$$

Thus, $\|K(\Phi, \tilde{u})(\tau)\| \leq e^{-a\tau}\delta$ by choosing $\delta > 0$ sufficiently small and $c > 0$ sufficiently large. For $\Phi, \Psi \in \mathcal{X}_\delta$ we use the fact that

$$\|N(\Phi(\tau)) - N(\Psi(\tau))\| \lesssim \delta e^{-2a\tau} \|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\tau > 0$ to infer that

$$\|P[K(\Phi, \tilde{u})(\tau) - K(\Psi, \tilde{u})(\tau)]\| \lesssim \delta e^{-2a\tau} \|\Phi - \Psi\|_{\mathcal{X}}$$

and

$$\|(1 - P)[K(\Phi, \tilde{u})(\tau) - K(\Psi, \tilde{u})(\tau)]\| \lesssim \delta e^{-a\tau} \|\Phi - \Psi\|_{\mathcal{X}}.$$

This implies that there is a $0 < k < 1$ such that

$$\|K(\Phi, \tilde{u}) - K(\Psi, \tilde{u})\|_{\mathcal{X}} \leq k \|\Phi - \Psi\|_{\mathcal{X}},$$

provided $\delta > 0$ is sufficiently small. Since $\mathcal{X}_\delta \subset \mathcal{X}$ is closed, we can apply the Banach fixed point theorem to infer the existence of a unique solution $\Phi_{\tilde{u}}$ to the equation $\Phi = K(\Phi, \tilde{u})$. Standard arguments show that the mapping $\tilde{u} \mapsto \Phi(\tilde{u}) := \Phi_{\tilde{u}}$ is continuous. q.e.d.

5.5. Proof of Theorem 1.3. Let $\tilde{v} \in \mathcal{H}$ with $\|\tilde{v}\| \leq \frac{\delta}{M^2}$, for $0 < \delta \leq \frac{T_0}{2}$. By Lemma 5.2,

$$\|U(\tilde{v}, T)\| \leq \tfrac{K\delta}{M},$$

for all $T \in I_{M,\delta} := [T_0 - \frac{\delta}{M}, T_0 + \frac{\delta}{M}]$ and some $K > 0$. By choosing M sufficiently large, we obtain

$$\|U(\tilde{v}, T)\| \leq \tfrac{\delta}{c},$$

for all $T \in I_{M,\delta}$, where $c > 0$ is the constant from Lemma 5.3. Let $\delta > 0$ be sufficiently small such that Lemma 5.3 applies. Hence, for every $T \in I_{M,\delta}$ there exists a unique solution $\Phi_T := \Phi(U(\tilde{v}, T)) \in \mathcal{X}_\delta$ to the equation

$$\Phi_T(\tau) = S(\tau)U(\tilde{v}, T) + \int_0^\tau S(\tau - \tau')N(\Phi_T(\tau'))d\tau' - e^\tau C(\Phi_T, U(\tilde{v}, T)).$$

Furthermore, since the mappings $T \mapsto U(\tilde{v}, T)$ and $\tilde{u} \mapsto \Phi(\tilde{u})$ are continuous, we see that $T \mapsto \Phi_T$ is continuous. Since $\operatorname{rg} P \subseteq \operatorname{span}(\tilde{\mathbf{g}})$, with $\tilde{\mathbf{g}}(\xi) = \mathbf{g}(|\xi|)$ denoting the symmetry mode, see Lemma 4.12, it suffices to show that

$$\big(C(\Phi_{T_{\tilde{v}}}, U(\tilde{v}, T_{\tilde{v}}))\big|\tilde{\mathbf{g}}\big) = 0, \tag{5.4}$$

for some $T_{\tilde{v}} \in I_{M,\delta}$. First, we estimate

$$\left(\int_0^\infty e^{-\tau} PN(\Phi_T(\tau)) d\tau \middle| \tilde{\mathbf{g}}\right) \lesssim \|\tilde{\mathbf{g}}\| \int_0^\infty e^{-\tau} \|PN(\Phi_T(\tau))\| d\tau$$
$$\lesssim \int_0^\infty e^{-\tau} \|\Phi_T(\tau)\|^2 d\tau \lesssim \delta^2.$$

Since $\partial_T \mathbf{W}(T, \cdot)|_{T=T_0} = \alpha \mathbf{g}$ for some $\alpha \in \mathbb{R}$, $\alpha \neq 0$, we have

$$\mathbf{W}(T, \cdot) = \mathbf{W} + \alpha(T - T_0)\mathbf{g} + (T - T_0)^2 R(T, \cdot),$$

where R depends continuously on T and $\|R(T, |\cdot|)\| \lesssim 1$ for all $T \in I_{M,\delta}$. Thus,

$$U(\tilde{v}, T) = T\tilde{v}(T^{\frac{1}{2}} \cdot) + \alpha(T - T_0)\tilde{\mathbf{g}} + (T - T_0)^2 R(T, |\cdot|),$$

which implies

$$(PU(\tilde{v}, T)|\tilde{\mathbf{g}}) = \alpha(T - T_0)\|\tilde{\mathbf{g}}\|^2 + f(T),$$

with $|f(T)| \lesssim \frac{\delta}{M^2} + \delta^2$. Consequently, we can write Eq. (5.4) as

$$T = T_0 + F(T)$$

for a continuous function F that satisfies $|F(T)| \lesssim \delta^2 + \frac{\delta}{M^2}$ for all $T \in I_{M,\delta}$. Choose M sufficiently large and δ sufficiently small to ensure that $|F(T)| \leq \frac{\delta}{M}$. By Brouwer's fixed point theorem, there exists a $T_{\tilde{v}} \in [T_0 - \frac{\delta}{M}, T_0 + \frac{\delta}{M}]$ such that $T_{\tilde{v}} = T_0 + F(T_{\tilde{v}})$. Hence, $\Phi_{T_{\tilde{v}}} \in \mathcal{X}_\delta$ satisfies Eq. (5.3). For the uniqueness of the solution in $C([0, \infty), \mathcal{H})$, we refer the reader for example to the proof of Theorem 4.11 in [8].

5.6. Theorem 1.3 implies Theorem 1.1. For fixed $T_0 > 0$, we choose $\delta, M > 0$ such that Theorem 1.3 holds. Set $\delta' = \frac{\delta}{M}$. For $u_0 \in \mathcal{E}$ and $\mathbf{w}_T$ defined as in Eq. (1.5), we have

$$\|u_0(|\cdot|) - \mathbf{w}_{T_0}(0, |\cdot|)\| = C\|u_0 - \mathbf{w}_{T_0}(0, \cdot)\|_{\mathcal{E}},$$

where the constant comes from the integration over $\mathbb{S}^6$. Assume that $\|u_0 - \mathbf{w}_{T_0}(0, \cdot)\|_{\mathcal{E}} \leq \frac{\delta'}{K}$ for $K = CM$ and set $\tilde{v}_0 := u_0(|\cdot|) - \mathbf{w}_{T_0}(0, |\cdot|)$. By definition of the initial data operator, we have

$$U(\tilde{v}_0, T) = Tu_0(T^{\frac{1}{2}}|\cdot|) - \mathbf{W}(|\cdot|) =: \Phi_0^T.$$

Obviously, $\Phi_0^T \in C^\infty_{\mathrm{rad}}(\mathbb{R}^7)$ for all $T \in [T_0 - \delta', T_0 + \delta']$, $\|\Phi_0^T\| < \infty$, and the decay of $\mathbf{W}$ implies $\Phi_0^T \in \mathcal{H}$ by an approximation argument. It is also easy to check that $\Phi_0^T \in \mathcal{D}(\tilde{L}_0) \subset \mathcal{D}(L)$. The smallness condition on the data implies that $\tilde{v}_0$ satisfies the assumptions of Theorem 1.3.

Hence, there is a $T \in [T_0 - \delta', T_0 + \delta']$ such that there exists a unique solution $\Phi \in C([0,\infty), \mathcal{H})$ to

$$\Phi(\tau) = S(\tau)\Phi_0^T + \int_0^\tau S(\tau - \tau')N(\Phi(\tau'))d\tau', \quad \tau \geq 0,$$

where $\Phi(\tau)(\cdot) = \varphi(\tau, |\cdot|)$, $\varphi(\tau, \cdot) \in C[0,\infty) \cap C^2(0,\infty)$ and

$$\|\Phi(\tau)\| \lesssim e^{-\frac{1}{150}\tau}, \quad \forall \tau \geq 0.$$

Lemma 5.1 implies that Φ is also a classical solution, see Theorem 6.1.5 in [**25**]. This means that $\Phi : (0,\infty) \to \mathcal{H}$ is continuously differentiable, $\Phi(\tau) \in \mathcal{D}(L)$ for all $\tau > 0$ and

$$\frac{d}{d\tau}\Phi(\tau) = (L_0 + L')\Phi(\tau) + N(\Phi(\tau)) \quad \tau > 0,$$

with $\Phi(0) = \Phi_0^T$. Recall that L_0 acts as a classical differential operator on functions in $\mathcal{D}(L)$. By setting $\psi(\tau, \rho) := \mathbf{W}(\rho) + \varphi(\tau, \rho)$, we obtain a classical solution to Eq. (1.10) corresponding to the initial condition $\psi(0, \cdot) = Tu_0(T^{\frac{1}{2}} \cdot)$. As a consequence,

$$u(t, r) := (T - t)^{-1}\psi(-\log(T - t) + \log T, \tfrac{r}{\sqrt{T-t}})$$

solves Eq. (1.6) for all $0 < t < T$. Furthermore, $u(0, r) = u_0(r)$ for all $r \in [0,\infty)$. Note that

$$\|\Delta \mathbf{w}_T(t, |\cdot|)\|_{L^2(\mathbb{R}^7)} = c_1(T-t)^{-\frac{1}{4}}, \quad \|\Delta^2 \mathbf{w}_T(t, |\cdot|)\|_{L^2(\mathbb{R}^7)} = c_2(T-t)^{-\frac{5}{4}},$$

for some constants $c_1, c_2 > 0$. By definition and rescaling, we get that for $k = 1, 2$,

$$\begin{aligned}(T - t)^{-\frac{3}{4}+k}\|\Delta^k u(t, |\cdot|) - \Delta^k \mathbf{w}_T(t, |\cdot|)\|_{L^2(\mathbb{R}^7)} \\ \leq \|\Phi(-\log(T - t) + \log T)\| \lesssim (T - t)^a,\end{aligned}$$

which implies Eq. (1.8). Furthermore,

$$\sup_{r>0} |\mathbf{w}_T(t, \cdot)| = \frac{1}{T - t} \sup_{r>0} \mathbf{W}\left(\tfrac{r}{\sqrt{T-t}}\right) = \frac{1}{T - t}\|\mathbf{W}\|_{L^\infty(\mathbb{R}^+)} = \frac{c_3}{T - t},$$

for some $c_3 > 0$. By Lemma 2.2,

$$\begin{aligned}(T - t)\|u(t, \cdot) - \mathbf{w}_T(t, \cdot)\|_{L^\infty(\mathbb{R}^+)} &\lesssim \|\Phi(-\log(T - t) + \log T)\| \\ &\lesssim (T - t)^a.\end{aligned}$$

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