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N-DIMENSION CENTRAL AFFINE CURVE FLOWS

Chuu-Lian Terng† & Zhiwei Wu*

Abstract

For n-dimensional central affine curve flows, we

1) solve the Cauchy problem with periodic initial data and with initial data having rapidly decaying central affine curvatures,
2) construct Bäcklund transformations, a Permutability formula, and explicit solutions,
3) write down formulas for the Bi-Hamiltonian structure and conservation laws.

1. Introduction

In this paper, we study the Cauchy problem, Bäcklund transformations, and the bi-Hamiltonian structure for central affine curve flows on $\mathbb{R}^n\backslash\{0\}$.

The group $SL(n,\mathbb{R})$ acts transitively on $\mathbb{R}^n\backslash\{0\}$ by the natural action $g\cdot y = gy$ for $g \in SL(n,\mathbb{R})$ and $y \in \mathbb{R}^n$. First we review the central affine arc-length parameter, central affine moving frames, and central affine curvatures for curves in $\mathbb{R}^n\backslash\{0\}$ (cf. [3], [17]). Given a curve γ in $\mathbb{R}^n\backslash\{0\}$, if $\det(\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)})$ is positive, then there is an orientation preserving parameter x unique up to translation, $\frac{\mathrm{d}x}{\mathrm{d}s} = \det(\gamma, \gamma_s, \ldots, \gamma_s^{(n-1)})^{\frac{2}{n(n-1)}}$, such that

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) = 1, \tag{1.1}$$

where $\gamma_x^{(i)} = \frac{\mathrm{d}^i\gamma}{\mathrm{d}x^i}$. Such parameter x is called the *central affine arc-length parameter*, and

$$g = (\gamma, \ldots, \gamma_x^{(n-1)})$$

is called the *central affine moving frame* along γ.

Take the x derivative of (1.1) to get

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-2)}, \gamma_x^{(n)}) = 0.$$

Hence, we have

$$\gamma_x^{(n)} = u_1\gamma + u_2\gamma_x + \cdots + u_{n-1}\gamma_x^{(n-2)}, \tag{1.2}$$

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where $u_i = \det(\gamma, \gamma_x, \ldots, \gamma_x^{(i-2)}, \gamma_x^{(n)}, \gamma_x^{(i)}, \ldots, \gamma_x^{(n-1)})$ is the *i-th central affine curvature* of γ for $1 \leq i \leq n-1$. Note that

$$g_x = g(b+u),$$

where $b = \sum_{i=1}^{n-1} e_{i+1,i}$ and $u = \sum_{i=1}^{n-1} u_i e_{in}$. Here we use e_{ij} to denote the $n \times n$ matrix whose entries are all zero except 1 on the ij-th entry. We also call u the *central affine curvature* of γ.

Let $\mathrm{I} = S^1$ or $\mathbb{R}$, and

$$\mathcal{M}_n(\mathrm{I}) = \{\gamma : \mathrm{I} \to \mathbb{R}^n \backslash \{0\} \,|\, \det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) = 1\},$$
$$V_n = \oplus_{i=1}^{n-1} \mathbb{R} e_{in} \subset sl(n, \mathbb{R}).$$

Let $\Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n)$ be the *central affine curvature map* defined by

$$\Psi(\gamma) = u = \sum_{i=1}^{n-1} u_i e_{in} = g^{-1} g_x - b,$$

where g and u are the central affine moving frame and curvature along γ. It follows from the Existence and Uniqueness Theorem of ordinary differential equations that $\{u_1, \ldots, u_{n-1}\}$ forms a complete set of differential invariants for $\gamma \in \mathcal{M}_n(\mathbb{R})$.

Marí Beffa proved in [**13**] that central affine curvatures for curves $\gamma \in \mathcal{M}_n(\mathbb{R})$ are linear combinations of Wilczynski's projective invariants for curves $\pi(\gamma)$ in $\mathbb{R}P^{n-1}$ in [**24**], where $\pi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}P^{n-1}$ is the natural projection.

We say $\eta(y) = (\eta_1, \ldots, \eta_m) : \mathbb{R} \to \mathbb{R}^m$ is a *differential polynomial of order k in $y = (y_1, \ldots, y_n)^t \in C^\infty(\mathbb{R}, \mathbb{R}^n)$* if each η_ℓ is a polynomial in $(y_i)_x^{(j)}$ for $1 \leq i \leq n$ and $j \leq k$.

A *central affine curve flow* is an evolution equation on $\mathcal{M}_n(\mathbb{R})$ of the form

$$\gamma_t = X(\gamma) = g\xi(u) = \xi_0(u)\gamma + \xi_1 \gamma_x + \cdots + \xi_{n-1} \gamma_x^{(n-1)}, \tag{1.3}$$

where $X(\gamma)$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ, $g(\cdot, t)$ is the central affine moving frame along $\gamma(\cdot, t)$, and $\xi(u) = (\xi_0(u), \ldots, \xi_{n-1}(u))^t$ is a differential polynomial in the central affine curvature $u(\cdot, t)$ of $\gamma(\cdot, t)$.

It is easy to see that central affine curve flows are invariant under the action of $SL(n, \mathbb{R})$ on $\mathbb{R}^n \backslash \{0\}$ and translations in the (x, t)-plane. In other words, if $\gamma(x, t)$ is a solution of (1.3), then so is $\tilde{\gamma}(x, t) = c\gamma(x + r_1, t + r_2)$, where $c \in SL(n, \mathbb{R})$ and $r_1, r_2 \in \mathbb{R}$ are constants.

Drinfel'd and Sokolov associated to each affine Kac–Moody algebra a KdV type soliton hierarchy in [**6**]. The j-th flow of the hierarchy associated to $A_{n-1}^{(1)}$ (i.e., the *$A_{n-1}^{(1)}$-KdV hierarchy*) is of the form

$$u_t = [\partial_x + b + u, Z_{j,0}(u)] = (Z_{j,0}(u))_x + [b + u, Z_{j,0}(u)], \tag{1.4}$$

for some $sl(n, \mathbb{R})$-valued differential polynomial $Z_{j,0}(u)$.

It was proved in [**6**] that the j-th $A^{(1)}_{n-1}$-KdV flow (1.4) is *the j-th Gel'fand–Dickey (GD_n) flow*,

$$L_t = [(L^{\frac{j}{n}})_+, L], \tag{1.5}$$

when $u = \sum_{i=1}^{n-1} u_i e_{in}$ is identified as $L_u = \partial_x^n - \sum_{i=1}^{n-1} u_i \partial_x^{(i-1)}$.

A natural hierarchy of curve flows on $\mathbb{R}P^{n-1}$, whose projective invariants satisfies the $A^{(1)}_{n-1}$-KdV hierarchy, was given by Marí Beffa in [**11**] and also by Ovsienko and Khesin in [**16**]. When $n = 3$, the second projective flow on $\mathbb{R}P^2$ was also noted by Chou and Qu in [**4**]. These curve flows on $\mathbb{R}P^{n-1}$ lift to the following hierarchy of central affine curve flows on $\mathcal{M}_n(\mathbb{R})$,

$$\gamma_t = g Z_{j,0}(u) e_1, \tag{1.6}$$

where $g(\cdot, t)$ and $u(\cdot, t)$ are the central affine moving frame and curvature along $\gamma(\cdot, t)$, $Z_{j,0}(u)$ is as in (1.4) and $e_1 = (1, 0, \ldots, 0)^t \in \mathbb{R}^n$. It was proved by Marí Beffa in [**13**] that the central affine curvature map Ψ gives a one-to-one correspondence between the space of solutions of (1.6) modulo $SL(n, \mathbb{R})$ and the space of solutions of (1.4).

For example, the second and the third central affine curve flow on $\mathcal{M}_n(\mathbb{R})$ are

$$\gamma_t = -\frac{2}{n} u_{n-1} \gamma + \gamma_{xx}, \quad n \neq 2. \tag{1.7}$$

$$\gamma_t = \left(-\frac{3}{n} u_{n-2} + \frac{3(n-3)}{2n} (u_{n-1})_x \right) \gamma - \frac{3}{n} u_{n-1} \gamma_x + \gamma_{xxx}, \quad n \neq 3. \tag{1.8}$$

Note that when $n = 2$, we have $\gamma_{xx} = u_1 \gamma$. So (1.8) becomes

$$\gamma_t = \frac{1}{4} (u_1)_x \gamma - \frac{1}{2} u_1 \gamma_x. \tag{1.9}$$

Pinkall proved in [**17**] that if γ is a solution of (1.9) on $\mathcal{M}_2(\mathbb{R})$, then u_1 is a solution of the KdV equation,

$$(u_1)_t = \frac{1}{4} ((u_1)_{xxx} - 6 u_1 (u_1)_x).$$

Next we explain some of our results:

Cauchy problem

The Cauchy problem for the $A^{(1)}_{n-1}$-KdV hierarchy with rapidly decaying initial data was solved by the method of inverse scattering (cf. [**2**]). We use the central affine curvature map Ψ to show that the solution of the Cauchy problem of the $A^{(1)}_{n-1}$-KdV hierarchy gives the solution of the Cauchy problems for the central affine curve flows with initial data having rapidly decaying central affine curvatures and with periodic initial data. The proof of the rapidly decaying case follows directly from the correspondence between the spaces of solutions of (1.4) and (1.6) under the central affine curvature map Ψ. For the periodic case, we need to solve the period problem.

Bäcklund transformation (BT)

Adler used Miura transform to construct a class of BTs for the GD_n-hierarchy in [**1**]. We constructed in [**23**] a class of BTs with a parameter k for the GD_n hierarchy and showed that Adler's BTs are our BTs with parameter $k = 0$. So we can use the central affine curvature map Ψ to construct BTs for the central affine curve flow (1.6) from BTs for (1.4). But this direct approach does not seem to give a geometric formula relating the new solutions and the given solution of (1.6). We need more results in our paper [**23**] to give a geometric formula. Moreover, BTs with parameter $k \neq 0$ and $k = 0$ for the central affine curve flows are given by different formulas. We also use BTs to construct infinitely many explicit soliton solutions and rational solutions of the central affine curve flow (1.6).

Bi-Hamiltonian structure

The well-known Adler–Gel'fand–Dickey (AGD) bi-Hamiltonian structure for the $A^{(1)}_{n-1}$-KdV hierarchy (cf. [**5**]) is a pair of compatible Poisson structures $\{\,,\}_1$ and $\{\,,\}_2$ on $C^\infty(S^1, V_n)$. Here compatibility means that $c_1\{\,,\}_1 + c_2\{\,,\}_2$ is again a Poison structure for all constants $c_1, c_2 \in \mathbb{R}$.

Note that if $y(u)$ is a differential polynomial in u then $\oint (y(u))_x \mathrm{d}x = 0$ for $u \in C^\infty(S^1, V_n)$ and $\int_{-\infty}^{\infty} (y(u))_x \mathrm{d}x = 0$ for rapidly decaying $u : \mathbb{R} \to V_n$. So the AGD brackets and their Hamiltonian theory works for both the periodic and rapidly decaying cases.

Ovsienko and Khesin in [**16**] determined all symplectic leaves of the AGD brackets $\{\,,\}_1$ and $\{\,,\}_2$ on $C^\infty(S^1, V_n)$. In particular, they showed that the space $C^\infty_e(S^1, V_n)$ of $u \in C^\infty(S^1, V_n)$ with trivial monodromy for the linear operator $\phi^{-1}\phi_x = b + u$ is a symplectic leaf for $\{\,,\}_2$. Since the central affine curvature map Ψ induces a bijection from $\frac{\mathcal{M}_n(S^1)}{SL(n,\mathbb{R})}$ to $C^\infty_e(S^1, V_n)$, the pull back of $\{\,,\}_2$ by Ψ is symplectic on $\frac{\mathcal{M}_n(S^1)}{SL(n,\mathbb{R})}$.

It is known that the two AGD brackets give rise to a sequence $\{\,,\}_i$ of compatible Poisson structures for $i \geq 1$ (cf. [**5**], [**9**]) and the Poisson operator of $\{\,,\}_i$ is $J_i = (J_2 J_1^{-1})^{i-2} J_2$.

Let $\{\,,\}^\wedge_i$ be the Poisson bracket on $\mathcal{M}_n(S^1)$ pulled back by the central affine curvature map Ψ from $\{\,,\}_i$ for $i \geq 1$. So we can use Ψ and formulas for J_i to obtain properties of Poisson structures $\{\,,\}^\wedge_i$ on $\mathcal{M}_n(S^1)$. But this translation of the bi-Hamiltonian theory for the $A^{(1)}_{n-1}$-KdV hierarchy to that of the central affine curve flow hierarchy involves some delicate and long computations. We also prove the following results:

(i) We identify the kernels of $\{\,,\}_2^\wedge$ and $\{\,,\}_3^\wedge$ on $\mathcal{M}_n(S^1)$.
(ii) We find two embeddings from $\mathcal{M}_n(S^1)$ to certain co-adjoint orbits such that the restrictions $\hat{w}_2$ and $\hat{w}_3$ of the co-adjoint orbit symplectic forms are the 2-forms defined by the Poisson structures $\{\,,\}_2^\wedge$ and $\{\,,\}_3^\wedge$.
(iii) The 2-form $\hat{w}_2$ induces a symplectic form on $\frac{\mathcal{M}_n(S^1)}{SL(n,\mathbb{R})}$.
(iv) The group $\mathbb{R}^{n-1}$ acts on $\mathcal{M}_n(S^1)$ generated by the first $(n-1)$ central affine curve flows. Since the central affine curve flows commute with the $SL(n,\mathbb{R})$-action, the direct product $SL(n,\mathbb{R})\times\mathbb{R}^{n-1}$ acts on $\mathcal{M}_n(S^1)$. We show that $\hat{w}_3$ induces a symplectic form on $\frac{\mathcal{M}_n(S^1)}{SL(n,\mathbb{R})\times\mathbb{R}^{n-1}}$.
(v) We write down the formulas for the closed forms $\hat{w}_2$ and $\hat{w}_3$ in terms of tangent vectors of $\mathcal{M}_n(S^1)$.

Result (iii) was proved by Fujioka and Kurose for the case $n=2$ in [**8**], and the formula for $\hat{w}_3$ on $\mathcal{M}_n(S^1)$ was obtained by Calini, Ivey, and Marí Beffa in [**3**] when $n=3$.

Let $\mathcal{G}$ be the Lie algebra of a finite dimensional semi-simple Lie group G. If the Lax pair of a soliton equation is $\mathcal{G}$-valued, then it is often possible to find a homogeneous space $\frac{G}{H}$, moving frames for curves in $\frac{G}{H}$, and a G-invariant geometric curve flow on $\frac{G}{H}$ so that the evolution of differential invariants for a solution of this curve flow is the given soliton equation. In [**12**], [**13**], [**14**], and [**15**], Marí Beffa used Fels and Olver's moving frame method (cf. [**7**]) to construct moving frames for curves in $\frac{G}{H}$ and then used reduction of the natural bi-Hamiltonian structure on $C^\infty(S^1,\mathcal{G})$ to the space of differential invariants to give a general construction of integrable curve flows on $\frac{G}{H}$. The first author in [**21**] used slices of the gauge action of $C^\infty(\mathbb{R},H)$ on $C^\infty(\mathbb{R},\mathcal{G})$ to construct moving frames for curves in $\frac{G}{H}$ so that the space of differential invariants is the phase space of some soliton equation, then used the Lax pair of the soliton equation to construct an integrable curve flow on $\frac{G}{H}$. We refer the readers to these papers for more examples and references concerning integrable curve flows on homogeneous spaces and soliton equations.

This paper is organized as follows: In Section 2, we review the construction of the $A_{n-1}^{(1)}$-KdV hierarchy and introduce the operator P_u that is needed to write down the formula for the AGD brackets. In Section 3, we review the relation between the $A_{n-1}^{(1)}$-KdV and the central affine curve flow hierarchies and study the Cauchy problems with periodic initial data and with initial data having rapidly decaying central affine curvature. We construct Bäcklund transformations and a Permutability formula and families of explicit solutions for the central affine curve flows in Section 4. In Section 5, we review the Hamiltonian aspects of the $A_{n-1}^{(1)}$-KdV hierarchy and write down the AGD brackets in terms of the operator P_u. Results (i)–(v) stated above are proved in Section 6. We use $n = 3$ through out the paper as our running example.

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2. The $A^{(1)}_{n-1}$-KdV hierarchy

In this section, we

1) review the construction of the $A^{(1)}_{n-1}$-KdV hierarchy (cf. [**6**], [**22**]),
2) solve the Cauchy problem for the central affine curve flows with periodic initial data or initial data with rapidly decaying central affine curvatures,
3) define and give properties of operator P_u, which will be needed to write down the formula for the bi-Hamiltonian structure in Section 5.

First, we set up some notations. Let

$$\mathcal{B}_n^+ = \{y = (y_{ij}) \in sl(n, \mathbb{R}) \,|\, y_{ij} = 0, i > j\},$$
$$\mathcal{B}_n^- = \{y = (y_{ij}) \in sl(n, \mathbb{R}) \,|\, y_{ij} = 0, i < j\},$$
$$\mathcal{N}_n^+ = \{y = (y_{ij}) \in sl(n, \mathbb{R}) \,|\, y_{ij} = 0, i \geq j\},$$
$$\mathcal{D}_n = \{y \in gl(n, \mathbb{R}) \,|\, y_{ij} = 0, i \neq j\}$$

denote the subalgebras of upper triangular, lower triangular, strictly upper triangular matrices in $sl(n, \mathbb{R})$ and diagonal matrices in $gl(n, \mathbb{R})$, respectively. Let N_n^+ denote the corresponding Lie subgroup of $\mathcal{N}_n^+$.

Construction of hierarchy on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$

Let

$$\mathcal{L}(sl(n, \mathbb{R})) = \left\{ \xi(\lambda) = \sum_{j \leq n_0} \xi_j \lambda^j \middle| \xi_j \in sl(n, \mathbb{R}), n_0 \text{ integer} \right\}.$$

For $\xi \in \mathcal{L}(sl(n, \mathbb{R}))$, we use the following notation:

$$\xi_+ = \sum_{j \geq 0} \xi_j \lambda^j, \quad \xi_- = \sum_{j<0} \xi_j \lambda^j.$$

Let

$$\Lambda = e_{1n}\lambda + b, \quad b = \sum_{i=1}^{n-1} e_{i+1,i}. \tag{2.1}$$

A direct computation implies that

$$\Lambda^i = (b^t)^{n-i}\lambda + b^i, \quad \Lambda^{-i} = (b^t)^i + b^{n-i}\lambda^{-1}, \quad 1 \leq i \leq n-1, \tag{2.2}$$

$$\Lambda^n = \lambda \mathrm{I}_n. \tag{2.3}$$

Use (2.2), (2.3) and a direct computation to get the following.

Proposition 2.1. *([6]) Given $\xi \in \mathcal{L}(sl(n,\mathbb{R}))$, then there exist unique $\xi_i \in \mathcal{D}_n$ such that $\xi = \sum_i \xi_i \Lambda^i$.*

Given $y = \mathrm{diag}(y_1, \ldots, y_n)$ and a permutation τ of $\{1, 2, \ldots, n\}$, let $y^\tau = \mathrm{diag}(y_{\tau(1)}, \ldots, y_{\tau(n)})$. Let σ denote the permutation defined by $\sigma(1) = n$ and $\sigma(i) = i - 1$ for $2 \le i \le n$. Then

$$y^\sigma = \mathrm{diag}(y_n, y_1, \ldots, y_{n-1}).$$

A direct computation implies that

$$\Lambda y = y^\sigma \Lambda, \quad \Lambda^i y = y^{\sigma^i} \Lambda^i. \tag{2.4}$$

Proposition 2.2. *([6]) Given $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, then there exists a unique $T = T(q,\lambda) = \mathrm{I} + \sum_{i>0} T_i(q)\Lambda^{-i}$ with $T_i(q) \in C^\infty(\mathbb{R}, \mathcal{D}_n)$, $T(q,\lambda)e_1 = e_1$ and*

$$T(\partial_x + \Lambda + q)T^{-1} = \partial_x + \Lambda + \sum_{i\ge 0} f_i \Lambda^{-i}, \tag{2.5}$$

for some f_i in $C^\infty(\mathbb{R},\mathbb{R})$. Moreover, T_i and f_i are differential polynomials of q.

Example 2.3. ([6]) We explain how to compute $T(u,\lambda) = \mathrm{I} + \sum_{i>0} T_i(u)\Lambda^{-i}$ and f_i of Proposition 2.2 for $u = \sum_{i=1}^{n-1} u_i e_{in}$. It is easy to see that $u = \sum_{i=1}^{n-1} v_i \Lambda^{-i}$, where $v_i = u_{n-i} e_{n-i,n-i}$. The condition $Te_1 = 0$ means that $T_{nk} = 0$ and the $(n-i+1, n-i+1)$ entry of T_{nk+i} is zero for $1 \le i \le n-1$. Equation (2.5) gives

$$T(\Lambda + u) - T_x - \Lambda T - \sum_{i\ge 0} f_i \Lambda^{-i} T = 0. \tag{2.6}$$

Substitute the power series of u and T in Λ into (2.6) and use (2.4) to express the left hand side of (2.5) as $\sum_{i\ge 0} \eta_i \Lambda^{-i}$ for some $\eta_i \in C^\infty(\mathbb{R}, \mathcal{D}_n)$. We use induction on i to show that we can find unique T_i to make $\eta_i = 0$. For example, $\eta_0 = T_1 - T_1^\sigma - f_0 = 0$. Take trace to see that $f_0 = 0$. So $T_1 - T_1^\sigma = 0$, which implies that $T_1 = h_1 \mathrm{I}_n$ for some h. But $(T_1)_{nn} = 0$, so $T_1 = 0$. The coefficient of Λ^{-1} gives $\eta_1 = T_2 - T_2^\sigma + v_1 - f_1 = 0$. Take trace to see that $nf_1 = \mathrm{tr}(v_1) = u_{n-1}$. So T_2 can be solved uniquely from $T_2 - T_2^\sigma = \frac{1}{n} u_{n-1} \mathrm{I}_n - v_1$ and $(T_2)_{n-1,n-1} = 0$. The coefficient of Λ^{-2} gives $T_3 - T_3^\sigma + v_2 - (T_2)_x - f_2 = 0$. Take trace to get $nf_2 = u_{n-2} - \mathrm{tr}(T_2)_x$. Continue this computation to get T_3, f_3, ... etc. In particular, we obtain

$$\begin{aligned} f_1(u) &= \frac{1}{n} u_{n-1}, \\ f_2(u) &= \frac{1}{n} u_{n-2} - \mathrm{tr}(T_2(u))_x, \\ f_3(u) &= \frac{1}{n}\left(u_{n-3} + \frac{n-3}{2n} u_{n-1}^2\right) - \mathrm{tr}(T_3(u))_x. \end{aligned}$$

Since $[\partial_x + \Lambda + \sum_{i\geq 0} f_i\Lambda^{-i}, \Lambda^j] = 0$, bracket (2.5) with Λ^j gives

$$[T(\partial_x + \Lambda + q)T^{-1}, \Lambda^j] = 0.$$

Conjugate this by T^{-1} to see that $[\partial_x + \Lambda + q, T^{-1}\Lambda^j T] = 0$. Hence, we get the following known result:

Theorem 2.4. *([6], [22]) Let q and $T(q,\lambda)$ be as in Proposition 2.2, and*

$$Y(q,\lambda) = T^{-1}(q,\lambda)\Lambda T(q,\lambda).$$

Then

1) $Y(q,\lambda) = \Lambda + \sum_{i\geq 0} y_i(q)\Lambda^{-i}$ *for some differential polynomial* $y_i(q) \in C^\infty(\mathbb{R}, \mathcal{D}_n)$,
2) Y *satisfies*

$$\begin{cases} [\partial_x + \Lambda + q, Y(q,\lambda)] = 0, \\ Y(q,\lambda)^n = \Lambda^n = \lambda \mathrm{I}_n, \end{cases} \tag{2.7}$$

3) $Y(q,\lambda)$ *can be solved uniquely from* (2.7),
4)

$$[\partial_x + \Lambda + q, Y^j(q,\lambda)] = 0. \tag{2.8}$$

The second equation of (2.7) gives

$$Y^{n+j}(q,\lambda) = \lambda Y^j(q,\lambda). \tag{2.9}$$

Write

$$Y^j(q,\lambda) = \sum_{-\infty}^{[\frac{j}{n}]+1} Y_{j,i}(q)\lambda^i. \tag{2.10}$$

It follows from (2.9) and (2.8) that we have

$$Y_{n+j,0}(q) = Y_{j,-1}(q), \tag{2.11}$$

$$[\partial_x + b + q, Y_{j,i}(q)] = [Y_{j,i-1}(q), e_{1n}]. \tag{2.12}$$

Since the right hand side of (2.12) lies in $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$,

$$q_t = [\partial_x + b + q, Y_{j,0}(q)] \tag{2.13}$$

is an evolution on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$, which is *the j-th flow* on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$. It can be checked that these flows commute and u is a solution of (2.13) if and only if

$$[\partial_x + \Lambda + q, \partial_t + (Y^j(q,\lambda))_+] = 0,$$

for all parameter $\lambda \in \mathbb{C}$.

The $A_{n-1}^{(1)}$-KdV flows as quotient flows under gauge action

Next we review the gauge invariance of (2.13) ([6]). The group $C^\infty(\mathbb{R}, N_n^+)$ acts on $C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ by gauge transformations

$$\triangle * q = \triangle(b+q)\triangle^{-1} - \triangle_x \triangle^{-1} - b,$$

for $\triangle \in C^\infty(\mathbb{R}, N_n^+)$ and $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$. Or equivalently,

$$\triangle(\partial_x + b + q)\triangle^{-1} = \partial_x + b + \triangle * q.$$

It can be checked (cf. [**22**]) that if $Y(q,\lambda)$ is a solution of (2.7) and $\triangle \in C^\infty(\mathbb{R}, N_n^+)$, then $Y(\triangle * q, \lambda) = \triangle Y(q,\lambda)\triangle^{-1}$. Hence, the flow (2.13) is invariant under the gauge action of $C^\infty(\mathbb{R}, N_n^+)$.

It was proved in [**6**] that given $q \in C^\infty(\mathbb{R}, \mathcal{B}_n^+)$ there exist unique $\triangle \in C^\infty(\mathbb{R}, N_n^+)$ and $u \in C^\infty(\mathbb{R}, V_n)$ such that $\triangle * q = u$. In other words, each gauge orbit of $C^\infty(\mathbb{R}, N_n^+)$ meets $C^\infty(\mathbb{R}, V_n)$ exactly once and $C^\infty(\mathbb{R}, V_n)$ is a cross section of the gauge action. So $C^\infty(\mathbb{R}, V_n)$ is isomorphic to the orbit space $\frac{C^\infty(\mathbb{R}, \mathcal{B}_n^+)}{C^\infty(\mathbb{R}, N_n^+)}$, and (2.13) induces a quotient flow on the cross section $C^\infty(\mathbb{R}, V_n)$ by projecting down along the orbits. Hence, given $u \in C^\infty(\mathbb{R}, V_n)$, there is a unique $\mathcal{N}_n^+$-valued differential polynomial $\zeta_j(u)$ in u, such that

$$[\partial_x + b + u, Y_{j,0}(u) - \zeta_j(u)] \in V_n = \oplus_{i=1}^{n-1} \mathbb{R} e_{in}. \tag{2.14}$$

Set

$$Z_j(u,\lambda) = (Y^j(u,\lambda))_+ - \zeta_j(u) = \sum_{0 \le i \le [\frac{j}{n}]+1} Z_{j,i}(u)\lambda^i. \tag{2.15}$$

Then

$$Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u), \tag{2.16}$$

and (2.14) can be written as

$$[\partial_x + b + u, Z_{j,0}(u)] \in C^\infty(\mathbb{R}, V_n). \tag{2.17}$$

The *j-th $A_{n-1}^{(1)}$-KdV flow* ($j \ge 1$ and $j \not\equiv 0 \pmod n$) constructed by Drinfel'd–Sokolov in [**6**] is the flow (1.4) on $C^\infty(\mathbb{R}, V_n)$, i.e.,

$$u_t = [\partial_x + b + u,\ Z_{j,0}(u)].$$

Proposition 2.5. *([**6**]) Let $\Lambda = e_{1n}\lambda + b$ be as in (2.1). Then the following statements are equivalent for $u \in C^\infty(\mathbb{R}^2, V_n)$:*

(i) *u is a solution of the j-th $A_{n-1}^{(1)}$-KdV flow, (1.4).*

(ii) *System*

$$\begin{cases} g^{-1}g_x = b + u, \\ g^{-1}g_t = Z_{j,0}(u) \end{cases} \tag{2.18}$$

is solvable for $g : \mathbb{R}^2 \to SL(n,\mathbb{R})$.

(iii) *For all parameter $\lambda \in \mathbb{C}$, we have*

$$[\partial_x + \Lambda + u, \partial_t + Z_j(u,\lambda)] = 0. \tag{2.19}$$

(iv) *Given any constant $\lambda \in \mathbb{C}$, the following system is solvable for $E(x,t,\lambda) \in GL(n,\mathbb{C})$:*

$$\begin{cases} E^{-1}E_x = \Lambda + u, \\ E^{-1}E_t = Z_j(u,\lambda), \\ \overline{E(x,t,\bar{\lambda})} = E(x,t,\lambda). \end{cases} \tag{2.20}$$

Definition 2.6. Let u be a solution of the j-th $A^{(1)}_{n-1}$-KdV flow (1.4), and $\mathcal{O}$ an open subset of $\mathbb{C}$. We call a smooth map $E : \mathbb{R}^2 \times \mathcal{O} \to GL(n,\mathbb{C})$ a *frame* of u if $E(x,t,\lambda)$ is holomorphic for $\lambda \in \mathcal{O}$ and $E(x,t,\lambda)$ satisfies (2.20).

Remark 2.7.

(i) Equation (2.19) gives a flat $sl(n,\mathbb{C})$ connection on the (x,t)-plane for each parameter λ and is called the *Lax pair* of the solution u of (1.4). A solution $E(\cdot,\cdot,\lambda)$ of (2.20) is a *parallel frame* of the flat connection (2.19). The parameter λ is often called the *spectral parameter* in the literature of soliton theory.

(ii) The linear system (2.20) depends on the parameter $\lambda \in \mathbb{C}$ polynomially. So given a holomorphic map $c : \mathbb{C} \to GL(n,\mathbb{C})$ satisfying $\overline{c(\bar{\lambda})} = c(\lambda)$, the solution $E(x,t,\lambda)$ of (2.20) with initial data $E(0,0,\lambda) = c(\lambda)$ is holomorphic in parameter $\lambda \in \mathbb{C}$.

(iii) The third condition of (2.20) implies that $E(x,t,\lambda) \in GL(n,\mathbb{R})$ for $\lambda \in \mathbb{R}$.

(iv) If $E(x,t,\lambda)$ is a frame of a solution u of (1.4) and $c : \mathbb{C} \to GL(n,\mathbb{C})$ is holomorphic and satisfies $\overline{c(\bar{\lambda})} = c(\lambda)$, then $c(\lambda)E(x,t,\lambda)$ is again a frame of u.

From $\mathrm{d}(\ln\det(E)) = \mathrm{tr}(E^{-1}\mathrm{d}E)$ and $\mathrm{tr}(\Lambda + u) = \mathrm{tr}(Z_j(u,\lambda)) = 0$, we have

Corollary 2.8. *If $E(x,t,\lambda)$ is a frame of a solution u of the j-th $A^{(1)}_{n-1}$-KdV flow, then $\det(E(x,t,\lambda))$ is independent of x, t.*

The operator P_u

Recall that $Z_{j,0}(u)$ satisfies (2.17). Next we give some properties of $C \in C^\infty(\mathbb{R}, sl(n,\mathbb{R}))$ that satisfies the same condition.

Theorem 2.9. *Given $u \in C^\infty(\mathbb{R}, V_n)$, if $C = (C_{ij}) \in C^\infty(\mathbb{R}, sl(n,\mathbb{R}))$ satisfying*

$$[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n), \tag{2.21}$$

then

(i) *entries of C are differential polynomials in $u, C_{21}, \ldots, C_{n1}$,*

(ii) *there exist differential polynomials ϕ_i in u and $C_{n1}, \ldots, C_{n-i+3,1}$ for $1 \le i \le n-1$ such that*

$$C_{ni} = C_{n-i+1,1} + (i-1)(C_{n-i+2,1})_x + \phi_i(u, C_{n1}, \ldots, C_{n-i+3,1}), \tag{2.22}$$

(iii) *C_{ij}'s are differential polynomials in $u, C_{n1}, \ldots, C_{n,n-1}$.*

Proof. Let C_j denote the j-th column of C. It follows from $C_x + [b + u, C] \in V_n$ and a direct computation that $C_{j+1} = (C_j)_x + (b+u)C_j$ for $1 \leq j \leq n-1$. Using this formula and $\sum_{i=1}^n C_{ii} = 0$, we can prove (i). Statements (ii) and (iii) can be proved using (i) and induction. q.e.d.

Theorem 2.9 implies that if $C = (C_{ij})$ satisfies (2.21) then C_{ij}'s are determined by the first column or the last row of C. So we have

Corollary 2.10. *Let $u \in C^\infty(\mathbb{R}, V_n)$, and $B, C \in C^\infty(\mathbb{R}, sl(n,\mathbb{R}))$. Suppose B and C satisfies (2.21). If B and C have the same first column or the last row, then $B = C$.*

Definition 2.11. Let $V_n^t = \oplus_{i=1}^{n-1} \mathbb{R} e_{ni}$. Fix $u \in C^\infty(\mathbb{R}, V_n)$, let

$$P_u : C^\infty(\mathbb{R}, V_n^t) \to C^\infty(\mathbb{R}, sl(n,\mathbb{R}))$$

denote the map defined by $P_u(v) = C$, where C is the unique $sl(n,\mathbb{R})$-valued map satisfies $\pi_0(C) = v$ and $[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)$. Here π_0 is the projection of $sl(n,\mathbb{R})$ onto V_n^t defined by

$$\pi_0 : sl(n,\mathbb{C}) \to V_n^t, \quad \pi_0(y) = \sum_{i=1}^{n-1} y_{ni} e_{ni}, \quad \text{for} \quad y = (y_{ij}). \tag{2.23}$$

It follows from Theorem 2.9 that P_u is well-defined and the entries of $P_u(v)$ are differential polynomials in u and v.

Since $[\partial_x + b + u, Z_{j,0}(u)] \in V_n$, we have $Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u)))$. It follows from $\zeta_j(u) \in \mathcal{N}_n^+$ and $Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u)$ that we have the following.

Corollary 2.12. *Let π_0 be the projection onto V_n^t defined by (2.23), $Y(u,\lambda)$ the solution of (2.7), and $Y_{j,0}(u), Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u)$ as in (2.10) and (2.15). Then $Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u))) = P_u(\pi_0(Y_{j,0}(u)))$.*

So the j-th $A_{n-1}^{(1)}$-KdV flow (1.4) on $C^\infty(\mathbb{R}, V_n)$ can be written as

$$u_{t_j} = [\partial_x + b + u, P_u(\pi_0(Y_{j,0}(u)))].$$

Example 2.13. We use the algorithm given in the proof of Theorem 2.9 to compute $P_u(v)$ for $n = 2, 3$. For $n = 2$ and $u = q e_{12}$, $v = \xi e_{21}$, we get

$$P_u(v) = \begin{pmatrix} -\frac{\xi_x}{2} & -\frac{1}{2}\xi_{xx} + q\xi \\ \xi & \frac{1}{2}\xi_x \end{pmatrix}. \tag{2.24}$$

For $n = 3$, $u = u_1 e_{13} + u_2 e_{23}$, and $v = v_1 e_{31} + v_2 e_{32}$, we have

$$P_u(v) = \begin{pmatrix} -v_2' + \frac{2}{3}v_1'' - \frac{2}{3}u_2 v_1 & p_{12} & p_{13} \\ v_2 - v_1' & p_{22} & p_{23} \\ v_1 & v_2 & v_2' - \frac{1}{3}v_1'' + \frac{1}{3}u_2 v_1 \end{pmatrix}, \tag{2.25}$$

where

$$\begin{aligned}
p_{12} &= -v_2'' + \frac{2}{3}(v_1)_x^{(3)} - \frac{2}{3}(u_2v_1)' + u_1v_1,\\
p_{13} &= -(v_2)_x^{(3)} + \frac{2}{3}(v_1)_x^{(4)} - \frac{2}{3}(u_2v_1)'' + (u_1v_1)' + u_1v_2,\\
p_{22} &= -\frac{1}{3}v_1'' + \frac{1}{3}u_2v_1,\\
p_{23} &= -v_2'' + \frac{1}{3}(v_1)_x^{(3)} - \frac{1}{3}(u_2v_1)' + u_2v_2 + u_1v_1.
\end{aligned}$$

Example 2.14. ([**3**], [**6**]) We write down the formula for $Z_{2,0}(u)$ and the second $A_{n-1}^{(1)}$-KdV flow with $n = 3$. We have computed T_i's for small i in Example 2.3, so we can compute the first column of $Y^j(u, \lambda)$ for small j. We see that the first column of $Y_{2,0}(u)$ is $(-\frac{2}{3}u_2, 0, 1)^t$. Use formula (2.25) for $P_u(v)$ to see that $v_1 = 1$ and $v_2 = 0$. By Corollary 2.12, $Z_{2,0}(u) = P_u(v)$. Use Example 2.13 to see that

$$Z_{2,0}(u) = \begin{pmatrix} -\frac{2}{3}u_2 & u_1 - \frac{2}{3}(u_2)_x & (u_1)_x - \frac{2}{3}(u_2)_{xx} \\ 0 & \frac{1}{3}u_2 & u_1 - \frac{1}{3}(u_2)_x \\ 1 & 0 & \frac{1}{3}u_2 \end{pmatrix}. \tag{2.26}$$

Therefore, the second $A_2^{(1)}$-KdV flow, $u_t = [\partial_x + b + u, Z_{2,0}(u)]$, is

$$\begin{cases} (u_1)_t = (u_1)_{xx} - \frac{2}{3}(u_2)_{xxx} + \frac{2}{3}u_2(u_2)_x, \\ (u_2)_t = -(u_2)_{xx} + 2(u_1)_x. \end{cases} \tag{2.27}$$

3. The central affine curve flow

In this section, we review the relation between the $A_{n-1}^{(1)}$-KdV hierarchy and central affine curve flows (cf. [**3**], [**13**]) and use the solution of the Cauchy problem of the $A_{n-1}^{(1)}$-KdV flows to solve the Cauchy problem for the central affine curve flows.

Descriptions of tangent vectors of $\mathcal{M}_n(\mathbb{R})$

Let $\gamma \in \mathcal{M}_n(\mathbb{R})$, and $u \in C^\infty(\mathbb{R}, V_n)$ its central affine curvature. We first derive some conditions for $\delta\gamma$ being a tangent vector of $\mathcal{M}_n(\mathbb{R})$ at γ. Since $\det(\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)}) = 1$, $\{\gamma, \ldots, \gamma_x^{(n-1)}\}$ is a basis of $\mathbb{R}^n$ for each $x \in \mathbb{R}$. So given $X \in C^\infty(\mathbb{R}, \mathbb{R}^n)$, there exist $\xi_i \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $X = \sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$. Take variation of the equation $\det(\gamma, \ldots, \gamma_x^{(n-1)}) = 1$ to see that $\delta\gamma = \sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ if and only if

$$\sum_{i=0}^{n-1} \det(\gamma, \ldots, \gamma_x^{(i-1)}, (\delta\gamma)_x^{(i)}, \gamma_x^{(i+1)} \ldots, \gamma_x^{(n-1)}) = 0. \tag{3.1}$$

Use (1.2), (3.1) and a direct computation to obtain the result noted in [**13**]:

$$\xi_0 = \phi_n(u, \xi_1, \ldots, \xi_{n-1}),$$

for some differential polynomial ϕ_n in $u, \xi_1, \ldots, \xi_{n-1}$.

Example 3.1. $X(\gamma) = \sum_{i=0}^{n-1} \xi_i \gamma_x^{(i)}$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ if

$$\xi_0 = -\frac{1}{2}\xi_1', \quad \text{for } n = 2,$$

$$\xi_0 = -\frac{1}{3}(\xi_2'' + 3\xi_1 + 2u_2\xi_2), \quad \text{for } n = 3,$$

$$\xi_0 = -\frac{1}{4}(\xi_3''' + 4\xi_2'' + 6\xi_1' + 3u_3'\xi_3 + 5u_3\xi_3' + 2u_2\xi_3 + 2u_3\xi_2), \quad \text{for } n = 4.$$

Here we use $y' = y_x$, $y'' = y_x^{(2)}$, $y''' = y_x^{(3)}$. For general n, we have

$$\xi_0 = -\frac{1}{n}((\xi_{n-1})_x^{(n-1)} + \cdots).$$

Proposition 3.2. *([**13**]) Let $\Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n)$ be the central affine curvature map and $u := \Psi(\gamma)$. Then the differential*

$$\mathrm{d}\Psi_\gamma(\delta\gamma) = [\partial_x + b + u, g^{-1}\delta g],$$

where $g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)})$ is the central affine frame for γ and

$$\delta g = (\delta\gamma, (\delta\gamma)_x, \ldots, (\delta\gamma)_x^{(n-1)}).$$

Proof. Take variation of $g^{-1}g_x = b + u$ to get

$$\delta u = -(g^{-1}\delta g)(b + u) + g^{-1}(\delta g)_x.$$

Set $\eta = g^{-1}\delta g$ and compute directly η_x to get $\eta_x = -[b+u, \eta] + \delta u$. But $\Psi(\gamma) = g^{-1}g_x - b$. Hence, $\mathrm{d}\Psi_\gamma(\delta\gamma) = \delta u = [\partial_x + b + u, g^{-1}\delta g]$. q.e.d.

As a consequence of Proposition 3.2, we see that if $\delta\gamma \in (T\mathcal{M}_n(\mathbb{R}))_\gamma$, then $C = g^{-1}\delta g$ satisfies (2.21). In fact, this is also a sufficient condition.

Proposition 3.3. *([**13**]) Let $\gamma \in \mathcal{M}_n(\mathbb{R})$, g and u the central affine moving frame and curvature along γ. If $C = (C_{ij}) : \mathbb{R} \to sl(n, \mathbb{R})$ satisfying (2.21), then*

$$\xi(\gamma) = gCe_1 = \sum_{i=1}^{n} C_{i1}\gamma_x^{(i-1)}$$

is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ.

Proof. Let $g = (\gamma, \ldots, \gamma_x^{(n-1)})$ be the central affine moving frame along γ, C_i the i-th column of C, and $\eta_i = gC_i$. Then $g^{-1}g_x = b + u$ and $\xi(\gamma) = gC_1 = \eta_1$. Let $\rho = [\partial_x + b + u, C] = C_x + [b + u, C]$. Then $\rho \in V_n$, and

$$(gC)_x = g_xC + gC_x = gC(b + u) + g\rho. \tag{3.2}$$

Since the first $(n-1)$ columns of ρ are zero, (3.2) implies that $\eta_{i+1} = (\eta_i)_x$ for $1 \leq i \leq n-1$. So we have

$$(\xi(\gamma))_x^{(i-1)} = (\eta_1)_x^{(i-1)} = \eta_i = gC_i.$$

Hence, $\det(\gamma, \ldots, \gamma_x^{(i-2)}, (\xi(\gamma))_x^{(i-1)}, \gamma_x^{(i)}, \ldots, \gamma_x^{(n-1)}) = C_{ii}$. Since C is in $sl(n, \mathbb{R})$, we have $\sum_{i=1}^n C_{ii} = 0$. So $\xi(\gamma)$ satisfies (3.1), i.e.,

$$\sum_{i=1}^{n} \det(\gamma, \ldots, \gamma_x^{(i-2)}, (\xi(\gamma))_x^{(i-1)}, \gamma_x^{(i)}, \ldots, \gamma_x^{(n-1)}) = 0.$$

Hence, $\xi(\gamma)$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ. q.e.d.

By (2.17), $Z_{j,0}(u)$ satisfies (2.21). So we have the following.

Corollary 3.4. *([**13**]) Let γ, g, u be as in Corollary 2.10, and $Z_{j,0}(u)$ as in (2.16). Then $\xi(\gamma) = gZ_{j,0}(u)e_1$ is a tangent vector field of $\mathcal{M}_n(\mathbb{R})$ and (1.6), $\gamma_t = gZ_{j,0}(u)e_1$, is a flow on $\mathcal{M}_n(\mathbb{R})$ (the j-th central affine curve flow).*

Since $Y_{j,0}(u)e_1 = Z_{j,0}(u)e_1$, the j-th central affine curve flow (1.6) can be written as

$$\gamma_t = gY_{j,0}(u)e_1.$$

Example 3.5. We explained in Example 2.3 how to compute T. Since $Y = T^{-1}\Lambda T$, we can compute $Y_{j,0}(u)$. In particular, we get formulas for small j.

(i) The first column of $Y_{2,0}(u)$ is $(-\frac{2}{n}u_{n-1}, 0, 1, 0, \ldots, 0)^t$ for general n. So the second central affine curve flow on $\mathcal{M}_3(\mathbb{R})$ (cf. [**3**]) is

$$\gamma_t = -\frac{2}{3}u_2\gamma + \gamma_{xx}. \tag{3.3}$$

(ii) For $n \neq 3$, the third central affine curve flow on $\mathcal{M}_n(\mathbb{R})$ is (1.8).

(iii) The fourth and the fifth central affine curve flows on $\mathcal{M}_3(\mathbb{R})$ are

$$\gamma_t = -\frac{1}{9}(2u_2'' - 3u_1' - 2u_2^2)\gamma + \frac{1}{3}(u_2' - u_1)\gamma_x - \frac{u_2}{3}\gamma_{xx},$$
$$\gamma_t = \frac{1}{9}(-u_1'' + u_1u_2)\gamma - \frac{1}{9}(u_2'' - 3u_1' + u_2^2)\gamma_x + \frac{1}{3}(u_2' - 2u_1)\gamma_{xx}.$$

The following is a consequence of Proposition 3.3.

Corollary 3.6. *Suppose g, u are the central affine moving frame and curvature along $\gamma \in \mathcal{M}_n(\mathbb{R})$, respectively. Let $\delta g = (\delta\gamma, \ldots, (\delta\gamma)_x^{(n-1)})$ for $\delta\gamma \in T_\gamma\mathcal{M}_n(\mathbb{R})$. Then:*

1) *$g^{-1}\delta g = P_u(\pi_0(g^{-1}\delta g))$, where π_0 is the projection onto V_n^t defined by (2.23).*
2) *Given $v \in C^\infty(\mathbb{R}, V_n^t)$, then $\delta\gamma := gP_u(v)e_1$ is in $T_\gamma\mathcal{M}_n(\mathbb{R})$ and $g^{-1}\delta g = P_u(v)$.*

Proof. By Proposition 3.2, $\mathrm{d}\Psi_\gamma(\delta\gamma) = [\partial_x + b + u, g^{-1}\delta g] \in C^\infty(\mathbb{R}, V_n)$. Then (1) follows from the Definition of P_u, and (2) follows from Proposition 3.3. q.e.d.

Next we identify the kernel of $\mathrm{d}\Psi$. Recall that $\Psi(\gamma_1) = \Psi(\gamma_2)$ if and only if there exists $c \in SL(n,\mathbb{R})$ such that $\gamma_2 = c\gamma_1$, where Ψ is the central affine curvature map. So we have

Proposition 3.7. *Let $\Psi : \mathcal{M}_n(\mathbb{R}) \to C^\infty(\mathbb{R}, V_n)$ be the central affine curvature map, $\Psi(\gamma) = u$, and g the central affine moving frame along $\gamma \in \mathcal{M}_n(S^1)$. Then*

(i) $\mathrm{Ker}(\mathrm{d}\Psi_\gamma) = \{c_0\gamma \mid c_0 \in sl(n,\mathbb{C})\}$,
(ii) $\Psi^{-1}(\Psi(\gamma))$ *is the $SL(n,\mathbb{R})$-orbit at γ.*

Corollary 3.8. *Let g and u be the central affine moving frame and curvature along $\gamma \in \mathcal{M}_n(\mathbb{R})$, and $\xi \in C^\infty(\mathbb{R}, sl(n,\mathbb{R}))$. If $[\partial_x + b + u, \xi] = 0$, then there exists $c_0 \in sl(n,\mathbb{R})$ such that $\xi = g^{-1}c_0 g = P_u(\pi_0(g^{-1}c_0 g))$ for some $c_0 \in sl(n,\mathbb{R})$ and $g\xi e_1 = c_0\gamma$ is tangent to $\mathcal{M}_n(\mathbb{R})$ at γ.*

Proof. By definition of P_u, $\xi = P_u(\pi_0(\xi))$. Proposition 3.3 implies that $\delta\gamma := g\xi e_1 \in T_\gamma\mathcal{M}_n(\mathbb{R})$. Corollary 3.6 implies that $\xi = P_u(\pi_0(\xi)) = g^{-1}\delta g$. By assumption, $\delta\Psi_\gamma(\delta\gamma) = 0$. Conclusions follow from Proposition 3.7 (i). q.e.d.

Remark 3.9. Note that the image of the central affine curvature map $\Psi : \mathcal{M}_n(S^1) \to C^\infty(S^1, V_n)$ is the space $C_e(S^1, V_n)$ of all $u \in C^\infty(S^1, V_n)$ such that the solution $g^{-1}g_x = b + u$ is periodic, i.e., u has zero monodromy. Moreover, all formulas are given by differential polynomials. Hence, results of this section work for $\mathcal{M}_n(S^1)$.

Correspondence between central affine curve flows and $A^{(1)}_{n-1}$-KdV

The following correspondence between solutions of the central affine curve flows and of the $A^{(1)}_{n-1}$-KdV flows was given in [**13**]. Since the proof is short, we include here.

Theorem 3.10. *([**13**])*

(i) *Let $u = \sum_{i=1}^{n-1} u_i e_{in}$ be a solution of the j-th $A^{(1)}_{n-1}$-KdV flow (1.4), and $g : \mathbb{R}^2 \to SL(n,\mathbb{R})$ a solution of (2.18). Then $\gamma(x,t) := g(x,t)e_1$ is a solution of the j-th central affine curve flow (1.6) with central affine curvature $u(x,t)$.*
(ii) *Let $\gamma(x,t)$ be a solution of (1.6) on $\mathcal{M}_n(\mathbb{R})$, and $u(\cdot,t)$ the central affine curvature along $\gamma(\cdot,t)$. Then u is a solution of the j-th $A^{(1)}_{n-1}$-KdV flow (1.4).*

Proof. (1) Note that $g^{-1}g_x = b + u$ implies $g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)})$ and $\gamma_x^{(n)} = u_1\gamma + \ldots + u_{n-1}\gamma_x^{(n-2)}$. So $\gamma(\cdot,t) \in \mathcal{M}_n(\mathbb{R})$ and $u_1, \ldots, u_{n-1}$

are the central affine curvatures of $\gamma(\cdot,t)$. Since $g_t = gZ_{j,0}(u)$, we get $\gamma_t = g_t e_1 = gZ_{j,0}(u)e_1$, which is the j-th central affine curve flow.

(2) Let $g(\cdot,t)$ be the central affine moving frame along $\gamma(\cdot,t)$. Then $g^{-1}g_x = b+u$. Set $B = g^{-1}g_t$. It follows form $g^{-1}g_x = b+u$ and $g^{-1}g_t = B$ that $u_t = [\partial_x + b + u, B]$. Since $u(\cdot,t) \in V_n$, B satisfies (2.21). By (2.17), $Z_{j,0}(u)$ satisfies (2.21). But $Be_1 = Z_{j,0}(u)e_1$. By Corollary 2.10, we get $B = Z_{j,0}(u)$ and u is a solution of (1.4) q.e.d.

As a consequence of Theorem 3.10, we have

Corollary 3.11. *Let Ψ denote the central affine curvature map, and γ_1, γ_2 solutions of* (1.6) *on $\mathcal{M}_n(\mathbb{R})$. Then*

(i) *$\Psi(\gamma_1(\cdot,t)) = \Psi(\gamma_2(\cdot,t))$ if and only if there is a constant c_0 in $SL(n,\mathbb{R})$ such that $\gamma_2 = c_0\gamma_1$,*

(ii) *Ψ induces a bijection between the space of solutions of* (1.6) *modulo $SL(n,\mathbb{R})$ and the space of solutions of* (1.4)*.*

Example 3.12. Note that $E(x,t,\lambda) = e^{x\Lambda + t\Lambda^j}$ is a frame of the solution $u=0$ of (1.4), $\gamma(x,t) = \exp(bx + b^j t)e_1$ is a solution of the j-th central affine curve flow (1.6) with central affine moving frame $g(x,t) = \exp(bx + b^j t)$ and zero central affine curvature, where $b = \sum_{i=1}^{n-1} e_{i+1,i}$.

We need the following simple Proposition later.

Proposition 3.13. *Let $\gamma(x,t)$ be a solution of the j-th central affine curve flow* (1.6)*, $g(\cdot,t)$ and $u(\cdot,t)$ the central affine moving frame and curvature along $\gamma(\cdot,t)$. Let $E(x,t,\lambda)$ be the frame of the solution u of the j-th $A_{n-1}^{(1)}$-KdV flow (i.e., E satisfies* (2.20)*) with initial data $E(0,0,\lambda) = g(0,0)$ for all $\lambda \in \mathbb{C}$. Then $E(x,t,0) = g(x,t)$.*

Proof. Let $g_1(x,t) = E(x,t,0)$. Since E satisfies (2.20), g_1 satisfies (2.18). But g is a solution of (2.18) and $g_1(0,0) = g(0,0)$. Uniqueness of solutions of (2.18) implies that $g_1 = g$. q.e.d.

Cauchy Problem

Next we discuss the Cauchy problem for the j-th central affine curve flow (1.6). The Cauchy problem for the $A_{n-1}^{(1)}$-KdV hierarchy (1.4) is solved for an open dense subset of rapidly decaying smooth initial data using the method of inverse scattering (cf. [**2**]). As a consequence, we get the solution for the Cauchy problem for the curve flow (1.6):

Theorem 3.14. [Cauchy problem with initial data having rapidly decaying central affine curvatures]
Let $j \geq 1$ and $j \not\equiv 0 \,(\mathrm{mod}\, n)$. Given $\gamma_0 \in \mathcal{M}_n(\mathbb{R})$ with rapidly decaying central affine curvatures $u_1^0, \ldots, u_{n-1}^0$, let g_0 be the central affine moving frame along γ_0. Suppose $u = \sum_{i=1}^{n-1} u_i e_{in}$ is the solution of the j-th $A_{n-1}^{(1)}$-KdV flow (1.4) *with $u(x,0) = \sum_{i=1}^{n-1} u_i^0(x) e_{in}$. Let $g(x,t) : \mathbb{R}^2 \to$*

$SL(n,\mathbb{R})$ be the solution of (2.18) with initial data $g(0,0) = g_0(0)$. Then $\gamma = ge_1$ is the solution of the j-th central affine curve flow (1.6) with $\gamma(x,0) = \gamma_0(x)$. Moreover, the central affine curvatures of $\gamma(\cdot,t)$ are also rapidly decaying.

We use the solution of Cauchy problem of the second $A^{(1)}_{n-1}$-KdV flow with periodic initial data to solve the Cauchy problem for the curve flow (1.7) with periodic initial data. By Theorem 3.10, we only need to solve the period problem of (2.18). In fact, we have the following.

Theorem 3.15. [Cauchy Problem with periodic initial data] *Let $\gamma_0 \in \mathcal{M}_n(S^1)$, g_0 and $u^0_1, \ldots, u^0_{n-1}$ the central affine moving frame and central affine curvatures of γ_0. Suppose $u = \sum_{i=1}^{n-1} u_i e_{in}$ is the solution of the periodic Cauchy problem of (1.4) with initial data $u(x,0) = \sum_{i=1}^{n-1} u^0_i e_{in}$. Let $g : \mathbb{R}^2 \to SL(n,\mathbb{R})$ be the solution of (2.18) with initial data $c_0 = g_0(0)$. Then $\gamma = ge_1$ is a solution of (1.6) with initial data $\gamma(x,0) = \gamma_0(x)$. Moreover, $\gamma(x,t)$ is periodic in x and $\{u_i(\cdot,t), 1 \leq i \leq n-1\}$ are the central affine curvatures for $\gamma(\cdot,t)$.*

Proof. Note that both g_0 and $g(\cdot,0)$ satisfy the same ordinary differential equation, $g^{-1}g_x = b + u(x,0)$, and have the same initial data. So the uniqueness of ordinary differential equations implies that $g(x,0) = g_0(x)$. It follows from Theorem 3.10 that $\gamma(x,t) = g(x,t)e_1$ is a solution of the curve flow (1.6). Moreover, $\gamma(x,0) = g(x,0)e_1 = \gamma_0(x)$. It remains to prove that γ is periodic in x.

Since γ_0 is periodic with period 2π, g_0 and u_0 are periodic in x with period 2π. And $Z_{j,0}(u)$ is periodic because $u(x,t)$ is periodic. It suffices to prove

$$y(t) = g(2\pi,t) - g(0,t)$$

is identically zero. To show this, we calculate

$$\begin{aligned} y_t &= g_t(2\pi,t) - g_t(0,t) \\ &= (gZ_{j,0}(u))(2\pi,t) - (gZ_{j,0}(u))(0,t) = (g(2\pi,t) - g(0,t))Z_{j,0}(u(0,t)) \\ &= y(t)Z_{j,0}(u(0,t)). \end{aligned}$$

Since g_0 is periodic in x with period 2π, $y(0) = g(2\pi,0) - g(0,0) = 0$. Note that $Z_{j,0}(u(0,t))$ is given and 0 is the solution of $y_t = yZ_{j,0}(u(0,t))$ with the same initial condition $y(0) = 0$. So it follows from the uniqueness of ordinary differential equations that y is identically zero. q.e.d.

4. Bäcklund transformations

In this section, we use Bäcklund transformations (BTs) and a Permutability formula for the $A^{(1)}_{n-1}$-KdV hierarchy given in our paper [**23**] to construct BTs and a Permutability formula for the j-th central affine curve flow (1.6). We also apply BTs to the trivial solution to obtain families of explicit solutions of (1.6). We proceed as follows:

1) We try to find a transformation $\gamma \mapsto \tilde{\gamma}$ such that $\gamma = D(\partial_x + h)\tilde{\gamma}$ for some constant $D \in GL(n, \mathbb{R})$ and $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$. If we use $D = \mathrm{I}_n$ then we only get BT with parameter $k = (-1)^n$.
2) We show that such h in (1) must satisfy system (4.3).
3) We use results in [**23**] to construct all solutions of (4.3).
4) Given a solution γ of (1.6) and a solution h of (4.3), we construct a new solution $\tilde{\gamma}$ in terms of γ and h.

Theorem 4.1. *Let $\gamma, \tilde{\gamma}$ be solutions of (1.6), and $g, \tilde{g}$, $u, \tilde{u}$ the central affine moving frames and curvatures of $\gamma, \tilde{\gamma}$, respectively. Suppose there exists $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$ and a constant $D \in GL(n, \mathbb{R})$ such that $\gamma = D(h\tilde{\gamma} + \tilde{\gamma}_x)$. Then we have the following:*

1) *$g = D\tilde{g}\tilde{f}$, where $\tilde{f} = b + h\mathrm{I}_n + N(u, h)$, and $N = (N_{ij})$ is strictly upper triangular with*

$$\begin{cases} N_{ij}(u,h) = \frac{(j-1)!}{(i-1)!(j-i)!} h_x^{(j-i)}, & 1 \le i < j < n, \\ N_{in}(u,h) = \tilde{u}_i + \frac{(n-1)!}{(i-1)!(n-i)!} h_x^{(n-i)}, & 1 \le i \le n-1. \end{cases} \tag{4.1}$$

2) *There exist differential polynomials $s_i(u, h)$ of order $(n - i)$ in h such that*

$$\tilde{u}_i = u_i + s_i(u, h), \quad 1 \le i \le n - 1. \tag{4.2}$$

3) *$\det(\tilde{f}) = (-1)^{n-1}(h_x^{(n-1)} - \xi_n(u, h)) = \det(D)^{-1}$ for some differential polynomial $\xi_n(u, h)$ of order $(n - 2)$ in h.*
4) *h satisfies*

$$(\mathrm{BT})_{u,k} \quad \begin{cases} h_x^{(n-1)} = \xi_n(u,h) - k, \\ h_t = \eta_{n,j}(u,h), \end{cases} \tag{4.3}$$

where $\eta_{n,j}(u, h)$ is a polynomial differential of order j in h and $k = (-1)^{(n-1)}\det(D)^{-1}$.

Proof. (1) Since $g = (\gamma, \gamma_x, \ldots, \gamma_x^{(n-1)})$, we have

$$D(\gamma_x^{(i-1)})_t = D(\gamma_t)_x^{(i-1)} = D\partial_x^{(i-1)}(\partial_x + h)\gamma.$$

Compute directly to get the formula for $\tilde{f}$ and $g = D\tilde{g}\tilde{f}$.

(2) By assumption, we have $g^{-1}g_x = b + u$ and $\tilde{g}^{-1}\tilde{g}_x = b + \tilde{u}$. Use $g = D\tilde{g}\tilde{f}$ to get

$$g^{-1}g_x = b + u = \tilde{f}^{-1}(b + \tilde{u})\tilde{f} + \tilde{f}^{-1}\tilde{f}_x.$$

Multiply the above equation by $\tilde{f}$ on the left to get

$$\tilde{f}_x = \tilde{f}(b + u) - (b + \tilde{u})\tilde{f}.$$

Compare coefficient of the nn-th entry of the above equation and use the formula for $\tilde{f}$ to obtain $\tilde{u}_{n-1} = u_{n-1} - nh_x$. Compare coefficient of the $(n, n - i)$-th entry to get the formula relating $\tilde{u}_{n-i-1}$ and u_{n-i-1}.

(3) Since $\det(g) = \det(\tilde{g}) = 1$ and $g = D\tilde{g}\tilde{f}$, we have $\det(\tilde{f}) = \det(D)^{-1}$. Use (4.1) and (4.2) to see that $\det(\tilde{f}) = (-1)^{n-1}(h_x^{(n-1)} - \xi_n(u,h))$ for some differential polynomial $\xi_n(u,h)$ of order $(n-2)$ in h.

(4) Note that $g = D\tilde{g}\tilde{f}$ implies

$$g^{-1}g_t = Z_{j,0}(u) = \tilde{f}^{-1}\tilde{f}_t + \tilde{f}^{-1}Z_{j,0}(\tilde{u})\tilde{f}.$$

So we have

$$\tilde{f}_t = \tilde{f}Z_{j,0}(u) - Z_{j,0}(\tilde{u})\tilde{f}.$$

Compare the 11-th entry of the above equation to see that h must satisfies the second equation of (4.3) for some $\eta_{n,j}(u,h)$. q.e.d.

Definition 4.2. Given $u \in C^\infty(\mathbb{R}^2, V_n)$ and $h \in C^\infty(\mathbb{R}^2, \mathbb{R})$, let

$$f_{u,h}(x,t,\lambda) = \Lambda + h(x,t)\mathrm{I}_n + N(u,h),$$

where $\Lambda = e_{1n}\lambda + b$ is defined by (2.1) and $N = (N_{ij})$ is strictly upper triangular such that N_{ij} is defined by (4.1) for $j < n$ and $N_{in} = u_i + s_i(u,h) + \frac{(n-1)!}{(i-1)!(n-i)!}h_x^{(n-i)}$. (Note that if we substitute (4.2) to (4.1), then $\tilde{f}$ in Theorem 4.1 is equal to $f_{u,h}(\cdot,\cdot,0)$).

Example 4.3. We use the algorithm given in the proof of Theorem 4.1 to compute $s_i(u,h)$ and $f_{u,h}$ for the case $n = 3$ to get

$$\begin{cases} s_1(u,h) = u_1 - (u_2)_x + 3hh_x, \\ s_2(u,h) = u_2 - 3h_x, \end{cases} \tag{4.4}$$

$$f_{u,h}(x,t,\lambda) = e_{13}\lambda + \begin{pmatrix} h & h_x & u_1 - (u_2)_x + h_{xx} + 3hh_x \\ 1 & h & u_2 - h_x \\ 0 & 1 & h \end{pmatrix}. \tag{4.5}$$

Theorem 4.1 gives a necessary condition on h for the existence of a BT $\gamma \to \tilde{\gamma}$ between solutions of (1.6). The proofs that (4.3) is solvable and the converse of Theorem 4.1 holds are quite complicated and long (cf. [**23**]).

Theorem 4.4. [BT with parameter k for the j-th $A_{n-1}^{(1)}$-KdV] *(*[**23**]*)* *Let u be a solution u of* (1.4)*, and $k \in \mathbb{R}$ a constant. Then the system* $(\mathrm{BT})_{u,k}$ (4.3) *is solvable for h. Moreover, if h is a solution of* $(\mathrm{BT})_{u,k}$*, then we have the following:*

1) $\det(f_{u,h}(x,t,\lambda)) = (-1)^{n-1}(\lambda - k)$.
2) $\tilde{u} = \sum_{i=1}^{n-1}\tilde{u}_i e_{in}$ *is a solution of* (1.6)*, where* $\tilde{u}_i = u_i + s_i(u,h)$ *and* $s_i(u,h)$ *is obtained as in* (4.2).

Definition 4.5. Given a solution u of (1.4) and a solution h of (4.3), let $h \bullet u$ denote the new solution of (1.4) constructed in Theorem 4.4.

The $n \times n$ *mKdV hierarchy* constructed in [**6**] is a hierarchy on $C^\infty(\mathbb{R}, \mathcal{T}_n)$, where $\mathcal{T}_n$ is the space of diagonal matrices in $sl(n,\mathbb{R})$. It is known that if $q = \mathrm{diag}(q_1, \ldots, q_n)$ is a solution of the j-th $n \times n$ mKdV flow, then $L = (\partial_x - q_n) \cdots (\partial_x - q_1)$ is a solution of the j-th GD_n flow. This is the Miura transform. Adler constructed a class of Bäcklund transformations for the GD_n hierarchy via Miura transform as follows:

Theorem 4.6. *([1]) Suppose $L = \partial_x^n - \sum_{i=1}^{n-1} u_i \partial_x^{i-1}$ is a solution of the j-th GD_n-flow (1.5) and we can factor L as the product $(\partial_x - q_n) \cdots (\partial_x - q_1)$ such that $q = \sum_{i=1}^{n-1} q_i e_{ii}$ is a solution of the j-th $n \times n$ mKdV flow. Then $\tilde{L} := (\partial - q_{n-1}) \cdots (\partial - q_1)(\partial - q_n)$ is again a solution of (1.5).*

Theorem 4.7. *([23]) The class of Bäcklund transformations constructed by Adler's Theorem 4.6 is the class of Bäcklund transformations with parameter $k = 0$ constructed in Theorem 4.4. In fact, if we use the same notations as in Theorem 4.6, then $h := -q_n$ is a solution of $(\mathrm{BT})_{u,0}$.*

Theorem 4.8. [Darboux transform] *([23])*
Let $E(x, t, \lambda)$ be a frame of a solution u of (1.6) holomorphic for $\lambda \in \mathcal{O}$, $k \in \mathbb{R}$ a constant, and $\mathbf{c} = (c_1, \ldots, c_{n-1}, -1)^t$ a constant vector. Let v_i denote the i-th coordinate of $E^{-1}(x, t, k)(\mathbf{c})$ for $1 \leq i \leq n$. Then we have the following.

1) *$h := -\frac{v_{n-1}}{v_n}$, is a solution of (4.3), and all solutions of (4.3) can be obtained this way.*
2) *If $k \neq 0$, then $\tilde{E} = E f_{u,h}^{-1}$ is a frame of $h \bullet u$ and $\tilde{E}(x, t, \lambda)$ is holomorphic for $\lambda \in \mathbb{C} \backslash \{k\}$.*
3) *If $k = 0$, then $F(x, t, \lambda) = C(\lambda) E(x, t, \lambda) f_{u,h}^{-1}(x, t, \lambda)$ is a frame for $h \bullet u$ and is holomorphic for $\lambda \in \mathbb{C}$, where $C(\lambda) = e_{1n}\lambda + b + \sum_{i=1}^{n-1} c_i e_{i+1,n}$.*

Theorem 4.9. [BT for the j-th central affine curve flow with $k \neq 0$]
Let $\gamma(x, t)$ be a solution of the j-th central affine curve flow (1.6), and $g(\cdot, t)$, $u(\cdot, t)$ the central affine moving frame and curvature along $\gamma(\cdot, t)$ (so u is a solution of (1.4)). Let $E(x, t, \lambda)$ be a frame of u with $E(0, 0, \lambda) = g(0, 0)$. Let h be the solution of $(\mathrm{BT})_{u,k}$ constructed from E, $k \neq 0$ and $\mathbf{c}$ as in Theorem 4.8, and $\eta_i(u, h)$ the i-th coordinate of $f_{u,h}^{-1}(x, t, 0) e_1$ for $1 \leq i \leq n$ (so they are differential polynomials in u, h). Then

$$h \sharp \gamma := (-1)^n k \eta_1(u, h) \gamma + \cdots + \eta_n(u, h) \gamma_x^{(n-1)}$$

is a solution of (1.6) with central affine moving frame

$$h \sharp g(x, t) := d(k) g(x, t) f_{u,h}(x, t, 0)^{-1},$$

and central affine curvature $h \bullet u$, where $f_{u,h}$ is defined by Definition 4.2 and

$$d(k) = \mathrm{diag}((-1)^n k, 1, \ldots, 1).$$

Proof. First note that both g and $E(\cdot,\cdot,0)$ satisfies (2.18) with the same initial condition. So $g(x,t) = E(x,t,0)$.

By Theorem 4.4, $\det(f_{u,h}(x,t,0)) = (-1)^n k \neq 0$. So $f_{u,h}(x,t,0)$ is invertible for all x,t. By Theorem 4.8, $\tilde{g} = g f_{u,h}(\cdot,\cdot,0)^{-1}$ solves (2.18) for $\tilde{u} = h \bullet u$. Note that $\det(\tilde{g}) = \det(f_{u,h}^{-1}(x,t,0)) = (-1)^n k^{-1}$. So $h\sharp g = d(k)\tilde{g}$ lies in $SL(n,\mathbb{R})$ and is a solution of (2.18) for $\tilde{u}$. By Theorem 3.10, $d(k)\tilde{g}e_1$ is a solution of (1.6) with central affine moving frame $h\sharp g$ and curvature $h \bullet u$. q.e.d.

Remark 4.10. In Theorem 4.9, we can replace $d(k)$ by any constant matrix $C \in GL(n,\mathbb{R})$ with $\det(C) = (-1)^n k$. Then $Cg\eta$ is still a solution of (1.6).

Example 4.11. [*Bäcklund transformations for the 2nd central affine curve flow on $\mathcal{M}_3(\mathbb{R})$*] Suppose γ is a solution of the second central affine curve flow (3.3) on $\mathcal{M}_3(\mathbb{R})$ with central affine curvature u, and h is a solution of $(\mathrm{BT})_{u,k}$ with $k \neq 0$. Recall that $f_{u,h}$ for $n = 3$ is given by (4.5). By Theorem 4.4 we have $\det(f_{u,h}(x,t,0)) = -k$. It follows from (4.5) that

$$f_{u,h}^{-1}(x,t,0) = -k^{-1}(h^2 + h_x - u_2, -h, 1)^t. \tag{4.6}$$

Then Theorem 4.9 implies that

$$h\sharp\gamma = (h^2 + h_x - u_2)\gamma + \frac{1}{k}h\gamma_x - \frac{1}{k}\gamma_{xx} \tag{4.7}$$

is a solution of (3.3) with central affine moving frame

$$h\sharp g = d(k)g(x,t)f_{u,h}(x,t,0)^{-1},$$

and central affine curvature $h \bullet u$, where $d(k) = \mathrm{diag}(-k,1,1)$.

Example 4.12. We have seen in Example 3.12 that $E(x,t,\lambda) = e^{\Lambda x + \Lambda^2 t}$ is a frame of the trivial solution $u = 0$ of (2.27),

$$\gamma(x,t) = E(x,t,0)e_1 = (1, x, \frac{x^2}{2} + t)^t \tag{4.8}$$

is the *trivial solution* of the second central affine flow (3.3) on $\mathcal{M}_3(\mathbb{R})$ with central affine curvature $u = 0$. We apply BTs with parameter $k \neq 0$ to this solution. It is not easy to express the entries of $E(x,t,\lambda) = e^{x\Lambda + t\Lambda^2}$ as known functions. But if we introduce $\lambda = z^3$, then entries of $E(x,t,z)$ are given by exponentials. First note that $E_x = E(b + z^3 e_{13}), E_t = E(b^2 + z^3(e_{12} + e_{23}))$, which is equivalent to the following linear system of function y:

$$(\partial_x^3 - z^3)y = 0, \quad (\partial_t^6 - z^6)y = 0. \tag{4.9}$$

So the first column η of $E(x,t,z^3)$ can be constructed by finding fundamental solutions for (4.9). We obtain

$$E(x,t,z^3) = \frac{1}{3}\begin{pmatrix} m_1(x,t,z) & zm_2(x,t,z) & z^2m_3(x,t,z) \\ \frac{1}{z}m_3(x,t,z) & m_1(x,t,z) & zm_2(x,t,z) \\ \frac{1}{z^2}m_2(x,t,z) & \frac{1}{z}m_3(x,t,z) & m_1(x,t,z) \end{pmatrix}, \tag{4.10}$$

where

$$\begin{pmatrix} m_1(x,t,z) \\ m_2(x,t,z) \\ m_3(x,t,z) \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} e^{zx+z^2t} \\ e^{\alpha(zx+\alpha z^2 t)} \\ e^{\alpha^2(zx+\alpha^2 z^2 t)} \end{pmatrix},$$

and $\alpha = e^{2\pi i/3}$.

Although the entries of $E(x,t,z^3)$ involves z^i in the denominators, a simple computation implies that they are holomorphic at $z=0$. To see m_i's are holomorphic in $\lambda = z^3$, we use $1+\alpha+\alpha^2 = 0$ and a direct computation to see that the coefficients of z^{3k+1} and z^{3k+2} of $m_1(x,t,z) = e^{zx+z^2t} + e^{\alpha(zx+\alpha z^2t)} + e^{\alpha^2(zx+\alpha^2 z^2 t)}$ as power series in z are zero. So $m_1(x,t,z)$ is holomorphic in $\lambda = z^3$. Similarly, $m_2(x,t,z)$ and $m_3(x,t,z)$ are holomorphic in $\lambda = z^3$.

We use E given by (4.10) to apply Theorem 4.8 with $k = -8c^3$ and $p_0 = (2c^2, -c, -1)^t$ to compute function h. We obtain

$$h(x,t) = -\sqrt{3}c\tan(\sqrt{3}c(x+2ct)) - c.$$

We use (4.4) to see that

$$\begin{cases} \tilde{u}_1 = 9c^3\sec^2(\sqrt{3}c(x+2ct))(1+\sqrt{3}\tan(\sqrt{3}c(x+2ct))), \\ \tilde{u}_2 = 9c^2\sec^2(\sqrt{3}c(x+2ct)) \end{cases}$$

is a solution of (2.27). By (4.7),

$$\tilde{\gamma} = \begin{pmatrix} 2c^2(\xi-1) \\ 2c^2x(\xi-1) + \frac{1}{8c^2}(\xi+1) \\ c^2(x^2+2t)(\xi-1) + \frac{1}{8c^2}x(\xi+1) + \frac{1}{8c^3} \end{pmatrix}$$

is a solution of the second central affine curve flow (3.3), where

$$\xi(x,t) = \sqrt{3}\tan(\sqrt{3}c(x+2ct)).$$

Below we use M^a to denote the *adjoint* of an $n\times n$ matrix M, i.e., M^a, is the transpose of the cofactor matrix of M and

$$MM^a = \det(M)\mathrm{I}_n.$$

Theorem 4.13. [BT for the central affine curve flow with $k=0$] *Let γ, g, u, E be as in Theorem 4.9, and h the solution of* $(\mathrm{BT})_{u,0}$ *constructed from E, $k=0$ and* $\mathbf{c}$ *as in Theorem 4.8. Let $E_1(x,t) = \frac{\partial}{\partial\lambda}\big|_{\lambda=0}E(x,t,\lambda)$, $A(x,t) = \frac{\partial}{\partial\lambda}\big|_{\lambda=0}f^a_{u,h}(x,t,\lambda)$, $b_0 = b + \sum_{i=1}^{n-1} c_i e_{i+1,n}$. Then*

$$\tilde{\gamma} = (-1)^{n-1}(e_{1n}g + b_0E_1)f^a_{u,h}(\cdot,\cdot,0)e_1 + b_0 gA \tag{4.11}$$

is a solution of (1.6) *with central affine curvature* $h\bullet u$ *and central affine moving frame*

$$\tilde{g} = (-1)^{n-1}(e_{1n}g + b_0E_1)f^a_{u,h}(\cdot,\cdot,0) + b_0 gA, \tag{4.12}$$

where $f^a_{u,h}$ *is the adjoint of* $f_{u,h}$.

Proof. We have seen in the proof of Theorem 4.9 that $E(x,t,0) = g(x,t)$. By Theorem 4.4 (i), we have $\det(f_{u,h}(x,t,\lambda)) = (-1)^{n-1}\lambda$. So

$$f^{-1}_{u,h} = \frac{1}{\det(f_{u,h})} f^a_{u,h} = (-1)^{n-1}\lambda^{-1} f^a_{u,h},$$

where $f^a_{u,h}$ is the adjoint of $f_{u,h}$. Theorem 4.8 (3) implies that

$$\begin{aligned} F(x,t,\lambda) &= C(\lambda)E(x,t,\lambda)f^{-1}_{u,h}(x,t,\lambda) \\ &= (-1)^{n-1}\lambda^{-1}C(\lambda)E(x,t,\lambda)f^a_{u,h}(x,t,\lambda) \end{aligned}$$

is a frame for the new solution $\tilde{u} = h\bullet u$ and is holomorphic at $\lambda = 0$, where $C(\lambda) = \Lambda + \sum_{i=1}^{n-1} c_i e_{i+1,n}$.

Set $\xi(x,t,\lambda) = (-1)^{n-1}C(\lambda)E(x,t,\lambda)f^a_{u,h}(x,t,\lambda)$. It was proved in [**23**] that $\xi(x,t,\lambda)$ is holomorphic at $\lambda = 0$ and $\xi(x,t,0) = 0$. So $F(x,t,0) = \frac{\partial}{\partial\lambda}\mid_{\lambda=0}\xi(x,t,\lambda)$. Compute directly to get

$$\begin{aligned} F(x,t,0) &= (-1)^{n-1}\frac{\partial}{\partial\lambda}\Big|_{\lambda=0} C(\lambda)E(x,t,\lambda)f^a_{u,h}(x,t,\lambda), \\ &= (-1)^{n-1}(e_{1n}g + b_0E_1(x,t))f^a_{u,h}(x,t,0) + b_0 g\left(\frac{\partial}{\partial\lambda}\Big|_{\lambda=0} f^a_{u,h}(x,t,\lambda)\right), \end{aligned}$$

where $b_0 = C(0) = b + \sum_{i=1}^{n-1} c_i e_{i+1,n}$. So we have

$$\tilde{g}(x,t) := F(x,t,0) = (-1)^{n-1}(e_{1n}g + b_0E_1)f^a_{u,h}(x,t,0) + b_0 g(x,t)A(x,t).$$

Since $\det(C(\lambda)) = (-1)^{n-1}\lambda = \det(f_{u,h}(x,t,\lambda))$ and $\det(E(x,t,\lambda)) = 1$, we have $\det(F(x,t,\lambda)) = 1$. This implies that $\det(\tilde{g}(x,t)) = 1$. Since $F(x,t,\lambda)$ is a frame for $\tilde{u}$, $\tilde{g}(x,t) = \tilde{E}(x,t,0)$ satisfies

$$\tilde{g}^{-1}\tilde{g}_x = b + \tilde{u}, \quad \tilde{g}^{-1}\tilde{g}_t = Z_{j,0}(\tilde{u}).$$

By Theorem 3.10 (i), $\tilde{\gamma} = \tilde{g}e_1$ is a solution of the curve flow (1.6).

Since the $(1,i)$-th cofactor of $f_{u,h}$ is independent of λ, the first column of $A(x,t)$ is zero, i.e., $A(x,t)e_1 = 0$. Hence, $\tilde{\gamma}$ is given by (4.11). q.e.d.

Example 4.14. [Rational solutions for (3.3)]

We apply BT with parameter $k = 0$ to the trivial solution γ defined by (4.8) of the second central affine curve flow (3.3) on $\mathcal{M}_3(\mathbb{R})$. We use $E(x,t,\lambda) = \exp(x\Lambda + t\Lambda^2)$ as a frame of $u = 0$ and apply Theorem 4.13

to obtain

$$E_1(x,t) = \frac{\partial}{\partial\lambda}\Big|_{\lambda=0} E(x,t,\lambda) = (e_{13}x + (e_{12}+e_{23})t)\exp(bx+b^2t)$$
$$= \begin{pmatrix} xt + \frac{1}{6}x^3 & t + \frac{1}{2}x^2 & x \\ \frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4 & xt + \frac{1}{6}x^3 & t + \frac{1}{2}x^2 \\ \frac{1}{2}xt^2 + \frac{1}{6}x^3t + \frac{1}{5!}x^5 & \frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4 & xt + \frac{1}{6}x^3 \end{pmatrix}.$$

We apply Theorem 4.13 to $u=0$ and $v_0 = (a_1, a_2, 1)^t$ to get

$$h = \frac{a_1x - a_2}{1 + a_1(\frac{x^2}{2} - t) - a_2x},$$

where a_1, a_2 are real constants. By (4.7), we get

$$\tilde{\gamma} = \begin{pmatrix} (h^2+h_{xx})(\frac{x^2}{2}+t) - hx + 1 \\ (h^2+h_{xx})(\frac{1}{6}x^3 + xt) - h(\frac{1}{2}x^2+t) + x \\ (h^2+h_{xx})(\frac{1}{2}t^2 + \frac{1}{2}x^2t + \frac{1}{24}x^4) - h(\frac{1}{6}x^3+xt) + \frac{1}{2}x^2 + t \end{pmatrix}$$

is a rational solution of (3.3) with central affine curvatures

$$\begin{cases} \tilde{u}_1 = -\frac{3(a_1x-a_2)(\frac{1}{2}a_1^2x^2 - a_1a_2x + a_1^2t + a_2^2 - a_1)}{(1+a_1(\frac{x^2}{2}-t)-a_2x)^3}, \\ \tilde{u}_2 = \frac{3(\frac{1}{2}a_1^2x^2 - a_1a_2x + a_1^2t + a_2^2 - a_1)}{(1+a_1(\frac{x^2}{2}-t)-a_2x)^2}. \end{cases}$$

Theorem 4.15. *([**23**])* [Permutability formula for BTs of the $A^{(1)}_{n-1}$-KdV]
Let u be a solution of (1.4), $E(x,t,\lambda)$ a frame of u, and $k_1 \neq k_2$ non-zero real constants. Let h_i denote the function constructed from $E(x,t,k_i)$ and a constant vector $p_i \in \mathbb{R}^n$, and $h_i \bullet u$ the solution of (1.4) as in Theorem 4.8 for $i = 1,2$. Suppose $h_1 \neq h_2$. Set

$$\begin{cases} \tilde{h}_1 = h_1 + \frac{(h_1-h_2)_x}{h_1-h_2}, \\ \tilde{h}_2 = h_2 + \frac{(h_1-h_2)_x}{h_1-h_2}. \end{cases}$$

Then $\tilde{h}_1 \bullet (h_2 \bullet u) = \tilde{h}_2 \bullet (h_1 \bullet u)$ is again a solution of (1.4). In particular, a new solution of (1.4) can be constructed algebraically from $h_1 \bullet u$ and $h_2 \bullet u$.

Corollary 4.16. [Permutability for the j-th central affine curve flow]
Let γ, u, g be as in Theorem 4.9, and $k_1, k_2, h_1, h_2, \tilde{h}_1, \tilde{h}_2$, as in Theorem 4.15. Let $h_i \sharp \gamma$ denote the solution of (1.6) constructed from γ and h_i for $i = 1,2$ as in Theorem 4.9. Then $\tilde{h}_1 \sharp (h_2 \sharp \gamma) = \tilde{h}_2 \sharp (h_1 \sharp \gamma)$ is again a solution of (1.6).

Remark 4.17.
(1) Suppose we have an explicit formula for a frame $E(x,t,\lambda)$ of a solution u of the j-th $A^{(1)}_{n-1}$-KdV flow (1.4). Given non-zero distinct constants $k_1, \ldots, k_m$, we apply Theorems 4.8 and 4.4 to construct an explicit solution $h_i \bullet u$ of (1.4) for $1 \leq i \leq m$. The Permutability

formula gives us a new solution $\tilde{h}_2 \bullet (h_1 \bullet u)$ of (1.4), which is written explicitly in terms of u, h_1 and h_2. We can continue this process to obtain infinitely many explicit solutions of (1.4).

(2) Suppose u is the central affine curvature of a solution γ of (1.6) and $E(x, t, \lambda)$ is an explicit frame for u such that $g(x, t) = E(x, t, 0)$ is the central affine moving frame along γ. Let k_i and h_i be as above. We use Theorems 4.9 to compute $h_i \sharp \gamma$ and Corollary 4.16 to get infinitely many family of solutions of (1.6).

5. Bi-Hamiltonian structure for the $A^{(1)}_{n-1}$-KdV hierarchy

In this section, we

(i) review the AGD brackets and conservation laws,

(ii) write down formulas for the AGD brackets in terms of the operator P_u defined in Definition 2.11,

(iii) give formulas for the Poisson operators J_1, J_2 defined by the AGD brackets and compute the kernels of J_1, J_2.

As explained in Introduction that the Hamiltonian theory of the $A^{(1)}_{n-1}$-KdV hierarchy works for both the space $\mathcal{S}(\mathbb{R}, V_n)$ of rapidly decaying smooth maps and $C^\infty(S^1, V_n)$.

Let $\langle\,,\rangle$ be the bi-linear form on $C^\infty(S^1, sl(n, \mathbb{R}))$ defined by

$$\langle y_1, y_2\rangle = \oint \mathrm{tr}(y_1(x) y_2(x))\mathrm{d}x. \tag{5.1}$$

The gradient of $\mathcal{F} : C^\infty(S^1, \mathcal{B}^+_n) \to \mathbb{R}$ at $q \in C^\infty(S^1, \mathcal{B}^+_n)$ is the unique element $\nabla\mathcal{F}(q)$ in $C^\infty(S^1, \mathcal{B}^-_n)$ defined by

$$\mathrm{d}\mathcal{F}_q(y) = \langle \nabla\mathcal{F}(q), y\rangle,$$

for all $y \in C^\infty(S^1, \mathcal{B}^+_n)$.

The gradient $\nabla F(u)$ of a functional $F : C^\infty(S^1, V_n) \to \mathbb{R}$ at u is the unique element in $C^\infty(S^1, V_n^t)$ satisfying

$$\mathrm{d}F_u(v) = \langle \nabla F(u), v\rangle,$$

for all $v \in C^\infty(S^1, V_n)$.

The AGD brackets for the $A^{(1)}_{n-1}$-KdV

The two Poisson structures on $C^\infty(S^1, \mathcal{B}^+_n)$ given in [**6**] are:

$$\{\mathcal{F}_1, \mathcal{F}_2\}_1(u) = \langle [\nabla\mathcal{F}_1(u), e_{1n}], \nabla\mathcal{F}_2(u)\rangle,$$
$$\{\mathcal{F}_1, \mathcal{F}_2\}_2(u) = \langle [\partial_x + b + u, \nabla\mathcal{F}_1(u)], \nabla\mathcal{F}_2(u)\rangle.$$

Recall that $C^\infty(S^1, V_n)$ is isomorphic to the orbit space $\frac{C^\infty(S^1,\mathcal{B}^+_n)}{C^\infty(S^1,N^+_n)}$. So given a functional F on $C^\infty(S^1, V_n)$, there is a unique $C^\infty(S^1, N^+_n)$-invariant function $\tilde{F}$ on $C^\infty(S^1, \mathcal{B}^+_n)$ whose restriction to $C^\infty(S^1, V_n)$ is F, i.e., $\tilde{F}(q) = F(u)$ if q lies in the same $C^\infty(S^1, N^+_n)$-orbit as $u \in C^\infty(S^1, V_n)$.

The Adler–Gel'fand–Dickey (AGD) Poisson brackets on $C^\infty(S^1, V_n)$ given in [**6**] are

$$\{F_1, F_2\}_1(u) = \langle [\nabla \tilde{F}_1(u), e_{1n}], \nabla \tilde{F}_2(u)\rangle, \tag{5.2}$$

$$\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, \nabla \tilde{F}_1(u)], \nabla \tilde{F}_2(u)\rangle, \tag{5.3}$$

where $\tilde{F}_i$ is the unique $C^\infty(S^1, N_n^+)$-invariant functional on $C^\infty(S^1, \mathcal{B}_n^+)$ defined by $F_i : C^\infty(S^1, V_n) \to \mathbb{R}$.

We need the relation between $\nabla \tilde{F}(u)$ and $\nabla F(u)$ for $u \in C^\infty(S^1, V_n)$ below to write down the formulas for $\{\,,\}_i$ $(i = 1, 2)$ on $C^\infty(S^1, V_n)$.

Proposition 5.1. *Let F be a functional on $C^\infty(S^1, V_n)$, and $\tilde{F}$ the functional on $C^\infty(S^1, \mathcal{B}_n^+)$ invariant under $C^\infty(S^1, N_n^+)$ defined by F. Then*

$$\nabla \tilde{F}(u) = \pi_{\mathcal{B}_n^-}(P_u(\nabla F(u))),$$

where $u \in C^\infty(S^1, V_n)$, P_u is the operator defined Definition 2.11, and $\pi_{\mathcal{B}_n^-}$ is the projection of $sl(n, \mathbb{R})$ onto $\mathcal{B}_n^-$ along $\mathcal{N}_n^+$.

Proof. Note that the infinitesimal vector field $\tilde{\xi}$ for the gauge action defined by ξ in $C^\infty(S^1, \mathcal{N}_n^-)$ is

$$\tilde{\xi}(u) = -[\partial_x + b + u, \xi],$$

where $u \in C^\infty(S^1, V_n)$. By assumption, $\tilde{F}(f * u) = \tilde{F}(u)$ for all $u \in C^\infty(S^1, V_n)$ and $f \in C^\infty(S^1, N_n^+)$. So $\mathrm{d}\tilde{F}_u(\tilde{\xi}(u)) = 0$. But

$$\begin{aligned}\mathrm{d}\tilde{F}_u(\tilde{\xi}(u)) &= \langle \nabla \tilde{F}(u), \tilde{\xi}(u)\rangle = -\langle \nabla \tilde{F}(u), [\partial_x + b + u, \xi]\rangle \\ &= \langle [\partial_x + b + u, \nabla \tilde{F}(u)], \xi\rangle,\end{aligned}$$

for all $\xi \in C^\infty(S^1, \mathcal{N}_n^+)$. So

$$[\partial_x + b + u, \nabla \tilde{F}(u)] \in C^\infty(S^1, \mathcal{B}_n^-).$$

Note that $\langle \mathcal{N}_n^+, \mathcal{B}_n^+\rangle = 0$. So to prove $\nabla \tilde{F}(u) = \pi_{\mathcal{B}_n^-}(P_u(\nabla F(u)))$ for $u \in C^\infty(S^1, V_n)$, it is equivalent to prove

$$\mathrm{d}\tilde{F}_u(y) = \langle P_u(\nabla F(u)), y\rangle, \tag{5.4}$$

for all $y \in C^\infty(S^1, \mathcal{B}_n^+)$.

We first prove (5.4) for $y \in C^\infty(S^1, V_n)$. Given $u, v \in C^\infty(S^1, V_n)$, we have

$$\mathrm{d}F_u(v) = \langle \nabla F(u), v\rangle = \mathrm{d}\tilde{F}_u(v) = \langle \nabla \tilde{F}(u), v\rangle.$$

So $\langle \nabla F(u) - \nabla \tilde{F}(u), v\rangle = 0$ for all $v \in C^\infty(S^1, V_n)$. This implies that

$$\pi_0(\nabla \tilde{F}(u)) = \nabla F(u),$$

where $\pi_0 : sl(n, \mathbb{R}) \to V_n^t$ is the projection defined by (2.23). By definition of P_u, we have $\pi_0(P_u(\nabla F(u))) = \nabla F(u)$. So we obtain $\mathrm{d}\tilde{F}_u(v) = \mathrm{d}F_u(v) = \langle P_u(\nabla F(u)), v\rangle$, i.e., (5.4) is true for $y \in C^\infty(S^1, V_n)$.

Since $C^\infty(S^1, V_n)$ is a cross section of the gauge action of $C^\infty(S^1, N_n^+)$ on $C^\infty(S^1, \mathcal{B}_n^+)$, the tangent space of $C^\infty(S^1, \mathcal{B}_n^+)$ at $u \in C^\infty(S^1, V_n)$ can be written as a direct sum of $C^\infty(S^1, V_n)$ and the tangent space of the $C^\infty(S^1, N_n^+)$-orbit at u. Since $\tilde{F}$ is invariant under $C^\infty(S^1, N_n^+)$, we have $d\tilde{F}_u(\tilde{\xi}(u)) = 0$ for all $\xi \in C^\infty(S^1, \mathcal{N}_n^+)$. So

$$\begin{aligned}\langle P_u(\nabla F(u)), \tilde{\xi}(u)\rangle &= \langle P_u(\nabla F(u)), -[\partial_x + b + u, \xi]\rangle \\ &= \langle [\partial_x + b + u, P_u(\nabla F(u))], \xi\rangle.\end{aligned}$$

By definition of P_u, we have $[\partial_x + b + u, P_u(\nabla F(u))] \in C^\infty(S^1, V_n)$. Since $\xi \in C^\infty(S^1, \mathcal{N}_n^+)$, we conclude that $\langle P_u(\nabla F(u)), \tilde{\xi}(u)\rangle = 0$. This proves $d\tilde{F}_u(\tilde{\xi}(u)) = \langle P_u(\nabla F(u)), \tilde{\xi}(u)\rangle = 0$. So (5.4) is true for y in the tangent space of $C^\infty(S^1, N_n^+)$-orbit at u. This completes the proof. q.e.d.

Theorem 5.2. *The Poisson structures on $C^\infty(S^1, V_n)$ defined by (5.2) and (5.3) can be written as follows:*

$$\{F_1, F_2\}_1(u) = \langle [P_u(\nabla F_1(u)), e_{1n}], P_u(\nabla F_2(u))\rangle, \tag{5.5}$$

$$\{F_1, F_2\}_2(u) = \langle [\partial_x + b + u, P_u(\nabla F_1(u))], \nabla F_2(u)\rangle, \tag{5.6}$$

where $P_u : C^\infty(S^1, V_n^t) \to C^\infty(S^1, sl(n, \mathbb{R}))$ is defined in Definition 2.11.

Proof. Since $\langle \mathcal{B}_n^+, \mathcal{N}_n^+\rangle = 0$, we have

$$\langle \pi_{\mathcal{B}_n^-}(\xi), \eta\rangle = \langle \xi, \eta\rangle, \quad \eta \in \mathcal{B}_n^+. \tag{5.7}$$

By Proposition 5.1 and (5.2), we have

$$\{F_1, F_2\}_1(u) = \langle [\pi_{\mathcal{B}_n^-}(P_u(\nabla F_1(u))), e_{1n}], \pi_{\mathcal{B}_n^-}(P_u(\nabla F_2(u)))\rangle.$$

Because $[e_{1n}, \mathcal{N}_n^+] = 0$, we continue the above computation and it equals

$$\langle [P_u(\nabla F_1(u)), e_{1n}], \pi_{\mathcal{B}_n^-}(P_u(\nabla F_2(u)))\rangle.$$

But $[e_{1n}, sl(n, \mathbb{R})] \in \mathcal{B}_n^+$. Use (5.7) to see that the above quantity is equal to

$$\langle [P_u(\nabla F_1(u)), e_{1n}], P_u(\nabla F_2(u))\rangle.$$

This proves (5.5).

To prove (5.6), we first note that if $\eta \in C^\infty(S^1, \mathcal{N}_n^+)$ then

$$[\partial_x + b + u, \eta] \in \mathcal{B}_n^+.$$

By definition of P_u, we have $[\partial_x + b + u, P_u(v)] \in C^\infty(S^1, V_n)$. So

$$[\partial_x + b + u, \pi_{\mathcal{B}_n^-}(P_u(\nabla F_1(u)))] \in C^\infty(S^1, \mathcal{B}_n^+).$$

Use $\langle \mathcal{B}_n^+, \mathcal{N}_n^+\rangle = 0$ to see that

$$\{F_1, F_2\}_2 = -\langle \pi_{\mathcal{B}_n^-}(P_u(\nabla F_1(u))), [\partial_x + b + u, P_u(\nabla F_2(u))]\rangle.$$

By definition of P_u, the second term is in $C^\infty(S^1, V_n)$. But $\langle V_n, \mathcal{N}_n^+\rangle = 0$. So we can continue the computation and get

$$\begin{aligned}\{F_1, F_2\}_2(u) &= -\langle P_u(\nabla F_1(u)), [\partial_x + b + u, P_u(\nabla F_2(u))]\rangle \\ &= \langle [\partial_x + b + u, P_u(\nabla F_1(u))], P_u(\nabla F_2(u))\rangle,\end{aligned}$$

which is (5.6) since $[\partial_x + b + u, P_u(\nabla F_1(u))] \in C^\infty(S^1, V_n)$. q.e.d.

Poisson operators and their kernels

Let

$$(J_i)_u : C^\infty(S^1, V_n^t) \to C^\infty(S^1, V_n)$$

be the *Poisson operator* corresponding to $\{\,,\}_i$ at u for $i = 1, 2$, i.e., $(J_i)_u$ is defined by

$$\{F_1, F_2\}_i(u) = \langle (J_i)_u(\nabla F_1(u)),\ \nabla F_2(u)\rangle.$$

Then the Hamiltonian equation for a functional $H : C^\infty(S^1, V_n) \to \mathbb{R}$ with respect to $\{\,,\}_i$ is

$$u_t = (J_i)_u(\nabla H(u)).$$

Next we compute the formula for the Poisson operator J_1.

Proposition 5.3.
The Poisson operator $(J_1)_u : C^\infty(S^1, V_n^t) \to C^\infty(S^1, V_n)$ is of the form $(J_1)_u(\xi) = -\sum_{i=1}^{n-1}(L_i)_u(\xi)e_{in}$ with

$$(L_i)_u(\xi) = n(\xi_{n-i})_x + k_i(\xi_1, \dots, \xi_{n-i-1}),$$

where $\xi = \sum_{i=1}^{n-1}\xi_i e_{ni}$ and k_i's are linear differential operators with differential polynomials of u as coefficients.

Proof. By Theorem 2.9, entries of $P_u(v)$ are differential polynomials of u and v. So we can use integration by parts to compute $(J_1)_u$. We proceed as follows: Let $u = \sum_{i=1}^{n-1} u_i e_{in}$, and

$$\begin{aligned}&\xi = \sum_{i=1}^{n-1}\xi_i e_{ni} := \nabla F_1(u), \quad \eta = \sum_{i=1}^{n}\eta_i e_{ni} := \nabla F_2(u),\\ &C = (C_{ij}) = P_u(\xi), \quad D = (D_{ij}) = P_u(\eta).\end{aligned}$$

Then $C_{ni} = \xi_i$ and $D_{ni} = \eta_i$ for $1 \le i \le n-1$. By (5.5) we have

$$\{F_1, F_2\}_1(u) = \langle [C, e_{1n}], D\rangle = \oint \sum_{i=1}^{n} C_{i1}D_{ni} - C_{ni}D_{i1}\mathrm{d}x.$$

From the definition of P_u, we have $[\partial_x + b + u, C] \in C^\infty(\mathbb{R}, V_n)$ and $[\partial_x + b + u, D] \in C^\infty(\mathbb{R}, V_n)$. It follows from Theorem 2.9 that there exists a differential polynomial ϕ_0 such that

$$C_{11} = \phi_0(C_{21}, \dots, C_{n1}) = f(\xi_1, \dots, \xi_{n-1}).$$

By Theorem 2.9, there is a differential polynomial f_n such that

$$C_{nn} = f_n(\xi_1, \ldots, \xi_{n-1}), \quad D_{nn} = f_n(\eta_1, \ldots, \eta_{n-1}).$$

We then use integration by parts to write down the Poisson operator $(J_1)_u$. To get $(L_j)_u(\xi)$, we only need to calculate the terms involving $\eta_j = D_{nj}$ in $\sum_{i=1}^n C_{i1}D_{ni} - C_{ni}D_{i1}$. We use (2.22) to compute these terms as follows:

$$\begin{aligned}
&D_{nj}C_{j1} + C_{n1}(D_{nn} - D_{11}) - \sum_{i=1}^{n+1-j} C_{ni}D_{i1} \\
&= \eta_j C_{j1} + \xi_1 \left(\sum_{i=1}^{n-1} D'_{i+1,i} \right) - C_{n,n+1-j}D_{n+1-j,1} - \sum_{i=1}^{n-j} C_{ni}D_{i1} \\
&= \eta_j(\xi_{n+1-j} - (n-j)\xi'_{n-j} + \phi_{n-j}(u, \xi_1, \ldots, \xi_{n-j-1})) + \xi_1 \sum_{i=1}^{n-1} D'_{i+1,i} \\
&\quad - \xi_{n+1-j}(\eta_j - (j-1)\eta'_{j-1} + \phi_{j-1}(u, \eta_1, \ldots, \eta_{j-2})) - \sum_{i=1}^{n-j} \xi_i D_{i1}.
\end{aligned}$$

Note that $\xi_1 \sum_{i=1}^{n-1} D'_{i+1,i}$ only depends on $\xi_1, \eta_1, \ldots, \eta_{n-1}$, and $\xi_i D_{i1}$ is a differential polynomial in u, ξ_i and $\eta_1, \ldots, \eta_{n+1-i}$ for each i. Therefore, to consider the term ξ_{n-j} in the coefficient of η_j in $\sum_{i=1}^{n-j} \xi_i D_{i1}$, we only need to calculate $\xi_{n-j}D_{n-j,1}$. Again, use $D_{n-j,1} = \eta_{j+1} - j\eta'_j + \phi_j(u, \eta_1, \ldots, \eta_{j-1})$ and integration by parts to see that the coefficients of η_j is $-n\xi'_{n-j}$ plus a differential operator depending on $u, \xi_1, \ldots, \xi_{n-j-1}$. q.e.d.

Corollary 5.4. *The dimension of the kernel of $(J_1)_u$ is $n-1$.*

Proof. It suffices to prove that solutions of the linear ordinary differential equations $(J_1)_u(\xi) = 0$ are determined by arbitrary $n-1$ constants. Suppose $\xi = \sum_{i=1}^{n-1} \xi_i e_{ni}$ and $(J_1)_u(\xi) = 0$. By Proposition 5.3, $(L_{n-1})_u(\xi) = (\xi_1)_x = 0$. So $\xi_1 = c_1$ a constant. Suppose we have solved $\xi_1, \ldots, \xi_i$ with initial data $\xi_j(0) = c_j$ for $1 \leq j \leq i$. Then by Proposition 5.3, we have

$$(L_{n-i-1})_u = n(\xi_{i+1})_x + k_{n-i-1}(\xi_1, \ldots, \xi_i) = 0.$$

So there is a unique ξ_{i+1} satisfies the above equation with $\xi_{i+1}(0) = c_{i+1}$. This proves the claim. q.e.d.

Proposition 5.5. *Let g and u denote the central affine moving frame and curvature along $\gamma \in \mathcal{M}_n(S^1)$. Then*

1) *the Poisson operator $(J_2)_u : C^\infty(S^1, V_n^t) \to C^\infty(S^1, V_n)$ is*

$$(J_2)_u(v) = [\partial_x + b + u, P_u(v)]. \tag{5.8}$$

2) $\mathrm{Ker}(J_2)_u = \{\pi_0(g^{-1}c_0 g) \,|\, c_0 \in sl(n, \mathbb{R})\}$.

Proof. Statement (1) follows from (5.6) and (2) follows from Corollary 3.8. q.e.d.

In the following examples, we compute explicit formulas for J_1 and J_2 for $n = 2, 3$.

Example 5.6. For $n = 2$, write $u = qe_{21}$, $\nabla F_1(u) = \xi e_{12}$, and $\nabla F_2(u) = \eta e_{21}$. By (2.24), we have

$$P_u(\nabla F_1(u)) = \begin{pmatrix} -\frac{1}{2}\xi_x & -\frac{1}{2}\xi_{xx} + q\xi \\ \xi & \frac{1}{2}\xi_x \end{pmatrix},$$
$$P_u(\nabla F_2(u)) = \begin{pmatrix} -\frac{1}{2}\eta_x & -\frac{1}{2}\eta_{xx} + q\eta \\ \eta & \frac{1}{2}\eta_x \end{pmatrix}.$$

Use (5.5), (5.6) and integration by part to get

$$\{F_1, F_2\}_1(u) = -2\oint \xi'\eta \mathrm{d}x,$$
$$\{F_1, F_2\}_2(u) = -\oint (\frac{1}{2}\xi_x^{(3)} - 2q\xi_x - q_x\xi)\eta \mathrm{d}x,$$

and the corresponding Poisson operators are

$$(J_1)_u(\xi e_{21}) = -2\xi_x e_{12},$$
$$(J_2)_u(\xi e_{21}) = (-\frac{1}{2}\xi_x^{(3)} + 2q\xi_x + q_x\xi)e_{12}.$$

These are the known Poisson structures for the KdV hierarchy.

Example 5.7. We write down the formula for J_1 and J_2 for $n = 3$. Let $u = u_1e_{13} + u_2e_{23}$, $\xi = \nabla F_1(u) = \xi_1e_{31} + \xi_2e_{32}$, and $\eta = \nabla F_2(u) = \eta_1e_{31} + \eta_2e_{32}$, Then $C = P_u(\xi) = (C_{ij})$, and $D = P_u(\eta) = (D_{ij})$ are given by (2.25). Integration by part gives

$$\{F_1, F_2\}_1(u) = \oint \sum_{i=1}^{3} C_{i1}D_{3i} - C_{3i}D_{i1}\mathrm{d}x = -3\oint (\xi_1'\eta_2 + \xi_2'\eta_1)\mathrm{d}x.$$

Hence,

$$(J_1)_u(\xi_1e_{31} + \xi_2e_{32}) = -3(\xi_2'e_{13} + \xi_1'e_{23}).$$

This formula was also obtained in [**3**].

We use (5.8) and a direct computation to see that

$$(J_2)_u(\xi) = (C_{13}' + u_1(C_{33} - C_{11}) - u_2C_{12})e_{13}$$
$$+ (C_{23}' + C_{13} + u_2(C_{33} - C_{22}) - u_1C_{21})e_{23}.$$

This gives

$$(J_2)_u(\xi_1e_{31} + \xi_2e_{32}) = (A_1)_u(\xi)e_{13} + (A_2)_u(\xi)e_{23},$$

where

$$(A_1)_u(\xi) = \frac{2}{3}\xi_1^{(5)} - \xi_2^{(4)} - \frac{2}{3}(u_2\xi_1)^{(3)} - \frac{2}{3}u_2\xi_1^{(3)} + u_1''\xi_1 + 2u_1'\xi_1' + 3u_1\xi_2' + u_1'\xi_2 + u_2\xi_2'' + \frac{2}{3}u_2(u_2\xi_1)',$$

$$(A_2)_u(\xi) = \xi_1^{(4)} - 2\xi_2^{(3)} - (u_2\xi_1)'' + 2u_2\xi_2' + u_2'\xi_2 + u_1\xi_1' + 2u_1'\xi_1.$$

Conservation laws

Next we review the conservation laws of the $A_{n-1}^{(1)}$-KdV hierarchy given in [**6**] and we include a proof here.

Theorem 5.8. *([**6**]) Let $u \in C^\infty(\mathbb{R}, V_n)$, and $T(u,\lambda)$ satisfying*

$$T(\partial_x + \Lambda + u)T^{-1} = \partial_x + \Lambda + \sum_{i>0} f_i\Lambda^{-i}, \tag{5.9}$$

for some $f_i \in C^\infty(\mathbb{R},\mathbb{R})$ as in Proposition 2.2. Let $H_j : C^\infty(S^1, V_n) \to \mathbb{R}$ be defined by

$$H_j(u) = n\oint f_j(u)\mathrm{d}x. \tag{5.10}$$

Then we have

$$\nabla H_j(u) = \pi_0(Y_{j,0}(u)). \tag{5.11}$$

Moreover, the j-th $A_{n-1}^{(1)}$-KdV flow is the Hamiltonian flow for H_j with respect to $\{\,,\}_2$ and the Hamiltonian flow for H_{n+j} with respect to $\{\,,\}_1$.

Proof. By Theorem 2.4, $Y = T^{-1}\Lambda T$ satisfies (2.7). Take variation of (5.9) when we vary u by δu to get

$$[T(\partial_x + \Lambda + u)T^{-1}, (\delta T)T^{-1}] + T\delta u T^{-1} = \sum_{i>0}\delta f_i\Lambda^{-i}.$$

Conjugate the above equation by T^{-1} to get

$$[\partial_x + \Lambda + u, T^{-1}\delta T] + \delta u = \sum_{i>0}\delta f_i T^{-1}\Lambda^{-i}T. \tag{5.12}$$

Take the inner product of (5.12) with $Y^j = T^{-1}\Lambda^j T$ to get

$$\langle[\partial_x + \Lambda + u, T^{-1}\delta T] + \delta u, Y^j\rangle = \sum_{i>0}\langle\delta f_i T^{-1}\Lambda^{-i}T, T^{-1}\Lambda^j T\rangle,$$

$$= \langle -T^{-1}\delta T, [\partial_x + \Lambda + u, Y^j]\rangle + \langle\delta u, Y^j\rangle = \sum_{i>0}\langle\delta f_i, \Lambda^{j-i}\rangle.$$

By (2.8), the first term on the left hand side is zero. So we obtain

$$\langle\delta u, Y^j\rangle = \sum_{i>0}\langle\delta f_i, \Lambda^{j-i}\rangle. \tag{5.13}$$

But $\operatorname{tr}(\Lambda^k) = 0$ for $k \not\equiv 0 \pmod{n}$ and $\operatorname{tr}(\Lambda^{nk}) = n\lambda^k$. Equate constant coefficients of both sides to get $\langle \delta u, Y_{j,0}(u)\rangle = n \oint \delta f_j \mathrm{d}x$. This proves that $\nabla H_j(u) = \pi_0(Y_{j,0}(u))$.

By Proposition 5.5, the Hamiltonian flow for F_j with respect to J_2 is

$$u_t = (J_2)_u(\nabla H_j(u)) = [\partial_x + b + u, P_u(\pi_0(Y_{j,0}(u)))] = [\partial_x + b + u, Z_{j,0}(u)],$$

which is the j-th $A^{(1)}_{n-1}$-KdV flow.

To compute $(J_1)_u(\nabla H_{n+j}(u))$, we first note that

$$Z_{k,0}(u) - Y_{k,0}(u) \in \mathcal{N}_n^+, \tag{5.14}$$

$$\langle \mathcal{N}_n^+, e_{1n}\rangle = 0. \tag{5.15}$$

Corollary 2.12 gives

$$P_u(\pi_0(Y_{j,0}(u))) = Z_{j,0}(u), \tag{5.16}$$

and (2.12) is

$$[Y_{n+j,0}(u), e_{1n}] = [\partial_x + b + u, Y_{j,0}(u)]. \tag{5.17}$$

Given any functional H on $C^\infty(S^1, V_n)$, we use (5.5) to compute

$$\begin{aligned}
\{H_{n+j}, H\}_1(u) &= \langle [P_u(\nabla H_{n+j}(u)), e_{1n}], P_u(\nabla H(u))\rangle, \quad \text{by (5.11)},\\
&= \langle [P_u(\pi_0(Y_{n+j,0}(u))), e_{1n}], P_u(\nabla H(u))\rangle, \ \text{by (5.16)},\\
&= \langle [Z_{n+j,0}(u), e_{1n}], P_u(\nabla H(u))\rangle, \ \text{by (5.14) and (5.15)},\\
&= \langle [Y_{n+j,0}(u), e_{1n}], P_u(\nabla H(u))\rangle, \ \text{by (5.17)},\\
&= \langle [\partial_x + b + u, Y_{j,0}(u)], P_u(\nabla H(u))\rangle\\
&= \langle -Y_{j,0}(u), [\partial_x + b + u, P_u(\nabla H(u))]\rangle.
\end{aligned}$$

By Definition of P_u (Definition 2.11), the second term is in $C^\infty(S^1, V_n)$. So only the V_n^t component of $Y_{j,0}(u)$ matters. Since $Y_{j,0}(u)$ and $Z_{j,0}(u)$ have the same V_n^t component, the above equality is equal to

$$\langle -Z_{j,0}(u), [\partial_x + b + u, P_u(\nabla H(u))]\rangle = \langle [\partial_x + b + u, Z_{j,0}(u)], P_u(\nabla H(u))\rangle.$$

Since the first term is in V_n and $\pi_o(P_u(\nabla H(u))) = \nabla H(u)$, we obtain

$$\{H_{n+j}, H\}_1(u) = \langle [\partial_x + b + u, Z_{j,0}(u)], \nabla H(u)\rangle,$$

which is equal to $\langle (J_1)_u(\nabla H_{n+j}(u)), \nabla H(u)\rangle$ for all functional H. So we have

$$(J_1)_u(\nabla H_{n+j}(u)) = [\partial_x + b + u, Z_{j,0}(u)],$$

i.e., the Hamiltonian flow for H_{n+j} with respect to $\{\,,\}_1$ is the j-th $A^{(1)}_{n-1}$-KdV flow. q.e.d.

Example 5.9. ([6], [5]) We have computed f_1, f_2, f_3 in Example 2.3. So the first three conservation laws for general n are

$$H_1(u) = \oint u_{n-1} \mathrm{d}x,$$
$$H_2(u) = \oint u_{n-2} \mathrm{d}x,$$
$$H_3(u) = \oint u_{n-3} + \frac{n-3}{2n} u_{n-1}^2 \mathrm{d}x.$$

Corollary 5.4 implies that $\dim(\mathrm{Ker}((J_1)_u)) = n-1$. The following Theorem, which will be used in Section 6, shows that the Kernel of $(J_1)_u$ is spanned by $\nabla H_1(u), \ldots, \nabla H_{n-1}(u)$.

Theorem 5.10. *Let H_j be the functional defined by (5.10) for the $A_{n-1}^{(1)}$-KdV hierarchy. Then $(J_1)_u(\nabla H_j(u)) = 0, 1 \leq j \leq n-1$. In other words, $H_1, \ldots, H_{n-1}$ are Casimirs of $\{\,,\}_1$. Moreover,* $\mathrm{Ker}((J_1)_u)$ *is equal to the span of $\{\nabla H_1(u), \ldots, \nabla H_{n-1}(u)\}$.*

Proof. Let $Y(u, \lambda)$ be the solution of (2.7). The power series of Y in λ is

$$Y(u, \lambda) = e_{1n}\lambda + Y_{1,0}(u) + Y_{1,1}(u)\lambda^{-1} + \cdots .$$

Let $Y_{j,0}(u)$ be defined by (2.10), i.e., the constant term of $Y^j(u, \lambda)$ as a power series in λ. We claim that $(J_1)_u(\pi_0(Y_{j,0}(u))) = 0$ for $1 \leq j \leq n-1$. It follows from the expansion of $Y(u, \lambda)$ that we have

$$Y(u, \lambda)^j = (b^t)^{n-j}\lambda + Y_{j,0}(u) + Y_{j,1}(u)\lambda^{-1} + \cdots, \quad 1 \leq j \leq n-1.$$

Since $[\partial_x + \Lambda + u, Y(u, \lambda)^j] = 0$, $[\partial_x + b + u, (b^t)^{n-j}] = [Y_{j,0}(u), e_{1n}]$. Recall that $Z_{j,0}(u) = Y_{j,0}(u) - \zeta_j(u)$ for some unique $\zeta_j(u) \in C^\infty(S^1, \mathcal{N}_n^+)$. But $[\zeta_j(u), e_{1n}] = 0$. So we get

$$[Z_{j,0}(u), e_{1n}] = [Y_{j,0}(u), e_{1n}] = [\partial_x + b + u, (b^t)^{n-j}].$$

By Corollary 2.12, $Z_{j,0}(u) = P_u(\pi_0(Z_{j,0}(u)))$. Let $\xi_1 = \pi_0(Z_{j,0}(u))$, $\xi_2 \in C^\infty(S^1, V_n^t)$. Then we have

$$\begin{aligned}\langle [Z_{j,0}(u), e_{1n}], P_u(\xi_2)\rangle &= \langle [\partial_x + b + u, (b^t)^{n-j}], P_u(\xi_2)\rangle \\ &= -\langle (b^t)^{n-j}, [\partial_x + b + u, P_u(\xi_2)]\rangle,\end{aligned}$$

which is zero because $[\partial_x + b + u, P_u(\xi_2)] \in V_n$ by definition of P_u and $(b^t)^{n-j}$ is in $\mathcal{N}_n^+$. This proves that $\langle (J_1)_u(\pi_0(Z_{j,0}(u))), \xi_2\rangle = 0$ for all $\xi_2 \in V_n^t$. Hence, $\pi_0(Z_{j,0}(u))$ lies in the kernel of $(J_1)_u$. By (5.11), $\nabla H_j(u) = \pi_0(Y_{j,0}(u)) = \pi_0(Z_{j,0}(u))$ for $1 \leq j \leq n-1$. So we get $(J_1)_u(\nabla H_j(u)) = 0, 1 \leq j \leq n-1$. q.e.d.

Next we write the conservation law H_j in terms of $n1$-th entry of $Y(u, \lambda)$.

Theorem 5.11. *Let $Y(u,\lambda)$ be the solution of* (2.7)*, $Y^j(u,\lambda) = \sum_{i\leq[\frac{j}{n}]+1} Y_{j,i}(u)\lambda^i$, and $H_j(u)$ defined by* (5.10)*. Then we have*

$$H_j(u) = -\frac{n}{j}\oint \mathrm{tr}(Y_{j,-1}(u)e_{1n})\mathrm{d}x.$$

Proof. By Theorem 2.4, $Y(u,\lambda) = T^{-1}(u,\lambda)\Lambda T(u,\lambda)$, where $T(u,\lambda)$ is as in Proposition 2.2. Take λ derivative of (5.9) to get

$$[T(\partial_x+\Lambda+u)T^{-1}, T_\lambda T^{-1}] + T(\Lambda)_\lambda T^{-1} = (\Lambda)_\lambda + \sum_{i>0} f_i(\Lambda^{-i})_\lambda.$$

Conjugate the above equation by T^{-1} to get

$$[\partial_x+\Lambda+u, T^{-1}T_\lambda] + e_{1n} = T^{-1}e_{1n}T + \sum_{i>0} f_i T^{-1}(\Lambda^{-i})_\lambda T.$$

We take the inner product of both sides of the above equation with $Y^j = T^{-1}\Lambda^j T$ and we get

$$\begin{aligned}
\mathrm{LHS} &= \langle[\partial_x+\Lambda+u, T^{-1}T_\lambda], Y^j\rangle + \langle e_{1n}, Y^j\rangle \\
&= \langle -T^{-1}T_\lambda, [\partial_x+\Lambda+u, Y^j]\rangle + \langle e_{1n}, Y^j\rangle \quad \text{by (2.8)} \\
&= \langle e_{1n}, Y^j\rangle, \\
\mathrm{RHS} &= \langle T^{-1}e_{1n}T, Y^j\rangle + \sum_{i>0}\langle f_i T^{-1}(\Lambda^{-i})_\lambda T, Y^j\rangle, \\
&= \langle e_{1n}, \Lambda^j\rangle + \sum_{i>0}\langle f_i, (\Lambda^{-i})_\lambda \Lambda^j\rangle.
\end{aligned}$$

So we have

$$\langle e_{1n}, Y^j\rangle = \langle e_{1n}, \Lambda^j\rangle + \sum_{i>0}\langle f_i, (\Lambda^{-i})_\lambda\Lambda^j\rangle.$$

The coefficients of λ^{-1} of the left hand side is $\langle e_{1n}, Y_{j,-1}(u)\rangle$ and the first term of the right hand side has no λ^{-1} term. We claim that the coefficient of λ^{-1} of $\mathrm{tr}((\Lambda^{-i})_\lambda\Lambda^j)$ is zero if $i\neq j$, and is $-j$ if $i=j$. To prove this, we write $i = nr+m$ and $j = ns+p$ with $r,s\geq 0$ and $0\leq m,p\leq n-1$. By (2.2) and (2.3), we have $\Lambda^{-i} = \lambda^{-r}((b^t)^m + \lambda^{-1}b^{n-m})$ and $\Lambda^j = \lambda^s((b^t)^{n-p}\lambda + b^p)$. A direct computation shows that

$$\mathrm{tr}((\Lambda^{-i})_\lambda\Lambda^j) = -\lambda^{s-r-1}(r\,\mathrm{tr}((b^t)^m b^p) + (r+1)\mathrm{tr}(b^{n-m}(b^t)^{n-p})). \tag{5.18}$$

So the coefficient of λ^{-1} of $\mathrm{tr}((\Lambda^{-i})_\lambda\Lambda^j) = 0$ if $r\neq s$ or $m\neq p$ (i.e., if $i\neq j$). If $i=j$, then (5.18) implies that the coefficient of λ^{-1} of $\mathrm{tr}((\Lambda^{-j})_\lambda\Lambda^j)$ is equal to $-(s\mathrm{tr}((b^t)^p b^p) + (s+1)\mathrm{tr}(b^{n-p}(b^t)^{n-p})) = -j$. This proves the claim and $\langle e_{1n}, Y(u,\lambda)\rangle = -j\oint f_j(u)\mathrm{d}x = -\frac{j}{n}H_j(u)$. q.e.d.

Theorem 5.8 implies that

$$(J_2)_u(\nabla H_j(u)) = (J_1)_u(\nabla H_{n+j}(u)).$$

So $(J_2)_u(\nabla H_j(u))$ is in the image of $(J_1)_u$. We write the above equation as $(J_1^{-1}J_2)_u(\nabla H_j(u)) = \nabla H_{n+j}(u)$. Then

$$(J_1^{-1}J_2)_u^k(\nabla H_j(u)) = \nabla H_{nk+j}(u). \tag{5.19}$$

Recall that $\{\,,\}_1$ and $\{\,,\}_2$ generate a sequence of Poisson brackets (cf. [**10**], [**5**]) on $C^\infty(S^1, V_n)$,

$$\{F_1, F_2\}_j(u) = \langle (J_j)_u(\nabla F_1(u)), \nabla F_2(u)\rangle, \tag{5.20}$$

where

$$J_j = J_2(J_1^{-1}J_2)^{j-2} = (J_2J_1^{-1})^{j-2}J_2. \tag{5.21}$$

Proposition 5.12. *([**5**]) The $(nk+j)$-th $A^{(1)}_{n-1}$-KdV flow is*

$$u_{t_{nk+j}} = (J_{k+2})_u(\nabla H_j(u)) = (J_2J_1^{-1})^kJ_2(\nabla H_j(u)),$$

where H_j is the functional on $C^\infty(S^1, V_n)$ defined by (5.10).

Proof. The Hamiltonian equation of $H_j(u)$ w.r.t. $\{\ ,\}_{k+2}$ is

$$\begin{aligned} u_t &= (J_{k+1})_u(\nabla H_j(u)) = J_2(J_1^{-1}J_2)^k(\nabla H_j(u)), \quad \text{by(5.19)} \\ &= J_2(\nabla H_{nk+j}(u)) = [\partial_x + b + u, Z_{nk+j,0}(u)], \end{aligned}$$

which is the $(nk+j)$-th $A^{(1)}_{n-1}$-KdV flow. q.e.d.

The above Proposition gives the well-known fact that the $A^{(1)}_{n-1}$-KdV hierarchy is given by the first $(n-1)$ flows and the recursive operator $J_2J_1^{-1}$.

6. Bi-Hamiltonian structure for central affine curve flows

In this section, we

1) give formulas for Hamiltonian vector fields for functional $\hat{H} = H \circ \Psi$ on $\mathcal{M}_n(S^1)$ with respect to the pull back bracket $\{\,,\}^\wedge_j$,
2) prove results (i)–(iv) stated in the introduction.

Recall that the pull back $\{\,,\}^\wedge_j$ of the Poisson structure $\{\,,\}_j$ to $\mathcal{M}_n(S^1)$ via the central affine curvature map Ψ is defined for functionals of the form $F \circ \Psi$, where F is a functional on $C^\infty(S^1, V_n)$. In other words, $\{\,,\}^\wedge_j$ is defined by

$$\{F_1 \circ \Psi, F_2 \circ \Psi\}^\wedge_j = \{F_1, F_2\}_j \circ \Psi,$$

for functionals F_1, F_2 on $C^\infty(S^1, V_n)$.

Proposition 6.1. *Let J_j be the j-th Poisson operator defined by (5.21), H a functional on $C^\infty(S^1, V_n)$, Ψ the central affine curvature map, and X the Hamiltonian vector field for $\hat{H} = H \circ \Psi$ with respect to the pull back Poisson structure $\{\,,\}^\wedge_j$ on $\mathcal{M}_n(S^1)$. Then for $\gamma \in \mathcal{M}_n(S^1)$ we have*

$$[\partial_x + b + u, g^{-1}\delta_X g] = (J_j)_u(\nabla H(u)),$$

where g and u are the central affine moving frame and central affine curvature along γ, respectively, and

$$\delta_X g = (X(\gamma), \ldots, (X(\gamma))_x^{(n-1)}). \tag{6.1}$$

Note that $X(\gamma)$ is a column vector and $\delta_X g$ is $sl(n, \mathbb{R})$-valued. We use $\delta_X g$ instead of $\delta_{X(\gamma)} g$ to simplify the notation.

Proof. Since $\{\,,\}_j^\wedge$ is the pull back of $\{\,,\}_j$, we have

$$\mathrm{d}\Psi_\gamma(X(\gamma)) = (J_j)_u(\nabla H(u)).$$

By Proposition 3.2, $\mathrm{d}\Psi_\gamma(X(\gamma)) = [\partial_x + b + u, g^{-1}\delta_X g]$. q.e.d.

Definition 6.2. A tangent vector field X on $\mathcal{M}_n(S^1)$ is $SL(n, \mathbb{R})$-*equivariant* if $X(c\gamma) = cX(\gamma)$ for all $c \in SL(n, \mathbb{R})$ and $\gamma \in \mathcal{M}_n(S^1)$.

Since both $\hat{H} = H \circ \Psi$ and $\{\,,\}_j^\wedge$ are invariant under the action of $SL(n, \mathbb{R})$, the Hamiltonian vector field X of $\hat{H}$ is equivariant under $SL(n, \mathbb{R})$. Although $\mathrm{d}\Psi_\gamma$ is not one-to-one, the following Lemma shows that it is injective on the space of $SL(n, \mathbb{R})$-equivariant tangent fields. We need this to show that the Hamiltonian flow for $\hat{H}_j$ with respect to $\{\,,\}_2^\wedge$ is the j-th central affine curve flow.

Lemma 6.3. *Let g and u denote the central affine moving frame and curvature along γ. If X is a $SL(n, \mathbb{R})$-equivariant tangent vector field on $\mathcal{M}_n(S^1)$ and $[\partial_x + b + u, g^{-1}\delta_X g] = 0$, then $X = 0$, where $\delta_X g$ is defined by* (6.1).

Proof. By Corollary 3.8, there exists a constant $c_o \in sl(n, \mathbb{R})$ such that $g^{-1}\delta_X g = g^{-1}c_0 g$. So $\delta_X g = c_0 g$ and $X(\gamma) = c_0\gamma$. Note that for $c \in SL(n, \mathbb{R})$, the central moving frame for $c\gamma$ is cg. Since $X(c\gamma) = cX(\gamma)$, we have $c_0 c\gamma = cc_0\gamma$ for all $\gamma \in \mathcal{M}_n(S^1)$. So $c_0 c = cc_0$ for all $c \in SL(n, \mathbb{R})$. Hence, $c_0 = \mu \mathrm{I}_n$ for some $\mu \in \mathbb{R}$. But $\mathrm{tr}(c_0) = 0$, so $c_0 = 0$. q.e.d.

Proposition 6.4. *Let H_j be functionals on $C^\infty(S^1, V_n)$ defined by* (5.10)*, Ψ the central affine curvature map, and $\hat{H}_j = H_j \circ \Psi$. Then the j-th central affine curve flow* (1.6) *is the Hamiltonian equation for $\hat{H}_j$ and $\hat{H}_{n+j}$ with respect to $\{\,,\}_2^\wedge$ and $\{\,,\}_1^\wedge$, respectively.*

Proof. Let X denote the Hamiltonian vector field of $\hat{H}_{n+j} = H_{n+j} \circ \Psi$ w.r.t. $\{\,,\}_1^\wedge$ and $\delta_X g$ as in (6.1). Since $\hat{H}_{n+j}$ is invariant under $SL(n, \mathbb{R})$, X is $SL(n, \mathbb{R})$-equivariant. By Proposition 6.1 and (5.19), we have

$$[\partial_x + b + u, g^{-1}\delta_X g] = (J_1)_u(\nabla H_{n+j}(u)) = [\partial_x + b + u, Z_{j,0}(u)].$$

Both X and $gZ_{j,0}(u)e_1$ are equivariant under $SL(n, \mathbb{R})$ and the above equality implies that we have $[\partial_x + b + u, g^{-1}\delta_X g - Z_{j,0}(u)] = 0$. Lemma 6.3 implies that $X(\gamma) = gZ_{j,0}(u)e_1$. So the Hamiltonian flow for $\hat{H}_{n+j}$ with respect to $\{\,,\}_1^\wedge$ is the j-th central affine curve flow.

Let Y denote the Hamiltonian vector field of $\hat{H}_j = H_j \circ \Psi$ w.r.t. $\{\ ,\ \}_2$ By Proposition 6.1, we have

$$[\partial_x + b + u, g^{-1}\delta_Y g] = (J_2)_u(\nabla H_j(u)) = [\partial_x + b + u, Z_{j,0}(u)].$$

Since both Y and $gZ_{j,0}(u)$ are equivariant under $SL(n, \mathbb{R})$, Lemma 6.3 implies that $Y(\gamma) = gZ_{j,0}(u)e_1$, and hence the Hamiltonian flow for $\hat{H}_j$ with respect to $\{\ ,\ \}_2^\wedge$ is the j-th central affine curve flow. q.e.d.

Proposition 6.5. *Let $1 \leq j \leq n-1$ and $k \geq 0$. Then the $(nk+j)$-th central affine curve flow on $\mathcal{M}_n(S^1)$ is*

$$\gamma_{t_{nk+j}} = gZ_{nk+j,0}(u)e_1 = g(P_u((J_1^{-1}J_2)_u^k(\nabla H_j(u))))e_1,$$

where g and u are the central affine moving frame and curvature of γ, respectively.

Proof. By Corollary 2.12, we have $Z_{nk+j,0}(u) = P_u(\pi_0(Y_{nk+j,0}(u)))$. From (5.11), this is equal to $P_u(\nabla H_{nk+j}(u))$.

By (5.19), we have $P_u(\nabla H_{nk+j}(u)) = P_u((J_1^{-1}J_2)_u^k(\nabla H_j(u)))$. This proves the formula for $\gamma_{t_{nk+j}}$. q.e.d.

Hence, the central affine curve flows can be generated by the functionals $H_1, \ldots, H_{n-1}$ and the operator $J_1^{-1}J_2$ recursively.

Proposition 6.6. *Let F_1 and F_2 be functionals on $C^\infty(S^1, V_n)$, and X_i the Hamiltonian vector field of $\hat{F}_i = F_i \circ \Psi$ with respect to $\{\ ,\ \}_j^\wedge$ for $i = 1, 2$. Then*

$$\{\hat{F}_1, \hat{F}_2\}_j^\wedge(\gamma) = -\langle g^{-1}\delta_{X_1}g, (J_2J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_{X_2}g))\rangle, \tag{6.2}$$

where g is the central affine moving frame along γ, $u = \Psi(\gamma)$, $\delta_X g$ is defined by (6.1), and π_0 is the projection defined by (2.23).

Proof. By Proposition 3.2, $\mathrm{d}\Psi_\gamma(\delta\gamma) = (J_2)_u(\pi_0(g^{-1}\delta g))$. Proposition 6.1 gives $\mathrm{d}\Psi(X_i(\gamma)) = (J_j)_u(\nabla F_i(u))$ for $i = 1, 2$. So we get

$$(J_2)_u(\pi_0(g^{-1}\delta_{X_i}g)) = (J_j)_u(\nabla F_i(u)), \tag{6.3}$$

$$\nabla F_i(u) = (J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_{X_i}g)). \tag{6.4}$$

By definition, we have

$$\{\hat{F}_1, \hat{F}_2\}_j^\wedge(\gamma) = \{F_1, F_2\}_j(u) = \langle (J_j)_u(\nabla F_1(u)), \nabla F_2(u)\rangle.$$

We substitute (6.3) to the first term and (6.4) to the second term and get

$$\begin{aligned} &= \langle J_2(\pi_0(g^{-1}\delta_{X_1}g)), (J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_{X_2}g))\rangle \\ &= -\langle \pi_0(g^{-1}\delta_{X_1}g), (J_2J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_{X_2}g))\rangle. \end{aligned}$$

Since $(J_2J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_{X_2}g)) \in C^\infty(\mathbb{R}, V_n)$, we obtain (6.2). q.e.d.

Definition 6.7. Let $\hat{w}_j$ be the 2-form on $\mathcal{M}_n(S^1)$ defined by

$$(\hat{w}_j)_\gamma(X_1(\gamma), X_2(\gamma)) = -\langle g^{-1}\delta_{X_1}g, (J_2J_j^{-1}J_2)_u(\pi_0(g^{-1}\delta_{X_2}g))\rangle, \tag{6.5}$$

where X_i is a tangent field on $\mathcal{M}_n(S^1)$ and $\delta_{X_i}g$ is given by (6.1) for $i = 1, 2$.

Let $X_{\hat{F}}$ denote the Hamiltonian vector field of the Hamiltonian $\hat{F} = F\circ\Psi$. By Proposition 6.6, we get $(\hat{w}_j)_\gamma(X_{\hat{F}_1}(\gamma), X_{\hat{F}_2}(\gamma)) = \{\hat{F}_1, \hat{F}_2\}_j(\gamma)$.

Note that $J_2J_2^{-1}J_2 = J_2$ and $J_2J_3^{-1}J_2 = J_1$. So (6.5) gives the following.

Proposition 6.8. *Let $\hat{w}_j$ be the 2-form on $\mathcal{M}_n(S^1)$ defined by (6.5). Then*

$$(\hat{w}_2)_\gamma(X_1(\gamma), X_2(\gamma)) = \langle[\partial_x + b + u, g^{-1}\delta_{X_1}g], g^{-1}\delta_{X_2}g\rangle, \tag{6.6}$$

$$(\hat{w}_3)_\gamma(X_1(\gamma), X_2(\gamma)) = \langle[g^{-1}\delta_{X_1}g, e_{1n}], g^{-1}\delta_{X_2}g\rangle. \tag{6.7}$$

Proof. By Proposition 5.5, we have

$$(J_2)_u(\pi_0(g^{-1}\delta_{X_2}g)) = [\partial_x + b + u, P_u(\pi_0(g^{-1}\delta_{X_2}g)).$$

It follows from Corollary 3.6 that $P_u(\pi_0(g^{-1}\delta_{X_2}g)) = g^{-1}\delta_{X_2}g$. So we have

$$\begin{aligned}(\hat{w}_2)_\gamma(X_1(\gamma), X_2(\gamma)) &= -\langle g^{-1}\delta_{X_1}g, [\partial_x + b + u, g^{-1}\delta_{X_2}g]\rangle \\ &\quad \langle[\partial_x + b + u, g^{-1}\delta_{X_1}g], g^{-1}\delta_{X_2}g\rangle.\end{aligned}$$

Similar computation gives the formula for $\hat{w}_3$. q.e.d.

Next we construct two embeddings of $\mathcal{M}_n(S^1)$ into certain co-adjoint orbits and show that $\hat{w}_2$ and $\hat{w}_3$ are the restrictions of co-adjoint orbit symplectic forms, hence they are closed.

Let M be the co-Adjoint orbit of G on the dual $\mathcal{G}^*$ of the Lie algebra $\mathcal{G}$ at $\ell_0 \in \mathcal{G}^*$. The orbit symplectic form on M is defined by

$$\tau_\ell(\tilde{\xi}(\ell), \tilde{\eta}(\ell)) = \ell([\xi, \eta]),$$

where $\ell \in M$, and $\tilde{\xi}$ and $\tilde{\eta}$ are infinitesimal vector fields corresponding to the co-Adjoint action generated by $\xi, \eta \in \mathcal{G}$.

We identify the Adjoint orbit of $C^\infty(S^1, SL(n,\mathbb{R}))$ on $C^\infty(\mathbb{R}, sl(n,\mathbb{R}))$ as the co-Adjoint orbit of $C^\infty(S^1, SL(n,\mathbb{R}))$ via the non-degenerate bilinear form (5.1), i.e.,

$$\langle\xi, \eta\rangle = \oint \mathrm{tr}(\xi(x)\eta(x))\mathrm{d}x.$$

Theorem 6.9. *Let $\mathcal{O}_1$ denote the Adjoint orbit of $C^\infty(S^1, SL(n,\mathbb{R}))$ on $C^\infty(S^1, sl(n,\mathbb{R}))$ at the constant loop e_{1n}, τ_1 the orbit symplectic form on $\mathcal{O}_1$, and $\mathfrak{k}_1$ the map from $\mathcal{M}_n(S^1)$ to $\mathcal{O}_1$ defined by $\mathfrak{k}_1(\gamma) = ge_{1n}g^{-1}$, where g is the central affine moving frame along γ. Let $\hat{w}_3$ be as in (6.7). Then $\mathfrak{k}_1^*\tau_1 = \hat{w}_3$.*

Proof. Given $\xi \in C^\infty(S^1, sl(n,\mathbb{R}))$, a direct computation implies that the infinitesimal vector field is

$$\tilde{\xi}(ge_{1n}g^{-1}) = [\xi, ge_{1n}g^{-1}].$$

So the orbit symplectic structure is

$$(\tau_1)_{ge_{1n}g^{-1}}([\xi, ge_{1n}g^{-1}], [\eta, ge_{1n}g^{-1}]) = \langle ge_{1n}g^{-1}, [\xi,\eta]\rangle.$$

The differential of $\mathfrak{k}_1$ at γ is

$$\mathrm{d}(\mathfrak{k}_1)_\gamma(\delta\gamma) = [\delta g g^{-1}, ge_{1n}g^{-1}].$$

Hence,

$$\begin{aligned}&(\mathfrak{k}_1^*\tau_1)_\gamma(\delta_1\gamma,\delta_2\gamma) = (\tau_1)_{ge_{1n}g^{-1}}([(\delta_1 g)g^{-1}, ge_{1n}g^{-1}], [(\delta_2 g)g^{-1}, ge_{1n}g^{-1}])\\ &= \langle ge_{1n}g^{-1}, [(\delta_1 g)g^{-1}, (\delta_2 g)g^{-1}]\rangle = \langle e_{1n}, [g^{-1}\delta_1 g, g^{-1}\delta_2 g]\rangle,\end{aligned}$$

which is equal to $(\hat{w}_3)_\gamma(\delta_1\gamma,\delta_2\gamma)$. q.e.d.

Let $\mathbb{R}\partial_x \oplus C^\infty(S^1, sl(n,\mathbb{R}))$ denote the Lie algebra with bracket defined by

$$[r_1\partial_x + u, r_2\partial_x + v] = r_1 v_x - r_2 u_x + [u,v], \quad r_1, r_2 \in \mathbb{R}.$$

It is known (cf. [**18**], [**19**]) that the dual of the central extension of the loop algebra $C^\infty(S^1, sl(n,\mathbb{R}))$ defined by the 2-cocycle

$$\rho(\xi,\eta) = \oint \mathrm{tr}(\xi_x(x)\eta(x))\mathrm{d}x,$$

can be identified as the Lie algebra $\mathbb{R}\partial_x + C^\infty(S^1, sl(n,\mathbb{R}))$ and the co-Adjoint action corresponds to the gauge action,

$$g\cdot(\partial_x + u) = g(\partial_x + u)g^{-1} = \partial_x + gug^{-1} - g_x g^{-1}.$$

Theorem 6.10. *Let $\mathcal{O}_2$ denote the gauge orbit of $C^\infty(S^1, SL(n,\mathbb{R}))$ at ∂_x, τ_2 the orbit symplectic form on $\mathcal{O}_2$, and $\mathfrak{k}_2 : \mathcal{M}_n(S^1) \to \mathcal{O}_2$ the map defined by $\mathfrak{k}_2(\gamma) = g^{-1}g_x$, where g is the central affine moving frame along γ. Let $\hat{w}_2$ be as in* (6.6)*. Then $\mathfrak{k}_2^*(\tau_2) = \hat{w}_2$.*

Proof. The infinitesimal vector field on $\mathcal{O}_2$ given by the gauge action for $\xi \in C^\infty(S^1, sl(n,\mathbb{R}))$ is $\tilde{\xi}(\partial_x + v) = -[\partial_x + v, \xi]$. Note that

$$\mathfrak{k}_2(\gamma) = \partial_x + g^{-1}g_x = \partial_x + b + u = \partial_x + b + \Psi(\gamma).$$

By Proposition 3.2,

$$\mathrm{d}(\mathfrak{k}_2)_\gamma(\delta\gamma) = \mathrm{d}\Psi_\gamma(\delta\gamma) = [\partial_x + b + u, g^{-1}\delta g].$$

Then

$$\begin{aligned}&(\mathfrak{k}_2^*\tau_2)_\gamma(\delta_1\gamma,\delta_2\gamma) = (\tau_2)_{\partial_x+b+u}(\mathrm{d}\mathfrak{k}_2(\delta_1\gamma), \mathrm{d}\mathfrak{k}_2(\delta_2\gamma))\\ &= (\tau_2)_{\partial_x+b+u}([\partial_x+b+u, g^{-1}\delta_1 g], [\partial_x+b+u, g^{-1}\delta_2 g])\\ &= \langle [\partial_x+b+u, g^{-1}\delta_1 g], g^{-1}\delta_2 g\rangle,\end{aligned}$$

which is equal to $(\hat{w}_2)_\gamma(\delta_1\gamma,\delta_2\gamma)$. q.e.d.

Recall that a *weak symplectic form* on M is a closed 2-form on M such that $w_x(v_1, v_2) = 0$ for all $v_2 \in TM_x$ implies that $v_1 = 0$. If M is of finite dimension, then a weak symplectic form is symplectic. When M is of infinite dimension, a weak symplectic form need not to be non-degenerated, but we can still have the Hamiltonian theory. Below we show that $\hat{w}_2$ and $\hat{\omega}_3$ induce weak symplectic forms on the orbit spaces $\mathcal{M}_n(S^1)/SL(n,\mathbb{R})$ and $\mathcal{M}_n(S^1)/(SL(n,\mathbb{R})\times\mathbb{R}^{n-1})$, respectively. As we mentioned in the introduction, the following two Theorems were proved by Fujioka and Kurose for $n = 2$ in [**8**]. We show that they are true for general n.

Theorem 6.11. *The* 2*-form* $\hat{w}_2$ *induces a weak symplectic form on the orbit space* $\mathcal{M}_n(S^1)/SL(n,\mathbb{R})$.

Proof. By Theorem 6.10, $\hat{w}_2$ is a closed 2-form. It follows from (6.6) that $(\hat{w}_2)_\gamma(\delta_1\gamma, \delta_2\gamma) = 0$ for all $\delta_2\gamma$ if and only if $[\partial_x + b + u, g^{-1}\delta_1 g] = 0$, where g is the central affine moving frame along γ and $u = \Psi(\gamma)$. By Proposition 3.2,

$$\mathrm{d}\Psi_\gamma(\delta_1\gamma) = [\partial_x + b + u, g^{-1}\delta_1 g] = (J_2)_u(\pi_0(g^{-1}\delta_1 g) = 0.$$

The theorem follows from Proposition 3.7. q.e.d.

We consider the $\mathbb{R}^{n-1}$-action on $\mathcal{M}_n(S^1)$ generated by the first $(n-1)$ central affine curve flow (1.6). Since the central affine curve flows commute with the $SL(n,\mathbb{R})$-action, the product group $SL(n,\mathbb{R})\times\mathbb{R}^{n-1}$ acts on $\mathcal{M}_n(S^1)$.

Theorem 6.12. *The* 2*-form* $\hat{w}_3$ *induces a weak symplectic form on the space* $\mathcal{M}_n(S^1)/(SL(n,\mathbb{R})\times\mathbb{R}^{n-1})$.

Proof. By Theorem 6.9, $\hat{w}_3$ is a closed 2-form. The formula of $\hat{w}_3$ implies that $(\hat{w}_3)_\gamma(\delta_1\gamma, \delta_2\gamma) = 0$ for all $\delta_2\gamma$ in $T\mathcal{M}(S^1)_\gamma$ if and only if

$$(J_1)_u(\pi_0(g^{-1}\delta_1 g)) = 0.$$

So $\pi_0(g^{-1}\delta_1 g)$ lies in the kernel of $(J_1)_u$. It follows from Theorem 5.10 that ξ lies in the span of $\nabla H_1(u), \ldots, \nabla H_{n-1}(u)$, where H_i's are the Hamiltonians defined by (5.10).

Let $X_1, \ldots, X_{n-1}$ denote the vector fields that generated the first $(n-1)$ central affine curve flows. Then $\mathrm{d}\Psi_\gamma(X_i) = (J_2)_u(\nabla H_i(u))$. If $\pi_0(g^{-1}\delta_1 g) = \nabla H_i(u)$ for some $1 \le i \le n-1$, then $\mathrm{d}\Psi(\delta_1\gamma) = (J_2)_u(\nabla H_i(u))$. So $\delta_1\gamma \in X_i(\gamma) + \mathrm{Ker}(\mathrm{d}\Psi_\gamma)$. By Proposition 3.7, $\mathrm{Ker}(\mathrm{d}\Psi_\gamma) =$ the tangent space of the $SL(n,\mathbb{R})$-orbit at γ. This finishes the proof. q.e.d.

We have written $\hat{w}_j(X_1, X_2)$ in terms of $g^{-1}\delta_{X_i} g$ and show that $g^{-1}\delta_X g = P_u(\pi_o(g^{-1}\delta_X g))$. It follows from Theorem 2.9 that entries of $g^{-1}\delta_X g$ are also differential polynomials of the first column. Hence, we can write $g^{-1}\delta_X g$ as differential polynomial in X. So we can write

$\hat{w}_j(X_1, X_2)$ in terms of X_i and their derivatives. In fact, Pinkall gave such formula for $\hat{w}_2$ in [**17**], Fujioka and Kurose for $\hat{w}_j$ in [**8**] on $\mathcal{M}_2(S^1)$, and Calini, Ivey, Marí Beffa wrote down the formula of $\hat{w}_3$ on $\mathcal{M}_3(S^1)$ in [**3**]. Below we work out the formulas for general n.

Theorem 6.13. *Let X, Y be tangent vectors of $\mathcal{M}_n(S^1)$ at γ. Then*

$$(\hat{w}_2)_\gamma(X,Y) = \sum_{i=1}^{n-1} \oint \det(\gamma, \ldots, \gamma_x^{i-2}, X_x^{(n)}, \gamma_x^{(i)}, \ldots, Y_x^{(i-1)})\mathrm{d}x$$
$$- \sum_{i,j=1}^{n-1} \oint u_j \det(\gamma, \ldots, \gamma_x^{(i-1)}, X_x^{(j-1)}, \gamma_x^{(i)}, \ldots, Y_x^{(i-1)})\mathrm{d}x,$$
$$(\hat{w}_3)_\gamma(X,Y) = \sum_{i=1}^{n-1} \oint \det(\gamma, \ldots, \gamma_x^{(i-2)}, X, \gamma_x^{(i)}, \ldots, Y_x^{(i-1)})\mathrm{d}x$$
$$+ \sum_{i=1}^{n-1} \oint \det(\gamma, \ldots, \gamma_x^{(i-2)}, X_x^{(i-1)}, \gamma_x^{(i)}, \ldots, Y)\mathrm{d}x.$$

Proof. Let $\gamma, g, u, \delta_i\gamma, \delta_i g$ be as in Proposition 6.6,

$$C = (C_{ij}) = g^{-1}\delta_1 g, \quad D = (D_{ij}) = g^{-1}\delta_2 g,$$

and C_i, D_i the i-th column of C and D, respectively. By Theorem 2.9, we can express C_i's as differential polynomials in C_1. Similarly, D_i's can be expressed as differential polynomials in D_1. Moreover, $C_i = (C_{1i}, \ldots, C_{ni})$ is the coordinate of $(\delta_1\gamma)_x^{(i-1)}$ with respect to the frame $g = (\gamma, \ldots, \gamma_x^{(n-1)})$, i.e., $(\delta_1\gamma)_x^{(i-1)} = \sum_{k=1}^n C_{ki}\gamma_x^{(k-1)}$.

If $Y = \sum_{i=1}^n y_i \gamma_x^{(i-1)}$, then $Y' = \sum_{i=1}^n (y_i' + y_{i-1} + u_i y_n)\gamma_x^{(i-1)}$. Write $(\delta_1\gamma)_x^{(n)} = \sum_{i=1}^n \xi_i (\delta_1\gamma)_x^{(i-1)}$. Then

$$\xi = (\xi_1, \ldots, \xi_n)^t = C_n' + (b+u)C_n.$$

By Proposition 6.8, we have

$$(\hat{w}_3)_\gamma(\delta_1\gamma, \delta_2\gamma) = \oint \sum_{i=1}^n C_{i1}D_{ni} - C_{ni}D_{i1}\mathrm{d}x,$$
$$(\hat{w}_2)_\gamma(\delta_1\gamma, \delta_2\gamma) = \oint \sum_{i=1}^n (C_x + (b+u)C)_{in}D_{ni} - (C(b+u))_{in}D_{ni}\mathrm{d}x$$
$$= \oint \sum_{i=1}^n \xi_i D_{ni} - \sum_{i=1}^n \sum_{j=1}^{n-1} C_{ij}u_j D_{ni}\mathrm{d}x.$$

We compute $(\hat{w}_3)_\gamma$ as follows: Let $X = \delta_1\gamma$, and $Y = \delta_2\gamma$. Then

$$\begin{aligned}(\hat{w}_3)_\gamma(X, Y) &= \oint \sum_{i=1}^{n} C_{i1}D_{ni} - C_{ni}D_{i1}\mathrm{d}x \\ &= \oint C_{n1}D_{nn} - C_{nn}D_{n1} + \sum_{i=1}^{n-1} C_{i1}D_{ni} - C_{ni}D_{i1}\mathrm{d}x \\ &= \oint (\sum_{i=1}^{n-1} C_{ii})D_{n1} - C_{n1}(\sum_{i=1}^{n-1} D_{ii}) + \sum_{i=1}^{n-1} C_{i1}D_{ni} - C_{ni}D_{i1}\mathrm{d}x.\end{aligned}$$

Note that $\det(\gamma, \ldots, \gamma_x^{(n-1)}) = 1$ and the k-th column of C and D are the coefficients of $X_x^{(k-1)}$ and $Y_x^{(k-1)}$ written as a linear combination of $\gamma, \ldots, \gamma_x^{(n-1)}$. So we have

$$\det(\gamma, \gamma_x, \ldots, \gamma_x^{(i-2)}, X_x^{(k-1)}, \gamma_x^{(i)}, \ldots, Y_x^{(\ell-1)}) = C_{ik}D_{n\ell} - C_{nk}D_{i\ell}. \tag{6.8}$$

Substitute (6.8) into the above formula for $w_\gamma(X, Y)$ to get the formula for $(\hat{w}_3)_\gamma$ as stated in the theorem.

Use $\mathrm{tr}(C) = \mathrm{tr}(D) = 0$ and (6.8) to get the formula for $\hat{w}_2$. q.e.d.

Example 6.14. For $n = 2$, Theorem 6.13 gives

$$\begin{aligned}(\hat{w}_2)_\gamma(X, Y) &= -\oint \det(X', Y') + u_1\det(X, Y)\mathrm{d}x, \\ (\hat{w}_3)_\gamma(X, Y) &= 2\oint \det(X, Y)\mathrm{d}x.\end{aligned}$$

The 2 form $\hat{w}_2$ was given in [**8**] and $\hat{w}_3$ in [**17**].

Example 6.15. For $n = 3$, we get

$$(\hat{w}_3)_\gamma(X, Y) = 3\oint \det(X, \gamma', Y)\mathrm{d}x,$$

which is the 2 form given in [**3**]. We also have

$$\begin{aligned}(\hat{w}_2)_\gamma(X, Y) = &\oint \det(X''', \gamma', Y) + \det(\gamma, X''', Y')\mathrm{d}x \\ &- \oint u_1(\det(X, \gamma', Y) + \det(\gamma, X, Y'))\mathrm{d}x \\ &- \oint u_2(\det(X', \gamma', Y) + \det(\gamma, X', Y'))\mathrm{d}x.\end{aligned}$$

Example 6.16. For $n = 4$, let $|v_1, v_2, v_3, v_4| = \det(v_1, v_2, v_3, v_4)$. Then we have

$$\begin{aligned}(\hat{w}_3)_\gamma(X,Y) = &\oint |X,\gamma',\gamma'',Y| + |\gamma,X,\gamma'',Y'| + |\gamma,\gamma',X,Y''|\mathrm{d}x \\ &+ \oint |X,\gamma',\gamma'',Y| + |\gamma,X',\gamma'',Y| + |\gamma,\gamma',X'',Y|\mathrm{d}x \\ &= 2\oint |X,\gamma',\gamma'',Y| + |\gamma,\gamma',X',Y'|\mathrm{d}x \\ &+ 2\oint |\gamma,\gamma',X'',Y| + |\gamma,\gamma',X,Y''|\mathrm{d}x.\end{aligned}$$

$$\begin{aligned}(\hat{w}_2)_\gamma(X,Y) = &\oint |X^{(4)},\gamma',\gamma'',Y| + |\gamma,X^{(4)},\gamma'',Y'| + |\gamma,\gamma',X^{(4)},Y''|\mathrm{d}x \\ &- \oint u_3(|X'',\gamma',\gamma'',Y| + |\gamma,X'',\gamma'',Y'| + |\gamma,\gamma',X'',Y''|)\mathrm{d}x \\ &- \oint u_2(|X',\gamma',\gamma'',Y| + |\gamma,X',\gamma'',Y'| + |\gamma,\gamma',X',Y''|)\mathrm{d}x \\ &- \oint u_1(|X,\gamma',\gamma'',Y| + |\gamma,X,\gamma'',Y'| + |\gamma,\gamma',X,Y''|)\mathrm{d}x.\end{aligned}$$

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Department of Mathematics
University of California
Irvine, CA 92697-3875
USA

E-mail address: cterng@math.uci.edu

Department of Mathematics
Ningbo University
Ningbo, Zhejiang, 315211
China

E-mail address: wuzhiwei@nbu.edu.cn