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ON THE BJÖRLING PROBLEM FOR WILLMORE SURFACES

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Abstract

We solve the analogue of Björling's problem for Willmore surfaces via a harmonic map representation. For the umbilic-free case the problem and solution are as follows: given a real analytic curve y_0 in $\mathbb{S}^3$, together with the prescription of the values of the surface normal and the dual Willmore surface along the curve, lifted to the light cone in Minkowski 5-space $\mathbb{R}^5_1$, we prove, using isotropic harmonic maps, that there exists a unique pair of dual Willmore surfaces y and $\hat{y}$ satisfying the given values along the curve. We give explicit formulae for the generalized Weierstrass data for the surface pair. For the three dimensional target, we use the solution to explicitly describe the Weierstrass data, in terms of geometric quantities, for all equivariant Willmore surfaces. For the case that the surface has umbilic points, we apply the more general half-isotropic harmonic maps introduced by Hélein to derive a solution: in this case the map $\hat{y}$ is not necessarily the dual surface, and the additional data of a derivative of $\hat{y}$ must be prescribed. This solution is generalized to higher codimensions.

1. Introduction

A *Willmore surface* in Euclidean 3-space $\mathbb{R}^3$ is an immersion S that is locally critical for the *Willmore functional*

$$\mathcal{W}(S) = \int_S H^2 \mathrm{d}A,$$

where H is the mean curvature of the surface. As such, these surfaces are generalizations of minimal surfaces, and also, from another point of view, of elastic curves. Hence, the interest in Willmore surfaces, which have attracted a lot of attention in recent decades. The governing equations are a fourth order nonlinear PDE, and they are, therefore, a challenging class of surfaces to get information about: for example, the Willmore conjecture, that the Clifford torus is the global minimizer of the Willmore energy among tori, proposed in the 1960's, took more than half a century to resolve [**26**].

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The property of being a Willmore surface is invariant under conformal transformations of the ambient space. Hence, from a theoretical point of view, the choice of conformally congruent target space is unimportant. In fact, the natural choice is the 3-sphere $\mathbb{S}^3$, because this case includes, up to Möbius equivalence, both $\mathbb{R}^3$ and the hyperbolic space $\mathbb{H}^3$ as proper subspaces. In this article, we generally regard the surfaces as living in $\mathbb{S}^3$, and more generally $\mathbb{S}^n$, $n \geq 3$. For further introduction and background on Willmore surfaces, especially relevant to this article, see Hélein [**16**].

Being one kind of generalization of minimal surfaces, it is natural to consider the extension of Björling's classical problem to Willmore surfaces. Björling's problem is to find the unique minimal surface that contains a given curve with surface normal prescribed along the curve. The solution can be found, in terms of the Weierstrass–Enneper representation, via analytic extension of the prescribed data. It is a useful tool in the study of minimal surfaces and has been generalized recently, through various means, to several other surface classes. An approach that can be expected to be fruitful among surfaces associated to harmonic maps can be found in the solution for non-minimal constant mean curvature surfaces given in [**4**]. Here one uses an infinite dimensional version of the Weierstrass–Enneper formula, the DPW method of Dorfmeister/Pedit/Wu [**11**], to again obtain the solution by holomorphic extension.

For Willmore surfaces, there are more than one type of harmonic maps one might consider employing. For example, it has long been known that the conformal Gauss map into the Grassmannian $Gr_{3,1}(\mathbb{R}^5_1)$ of Lorentzian 4-planes in $\mathbb{R}^5_1$ is harmonic. This is a certain lift of the surface normal into $\mathbb{R}^5_1$, and the harmonicity of this map has been used in [**12**] to study Willmore surfaces via the DPW method. The related flat connections also form the basis for some of the recent works on *constrained* Willmore surfaces: see, e.g., [**9**, **2**, **14**, **18**].

On the other hand, a different ("roughly") harmonic map, this time into $SO(1,4)/(SO(1,1) \times SO(3))$ was found by Hélein in [**16**] (See also [**17**]). In our distillation of Hélein's work, the basic object is the map $Y \wedge \hat{Y}$, where Y and $\hat{Y}$ are the surface and its dual, lifted to the light cone. Essentially, the projections of Y and $\hat{Y}$ are Willmore if and only if $Y \wedge \hat{Y}$ is what we call an *isotropic* harmonic map. The DPW method also works for isotropic harmonic maps, and this is the approach we will use.

1.1. Results of this article. If only the surface and surface normal are prescribed along a curve, then there is no hope of obtaining a unique solution for the Björling problem for Willmore surfaces (see Figures 1 and 6). One needs to prescribe something more, and it turns out that the value along the curve of the dual surface $\hat{Y}$ is enough. Hence, the

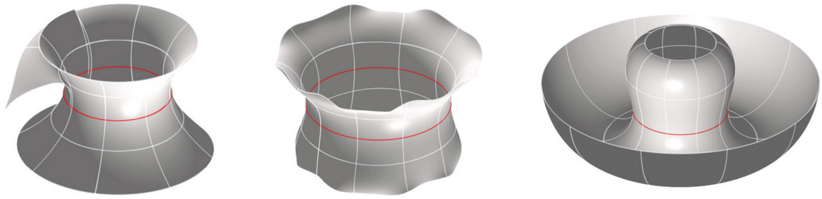

Figure 1. Three solutions to the Björling problem for Willmore surfaces in $\mathbb{S}^3$, all with the same initial curve (a circle) and the same normal along the curve. The prescribed dual surface data $\hat{Y}_0$ is different in each case. The surfaces are all given the same stereographic projection to $\mathbb{R}^3$.

representation in terms of $Y \wedge \hat{Y}$ seems canonical for this problem, rather than the conformal Gauss map representation.

In Section 2, we outline the projective light cone model for conformal surface theory, the basic theory of Willmore surfaces in this setting, and the relation with isotropic harmonic maps into $SO(1,4)/(SO(1,1) \times SO(3))$. In Section 3, we derive the DPW construction for isotropic harmonic maps. The DPW construction for harmonic maps $f : \Sigma \to G/K$ makes use of a *holomorphic frame* F_-^λ for the *extended frame* $F^\lambda : \Sigma \to \Omega G \cong \Lambda G^{\mathbb{C}}/\Lambda^+ G^{\mathbb{C}}$, a lift of f into the group of based loops in G. The Maurer–Cartan form η of F_-^λ is known as a *potential*, and this is the Weierstrass data for the problem. Given a potential η, which essentially consists of a series of arbitrary holomorphic functions, the equation $\mathrm{d}F_-^\lambda = F_-^\lambda \eta$ can be solved, and a frame $F^\lambda : \Sigma \to \Lambda G$ is obtained via the Iwasawa decomposition. If G is non-compact, all of this happens only on a large open set (the big cell) of the loop group, but otherwise the theory is the same. We need to verify that the theory restricts to *isotropic* harmonic maps (see Definition 2.10), and this is indeed the case because the isotropic condition is preserved by the loop group decompositions.

In Section 4, we present, in Theorems 4.1 and 4.3, a solution to the Björling problem for Willmore surfaces: given a real analytic sphere congruence ψ_0 (a lift of the surface normal) along a curve $\mathbb{I}$, with two enveloping curves Y_0 and $\hat{Y}_0$, there exists a unique dual pair of Willmore surfaces Y and $\hat{Y}$ that restrict, along $\mathbb{I}$, to Y_0 and $\hat{Y}_0$ (Figure 2). We also give an explicit formula for a holomorphic potential for the surface, in terms of the prescribed geometric data.

In Section 5, we apply this result to describe all $SO(4)$-*equivariant* Willmore surfaces in $\mathbb{S}^3$, that is surfaces invariant under the action of a 1-parameter subgroup of the isometry group. Our approach is to solve

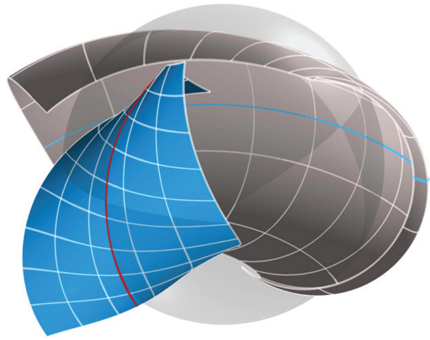

Figure 2. Dual solutions of Björling's problem for Willmore surfaces. The prescribed data is the pair of curves (one red, one blue) together with a family of 2-spheres tangent to both curves at the touching points. One 2-sphere is shown.

the Björling problem along a parallel. One can describe all $SO(1,3)$-*equivariant* Willmore surfaces in $\mathbb{H}^3$ in an analogous way, and we give the details for some of these, including hyperbolic rotational surfaces and the hyperbolic analogue of Hopf surfaces in Section 6. We remark that it is known [**6**, **22**] that Willmore surfaces of revolution in $\mathbb{R}^3$ can be obtained by revolving about the x-axis an elastic curve in $\mathbb{H}^2$, represented by the upper half plane model above the x-axis. General equivariant surfaces have not been described so explicitly, however, Ferus and Pedit [**15**] gave a description of all non-rotational $SO(4)$-equivariant Willmore tori.

Figure 3. An $SO(1,3)$-equivariant Willmore surface not congruent to a minimal surface in any space form (Section 6.1.2).

In Section 7, we extend the loop group representation to the case of isotropic and *half*-isotropic harmonic maps for general n. The half-isotropic case is a generalization of the isotropic case where $\hat{Y}$ is no longer required to be the dual (or geometric adjoint transform) of Y.

This section is partly motivated by the desire to give a uniform treatment of results of Hélein [**16**], Xia/Shen [**34**] and Ma [**24**], but it also allows us to deal with umbilics, which are ruled out in the isotropic case.

We end this paper in Section 8 with an application of the harmonic maps in Section 7 to the solution of the Björling problem for Willmore surfaces in $\mathbb{S}^{n+2}$. The half-isotropic setting is needed for Willmore surfaces in $\mathbb{S}^{n+2}$ with umbilics since at umbilics the isotropic harmonic map may have singularities. Since $\hat{Y}$ is no longer required to be the dual of Y in the half-isotropic setting or in the isotropic setting with higher codimension, there is now more freedom, and so an additional condition is needed to define a unique solution. The v derivative, $\hat{Y}_v$ turns out to be sufficient.

1.2. Concluding remarks. All the images in this article were produced by numerically implementing the DPW method for the problem at hand (available at: http://davidbrander.org/software.html at time of publication). In our examples, mainly working in the isotropic setting, the surfaces appear smooth when the boundary of the Iwasawa big cell is approached. One expects that these are points where the surface and its dual coincide, such as can happen at umbilics (see Lemma 2.7 below). Babich and Bobenko [**1**], constructed Willmore surfaces which contain lines of umbilics. For such solutions, one needs to use the general construction of Section 8.

Recently, Jensen, Musso and Nicolodi have provided a solution of the geometric Cauchy problem for the more general membrane shape equation [**20**]. This equation includes Willmore surfaces as a special case. Their solution, which needs an umbilic-free assumption, is quite different: the framework is differential systems, the problem is posed in principal coordinates, the Cauchy data are the curve y, the mean curvature h and the transverse derivative h_v along the curve $y(u)$, plus the value of the normal at a single point. Because of these major differences, the range of applications of their solution is fundamentally different – for example, the description of all equivariant surfaces we provide here does not seem feasible with their formulation.

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2. Willmore surfaces in $\mathbb{S}^{n+2}$

2.1. Conformal surface theory in the projective light cone model. We will review first the projective light cone model of the conformal geometry of $\mathbb{S}^{n+2}$ and derive the surface theory in this model.

Then we formulate it at the Lie algebra level. Our treatment here follows the surface theory in [**8**, **25**].

We denote by $\mathbb{R}^{n+4}_1$ the space $\mathbb{R}^{n+4}$ equipped with a Lorentzian metric

$$\langle x, y\rangle = -x_0y_0 + \sum_{j=1}^{n+3} x_jy_j = x^t I_{1,n+3}y, \quad I_{1,n+3} = \mathrm{diag}(-1, 1, \cdots, 1).$$

Let $\mathcal{C}^{n+3}_+$ be the forward light cone of $\mathbb{R}^{n+4}_1$, i.e., for any $x \in \mathcal{C}^{n+3}_+$, $x_0 > 0$. One can see that the projective light cone

$$Q^{n+2} = \{\ [x] \in \mathbb{R}P^{n+3} \mid x \in \mathcal{C}^{n+3}_+\},$$

with the induced conformal metric, is conformally equivalent to $\mathbb{S}^{n+2}$, and the conformal group of Q^{n+2} is exactly the orthogonal group $O(1, n + 3)/\{\pm 1\}$ of $\mathbb{R}^{n+4}_1$, acting on Q^{n+2} by $T([x]) = [Tx]$, $T \in O(1, n + 3)$. We denote by $SO^+(1, n + 3)$ the connected component of $O(1, n + 3)$ containing I, that is for any $T \in SO^+(1, n + 3)$, $\det T = 1$ and T preserves the signature of the first coordinate of any $x \in \mathbb{R}^{n+4}_1$ (i.e., it preserves the time direction).

Let $y : M^2 \to \mathbb{S}^{n+2}$ be a conformal immersion from a Riemann surface M. Let $U \subset M$ be an open subset. A local lift of y is a map $Y : U \to \mathcal{C}^{n+3}_+$ such that $\pi \circ Y = y$. Two different local lifts differ by a scaling, so with conformal induced metrics. Here we call y a conformal immersion, if $\langle Y_z, Y_z\rangle = 0$ and $\langle Y_z, Y_{\bar z}\rangle > 0$ for any local lift Y and any complex coordinate z on M. Then there is a decomposition $M \times \mathbb{R}^{n+4}_1 = V \oplus V^\perp$, where

$$V = \mathrm{Span}\{Y, \mathrm{Re}Y_z, \mathrm{Im}Y_z, Y_{z\bar z}\}$$

is a Lorentzian rank-4 sub-bundle independent of the choice of Y and z. Their complexifications are denoted separately as $V_{\mathbb{C}}$ and $V^\perp_{\mathbb{C}}$.

Fix a local coordinate z. There is a local lift Y satisfying $|\mathrm{d}Y|^2 = |\mathrm{d}z|^2$, called the canonical lift (with respect to z). Choose a frame $\{Y, Y_z, Y_{\bar z}, N\}$ of $V_{\mathbb{C}}$, where $N \in \Gamma(V)$ is uniquely determined by

$$\langle N, Y_z\rangle = \langle N, Y_{\bar z}\rangle = \langle N, N\rangle = 0, \langle N, Y\rangle = -1. \tag{2.1}$$

Now we define *the conformal Gauss map* of y as follows. See also [**5**, **8**, **13**, **25**].

Definition 2.1. For a conformally immersed surface $y : M \to \mathbb{S}^{n+2}$ with canonical lift Y (with respect to a local coordinate z), we define

$$G := Y \wedge Y_u \wedge Y_v \wedge N = -2i \cdot Y \wedge Y_z \wedge Y_{\bar z} \wedge N,\ z = u + iv,$$

where $N \equiv 2Y_{z\bar z}(\bmod Y)$ is the frame vector determined in (2.1). It is direct to see that G is well defined. We call $G : M \to Gr_{3,1}(\mathbb{R}^{n+4}_1)$ *the conformal Gauss map* of y.

Given frames as above, and noting that Y_{zz} is orthogonal to Y, Y_z and $Y_{\bar{z}}$, there exists a complex function s and a section $\kappa \in \Gamma(V_{\mathbb{C}}^{\perp})$ such that

$$Y_{zz} = -\frac{s}{2}Y + \kappa.$$

This defines two basic invariants κ and s depending on coordinates z, *the conformal Hopf differential* and *the Schwarzian* of y (for more discussion, see [**8**, **25**]). Let D denote the normal connection and $\psi \in \Gamma(V_{\mathbb{C}}^{\perp})$ any section of the normal bundle. The structure equations can be given as follows:

$$\begin{cases} Y_{zz} = -\frac{s}{2}Y + \kappa, \\ Y_{z\bar{z}} = -\langle \kappa, \bar{\kappa} \rangle Y + \frac{1}{2}N, \\ N_z = -2\langle \kappa, \bar{\kappa} \rangle Y_z - sY_{\bar{z}} + 2D_{\bar{z}}\kappa, \\ \psi_z = D_z\psi + 2\langle \psi, D_{\bar{z}}\kappa \rangle Y - 2\langle \psi, \kappa \rangle Y_{\bar{z}}. \end{cases}$$

The conformal Gauss, Codazzi and Ricci equations as integrable conditions are:

$$\begin{cases} \frac{1}{2}s_{\bar{z}} = 3\langle \kappa, D_z\bar{\kappa} \rangle + \langle D_z\kappa, \bar{\kappa} \rangle, \\ \mathrm{Im}(D_{\bar{z}}D_{\bar{z}}\kappa + \frac{\bar{s}}{2}\kappa) = 0, \\ R^D_{\bar{z}z} = D_{\bar{z}}D_z\psi - D_zD_{\bar{z}}\psi = 2\langle \psi, \kappa \rangle \bar{\kappa} - 2\langle \psi, \bar{\kappa} \rangle \kappa. \end{cases} \tag{2.2}$$

The conformal Hopf differential plays an important role in the study of Willmore surfaces. To see this, we first give the transformation formula of κ. For another complex coordinate w, $Y_1 = Y \cdot |\frac{\mathrm{d}w}{\mathrm{d}z}|$ is the canonical lift with respect to w. So the corresponding Hopf differential κ_1 with respect to (Y_1, w) is

$$\kappa_1 = \kappa \cdot \left(\frac{\mathrm{d}z}{\mathrm{d}w}\right)^2 / |\frac{\mathrm{d}z}{\mathrm{d}w}|. \tag{2.3}$$

Direct computation using the structure equations above shows that G induces a conformal-invariant metric

$$g := \frac{1}{4}\langle \mathrm{d}G, \mathrm{d}G \rangle = \langle \kappa, \bar{\kappa} \rangle |\mathrm{d}z|^2,$$

on M. Note this metric degenerates at umibilic points of y. We define the Willmore functional and Willmore surfaces by use of this metric.

Definition 2.2. *The Willmore functional* of y is defined as the area of M with respect to the metric above:

$$W(y) := 2i \int_M \langle \kappa, \bar{\kappa} \rangle \mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

An immersed surface $y : M \to \mathbb{S}^{n+2}$ is called a *Willmore surface* if it is a critical surface of the Willmore functional with respect to any variation of the map $y : M \to \mathbb{S}^{n+2}$.

It is direct to verify that $W(y)$ is well-defined from the formula (2.3). Willmore surfaces can be characterized as follows [**5**, **8**, **13**, **31**]:

Theorem 2.3. *For a conformal immersion $y : M \to \mathbb{S}^{n+2}$, the following three conditions are equivalent:*

(i) *The immersion y is Willmore.*
(ii) *The conformal Gauss map G is a harmonic map into $G_{3,1}(\mathbb{R}^{n+3}_1)$.*
(iii) *The conformal Hopf differential κ of y satisfies the following Willmore condition, which is stronger than the conformal Codazzi equation* (2.2)*:*

$$D_{\bar z}D_{\bar z}\kappa + \frac{\bar s}{2}\kappa = 0.$$

In the seminal paper [**5**], Bryant showed that every Willmore surface Y in $\mathbb{S}^3$ admits a dual Willmore surface $\hat{Y}$, i.e., another map $\hat{Y}$, which may have branch points or degenerate to a point, but, if immersed, has the same complex coordinate and the same conformal Gauss map as Y. This duality theorem, however, does not hold in general when the codimension is bigger than 1 ([**13**], [**8**], [**24**]). To characterize Willmore surfaces with dual surfaces, in [**13**] Ejiri introduced the notion of *S-Willmore surfaces*. Here we define it slightly differently to include all Willmore surfaces with dual surfaces:

Definition 2.4. A Willmore immersion $y : M^2 \to \mathbb{S}^{n+2}$ is called an S-Willlmore surface if its conformal Hopf differential satisfies

$$D_{\bar z}\kappa || \kappa,$$

i.e., there exists some function μ on M such that $D_{\bar z}\kappa + \frac{\mu}{2}\kappa = 0$.

A basic result of [**13**] states that a Willmore surface admits a dual surface if and only if it is S-Willmore. Moreover, the dual surface is also Willmore, when it is non-degenerate.

Example 2.5. 1. It is well known that minimal surfaces in Riemannian space forms are Willmore surfaces (see [**5**, **21**], for example). These surfaces give the basic examples of Willmore surfaces. Moreover, they are, in any codimension, S-Willmore surfaces, i.e., Willmore surfaces with a dual surface, see [**13**, **25**].

2. Using the Hopf bundle, Pinkall [**28**] obtained a family of non-minimal Willmore surfaces in $\mathbb{S}^3$ via the elastic curves.

2.2. Harmonic maps into $SO^+(1,4)/\left(SO^+(1,1)\times SO(3)\right)$ related to Willmore surfaces. In the classic paper [**16**], Hélein showed that there exists another family of flat connections associated with an umbilic free Willmore surface in $\mathbb{S}^3$, besides the one related to the conformal Gauss map. Hélein's connections yield many "roughly harmonic" maps $Y \wedge \hat{Y}$, that take values in $SO^+(1,4)/\left(SO^+(1,1)\times SO(3)\right)$. Here $\hat{Y}$ is an arbitrary lightlike vector other than Y in the mean curvature sphere V of Y. Moreover, he found that if $\hat{Y}$ is chosen suitably (which yields a Riccati equation), the roughly harmonic map $Y \wedge \hat{Y}$ will be truly harmonic [**16**]. A special choice is to set $\hat{Y}$ to be the dual surface of Y ([**16**], [**17**]). These results are generalized for Willmore surfaces in $\mathbb{S}^{n+2}$ in [**34**].

In a different approach Ma [**24**] proved that a Willmore surface in $\mathbb{S}^{n+2}$ locally always admits an adjoint transform (which in general may be non-unique). This is the generalization of the duality theorem of Willmore surfaces in $\mathbb{S}^3$. Furthermore, he found that a Willmore surface together with an adjoint transform, derives a new kind of harmonic map into $SO^+(1,n+3)/\left(SO^+(1,1)\times SO(n+2)\right)$, which turns out to be one of the harmonic maps found by Hélein [**16**] and Qiaoling Xia, Yibing Shen [**34**].

To avoid burdening the reader who may be primarily concerned with the $\mathbb{S}^3$ case with unnecessary information, we will restrict ourselves, in this subsection and the sections immediately following, to Willmore surfaces in $\mathbb{S}^3$. The general case of $\mathbb{S}^{n+2}$ includes more possibilities, which we discuss in Section 7.

Let $y : U \to \mathbb{S}^3$ be an umbilic free Willmore surface with canonical lift Y with respect to z as above. We introduce $\hat{Y}$ as

$$\hat{Y} = N + \bar{\mu}Y_z + \mu Y_{\bar{z}} + \frac{1}{2}|\mu|^2 Y, \tag{2.4}$$

with $\mu \mathrm{d}z = 2\langle \hat{Y}, Y_z\rangle \mathrm{d}z$ a complex connection 1-form. Direct computation yields

$$\hat{Y}_z = \frac{\mu}{2}\hat{Y} + \theta\left(Y_{\bar{z}} + \frac{\bar{\mu}}{2}Y\right) + \rho\left(Y_z + \frac{\mu}{2}Y\right) + 2\zeta,$$

with

$$\theta := \mu_z - \frac{\mu^2}{2} - s, \quad \rho := \bar{\mu}_z - 2\langle\kappa,\bar{\kappa}\rangle, \quad \zeta := D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa.$$

Then $\hat{Y}$ is the dual surface of Y if and only if $D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa = 0$ ([**5**], [**13**], [**25**], [**24**]). Note now the Willmore equation is equivalent to the Riccati equation

$$\mu_z - \frac{\mu^2}{2} - s = 0. \tag{2.5}$$

Theorem 2.6. [**16**], [**34**], [**24**] (Harmonicity of another map) *Let Y be an umbilic free Willmore surface in $\mathbb{S}^3$ with $\hat{Y}$ its dual surface. Set*

$$\begin{array}{cccc} f_h: & U & \to & SO^+(1,n+3)/\left(SO^+(1,1)\times SO(n+2)\right), \\ & p\in U & \mapsto & Y(p)\wedge\hat{Y}(p). \end{array}$$

Then f_h is a conformally harmonic map.

At umbilic points it is possible that there exists a limit of μ such that (2.5) holds. Due to the following lemma, the harmonic map f_h has no definition when μ tends to ∞.

Lemma 2.7. [**12**] *At the umbilic points of Y, the limit of μ goes to a finite number or infinity. When μ goes to infinity, $[\hat{Y}]$ tends to $[Y]$, and at the point in question we have $[\hat{Y}] = [Y]$.*

In order to use the machinery of loop groups, we need to examine the structure of the Maurer–Cartan form of a frame for $Y \wedge \hat{Y}$:

Proposition 2.8. *Let $f_h = Y \wedge \hat{Y}$ be a harmonic map, where Y and $\hat{Y}$ are a Willmore surface and its dual, as above. Chose a frame*

$$F = \left(\frac{1}{\sqrt{2}}(Y + \hat{Y}), \frac{1}{\sqrt{2}}(-Y + \hat{Y}), P_1, P_2, \psi \right) : U \to SO^+(1,4),$$

with $Y_z + \frac{\mu}{2} Y = \frac{1}{2}(P_1 - iP_2)$, and ψ a unit vector in the normal bundle $V^\perp$. Set $\kappa = k\psi$. Then the Maurer–Cartan form $\alpha = F^{-1}\mathrm{d}F = \alpha' + \alpha''$ of F is

$$\alpha' = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,1} & A_2 \end{pmatrix} \mathrm{d}z,$$

with

$$A_1 = \begin{pmatrix} 0 & \frac{\mu}{2} \\ \frac{\mu}{2} & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} \frac{1+\rho}{2\sqrt{2}} & \frac{-i-i\rho}{2\sqrt{2}} & 0 \\ \frac{1-\rho}{2\sqrt{2}} & \frac{-i+i\rho}{2\sqrt{2}} & 0 \end{pmatrix} = \begin{pmatrix} b_1^t \\ b_2^t \end{pmatrix}.$$

So

$$B_1 B_1^t = 0.$$

It is straightforward to see that this last condition on B_1 is independent of the choice of frame F for the harmonic map f_h. Conversely, this condition is also sufficient to characterize Willmore surfaces:

Theorem 2.9. [**16**], [**17**], [**34**], [**24**]. *Let f be a non-constant harmonic map $M \to SO^+(1,4)/(SO^+(1,1) \times SO(3))$, satisfying $B_1 B_1^t = 0$. Then Y and $\hat{Y}$ are a pair of dual (possibly degenerate) Willmore surfaces. Moreover, set*

$$B_1 = (b_1 \ b_2)^t \text{ with } b_1, b_2 \in \mathbb{C}^3.$$

Then Y is immersed at the points $(b_1^t + b_2^t)(\bar{b}_1 + \bar{b}_2) > 0$ and $\hat{Y}$ is immersed at the points $(b_1^t - b_2^t)(\bar{b}_1 - \bar{b}_2) > 0$.

Note that Y or $\hat{Y}$ may degenerate to a point, and in this case the dual ($\hat{Y}$ or Y) is Möbius equivalent to a minimal surface in $\mathbb{R}^3$.

Since $B_1 B_1^t = 0$ serves as some isotropic condition, we define:

Definition 2.10. Let $f : M \to SO^+(1,4)/(SO^+(1,1) \times SO(3))$ be a non-constant harmonic map. Then f is called an *isotropic* harmonic map if the Maurer–Cartan form of any frame of f, with the above notation, satisfies $B_1 B_1^t = 0$.

This characterization of Willmore surfaces in terms of isotropic harmonic maps essentially follows from the work of Hélein [**16**, **17**], although the name "isotropic" is not used there.

3. Isotropic harmonic maps into $SO^+(1,4)/(SO^+(1,1)\times SO(3))$

3.1. Harmonic maps into a Symmetric space. Let $N = G/K$ be a symmetric space with involution $\sigma : G \to G$ such that $G^\sigma \supset K \supset (G^\sigma)_0$. Let $\mathfrak{g}$ and $\mathfrak{k}$ denote the Lie algebras of G and K respectively. The Cartan decomposition shows that

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k},\mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}.$$

Denote $\pi : G \to G/K$ the projection of G into G/K.

Let $f : M \to G/K$ be a conformal harmonic map from a connected Riemann surface M. Let $U \subset M$ be an open, simply connected subset. Then there exists a frame $F : U \to G$ such that $f|_U = \pi \circ F$. So we have the Maurer–Cartan form and Maurer–Cartan equation

$$F^{-1}\mathrm{d}F = \alpha, \ \mathrm{d}\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Decomposing these with respect to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ amounts to:

$$\alpha = \alpha_0 + \alpha_1, \quad \alpha_0 \in \Gamma(\mathfrak{k} \otimes T^*M), \quad \alpha_1 \in \Gamma(\mathfrak{p} \otimes T^*M),$$

$$\begin{cases} \mathrm{d}\alpha_0 + \frac{1}{2}[\alpha_0 \wedge \alpha_0] + \frac{1}{2}[\alpha_1 \wedge \alpha_1] = 0, \\ \mathrm{d}\alpha_1 + [\alpha_0 \wedge \alpha_1] = 0. \end{cases}$$

Decomposing α_1 further into the $(1,0)$-part α_1' and the $(0,1)$-part α_1'', we then set

$$\alpha_\lambda = \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'', \quad \lambda \in \mathbb{S}^1.$$

We have the famous characterization in terms of one-parameter families:

Lemma 3.1. ([**11**]) *The map $f : M \to G/K$ is harmonic if and only if*

$$\mathrm{d}\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0 \ \ \textit{for all } \lambda \in \mathbb{S}^1.$$

Definition 3.2. The frame $F(z,\lambda)$, solving from the equation

$$\mathrm{d}F(z,\lambda) = F(z,\lambda)\,\alpha_\lambda,$$

with the initial condition $F(0,\lambda) = F(0)$, is called an *extended frame* of the harmonic map f. Note that it satisfies $F(z,1) = F(z)$.

3.2. The DPW construction of harmonic maps.

3.2.1. Two decomposition theorems. We denote by $SO^+(1,n+3)$ the connected component of the identity of the linear isometry group of $\mathbb{R}^{n+4}_1$, with the metric introduced in Section 2. Then

$$\mathfrak{so}(1,n+3) = \mathfrak{g} = \{X \in \mathfrak{gl}(n+4,\mathbb{R}) | X^t I_{1,n+3} + I_{1,n+3}X = 0\}.$$

Consider the involution

$$\begin{array}{llll} \sigma : SO^+(1,n+3) & \to SO^+(1,n+3) & & \\ \qquad A & \mapsto DAD^{-1}, & \text{where} & D = \begin{pmatrix} -I_2 & 0 \\ 0 & I_{n+2} \end{pmatrix}. \end{array}$$

We have $SO^+(1,n+3)^\sigma \supset SO^+(1,1)\times SO(n+2) = (SO^+(1,n+3)^\sigma)_0$. We also have

$$\mathfrak{g} = \left\{ \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,1} & A_2 \end{pmatrix} | A_1^t I_{1,1} + I_{1,1}A_1 = 0, A_2 + A_2^t = 0 \right\} = \mathfrak{k} \oplus \mathfrak{p},$$

with

$$\mathfrak{k} = \left\{ \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} | A_1^t I_{1,1} + I_{1,1}A_1 = 0, A_2 + A_2^t = 0 \right\},$$

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,1} & 0 \end{pmatrix} \right\}.$$

Let $G^{\mathbb{C}} = SO^+(1,n+3,\mathbb{C}) := \{X \in SL(n+4,\mathbb{C}) \mid X^t I_{1,n+3} X = I_{1,n+3}\}$, which has Lie algebra $\mathfrak{so}(1,n+3,\mathbb{C})$. Extend σ to an inner involution of $SO^+(1,n+3,\mathbb{C})$ with fixed point group $K^{\mathbb{C}} = S(O^+(1,1,\mathbb{C})\times O(n+2,\mathbb{C}))$.

Let $\Lambda G^{\mathbb{C}}_\sigma$ denote the group of loops in $G^C = SO^+(1,n+3,\mathbb{C})$ with the twisting by σ. Let $\Lambda^+ G^{\mathbb{C}}_\sigma$ denote the subgroup of loops which extend holomorphically to the unit disk $|\lambda| \leq 1$. We also use the subgroup

$$\Lambda^+_B G^{\mathbb{C}}_\sigma := \{\gamma \in \Lambda^+ G^{\mathbb{C}}_\sigma \mid \gamma|_{\lambda=0} \in \mathfrak{B}\}.$$

Here $\mathfrak{B} \subset K^{\mathbb{C}}$ is defined from the Iwasawa decomposition

$$K^{\mathbb{C}} = K \cdot \mathfrak{B}.$$

In this case,

$$\mathfrak{B} = \left\{ \begin{pmatrix} \mathrm{b}_1 & 0 \\ 0 & \mathrm{b}_2 \end{pmatrix} \mid \mathrm{b}_1 = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix}, \theta \in \frac{\mathbb{R}}{2\pi\mathbb{Z}}, \text{ and } \mathrm{b}_2 \in \mathfrak{B}_2 \right\}.$$

Here $\mathfrak{B}_2$ is the solvable subgroup of $SO(n+2,\mathbb{C})$. For more details, see Lemma 4 of [**16**]. Then we have:

Theorem 3.3. *Theorem 5 of* [**16**]*, see also* [**34**], [**11**], [**29**], [**3**] *(Iwasawa decomposition): The multiplication* $\Lambda G_\sigma \times \Lambda^+_B G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}_\sigma$ *is a real analytic diffeomorphism onto the open dense subset* $\Lambda G_\sigma \cdot \Lambda^+_B G^{\mathbb{C}} \subset \Lambda G^{\mathbb{C}}_\sigma$.

Let $\Lambda^-_* G^{\mathbb{C}}_\sigma$ denote the loops that extend holomorphically into ∞ and take the value I at infinity.

Theorem 3.4. *Theorem 7 of* [**16**]*, see also* [**34, 11, 29, 3**] *(Birkhoff decomposition): The multiplication* $\Lambda^-_* G^{\mathbb{C}}_\sigma \times \Lambda^+ G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}_\sigma$ *is a real analytic diffeomorphism onto the open subset* $\Lambda^-_* G^{\mathbb{C}}_\sigma \cdot \Lambda^+ G^{\mathbb{C}}$ *(the big cell) of* $\Lambda G^{\mathbb{C}}_\sigma$.

3.2.2. The DPW construction and Wu's formula. Here we recall the DPW construction for harmonic maps. Let $\mathbb{D} \subset \mathbb{C}$ be a disk or $\mathbb{C}$ itself, with complex coordinate z.

Theorem 3.5. [**11**]

(i) *Let $f : \mathbb{D} \to G/K$ be a harmonic map with an extended frame $F(z, \bar{z}, \lambda) \in \Lambda G_\sigma$ and $F(0, 0, \lambda) = I$. Then there exists a Birkhoff decomposition*

$$F_-(z, \lambda) = F(z, \bar{z}, \lambda) F_+(z, \bar{z}, \lambda), \quad \text{with} \;\; F_+ \in \Lambda^+ G_\sigma^{\mathbb{C}},$$

such that $F_-(z, \lambda) : \mathbb{D} \to \Lambda_*^- G_\sigma^{\mathbb{C}}$ is meromorphic. Moreover, the Maurer–Cartan form of F_- is of the form

$$\eta = F_-^{-1} \mathrm{d} F_- = \lambda^{-1} \eta_{-1}(z) \mathrm{d} z,$$

with η_{-1} independent of λ. The 1-form η is called the normalized potential of f.

(ii) *Let η be a $\lambda^{-1} \cdot \mathfrak{p}$-valued meromorphic 1-form on $\mathbb{D}$. Let $F_-(z, \lambda)$ be a solution to $F_-^{-1} \mathrm{d} F_- = \eta$, $F_-(0, \lambda) = I$. Then on an open subset $\mathbb{D}_{\mathfrak{I}}$ of $\mathbb{D}$ one has*

$$F_-(0, \lambda) = \tilde{F}(z, \bar{z}, \lambda) \cdot \tilde{F}^+(z, \bar{z}, \lambda), \quad \text{with} \;\; \tilde{F} \in \Lambda G_\sigma, \;\; \tilde{F} \in \Lambda_B^+ G_\sigma^{\mathbb{C}}.$$

This way, one obtains an extended frame $\tilde{F}(z, \bar{z}, \lambda)$ of some harmonic map from $\mathbb{D}_{\mathfrak{I}}$ to G/K with $\tilde{F}(0, \lambda) = I$. Moreover, all harmonic maps can be obtained in this way, since these two procedures are inverse to each other if the normalization at some base point is used.

The normalized potential can be determined in the following way. Let f and F be as above. Let $\alpha_\lambda = F^{-1} \mathrm{d} F$. Let δ_1 and δ_0 denote the sum of the holomorphic terms of z around $z = 0$ in the Taylor expansion of $\alpha_1'(\frac{\partial}{\partial z})$ and $\alpha_0'(\frac{\partial}{\partial z})$.

Theorem 3.6. [**33**] *(Wu's formula) We retain the notations of Theorem 3.5. Then the normalized potential of f with respect to the base point 0 is given by*

$$\eta = \lambda^{-1} \Delta_0 \delta_1 \Delta_0^{-1} \mathrm{d} z,$$

where $\Delta_0 : \mathbb{D} \to G^{\mathbb{C}}$ is the solution to $\Delta_0^{-1} \mathrm{d} \Delta_0 = \delta_0 \mathrm{d} z$, $\Delta_0(0) = I$.

For many applications, normalized potentials are too specific. Another type of *holomorphic* potential was also introduced in [**11**]:

Theorem 3.7. [**11**] *We retain the notations of f and $F(z, \bar{z}, \lambda)$ in Theorem 3.5. Then there exists some $V+ : \mathbb{D} \to \Lambda^+ G_\sigma^{\mathbb{C}}$ such that*

$$C(z, \lambda) = F(z, \bar{z}, \lambda) V_+(z, \bar{z}, \lambda)$$

is holomorphic in z and in $\lambda \in \mathbb{C}^*$. Moreover, the Maurer–Cartan form $\Xi = C^{-1} \mathrm{d} C$ is a holomorphic 1-form on $\mathbb{D}$ with $\lambda\eta$ holomorphic in λ for all $\lambda \in \mathbb{C}$. The 1-form Ξ is called a holomorphic potential of f.

Conversely, let Ξ be a $\Lambda\mathfrak{g}_{\sigma}^{\mathbb{C}}$-valued holomorphic 1-form on $\mathbb{D}$ such that $\lambda\Xi$ is holomorphic in λ for all $\lambda \in \mathbb{C}$. Let C be a solution to $C^{-1}\mathrm{d}C = \Xi$, $C(0,\lambda) = I$. Then on an open subset $\mathbb{D}_{\mathfrak{I}}$ of $\mathbb{D}$, one obtains

$$C(z,\lambda) = \hat{F}(z,\bar{z},\lambda) \cdot \hat{V}_{+}(z,\bar{z},\lambda), \quad \textit{with} \quad \tilde{F} \in \Lambda G_{\sigma}, \ \hat{V}_{+} \in \Lambda_{B}^{+} G_{\sigma}^{\mathbb{C}}.$$

Hence, one obtains an extended frame $\hat{F}(z,\bar{z},\lambda)$ of some harmonic map from $\mathbb{D}_{\mathfrak{I}}$ to G/K with $\hat{F}(0,\lambda) = I$. Moreover, all harmonic maps can be obtained in this way.

Note that there exist many different holomorphic potentials for a harmonic map.

3.3. Potentials of isotropic harmonic maps. Let $\mathbb{D}$ denote the unit disk of $\mathbb{C}$ or $\mathbb{C}$ itself.

Theorem 3.8. [**16**], [**17**]

(i) *Let $f : \mathbb{D} \to SO^{+}(1,4)/(SO^{+}(1,1) \times SO(3))$ be an isotropic harmonic map with complex coordinate z. Then its normalized potential satisfies*

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,1} & 0 \end{pmatrix} \mathrm{d}z, \quad \textit{with} \quad \hat{B}_1 \hat{B}_1^t = 0.$$

Conversely, let f be the harmonic map derived from a normalized potential η satisfying the above condition. Then $f = Y \wedge \hat{Y}$ is an isotropic harmonic map associated with the dual Willmore surfaces Y and $\hat{Y}$.

(ii) *Let $f : \mathbb{D} \to SO^{+}(1,4)/(SO^{+}(1,1) \times SO(3))$ be an isotropic harmonic map with complex coordinate z. Then any holomorphic potential of f satisfies*

$$\Xi = \sum_{j=-1}^{\infty} \lambda^j \xi_j \mathrm{d}z, \quad \textit{with} \quad \xi_{-1} = \begin{pmatrix} 0 & \tilde{B}_1 \\ -\tilde{B}_1^t I_{1,1} & 0 \end{pmatrix} \quad \textit{and} \quad \tilde{B}_1 \tilde{B}_1^t = 0.$$

Conversely, let f be the harmonic map derived from a holomorphic potential Ξ satisfying the condition above. Then $f = Y \wedge \hat{Y}$ is an isotropic harmonic map associated with the dual Willmore surfaces Y and $\hat{Y}$.

The proof comes directly from the decompositions $F = F_{-} \cdot F+$ and $F = C \cdot V_{+}$, and the fact that conjugation by some $T \in SO^{+}(1,1,\mathbb{C}) \times SO(3,\mathbb{C})$ does not change the isotropic condition $B_1 B_1^t = 0$.

In [**16**], there is an interesting description of Willmore surfaces Möbius equivalent to minimal surfaces in space forms. Here we restate it as:

Theorem 3.9. *(*[**16**]*) Let $f_h = Y \wedge \hat{Y}$ be a non-constant isotropic harmonic map.*

(i) *The map* $[Y]$ *is Möbius equivalent to a minimal surface in* $\mathbb{R}^3$ *if* $\hat{Y}$ *reduces to a point. In this case*

$$B_1 = \begin{pmatrix} b_1 & b_1 \end{pmatrix}^t.$$

(ii) *The map* $[Y]$ *is Möbius equivalent to a minimal surface in* $\mathbb{S}^3$ *if* f_h *reduces to a harmonic map into* $SO(4)/SO(3)$*. In this case*

$$B_1 = \begin{pmatrix} 0 & b_1 \end{pmatrix}^t.$$

(iii) *The map* $[Y]$ *is Möbius equivalent to a minimal surface in* H^3 *if* f_h *reduces to a harmonic map into* $SO^+(1,3)/SO^+(1,2)$*. In this case*

$$B_1 = \begin{pmatrix} b_1 & 0 \end{pmatrix}^t.$$

Here $b_1 \in \mathbb{C}^3$ *and* $b_1^t b_1 = 0$*.*

The converse of the above results also hold. That is, if B_1 *is (up to conjugation) of the form stated above, then* $[Y]$ *is Möbius equivalent to the corresponding minimal surface, wherever it is immersed.*

3.4. Examples. By implementing the Iwasawa decomposition numerically, one can compute solutions and plot the images of Willmore surfaces with the aid of a computer. Here are some simple examples, with images shown at Figure 4.

Example 3.10. Let

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,1} & 0 \end{pmatrix} \mathrm{d}z, \quad \text{with} \quad \hat{B}_1 = \begin{pmatrix} b_1 & b_2 \end{pmatrix}^t.$$

It is shown in [**16**], that if one chooses

$$b_1^t = 0,\ b_2^t = \frac{\sqrt{2}}{4}\left(1 - \frac{z^2}{8}, -i(1+\frac{z^2}{8})\,\frac{\sqrt{2}z}{2}\right),$$

one will obtain the Clifford torus in $\mathbb{S}^3$. Note that b_2 is exactly the Weierstrass-representation data of the Enneper surface.

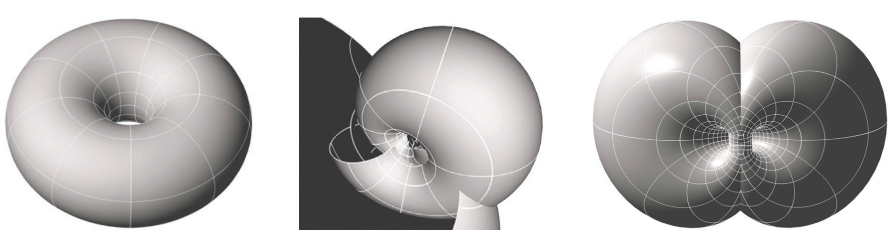


Figure 4. Willmore surfaces computed with a numerical implementation of DPW. Left: Example 3.10. Middle: Example 3.11. Right: Example 3.12.

Example 3.11. If we choose

$$b_1^t = \frac{i}{4}\left(\frac{8-z^2}{8},\ -i\frac{8+z^2}{8},\ \frac{\sqrt{2}z}{2}\right),\ b_2^t = \frac{\sqrt{3}}{4}\left(\frac{8-z^2}{8},\ -i\frac{8+z^2}{8},\ \frac{\sqrt{2}z}{2}\right),$$

we obtain the second image in Figure 4. Note that this Willmore surface is not Möbius equivalent to a minimal surface in any space form, by Theorem 3.9.

Example 3.12. Replacing z with $1/z$ in the Clifford torus potential:

$$b_1^t = 0,\ b_2^t = \left(1-\frac{1}{z^2},\ -i(1+\frac{1}{z^2}),\ \frac{1}{z}\right),$$

and integrating with initial condition $F(1) = I$, we obtain the third image in the figure. This Willmore surface is Möbius equivalent to a minimal surface in $\mathbb{S}^3$ by Theorem 3.9.

4. Björling's problem for Willmore surfaces in $\mathbb{S}^3$

We state the Björling problem for Willmore surfaces in $\mathbb{S}^3$ as: Given a sphere congruence together with two enveloping curves on an interval $\mathbb{I}$ of $\mathbb{S}^3$, does there exist a unique pair of dual Willmore surfaces such that their restrictions to the interval $\mathbb{I}$ coincide with the two enveloping curves separately and their mean curvature sphere coincides with the sphere congruence?

Concretely, we have the following result:

Theorem 4.1. *Let $\psi_0 = \psi_0(u) : \mathbb{I} \to \mathbb{S}^4_1$ denote a non-constant real analytic sphere congruence from $\mathbb{I}$ to $\mathbb{S}^3$, with enveloping curves $[Y_0]$ and $[\hat{Y}_0]$ such that $\langle Y_0, Y_0\rangle = \langle \hat{Y}_0, \hat{Y}_0\rangle = 0$, $\langle Y_0, \hat{Y}_0\rangle = -1$, and u is the arc-parameter of $Y_0 : \mathbb{I} \to \mathcal{C}^4_+$. Then there exists a unique pair of dual Willmore surfaces $y, \hat{y} : \Sigma \to \mathbb{S}^3$, with Σ some simply connected open subset containing $\mathbb{I}$, such that the lifts $Y, \hat{Y}$ of $y, \hat{y}$ satisfy*

$$Y|_{\mathbb{I}} = Y_0,\ \ \hat{Y}|_{\mathbb{I}} = \hat{Y}_0.$$

Moreover, let $\psi : \Sigma \to \mathbb{S}^4_1$ be the conformal Gauss map of Y, we have $\psi|_{\mathbb{I}} = \psi_0$.

For minimal surfaces in space forms, the mean curvature 2-sphere is the same as their tangent planes. For a minimal surface in $\mathbb{R}^n$, the dual surface $\hat{Y}$ is a point at infinity. For a minimal surface y in $\mathbb{S}^n$ or $\mathbb{H}^n$, the dual surface is exactly $-y$. So the Björling problem for minimal surfaces in space forms is a corollary of Theorem 4.1:

Corollary 4.2. *Let $y_0(u)$ be a real analytic curve in a space form defined on $\mathbb{I}$ and let $n_0(u)$ be a real analytic unit vector normal to y_0. Then there exists a unique minimal surface $y(u, v)$ in the space form such that $y(u, 0)|_{\mathbb{I}} = y_0(u)$ and $n(u, 0)|_{\mathbb{I}} = n_0(u)$. Here n is the unit normal of y.*

Theorem 4.1 is a straightforward corollary of the following:

Theorem 4.3. *We retain the assumptions and notations in Theorem* 4.1. *Choose two real analytic unit vector fields P_1 and P_2 on $\mathbb{I}$ such that*

$$Y_{0u} = P_1 \mod Y_0,\ P_2 \perp \{\psi_0, Y_0, \hat{Y}_0, P_1\} \text{ and } \det(Y_0, \hat{Y}_0, P_1, P_2, \psi_0) = 1.$$

There exist real analytical functions $\mu_1 = \mu_1(u)$, $\rho_1 = \rho_1(u)$, $\rho_2 = \rho_2(u)$, $k_1 = k_1(u)$, $k_2 = k_2(u)$ on $\mathbb{I}$ such that

$$\begin{cases} Y_{0u} = -\mu_1 Y_0 + P_1, \\ \hat{Y}_{0u} = \mu_1 \hat{Y}_0 + \rho_1 P_1 + \rho_2 P_2, \\ P_{1u} = \mu_2 P_2 + 2k_1\psi_0 + \hat{Y}_0 + \rho_1 Y_0, \\ P_{2u} = -\mu_2 P_1 - 2k_2\psi_0 + \rho_2 Y_0, \\ \psi_{0u} = -2k_1 P_1 + 2k_2 P_2 \end{cases} \tag{4.1}$$

holds. Set $\mu = \mu_1 + i\mu_2$, $k = k_1 + ik_2$ and $\rho = \rho_1 + i\rho_2$. For a real analytic function $x(u)$ on $\mathbb{I}$, denote its analytic extension to a simply connected open subset containing $\mathbb{I}$ by $x(z)$. Consider the holomorphic potential

$$\Xi = \left(\lambda^{-1}\mathcal{A}_1 + \mathcal{A}_0 + \lambda\mathcal{A}_{-1}\right) \mathrm{d}z,$$

with

$$\mathcal{A}_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\ \mathcal{A}_1 = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,1} & 0 \end{pmatrix},\ \mathcal{A}_{-1}(z) = \overline{\mathcal{A}_1(\bar{z})},$$

$$A_1(z) = \begin{pmatrix} 0 & \mu_1(z) \\ \mu_1(z) & 0 \end{pmatrix},\ A_2(z) = \begin{pmatrix} 0 & -\mu_2(z) & -2k_1(z) \\ \mu_2(z) & 0 & 2k_2(z) \\ 2k_1(z) & -2k_2(z) & 0 \end{pmatrix},$$

$$B_1(z) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\rho(z) & -i-i\rho(z) & 0 \\ 1-\rho(z) & -i+i\rho(z) & 0 \end{pmatrix}.$$

By DPW, Theorem 3.7, *the potential Ξ provides an isotropic harmonic map, together with a unique pair of dual Willmore surfaces $y, \hat{y} : \Sigma \to \mathbb{S}^3$, with Σ some open subset containing $\mathbb{I}$, such that the lifts $Y, \hat{Y}$ of $y, \hat{y}$ satisfy*

$$Y|_{\mathbb{I}} = Y_0,\ \hat{Y}|_{\mathbb{I}} = \hat{Y}_0.$$

Moreover, let $\psi : \Sigma \to \mathbb{S}^4_1$ be the conformal Gauss map of Y. Then $\psi|_{\mathbb{I}} = \psi_0$.

Proof. Set

$$F_0 = \left(\frac{Y_0 + \hat{Y}_0}{\sqrt{2}}, \frac{-Y_0 + \hat{Y}_0}{\sqrt{2}}, P_1, P_2, \psi_0\right).$$

Rewriting (4.1), we obtain

$$F_0^{-1}\mathrm{d}F_0 = (\hat{\alpha}_1 + \hat{\alpha}_0 + \hat{\alpha}_{-1})\mathrm{d}u,$$

with

$$\hat{\alpha}_0(u) = \mathcal{A}_0(u), \quad \hat{\alpha}_1(u) = \mathcal{A}_1(u),$$

and $\mathcal{A}_j$ are as in the statement of the theorem. Introducing λ, we set

$$\hat{\alpha}_\lambda = (\lambda^{-1}\hat{\alpha}_1 + \hat{\alpha}_0 + \lambda\hat{\alpha}_{-1})\mathrm{d}u.$$

Let $F_0(u, \lambda)$ be the solution to $\mathrm{d}F_0(u, \lambda) = F_0(u, \lambda)\hat{\alpha}_\lambda$, $F_0(u, \lambda)|_{\lambda=1} = F_0$.

Let $z = u + iv$ be the complex coordinate such that $u + i0$ parameterizes $\mathbb{I}$. As a consequence, the holomorphic 1-form Ξ coincides with $\hat{\alpha}_\lambda$ when restricted to $\mathbb{I}$, since on $\mathbb{I}$ $z = u + i0 = u$. Assume that $\mathfrak{F}$ is the solution to

$$\mathfrak{F}^{-1}d\mathfrak{F} = \Xi, \quad \mathfrak{F}(u_0 + i0, \lambda) = F_0(u_0 + i0, \lambda) \text{ for some } u_0 \in \mathbb{I}.$$

Then

$$\mathfrak{F}(z = u, \lambda) = F_0(u, \lambda), \quad \forall u \in \mathbb{I}.$$

Since $F_0(u, \lambda) \in \Lambda G_\sigma$ for all $u \in \mathbb{I}$, $\mathfrak{F}(z)$ is in the big cell for z in some open subset $\mathbb{D}_0$ containing $\mathbb{I}$. Performing the Iwasawa decomposition of Theorem 3.3, pointwise on $\mathbb{D}_0$, we obtain

$$\mathfrak{F} = \check{F}(z, \lambda) \cdot \check{F}_+(z, \lambda),$$

with $\check{F}(z, \lambda) = \overline{\check{F}(z, \lambda)}$ on $\mathbb{D}_0$ and $\check{F}_+(z, \lambda) \in \Lambda_B^+ G_\sigma^{\mathbb{C}}$. By the initial condition we have

$$\check{F}(z = u, \lambda) = F_0(u, \lambda), \quad \forall u \in \mathbb{I}.$$

By Theorem 3.5, $\check{F}$ is an extended frame of some harmonic map. It is straightforward to compute $\hat{B}_1\hat{B}_1^t \equiv 0$. By Theorem 3.8, $\check{F}$ is an isotropic harmonic map. Now set $\check{F} = (e_{-1}, e_0, e_1, e_2, \psi)$, then $Y = \frac{1}{\sqrt{2}}(e_{-1} - e_0)$, $\hat{Y} = \frac{1}{\sqrt{2}}(e_{-1} + e_0)$ and ψ are the desired dual Willmore surfaces and their conformal Gauss map, which are unique and coincide, by construction, with Y_0, $\hat{Y}_0$ and ψ_0 on $\mathbb{I}$. q.e.d.

The potential Ξ defined in the above theorem is a special type of holomorphic potential one can generally define by taking the Maurer–Cartan form of the extended frame F for a harmonic map, restricting to some curve in the domain, and then extending holomorphically. We call it the *boundary potential*.

4.1. Examples. In the following examples we let $E_0 = (1, 0, 0, 0, 0), \dots$ $E_4 = (0, 0, 0, 0, 1)$ denote an orthonormal basis for $\mathbb{R}^5_1$, with $\langle E_0\, , \, E_0 \rangle = -1$. For convenience, we write X' for X_u, and we abuse notation by dropping the subscripts on Y_0, $\hat{Y}_0$ and ψ_0.

Example 4.4. Let us consider a Willmore surface in $\mathbb{S}^3$ containing the circle $(\cos u, \sin u, 0, 0)$. A lift is $Y = (1, \cos u, \sin u, 0, 0)$. The simplest case is where the plane spanned by Y and $\hat{Y}$ is constant: without loss of generality we can take $\hat{Y} = (1/2)(1, -\cos u, -\sin u, 0, 0)$. From Equations (4.1), we have

$$P_1 = Y' + \mu_1 Y = (0, -\sin u, \cos u, 0, 0) + \mu_1(1, \cos u, \sin u, 0, 0).$$

The requirement that $\langle \hat{Y}, P_1 \rangle = 0$ gives us:

$$\mu_1 = 0, \quad P_1 = (0, -\sin u, \cos u, 0, 0).$$

The equation $\hat{Y}' = \mu_1 \hat{Y} + \rho_1 P_1 + \rho_2 P_2$ gives us

$$\rho_1 = -1/2, \quad \rho_2 = 0.$$

The third equation from (4.1) is

$$(0, -\cos u, -\sin u, 0, 0) = P_1' = \mu_2 P_2 + 2k_1 \psi + \hat{Y} + \rho_1 Y.$$

Since ψ and P_2 necessarily take values in $\mathrm{Span}\{E_3, E_4\}$, we conclude that $\mu_2 = k_1 = 0$. The only remaining parameter for the potential is k_2, and this is determined by our choice of ψ, which could be any vector field taking values in $\mathrm{Span}\{E_3, E_4\}$. For example, $k_2 = 0$ corresponds to ψ and P_2 being constant along the curve. The Willmore surface obtained is a round sphere. More generally, we must have

$$\psi = -\sin(\theta) E_3 + \cos(\theta) E_4, \quad P_2 = -\cos(\theta) E_3 - \sin(\theta) E_4,$$

where θ is any real analytic map $\mathbb{R} \to \mathbb{R}$. The last equation at (4.1), becomes $\theta' P_2 = \psi' = 2k_2 P_2$, and so we conclude that $k_2 = \theta'/2$. There are no further constraints, so we can say that all solutions corresponding to the pair Y and $\hat{Y}$ above are obtained from a choice of angle function θ with the boundary potential given by the data:

$$(\mu, k, \rho) = (0, i\theta'/2, -1/2).$$

Example 4.5. A special case of the previous example is when θ' is constant, and for this we can write down the solution explicitly: consider the immersion

$$y(u, \tilde{v}) = (\cos u \cos \tilde{v}, \, \sin u \cos \tilde{v}, \, \cos ru \; \sin \tilde{v}, \, \sin ru \; \sin \tilde{v}),$$

where r is a non-zero real number. Note that the case that $r = \ell/m$ is rational corresponds to Lawson's minimal tori and Klein bottles $\tau_{m,\ell}$ (see equation (7.1) of [**23**]). The surfaces $\tau_{m,\ell}$ are all distinct compact genus one surfaces for distinct relatively prime pairs of positive integers (m, ℓ). They are non-orientable if and only if 2 divides m or ℓ.

Conformal coordinates (u, v) for y are defined by setting $u = u$ and $v = \int_0^{\tilde{v}} (\cos^2 w + r^2 \sin^2 w)^{-1/2} \mathrm{d}w$. Setting $R = \sqrt{\cos^2 \tilde{v} + r^2 \sin^2 \tilde{v}}$, a

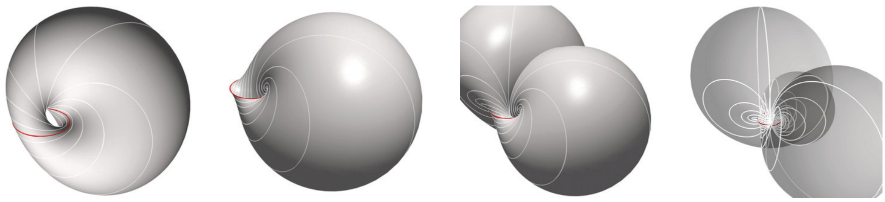


Figure 5. Conformally immersed Lawson surfaces. Left: Clifford torus. Middle: Klein bottle. Right: Torus.

canonical lift and frame are given by:

$$\begin{aligned}
Y &= (1, y), \qquad \hat{Y} = \frac{1}{2}(1, -y), \\
P_1 &= \frac{1}{R}\left(0, -\sin u \cos \tilde{v}, \cos u \cos \tilde{v}, -r \sin ru \, \sin \tilde{v}, r \cos ru \, \sin \tilde{v}\right), \\
P_2 &= (0, -\cos u \sin \tilde{v}, -\sin u \sin \tilde{v}, \cos ru \, \cos \tilde{v}, \sin ru \, \cos \tilde{v}), \\
\psi &= \frac{1}{R}\left(0, -r \sin u \sin \tilde{v}, r \cos u \sin \tilde{v}, \sin ru \, \cos \tilde{v}, -\cos ru \, \cos \tilde{v}\right).
\end{aligned}$$

The restriction of this frame to $v = \tilde{v} = 0$ is precisely the frame given in Example 4.4, with $\theta' = r$. In particular, the boundary potentials for Lawson's minimal surfaces $\tau_{m,\ell}$ are given by

$$(\mu, k, \rho) = (0, i\ell/(2m), -1/2).$$

For the case that r is not rational, one obtains an immersed cylinder. Figure 5 shows three examples computed from these potentials.

5. Equivariant surfaces

The Lawson-type surfaces of the previous example are special cases of Willmore surfaces invariant under the action of a 1-parameter subgroup of $SO(4)$. More generally, by an *equivariant* surface we mean one that is invariant under the action of a 1-parameter subgroup of the Möbius group $SO^+(1,4)$. Such a subgroup necessarily sits inside either a copy of $SO(4)$ or of $SO^+(1,3)$, the isometry groups of $\mathbb{S}^3$ and $\mathbb{H}^3$ respectively. We will consider the $SO(4)$ case first, which we will call $SO(4)$-*equivariant* surfaces. Up to conjugation in $SO(4)$, such a subgroup acts on $(z, w) \in \mathbb{S}^3 \subset \mathbb{C}^2$ by $(z, w) \mapsto (e^{it}z, e^{irt}w)$, where $r \in \mathbb{R}$. The case $r = 0$ corresponds to surfaces of revolution, and $r = 1$ corresponds to Hopf cylinders.

5.1. Criteria for minimality in space forms. We are interested to distinguish those Willmore surfaces that are "non-minimal" in the sense that they are not Möbius equivalent to a minimal surface in some space form. For equivariant surfaces, the criteria is given in the lemma below.

We first remark that a standard argument [**7**] shows that a surface is equivariant, with the curve $v=0$ an equivariant curve, if and only if the corresponding holomorphic potential depends only on v. This means that the Björling potentials corresponding to equivariant surfaces are exactly those with μ, k and ρ constant. See also the direct argument below in Section 5.2.

First we recall a well-known characterization of minimality in space forms for Willmore surfaces (see, for example, Page 377 of [**16**]):

Lemma 5.1. *Let y be a Willmore surface in $\mathbb{S}^n$ with Y and $\hat{Y}$ a lift of itself and its dual surface. Then y is Möbius equivalent to a minimal surface in some n-dimensional space form if and only if there exist two real functions a and b such that $aY+b\hat{Y}\neq 0$ is constant. Moreover, the space form is*

(i) $\mathbb{S}^n(r)$ *if and only if* $\langle aY+b\hat{Y}, aY+b\hat{Y}\rangle = -r^2$;
(ii) $\mathbb{R}^n$ *if and only if* $\langle aY+b\hat{Y}, aY+b\hat{Y}\rangle = 0$ *if only if* $[\hat{Y}]$ *is constant;*
(iii) $\mathbb{H}^n(r)$ *if and only if* $\langle aY+b\hat{Y}, aY+b\hat{Y}\rangle = r^2$.

Applying this to equivariant Willmore surfaces in $\mathbb{S}^3$, we have

Lemma 5.2. *Let y be an equivariant Willmore surface generated by the boundary potential corresponding to constants $(\mu_1,\mu_2,k_1,k_2,\rho_1,\rho_2)$. Then*

(i) *The surface y is Möbius equivalent to a minimal surface in $\mathbb{R}^3$ if and only if $\rho_1=\rho_2=0$;*
(ii) *The surface y is Möbius equivalent to a minimal surface in $\mathbb{S}^3$ if and only if $\mu_1=\rho_2=0$ and $\rho_1<0$;*
(iii) *The surface y is Möbius equivalent to a minimal surface in $\mathbb{H}^3$ if and only if $\mu_1=\rho_2=0$ and $\rho_1>0$.*

Proof. (i) By Lemma 5.1 if y is Möbius equivalent to a minimal surface in $\mathbb{R}^3$, then $[\hat{Y}]$ is constant. Hence, by (4.1), $\rho_1=\rho_2=0$.

Conversely, if $\rho_1=\rho_2=0$, then

$$\mathcal{A}_1 = \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,1} & 0 \end{pmatrix} \quad \text{with } \hat{B}_1 = \begin{pmatrix} \hat{b}_1^t \\ \hat{b}_1^t \end{pmatrix}.$$

By simple computation one will see that $\hat{B}_1$ being of the above form is conjugation invariant. So let $F=(e_0,e_1,e_2,e_3,e_4)$ be the extended frame derived from Ξ. Then the B_1 part of the Maurer–Cartan form of F has the same form, which means that $\hat{Y}_z = \frac{1}{\sqrt{2}}(e_{-1}+e_0)_z = 0 \mod \hat{Y}$. By Lemma 5.1, y is Möbius equivalent to a minimal surface in $\mathbb{R}^3$.

(ii) By Lemma 5.1, if y is Möbius equivalent to a minimal surface in $\mathbb{S}^3$, then there exist two real functions a and b such that

$$(aY+b\hat{Y})_u = 0, \quad \text{and} \quad \langle aY+b\hat{Y}, aY+b\hat{Y}\rangle = -2ab = -r^2.$$

By (4.1), $\mu_1 = \rho_2 = 0$, and $a + b\rho_1 = 0$. Since $ab = r^2 > 0$, $\rho_1 < 0$.

Conversely, if $\mu_1 = \rho_2 = 0$ and $\rho_1 < 0$, there exists a unique $\theta_0 \in \mathbb{R}$ such that $(1 + \rho_1)\cosh\theta_0 + (1 - \rho_1)\sinh\theta_0 = 0$. Let

$$T = \mathrm{diag}(T_1, I_3), \text{ with } T_1 = \begin{pmatrix} \cosh\theta_0 & \sinh\theta_0 \\ \sinh\theta_0 & \cosh\theta_0 \end{pmatrix}.$$

Then the first row and column of $\tilde{\Xi} = T\Xi T^{-1}$ are both zero. That is, $\tilde{\Xi}$ induces a conformal harmonic map into $SO(4)/SO(3) = \mathbb{S}^3$, which means that the surfaces induced by $\tilde{\Xi}$ are Möbius equivalent to some minimal surfaces in $\mathbb{S}^3$. Let F be the extended frame of Ξ. So $\tilde{F} = TFT^{-1}$ is the extended frame of $\tilde{\Xi}$ and hence y is Möbius equivalent to some minimal surface in $\mathbb{S}^3$.

The proof of (iii) is the same as (ii), and is left to the interested reader. q.e.d.

5.2. Surfaces of revolution in $\mathbb{S}^3$. A *rotational surface* in $\mathbb{S}^3$ is an equivariant surface where the 1-parameter subgroup fixes a geodesic in $\mathbb{S}^3$, or, equivalently fixes a plane in $\mathbb{R}^4$. Without loss of generality, we can take the geodesic to be the unit circle in the plane $E_3 \wedge E_4$, so that the action is $R_t(z, w) = (e^{it}z, w)$. A point on the surface that is not a fixed point of the action is (after a rotation in the fixed plane $E_3 \wedge E_4$) of the form $(a\cos\theta, a\sin\theta, b, 0)$, where $a^2 + b^2 = 1$, and $a \neq 0$. Applying R_t, the surface thus contains the curve $\gamma(t) = (a\cos t, a\sin t, b, 0)$, and we write our initial curve as

$$y(u) = af(u) + bE_3, \quad f(u) = \cos uE_1 + \sin uE_2, \quad a^2 + b^2 = 1, \quad a \neq 0.$$

The surface normal along this curve must be of the form

$$n(u) = -bcf + acE_3 + dE_4,$$

where $c^2 + d^2 = 1$, and the assumption that the surface is invariant under R_u means that c and d are constant. The starting point for the construction is the canonical lift Y of y and a general R_u-invariant lift ψ of n:

$$Y = \frac{1}{a}E_0 + f + \frac{b}{a}E_3, \qquad \psi = (0, n) + hY, \tag{5.1}$$

with

$$f(u) = \cos tE_1 + \sin tE_2, \; a^2 + b^2 = 1, \; c^2 + d^2 = 1, \; a \neq 0, \; h \in \mathbb{R},$$

where a, b, c, d and are constant, and the constant h will be the value of the mean curvature along the curve. We expect another parameter to appear because we have not yet chosen $\hat{Y}$, but we begin by finding all possible solutions to (5.1), and then identify those that are equivariant.

The last equation of (4.1) becomes:

$$(h - bc)f' = \psi' = -2k_1P_1 + 2k_2P_2.$$

If $h - bc = 0$ then we must have $k_1 = k_2 = 0$ along the whole curve, and hence the curve is a line of umbilics. If the surface is not totally umbilic, we can choose a different parallel curve as our initial curve for the Björling problem. Hence, we assume that $h \neq bc$. In this case, $\psi' \neq 0$, and we necessarily have $\mathrm{span}(P_1, P_2, \psi) = \mathrm{span}(\psi, \psi', V)$, where V depends on the choice of $\hat{Y}$. We can, therefore, choose $P_1 = \psi'/|\psi'|$, that is:

$$P_1 = f', \quad k_1 = \frac{\beta}{2}, \quad k_2 = 0, \quad \mu_1 = 0, \qquad \beta := bc - h.$$

From the second and third equation of (4.1), one obtains

$$\begin{aligned} \hat{Y}' &= \rho_1 P_1 + \rho_2 P_2, \\ P_1' &= -f = \mu_2 P_2 + \beta\psi + \hat{Y} + \rho_1 Y. \end{aligned}$$

Differentiating the expression $\hat{Y} = -\mu_2 P_2 - f - \beta\psi - \rho_1 Y$, we have

$$\rho_1 P_1 + \rho_2 P_2 = \hat{Y}' = -\mu_2' P_2 - \mu_2 P_2' - f' + \beta^2 f' - \rho_1' Y - \rho_1 P_1.$$

The fourth equation of (4.1) is $P_2' = -\mu_2 P_1 + \rho_2 Y$. Inserting this above, we end up with

$$P_1(2\rho_1 - \mu_2^2 + 1 - \beta^2) + P_2(\rho_2 + \mu_2') + Y(\rho_1' + \mu_2\rho_2) = 0.$$

The vanishing of the coefficients of P_1, P_2 and Y above implies that

$$\rho_1 = \frac{1}{2}\left(\mu_2^2 + \beta^2 - 1\right), \quad \rho_2 = -\mu_2'.$$

The third equation, $\rho_1' = -\mu_2\rho_2$, gives nothing new, and so we retain the function μ_2 as a parameter m.

In summary, all possible Willmore surfaces containing the curve and surface normal specified at (5.1) are given by the boundary potential data

$$(\mu, k, \rho) = \left(im(u), \ \frac{\beta}{2}, \ \frac{1}{2}(m(u)^2 + \beta^2 - 1) - im'(u)\right),$$

where $m(u)$ is an arbitrary function of u. Three examples are computed numerically and displayed in Figure 1. All have the same value for β, but with respectively $m(u) = e^{u-\pi/2}$, $m(u) = 2\cos^2(2u)$ and $m(u) = -1$. An interesting result of Palmer [**27**] shows that such a Willmore surface, i.e., containing a circle and intersecting the plane of the circle with constant contact angle, cannot enclose a topological disc, unless it is part of a sphere or a plane.

Only the last of our examples is a surface of revolution, because we have not yet taken into account that all the geometry of the surface should be invariant under the action of $T(u)$. In that case, the dual surface $\hat{Y}$, which is unique, must also be invariant. This, combined with the invariance of P_1 and ψ implies that the vector P_2 is invariant too. Noting that $\langle P_2, P_1\rangle = \langle P_2, f'\rangle = 0$, this means we can write

$$P_2 = AE_0 + Bf + CE_3 + DE_4,$$

where A, B, C and E are all constants. Differentiating this, the fourth equation from (4.1) is

$$BP_1 = Bf' = P_2' = -mP_1 + \rho_2 Y,$$

from which we conclude that $m = -B$ is constant and $\rho_2 = 0$. Hence, we have the characterization:

Theorem 5.3. *All Willmore surfaces of revolution in $\mathbb{S}^3$ are given by the boundary potentials with data:*

$$(\mu, k, \rho) = \left(im,\ \frac{\beta}{2},\ \frac{1}{2}(m^2 + \beta^2 - 1) \right), \quad \beta \in \mathbb{R}, \quad m \in \mathbb{R},$$

where $\beta = bc - h$ if b and c are chosen as described above, and h is the value of the mean curvature along the initial parallel.

Proof. We have already shown this for the case $\beta \neq 0$ and non-totally umbilic surfaces. If $\beta = 0$ then the last row and column of the potential are zero, and so the surface is an immersion into a totally geodesic sphere $\mathbb{S}^2 \subset \mathbb{S}^3$. Conversely, The only totally umbilic surface of revolution in $\mathbb{S}^3$ is the totally geodesic 2-sphere. q.e.d.

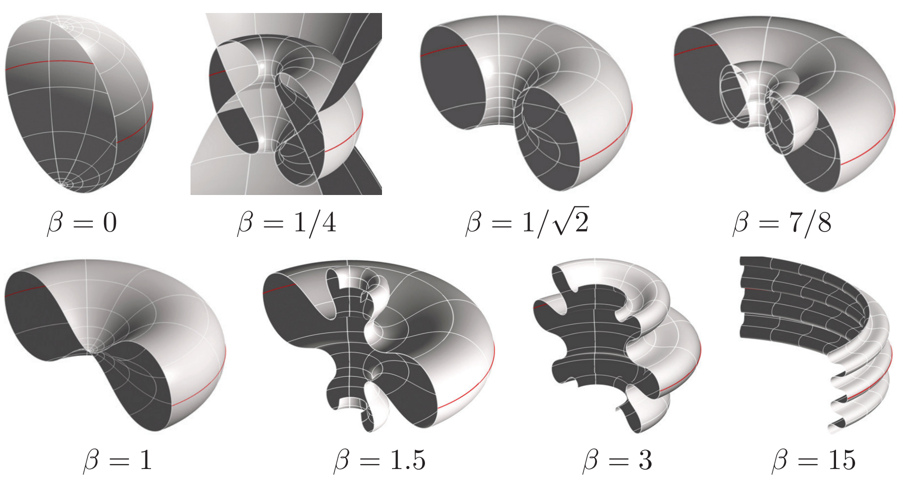
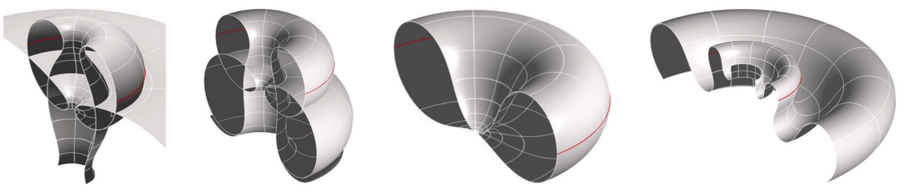


Figure 6. Examples of Willmore surfaces of revolution. All are computed with $m = 0$. Surfaces are stereographically projected from the point $(0, 0, 0, 1)$. The first four are congruent to minimal surfaces in $\mathbb{S}^3$, the fifth to a catenoid, and the last three to minimal surfaces in $\mathbb{H}^3$ (Theorem 5.6).

Remark 5.4. If one is only interested in rotational surfaces up to Möbius equivalent then all solutions are obtained by integrating the above potential with the identity as initial condition. To plot the surface with a suitable projection that shows the relevant symmetry, we then premultiply the solution by the initial condition $F_0(u_0) = \Big((Y_0+\hat{Y}_0)/\sqrt{2}, (-Y_0+\hat{Y}_0)/\sqrt{2}, P_1, P_2, \psi_0\Big)\big|_{u=u_0}$ corresponding to a definite choice of b and c. For the case $\beta = 0$ one only obtains totally geodesic spheres, so the initial condition is not important. Hence, all possible real values of β are covered by taking $b = c = 0$, $a = 1$, $d = -1$ and h arbitrary. The examples shown in Figures 6 and 7 are computed numerically, applying this initial condition, and then stereographically projected from the point $(0, 0, 0, 1)$.



$(0.5, 0.25)$ $(0.5, \sqrt{3}/4)$ $(0.5, \sqrt{3}/2)$ $(5, 1)$

Figure 7. Surfaces of revolution with various values of (m, β). The non-zero value of m means the surface normal along the initial curve is not perpendicular to the axis of revolution.

Remark 5.5. On the other hand, one can obtain all solutions up to *isometric* equivalence in $\mathbb{S}^3$, if one considers all possible values of b and c in the construction and uses the correct initial condition. To see that this is needed for isometric equivalence, consider that if $b = c = 0$ we necessarily have $\beta = h$. But then, for non-totally umbilic solutions (i.e., $\beta \neq 0$) we would need to have $h \neq 0$. Thus, the non-trivial solutions computed with this initial condition cannot be minimal in $\mathbb{S}^3$, only Möbius equivalent to a minimal surface.

5.2.1. Minimal surface representations for rotational surfaces. It has long been known that a Willmore surface of revolution is necessarily Möbius equivalent to a minimal surface in one of the three space forms ([**30**]). Applying Lemma 5.2, we immediately recover that result and characterize the corresponding potentials as follows:

Theorem 5.6. *The Willmore surface of revolution corresponding to the point $(m, \beta) \in \mathbb{R}^2$, with $\beta \neq 0$, is Möbius equivalent to a minimal surface in:*

(i) *Hyperbolic* 3*-space* $\mathbb{H}^3$ *if and only if* $m^2+\beta^2>1$,
(ii) *Euclidean* 3*-space if and only if* $m^2+\beta^2=1$,
(iii) *The* 3*-sphere* $\mathbb{S}^3$ *if and only if* $m^2+\beta^2<1$.

Note that if $\beta\neq 0$, then the corresponding Willmore surface is not totally umbilic.

5.3. Non-rotational $SO(4)$-equivariant surfaces. We now consider $SO(4)$-equivariant surfaces that are not surfaces of revolution, namely the isometries $(z,w)\mapsto(e^{it}z,e^{irt}w)$ where $r\neq 0$. Let $p=(z,w)\subset\mathbb{C}^2$, with $|z|^2+|w|^2=1$ be an arbitrary point on the surface. After a rotation of $\mathbb{S}^3$, we can assume that $z=(a,0)$ and $w=(b,0)$, where $a^2+b^2=1$. We can, therefore, take the initial curve as $y=(ae^{it},be^{irt})$, with $r\neq 0$. An $SO(1,4)$ frame for $\mathbb{R}^5_1=\mathbb{R}\times\mathbb{C}^2$ along the curve, invariant under the action of the subgroup, is given by

$$f_0=(1,0,0),\quad f_1=(0,e^{it},0),\quad f_2=(0,ie^{it},0),$$
$$f_3=(0,0,e^{irt}),\quad f_4=(0,0,ie^{irt}),$$

where, for computations, we note that $f_2=f_1'$ and $f_4=f_3'/r$. Writing all vectors as coordinate vectors in this frame, we have the canonical lift for y as

$$Y=\frac{1}{R}(1,a,0,b,0),\quad R=\sqrt{a^2+b^2r^2},\quad a^2+b^2=1,\quad r\neq 0.$$

The most general unit normal for the surface along y give us, in the frame f_i,

$$n=\left(-bc,-\frac{bdr}{R},ac,\frac{ad}{R}\right),\quad \psi=(0,n)+hY,\quad c^2+d^2=1,\ h\in\mathbb{R},$$

where h, c and d are constant. As with rotational surfaces, all of the vector fields, $\hat{Y}$, P_1 and P_2 can be chosen to be invariant, and thus have constant coefficients in the basis f_i. Hence, all possible solutions are obtained using linear algebra. We can write

$$P_1=Y'+\mu_1Y=\frac{1}{R}\left(\ell,\ell,a,\ell b,rb\right),$$

where $\mu_1=\ell$ is constant. As in the rotational case, we assume that the surface is not totally umbilic, implying that $\psi'\neq 0$ and $\operatorname{span}(P_1,P_2,\psi)=\operatorname{span}(P_1,\psi',\psi)$. To find P_2, we extend the orthonormal pair (ψ,P_1) to an orthonormal basis (ψ,P_1,P_2) for $\operatorname{span}(P_1,\psi,\psi')$, and find:

$$\begin{aligned}P_2 &= (0,bd,-\frac{bcR}{r},-ad,acR)+\frac{abc(1-r^2)}{rR}\left(\ell,a\ell,a,b\ell,br\right)\\ &\quad -\frac{h\ell}{r}(1,a,0,b,0).\end{aligned}$$

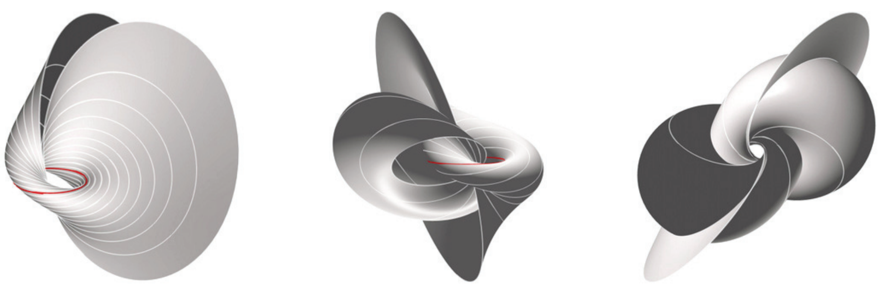
It is also straightforward algebra to find the unique null vector field $\hat{Y}$ that is orthogonal to P_1, P_2 and ψ and satisfies $\langle\hat{Y},Y\rangle=-1$. Substituting these expressions into (4.1), we, finally, obtain:

Theorem 5.7. *All non-rotational equivariant Willmore surfaces in* $\mathbb{S}^3$ *are obtained from the boundary potential* $\Xi_{r,\theta,\phi,\ell,h}$, *with* $r \in \mathbb{R} \setminus \{0\}$, $\ell, h \in \mathbb{R}$, *and* $\theta, \phi \in \mathbb{R} \mod 2\pi\mathbb{Z}$, *defined as follows: write*

$$a := \cos\theta, \quad b = \sin\theta, \quad c = \cos\phi, \quad d = \sin\phi, \quad R = \sqrt{a^2 + r^2 b^2}.$$

The potential $\Xi_{r,\theta,\phi,\ell,h}$ *is the boundary potential with the following data:*

$$\begin{aligned}
\mu_1 &= \ell, \qquad \mu_2 = \frac{ab\left(r^2-1\right)\left(c\ell R + dr\right)}{rR} + \frac{R\ell}{r}h, \\
k_1 &= \frac{abc\left(1-r^2\right)}{2R} - \frac{1}{2}h, \qquad k_2 = \frac{r}{2R}, \\
\rho_1 &= -\frac{R^2}{2} + \frac{a^2b^2cd\ell\left(r^2-1\right)^2}{rR} + \frac{\ell^2\left(a^2b^2c^2\left(r^2-1\right)^2 - r^2\right)}{2r^2} \\
&\quad + hab\left(r^2-1\right)\left(\frac{Rc\ell^2}{r^2} + \frac{d\ell}{r} + \frac{c}{R}\right) + \frac{h^2}{2}\left(\frac{R^2\ell^2}{r^2} + 1\right), \\
\rho_2 &= \frac{abd\ell\left(r^2-1\right)}{R} + \frac{abc\ell^2\left(r^2-1\right)}{r} + h\left(\frac{\ell^2 R}{r} + \frac{r}{R}\right).
\end{aligned}$$



$\ell = 0, \ h = 1$ $\qquad$ $\ell = 1, \ h = 1$ $\qquad$ $\ell = 1, \ h = 0$

Figure 8. Examples of Willmore Hopf cylinders.

5.4. Special classes of non-rotational surfaces.

5.4.1. Willmore Hopf cylinders, Case $r = 1$: Here the data simplifies to

$$(\mu_1, \mu_2, k_1, k_2, \rho_1, \rho_2) = \left(\ell, \ h\ell, \ -\frac{h}{2}, \ \frac{1}{2}, \ \frac{h^2(\ell^2+1) - \ell^2 - 1}{2}, \ h\left(\ell^2+1\right)\right),$$

which only depends on h and ℓ. Hence, there is a two parameter family of Willmore Hopf cylinders. According to Lemma 5.2, the surface is Möbius equivalent to a minimal surface in some space form if and only if $\ell = h = 0$, in which case the data is of the form $(0, 0, 0, 1/2, -1/2, 0)$, a Clifford torus in $\mathbb{S}^3$. Otherwise, the surface is not minimal. This re-derives Proposition 2 of [**28**].

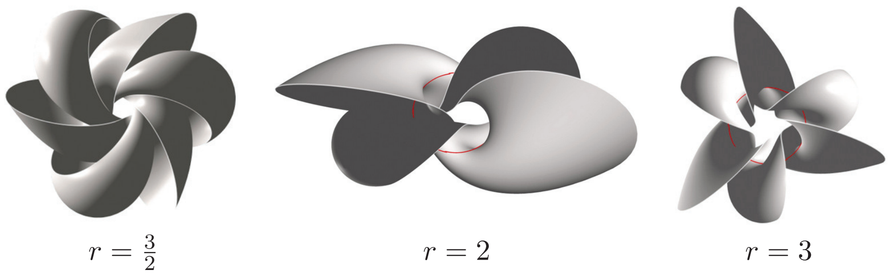


Figure 9. Equivariant Willmore cylinders containing an equator (Section 5.4.2). All have $\ell = h = 1$. The value of r is the number of times that the normal rotates around the circle in one revolution. The surface is a cylinder if r is rational.

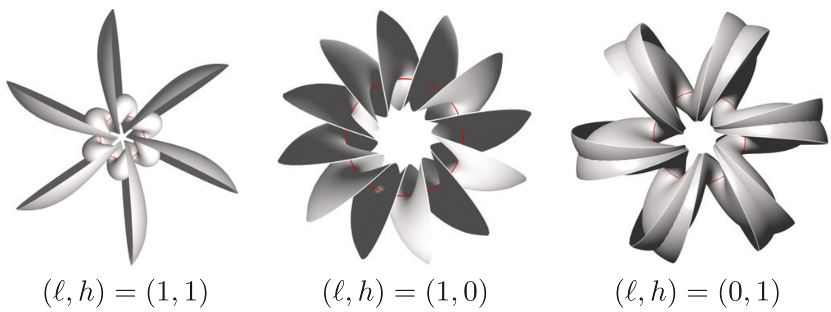


Figure 10. The effect of changing the value of h and ℓ. All have $r = 6$.

5.4.2. Equivariant surfaces containing an equator, Case $a = 1$, $b = 0$: In this case, the data $(\mu_1, \mu_2, k_1, k_2, \rho_1, \rho_2)$ are equal to

$$\left(\ell, \frac{\ell R h}{r}, -\frac{h}{2}, \frac{r}{2R}, \frac{h^2\left(\ell^2 + r^2\right) - r^2(1+\ell^2)}{2R^2 r^2}, \frac{\left(\ell^2 + r^2\right) h}{Rr}\right).$$

The surface is minimal if and only if $h = \ell = 0$, and then the data reduces to $(0, 0, 0, r/2, -1/2, 0)$, the Lawson-type surfaces of Example 4.5. Non-minimal examples are shown in Figures 9 and 10.

5.5. $SO(4)$-equivariant minimal surfaces.

Theorem 5.8. *If a non-rotational $SO(4)$ equivariant Willmore surface in $\mathbb{S}^3$ is Möbius equivalent to a minimal surface in some space form, that space form is necessarily $\mathbb{S}^3$. The boundary potential is given by the following data, where a, b, c, d, r and R are as in Theorem 5.7:*

$$(\mu_1, \mu_2, k_1, k_2, \rho_1, \rho_2) = \left(0, \frac{abd\left(r^2-1\right)}{R}, -\frac{abc\left(r^2-1\right)}{2R}, \frac{r}{2R}, -\frac{R^2}{2}, 0\right).$$

Proof. Considering Lemma 5.2, note that if the surface is minimal in $\mathbb{R}^3$, so that $[\hat{Y}]$ is constant, we can assume, at least locally, that $\hat{Y}$ is constant, so that μ_1 is zero, as it is also for minimal surfaces in the other two space forms. Inserting $\ell = \mu_1 = 0$ into the potential given at Theorem 5.7, we obtain the potential data:

$$\left(0, \frac{abd\left(r^2-1\right)}{R}, \frac{abc(1-r^2)}{2R} - \frac{H}{2}, \frac{r}{2R}, \frac{Habc(r^2-1)}{R} + \frac{H^2-R^2}{2}, \frac{rH}{R}\right),$$

in particular, $\rho_2 = rH/R$, and this is zero if and only if $H = 0$. With $\ell = H = 0$, the data reduce to that given in the statement of the theorem. Since $r \neq 0$, we have $\rho_1 < 0$ and so the surface can only be minimal in $\mathbb{S}^3$. q.e.d.

6. $SO(1,3)$-Equivariant surfaces

Given a lift Y of a Willmore surface y in $\mathbb{S}^3$ to the light cone in $\mathbb{R}^5_1$, any of the projections to $\mathbb{H}^3 \subset \mathbb{R}^4_1$, for example,

$$(Y_0, Y_1, ..., Y_4) \mapsto (Y_0, Y_1, Y_2, Y_3)/Y_4$$

gives a Willmore surface (possibly with singularities) in $\mathbb{H}^3$, Möbius equivalent to y. Each choice of subgroup $SO(1,3)$ in $SO(1,4)$ corresponds to one of these projections. For definiteness, we choose the projection above, which corresponds to the subgroup $SO(1,3) \times \{1\}$. Since we have already considered the subgroup $\mathbb{S}^1$, the only 1-parameter subgroups left are of the form

$$\exp\left\{\mathrm{diag}\left(\begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}, \begin{pmatrix} 0 & rt \\ -rt & 0 \end{pmatrix}, 0\right)\right\}, \quad r \in \mathbb{R}.$$

After an action of $SO(1,1) \times SO(2)$, and a rescaling so that $\langle Y', Y' \rangle = 1$, we can assume the initial curve $T(u)Y(0) =$ is of the form

$$\begin{pmatrix} \cosh u & \sinh u & 0 & 0 & 0 \\ \sinh u & \cosh u & 0 & 0 & 0 \\ 0 & 0 & \cos ru & -\sin ru & 0 \\ 0 & 0 & \sin ru & \cos ru & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ 0 \\ b \\ 0 \\ c \end{pmatrix}, \quad \begin{array}{c} a^2 + r^2 b^2 = 1, \\ c^2 = a^2 - b^2. \end{array}$$

The general solution can be found as in the $SO(4)$ case. To simplify matters, we discuss two interesting cases: one case which includes the hyperbolic analogue of rotational surfaces in the next subsection, and then the case $r = 1$ in the following subsection.

6.1. Case $a = 1$, $b = 0$, $c = 1$: This case includes, but is not restricted to, the case $r = 0$, because if r is zero then $a = \pm 1$, and the lower right part of $T(u)$ is the 3×3 identity matrix I_3. In this case, there are many possible hyperbolic spaces on which $T(u)$ acts isometrically, and we can freely rotate among the last three coordinates without losing any generality. Hence, we can assume that our initial point is $(1,0,0,0,1)$, that is, $a=c=1$ and $b=0$. A suitable invariant frame along the curve is given by $\xi_i := T(u)(E_i)$, namely:

$$\xi_0 = (\cosh u, \sinh u, 0,0,0), \quad \xi_1 = (\sinh u, \cosh u, 0,0,0),$$
$$\xi_2 = (0,0,\cos ru, \sin ru, 0), \quad \xi_3 = (0,0,-\sin ru, \cos ru, 0),$$
$$\xi_4 = (0,0,0,0,1).$$

Writing vectors as coordinate vectors in the frame ξ_i, we find, for $a=1$, $b=0$, the frame:

$$\begin{aligned} &Y = (1,0,0,0,1), \quad \hat{Y} = \left(\frac{h^2+1}{2}, 0, h\cos\theta, h\sin\theta, \frac{h^2-1}{2}\right), \\ &P_1 = (0,1,0,0,0), \quad P_2 = (0,0,\sin\theta, -\cos\theta, 0), \\ &\psi = (h, 0, \cos\theta, \sin\theta, h), \end{aligned} \tag{6.1}$$

where h and θ are arbitrary real constants. Using equations (4.1), we find the potential data:

$$(\mu_1,\ \mu_2,\ k_1,\ k_2,\ \rho_1,\ \rho_2) = \left(0,\ 0,\ -\frac{h}{2},\ -\frac{r}{2},\ \frac{h^2+1}{2},\ -hr\right).$$

Note that these surfaces are congruent to minimal surfaces in $\mathbb{H}^3$ if and only if $hr=0$. If $hr \neq 0$ then they are not congruent to a minimal surface in any space form.

6.1.1. The minimal case, $hr = 0$. Note that a discussion of rotational minimal surfaces in $\mathbb{H}^3$ can be found in [**10**]. If both h and r are zero then the surface is a totally umbilic sphere. Other than this there are two types: surfaces with $r=0$, which are a hyperbolic version of surfaces of revolution, and surfaces with $h=0$, a hyperbolic analogue of the Lawson-type surfaces in Example 4.5.

Note that in the case $r=0$, the action is by $SO(1,1)\times\{I_3\}$, so the surfaces $(Y_0,Y_1,Y_2,Y_4)/Y_3$, and $(Y_0,Y_1,Y_3,Y_4)/Y_2$ will also be rotationally invariant in $\mathbb{H}^3$. Some examples from the case $r=0$ are displayed in Figure 11. The initial curve is $Y(u,0) = (\cosh(u), \sinh(u), 0, 0, 1)$. We have plotted the projection $(Y_0,Y_1,Y_2,Y_3,Y_4) \mapsto (Y_1,Y_2,Y_4)/(Y_0-Y_3)$, which can be regarded either as the stereographic projection from $(0,0,1,0)$ of the surface $(Y_1,Y_2,Y_3,Y_4)/Y_0$ in $\mathbb{S}^3$, or a Poincaré ball image of the surface $(Y_0,Y_1,Y_2,Y_4)/Y_3$ in $\mathbb{H}^3$. As surfaces in $\mathbb{H}^3$ they have several pieces, as they pass through the boundary of the Poincaré ball. A different projection of the same surfaces in $\mathbb{S}^3$ is also shown. This corresponds to a different Willmore surface in $\mathbb{H}^3$, which is not isometrically equivalent, only Möbius equivalent. For certain values of h (the middle two surfaces), the numerics indicate that the surface closes up in the v direction.

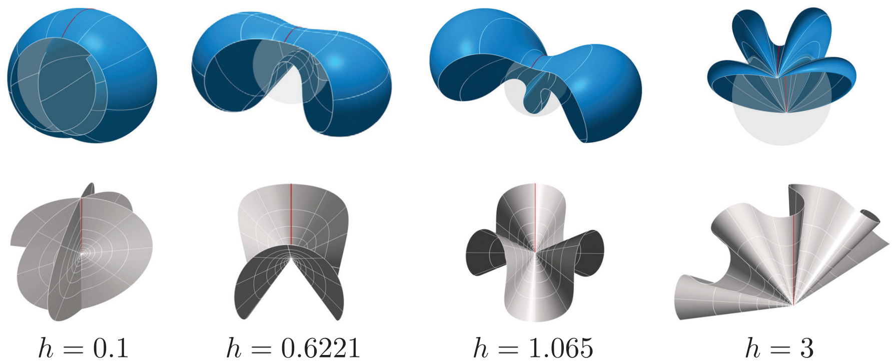


Figure 11. Top: hyperbolic Willmore surfaces of revolution in $\mathbb{H}^3$ (case $r = 0$), projected to the Poincaré ball. Bottom: Möbius equivalent surfaces in $\mathbb{S}^3$ projected from the point $(-1, 0, 0, 0)$. The latter are cones.

The other type of minimal surfaces in $\mathbb{H}^3$ here are those with $h = 0$, $r \neq 0$. The potential data is:

$$(\mu, k, \rho) = \left(0, -i\frac{r}{2}, \frac{1}{2}\right),$$

differing from the Lawson-type potentials of Example 4.5 only in the sign of ρ. Again we have an explicit form for the solutions: consider the surface in $\mathbb{H}^3 \subset \mathbb{R}^3_1 \subset \mathbb{R}^4_1$ given by:

(6.2)
$$f(u, \tilde{v}) = (\cosh\tilde{v}\cosh u,\ \cosh\tilde{v}\sinh u,\ \sinh\tilde{v}\cos ru,\ \sinh\tilde{v}\sin ru)\,.$$

This is an analogue in $\mathbb{H}^3$ of the Lawson type surfaces, and a geodesically ruled minimal surface, that appears in [**10**]. Consider now the lift to the light cone and associated frame given by, for $R = \sqrt{\cosh^2\tilde{v} + r^2\sinh^2\tilde{v}}$:

$$Y(u, \tilde{v}) = (f(u, \tilde{v}),\ 1)\,, \qquad \hat{Y}(u, \tilde{v}) = \frac{1}{2}\,(f(u, \tilde{v}),\ -1)\,,$$

$$P_1 = \frac{1}{R}\,(\cosh\tilde{v}\sinh u, \cosh\tilde{v}\cosh u, -r\sinh\tilde{v}\sin ru, r\sinh\tilde{v}\cos ru, 0)\,,$$

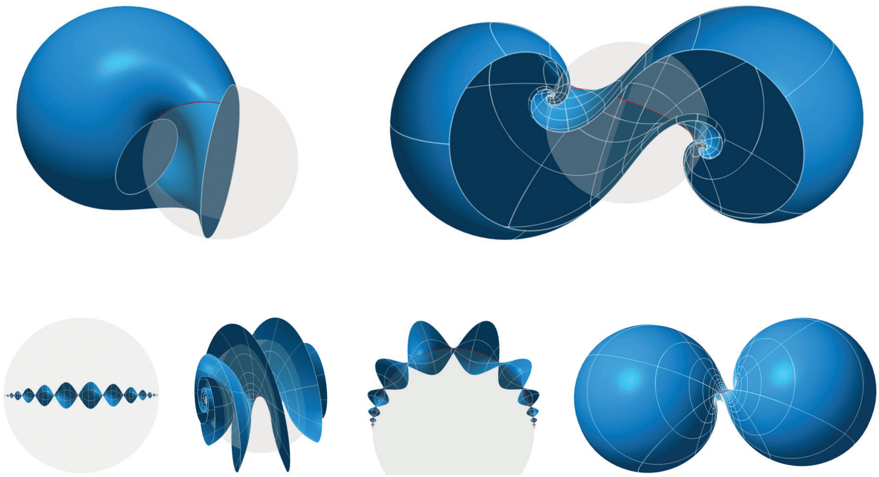
$$P_2 = (\sinh\tilde{v}\cosh u, \sinh\tilde{v}\sinh u, \cosh\tilde{v}\cos ru, \cosh\tilde{v}\sin ru, 0)\,,$$

$$\psi = -\frac{1}{R}\,(r\sinh\tilde{v}\sinh u, r\sinh\tilde{v}\cosh u, \cosh\tilde{v}\sin ru, -\cosh\tilde{v}\cos ru, 0)\,.$$

With respect to the coordinates (u, v), where v is given by

$$v(\tilde{v}) = \int_0^{\tilde{v}} \left(\cosh^2\nu + r^2\sinh^2\nu\right)^{-1/2} \mathrm{d}\nu,$$

the maps Y and $\hat{Y}$ are conformally immersed, and canonical lifts of f, by which we mean that $\langle Y, \hat{Y}\rangle = -1$, and $|\mathrm{d}Y|^2 = |\mathrm{d}z|^2$. Additionally, ψ_z is orthogonal to both Y and $\hat{Y}$, and the frame is orthonormal. Finally,



(0, 0, 0, 1) (0, 0, 0, 1) (0, 0, 1, 0) (0, 0, 1, 0)

Figure 12. Top: the hyperbolic Lawson surface (6.3) with $r = 2$, projected from the point $(0, 0, 1, 0)$. The v-curves are circles. Bottom: two different projections of the case $r = 5$. The projection from $(0, 0, 0, 1)$ is the Poincare ball image of a minimal surface in H^3. The projection from $(0, 0, 1, 0)$ is a topological cylinder.

along the curve $v = 0$, this frame is nothing other than the frame given above at (6.1), for the case $h = 0$, with $\theta = \pi/2$. The value of θ is not relevant, since it does not appear in the potential. Hence, the maps $Y(u, v)$, for $r \neq 0$, give all the solutions for this case.

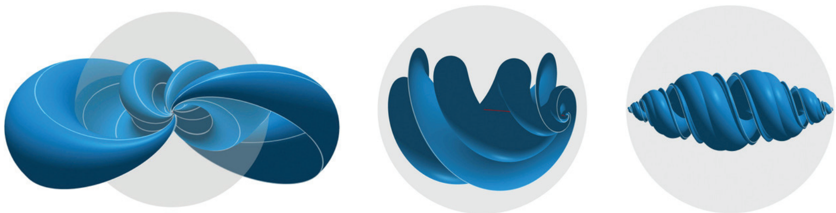
Note that the $\tilde{v}$ coordinate in (6.2) only gives a part of the surface, namely that part that lies in one copy of $\mathbb{H}^3$. The map $\tilde{v} \mapsto v$ takes the whole real line to a bounded open interval in $\mathbb{R}$. Computing the rest of the surface with the coordinate v, we find that the surface continues smoothly through the boundary. In fact, the curves $u =$ constant are closed curves, and the surface

(6.3)

$$y(u, v) = \frac{1}{\cosh \tilde{v} \cosh u} (\cosh \tilde{v} \sinh u,\ \sinh \tilde{v} \cos ru,\ \sinh \tilde{v} \sin ru,\ 1),$$

in $\mathbb{S}^3$ is apparently a topological cylinder.

6.1.2. The non-minimal case, $hr \neq 0$**.** Examples that are not congruent to minimal surfaces are shown in Figure 13, where we used the projection $(Y_0, Y_1, Y_2, Y_3, Y_4) \mapsto (Y_1, Y_2, Y_3)/(Y_0 + Y_4)$. The initial curve in this projection is the straight line segment $\{(x, 0, 0) \mid -1 < x < 1\}$. A different projection, $(Y_1, Y_2, Y_4)/(Y_0 - Y_3)$ of the case $r = 2$ is also shown in Figure 3.



$r = 0.5$ $r = 2$ $r = 5$

Figure 13. Nonminimal $SO(1,3)$-equivariant surfaces in $\mathbb{H}^3$. All have $b = 0$ and $h = 2$.

6.2. Hyperbolic Hopf surface analogues: Case $r = 1$: We again write vectors as coordinate vectors in the frame ξ_i. The initial curve is thus:

$$Y = (a, 0, b, 0, c), \quad a^2 + b^2 = 1, \quad c^2 = a^2 - b^2.$$

After suitable isometries of the ambient space, we can assume that a, b and c are all non-negative, so that there is a unique constant θ satisfying:

$$a = \cos\theta, \quad b = \sin\theta, \quad c = \sqrt{\cos 2\theta}, \qquad 0 \leq \theta \leq \frac{\pi}{4}.$$

The most general choice for ψ and $P_1 = \psi' + \mu_1 \psi$, invariant along the curve are, in the basis ξ_i:

$$\begin{aligned} \psi &= \left(0,\, -b^2 q,\, -\frac{c}{a}\sqrt{1 - b^2 q^2},\, abq,\, \frac{b}{a}\sqrt{1 - b^2 q^2}\right) + hY, \\ P_1 &= (0,\, a,\, 0,\, b,\, 0) + mY, \qquad h,\, m \in \mathbb{R}, \quad |q| \leq \frac{1}{|b|}, \end{aligned}$$

where m, q and h are all constant. We extend these using linear algebra to find the most general form for

$$P_2 = \left(-cq,\, -b\sqrt{1 - b^2 q^2},\, 0,\, a\sqrt{1 - b^2 q^2},\, -aq\right) + pY, \quad p \in \mathbb{R},$$

and, finally, find the unique null vector $\hat{Y}$ orthogonal to P_1, P_2, ψ and ψ' satisfying $\langle \hat{Y}, Y \rangle = -1$. The condition $\langle \hat{Y}, \psi' \rangle = 0$ gives a further constraint on the parameters:

$$amh + a^2 q - bcm\sqrt{1 - b^2 q^2} - acp = 0. \tag{6.4}$$

Substituting Y, $\hat{Y}$, ψ, P_1 and P_2 into (4.1), we obtain the boundary potential data:

$$(\mu_1,\, \mu_2,\, k_1,\, k_2) = \left(m,\; acq - p,\; \frac{bc\sqrt{1 - b^2 q^2}}{2a} - \frac{h}{2},\; -\frac{c}{2}\right), \tag{6.5}$$
$$\rho_1 = \frac{h^2 + p^2 - 2\,acpq - m^2}{2}$$

$$+\frac{-2\,abch\sqrt{1-q^2b^2}+c^4q^2+c^2}{2a^2},$$

$$\rho_2=\frac{a^2cmq-ach-amp-b\sqrt{1-q^2b^2}}{a}.$$

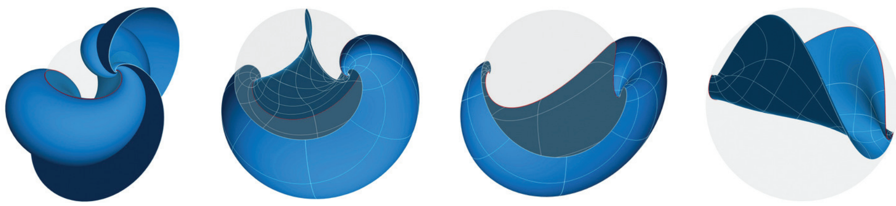


Figure 14. Several partial plots of a hyperbolic Hopf-type surface. Here $c=0$ and $h=m=p=1$. The third images shows one of the pieces outside the Poincaré sphere, the fourth image one of the inside pieces. When $c=0$, the initial curve lies on the sphere itself.

If c is non-zero, we can eliminate p by solving the constraint (6.4) for p, while if c is zero, all the data simplifies and we can eliminate q instead. We summarize this as

Theorem 6.1. *All $SO(1,3)$-equivariant surfaces with $r=1$ are determined by the boundary potentials with data as follows:*

(i) *If $c\neq 0$, then the boundary potential data is (6.5), where $a=\cos\theta$, $b=\sin\theta$, $c=\sqrt{\cos 2\theta}$, and*

$$p=\frac{amh+a^2q-bcm\sqrt{1-b^2q^2}}{ac}.$$

The real parameters θ, m, q and h are arbitrary, subject to the conditions:

$$0\leq\theta<\frac{\pi}{4},\quad |q|\leq\frac{1}{|\sin\theta|}.$$

(ii) *If $c=0$, the boundary potential $(\mu_1,\mu_2,k_1,k_2,\rho_1,\rho_2)$ is equal to*

$$\left(m,\ -p,\ -\frac{h}{2},\ 0,\ \frac{h^2+p^2-m^2}{2},\ -pm-\sqrt{1-h^2m^2}\right),$$

for h, m and p arbitrary real numbers subject to the condition $|hm|\leq 1$.

7. Isotropic and half-isotropic harmonic maps associated to Willmore surfaces in $\mathbb{S}^{n+2}$

Hélein's treatment [**16**] of Willmore surfaces has been generalized in [**34**] to $\mathbb{S}^{n+2}$. However, the geometry inside was unclear prior to the introduction of adjoint transforms by Xiang Ma [**24**]. One aim, in this section, is to clarify this interesting relationship between Willmore surfaces and isotropic harmonic maps using the language of [**8**] and [**24**]. In Section 8, we will use half-isotropic maps to solve the Björling problem for all Willmore surfaces in any codimension, with or without umbilics.

7.1. Adjoint transforms and harmonic maps associated to Willmore surfaces. Let Y be a Willmore surface as in Section 2.2. As before, consider another lightlike vector $\hat{Y}$ in the mean curvature sphere of Y, given by

$$\hat{Y} = N + \bar{\mu} Y_z + \mu Y_{\bar{z}} + \frac{1}{2}|\mu|^2 Y,$$

satisfying $\langle \hat{Y}, Y \rangle = -1$.

Definition 7.1. [**24**] The map into $\mathbb{S}^{n+2}$ determined by $\hat{Y}$, defined as above, is called an adjoint transform of the Willmore surface Y if the following two equations hold for μ:

$$\mu_z - \frac{\mu^2}{2} - s = 0, \tag{7.1}$$

$$\langle D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa, D_{\bar{z}}\kappa + \frac{\bar{\mu}}{2}\kappa \rangle = \sum_j \gamma_j^2 = 0. \tag{7.2}$$

Theorem 7.2. [**24**] Willmore property and existence of adjoint transform: *The adjoint transform $\hat{Y}$ of a Willmore surface y is also a Willmore surface (may degenerate). Moreover,*

(i) *When $\langle \kappa, \kappa \rangle \equiv 0$, any solution to equation* (7.1) *is a solution to both* (7.1) *and* (7.2).
(ii) *When $\langle \kappa, \kappa \rangle \neq 0$ and $\Omega \mathrm{d}z^6 := \left(\langle D_{\bar{z}}\kappa, \kappa \rangle^2 - \langle \kappa, \kappa \rangle \langle D_{\bar{z}}\kappa, D_{\bar{z}}\kappa \rangle \right) \mathrm{d}z^6 \neq 0$, there are exactly two different solutions to equation* (7.2), *which also solve* (7.1), *that is, exactly two adjoint surfaces of* $[Y]$.
(iii) *When $\langle \kappa, \kappa \rangle \neq 0$ and $\langle D_{\bar{z}}\kappa, \kappa \rangle^2 - \langle \kappa, \kappa \rangle \langle D_{\bar{z}}\kappa, D_{\bar{z}}\kappa \rangle \equiv 0$, there exists a unique solution to* (7.2), *which also solves* (7.1), *that is, a unique adjoint surface of* $[Y]$.

Theorem 7.3. *Let $[Y]$ be a Willmore surface. Let μ be a solution to the Riccati equation* (7.1) *on U, defining $\hat{Y}$ as above. Let $f_h : U \to SO^+(1, n+3)/(SO^+(1,1) \times SO(n+2))$ be the map taking p to $Y(p) \wedge \hat{Y}(p)$. Then:*

(i) ([**16**], [**34**]) *The map f_h is harmonic, and is called a* half-isotropic harmonic map with respect to Y.
(ii) ([**24**]) *If μ also solves* (7.2), *f_h is* conformally *harmonic, and is called an* isotropic harmonic map with respect to Y.

Proposition 7.4. *Let $f_h = Y \wedge \hat{Y}$ be a half-isotropic harmonic map. Choose e_1, e_2 with $Y_z + \frac{\mu}{2}Y = \frac{1}{2}(e_1 - ie_2)$, and a frame $\{\psi_j, j = 1, \cdots, n\}$ of the normal bundle $V^\perp$, so that*

$$\kappa = \sum_{j=1}^{n} k_j\psi_j,\ \zeta = \sum_{j=1}^{n} \gamma_j\psi_j,\quad D_z\psi_j = \sum_{l=1}^{n} b_{jl}\psi_l,\quad b_{jl} + b_{lj} = 0.$$

Set

$$F = \left(\frac{1}{\sqrt{2}}(Y + \hat{Y}), \frac{1}{\sqrt{2}}(-Y + \hat{Y}), e_1, e_2, \psi_1, \cdots, \psi_n\right).$$

Then the Maurer–Cartan form $\alpha = F^{-1}\mathrm{d}F = \alpha' + \alpha''$ of F has the structure:

$$\alpha' = \begin{pmatrix} A_1 & B_1 \\ -B_1^t I_{1,1} & A_2 \end{pmatrix} \mathrm{d}z, \tag{7.3}$$

with

$$A_1 = \begin{pmatrix} 0 & \frac{\mu}{2} \\ \frac{\mu}{2} & 0 \end{pmatrix},\ B_1 = \begin{pmatrix} \frac{1+\rho}{2\sqrt{2}} & \frac{-i-i\rho}{2\sqrt{2}} & \sqrt{2}\gamma_1 & \cdots & \sqrt{2}\gamma_n \\ \frac{1-\rho}{2\sqrt{2}} & \frac{-i+i\rho}{2\sqrt{2}} & -\sqrt{2}\gamma_1 & \cdots & -\sqrt{2}\gamma_n \end{pmatrix},$$

and

$$B_1 B_1^t = 2\left(\sum_{j=1}^{n} \gamma_j^2\right) \cdot \mathbf{E},\quad \textit{with } \mathbf{E} := \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{7.4}$$

Moreover, f_h is an isotropic harmonic map, and hence $\hat{Y}$ an adjoint transform of Y, if and only if f_h is a conformally harmonic map, if and only if

$$B_1 B_1^t = 0. \tag{7.5}$$

Lemma 7.5. *The maps $[Y]$ and $[\hat{Y}]$ associated to a half-isotropic harmonic map are a pair of dual (S-)Willmore surfaces if and only if rank$(B_1) = 1$.*

For any $\Psi_1 \in SO(1,1)$ there exists some $a \in \mathbb{R}^+$ such that

$$\Psi_1 \mathbf{E} \Psi_1^t = a^2 \cdot \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = a^2\mathbf{E}. \tag{7.6}$$

It follows that the condition (7.4) on B_1 is independent of the choice of frame F for f_h. The following theorem shows that Equation (7.4) is a good condition to characterize half-isotropic harmonic maps. We refer to [**16**], [**34**] for a proof (Lemma 3 of [**16**] and Proposition 2.1 of [**34**]).

Theorem 7.6. *Let $f : M \to SO^+(1, n+3)/(SO^+(1,1) \times SO(n+2))$ be a harmonic map satisfying (7.4). Then either $f = Y \wedge \hat{Y}$ is a half-isotropic harmonic map associated with the Willmore surface Y, or B_1 is of the form*

$$\begin{pmatrix} b_1 & -b_1 \end{pmatrix}^t, \tag{7.7}$$

for some b_1. In the latter case $[Y]$ is a constant point in $\mathbb{S}^{n+2}$.

Note that the condition (7.7) is invariant under conjugation.

7.2. Harmonic maps into $SO^+(1,n+3)/(SO^+(1,1)\times SO(n+2))$. Let $f: M \to SO^+(1,n+3)/(SO^+(1,1)\times SO(n+2))$ be a harmonic map with a (local) lift frame $F: M \to SO^+(1,n+3)$ and the Maurer–Cartan form $\alpha = F^{-1}\mathrm{d}F$. Let z be a local complex coordinate of M. Then

$$\alpha_0' = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}\mathrm{d}z, \quad \alpha_1' = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,1} & 0 \end{pmatrix}\mathrm{d}z.$$

To have a detailed discussion of half-isotropic and isotropic harmonic maps, we first take a look at their normalized potentials.

Theorem 7.7. ([**16**], [**17**], [**34**]) *The normalized potential of a half-isotropic harmonic map $f = Y\wedge\hat{Y}$ is of the form*

$$\eta = \lambda^{-1}\begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,1} & 0 \end{pmatrix}\mathrm{d}z,$$

with

$$\hat{B}_1\hat{B}_1^t = \hat{\gamma}\mathbf{E}. \tag{7.8}$$

And f is an isotropic harmonic map if and only if

$$\hat{B}_1\hat{B}_1^t = 0. \tag{7.9}$$

Moreover, $[Y]$ and $[\hat{Y}]$ forms a pair of dual (S-)Willmore surfaces if and only if rank$(\hat{B}_1)=1$ and f is an isotropic harmonic map.

Proof. Let $\tilde{A}_1$, $\tilde{A}_2$ and $\tilde{B}_1$ be the holomorphic part of A_1, A_2 and B_1 respectively, with respect to some base point z_0 such that $F(z_0,\lambda)=I$. So $\tilde{B}_1$ has the same form as B_1 and hence $\tilde{B}_1\tilde{B}_1^t = \tilde{\gamma}\mathbf{E}$ for some $\hat{\gamma}$. Let $\Psi = \mathrm{diag}\{\Psi_1,\Psi_2\}$ be a solution to

$$\Psi^{-1}d\Psi = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & \tilde{A}_2 \end{pmatrix}\mathrm{d}z, \ \Psi(z_0) = I.$$

By Wu's formula in Theorem 3.6,

$$\eta = \lambda^{-1}\Psi\begin{pmatrix} 0 & \tilde{B}_1 \\ \tilde{B}_1^t I_{1,1} & 0 \end{pmatrix}\Psi^{-1}\mathrm{d}z = \lambda^{-1}\begin{pmatrix} 0 & \hat{B}_1 \\ \hat{B}_1^t I_{1,1} & 0 \end{pmatrix}\mathrm{d}z,$$

with $\hat{B}_1 = \Psi_1\tilde{B}_1\Psi_2^{-1} = \Psi_1\tilde{B}_1\Psi_2^t$. So we have $\hat{B}_1\hat{B}_1^t = \Psi_1\tilde{B}_1\Psi_2^t\Psi_2\tilde{B}_1\Psi_1^t = \hat{\gamma}\Psi_1\mathbf{E}\Psi_1^t$. Then (7.9) follows directly. And (7.8) follows from (7.6). q.e.d.

Note that the isotropic condition $B_1B_1^t = 0$ is equivalent to the pair of equations $\langle Y_z, Y_z\rangle = \langle \hat{Y}_z,\hat{Y}_z\rangle = 0$. So if a non-constant harmonic map f is isotropic, by Theorem 4.8 of [**24**], Y and $\hat{Y}$ form a pair of adjoint Willmore surfaces. Then one has (compare also [**16**], [**17**] and [**34**]):

Theorem 7.8. [**24**], [**17**] *Let $f_h = Y \wedge \hat{Y}$ be an isotropic harmonic map. Then Y and $\hat{Y}$ form a pair of adjoint Willmore surfaces. Moreover, set*

$$B_1 = (b_1 \ b_2)^t \text{ with } b_1, b_2 \in \mathbb{C}^{n+2}.$$

Then Y is immersed at the points $(b_1^t + b_2^t)(\bar{b}_1 + \bar{b}_2) > 0$ and $\hat{Y}$ is immersed at the points $(b_1^t - b_2^t)(\bar{b}_1 - \bar{b}_2) > 0$. Especially, when $[Y]$ and $[\hat{Y}]$ are in $\mathbb{S}^3$, they are a pair of dual Willmore surfaces.

Theorem 7.9. *(*[**16**], [**17**], [**34**]*) Let $f = Y \wedge \hat{Y}$ be a harmonic map with normalized potential*

$$\eta = \lambda^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,1} & 0 \end{pmatrix} \mathrm{d}z$$

satisfying $\hat{B}_1 \hat{B}_1^t = \hat{\gamma}\mathbf{E}$. Then either f is a half-isotropic harmonic map (and Y is a Willmore surface), or

$$\hat{B}_1 = \begin{pmatrix} \hat{b}_1 & -\hat{b}_1 \end{pmatrix}^t.$$

Proof. By the DPW construction, an extended frame F of f is derived from the decomposition $F = F_- \cdot F_+$, for some F_- such that $F_-^{-1}\mathrm{d}F_- = \eta, F_-(0,\lambda) = I$. Assume that $F_+ = \sum_{j=0} \lambda^j F_{+j}$ is the Taylor expansion of F_+ with respect to $\lambda \in \mathbb{C}$. So $F_{+0} = \operatorname{diag}(F_{+01}, F_{+02})$, with $F_{+01} \in SO(1,1,\mathbb{C}), F_{+02} \in SO(n+2,\mathbb{C})$. Then let

$$F^{-1}\mathrm{d}F = \lambda^{-1}\alpha_1 + \alpha_0 + \lambda\alpha_{-1} \quad \text{with} \quad \alpha_1 = \begin{pmatrix} 0 & B_1 \\ B_1^t I_{1,1} & 0 \end{pmatrix} \mathrm{d}z.$$

We have

$$\begin{pmatrix} 0 & B_1 \\ \hat{B}_1^t I_{1,1} & 0 \end{pmatrix} = F_{+0}^{-1} \begin{pmatrix} 0 & \hat{B}_1 \\ \hat{B}_1^t I_{1,1} & 0 \end{pmatrix} F_{+0}.$$

So $B_1 = F_{+01}^{-1} \hat{B}_1 F_{+02}$. By (7.6), B_1 satisfies (7.8). The rest follows from Theorem 7.6. q.e.d.

Concerning holomorphic potentials, by similar methods, we have

Theorem 7.10. *Let $f : \mathbb{D} \to SO^+(1,n+3)/(SO^+(1,1) \times SO(n+2))$ be a non-constant harmonic map, with an extended frame $F(z,\bar{z},\lambda)$. Let*

$$\Xi = C^{-1}\mathrm{d}C = \sum_{j=-1}^{+\infty} \lambda^j \xi_j \mathrm{d}z$$

be a holomorphic potential of f given by a holomorphic frame $C = F \cdot V_+$. Assume that

$$\xi_{-1} = \begin{pmatrix} 0 & \hat{B}_1 \\ -\hat{B}_1^t I_{1,1} & 0 \end{pmatrix}.$$

Then

(i) $f = Y \wedge \hat{Y}$ *is an isotropic harmonic map if and only if*

$$\hat{B}_1 \hat{B}_1^t = 0. \tag{7.10}$$

Moreover, $[Y]$ *and* $[\hat{Y}]$ *forms a pair of dual (S-)Willmore surfaces if and only if* $rank(\hat{B}_1) = 1$.

(ii) *If* f *is a half-isotropic harmonic map, then* $\hat{B}_1$ *satisfies* $\hat{B}_1 \hat{B}_1^t = \widehat{\gamma}\mathbf{E}$. *Conversely, if* $\hat{B}_1$ *satisfies* $\hat{B}_1 \hat{B}_1^t = \widehat{\gamma}\mathbf{E}$, *then either* f *is a half-isotropic harmonic map, or*

$$\hat{B}_1 = \begin{pmatrix} \hat{b}_1 & -\hat{b}_1 \end{pmatrix}^t .$$

In the latter case, f *is not a half-isotropic harmonic map. But if* $\widehat{\gamma} \equiv 0$, *then* $\hat{Y}$ *is Möbius equivalent to a minimal surface in* $\mathbb{R}^{n+2}$ *and* $\tilde{f} := \hat{Y} \wedge Y$ *is the isotropic harmonic map given by* $\hat{Y}$ *and its dual surface* Y.

8. Generalized Björling problem for Willmore surfaces in $\mathbb{S}^{n+2}$

We are now in a position to solve a generalization of the Björling problem for all Willmore surfaces in $\mathbb{S}^{n+2}$, with or without umbilic points.

8.1. The $\mathbb{S}^3$ case. To address Willmore surfaces with umbilic points in $\mathbb{S}^3$, one needs to consider half-isotropic harmonic maps instead of the isotropic ones, because, at umbilic points, $[Y]$ and $[\hat{Y}]$ may coincide and then $Y \wedge \hat{Y}$ is not well-defined. In the half-isotropic case, if we only prescribe Y, $\hat{Y}$ and ψ, we will not have enough information on the tangent plane of $\hat{Y}$ to generate a unique solution. A solution is to additionally prescribe the v derivative $\hat{Y}_v$ along the curve.

Theorem 8.1. *Let* $\psi_0 = \psi_0(u) : \mathbb{I} \to \mathbb{S}^4_1$ *denote a non-constant real analytic sphere congruence from* $\mathbb{I}$ *to* $\mathbb{S}^3$, *with a real analytic enveloping curve* $[Y_0]$ *and* u *being the arc-parameter of* $Y_0 : \mathbb{I} \to \mathcal{C}^4_+ \subset \mathbb{R}^5_1$. *Let* $\hat{Y}_0 : \mathbb{I} \to \mathcal{C}^4_+$ *be a real analytic map such that* $\langle \psi_0, \hat{Y}_0 \rangle = 0$ *and* $\langle Y_0, \hat{Y}_0 \rangle = -1$. *Let* $\gamma_{12} : \mathbb{I} \to \mathbb{R}$ *be a real analytic function.*

Then there exists a unique Willmore surface $y : \Sigma \to \mathbb{S}^3$, *with conformal Gauss map* ψ, Σ *some simply connected open subset containing* $\mathbb{I}$ *and* $z = u + iv$ *a complex coordinate of* Σ, *such that:*

(i) *The canonical lift* Y *of* y *satisfies* $Y|_{\mathbb{I}} = Y_0$;
(ii) *The conformal Gauss map* ψ *satisfies* $\psi|_{\mathbb{I}} = \psi_0$ *and* $\langle \psi_v|_{\mathbb{I}}, \hat{Y}_0 \rangle = -\gamma_{12}$.

Theorem 8.1 is a straightforward corollary of the following:

Theorem 8.2. *We retain the assumptions and notations in Theorem* 8.1*. Choose two real analytic unit vector fields P_1 and P_2 on $\mathbb{I}$ such that*

$$Y_{0u} = P_1 \mod Y_0,\ P_2 \perp \{\psi_0, Y_0, \hat{Y}_0, P_1\} \text{ and } \det(Y_0, \hat{Y}_0, P_1, P_2, \psi_0) = 1.$$

There exist real analytical functions $\mu_1 = \mu_1(u)$, $\mu_2 = \mu_2(u)$, $\rho_1 = \rho_1(u)$, $\rho_2 = \rho_2(u)$, $k_1 = k_1(u)$, $k_2 = k_2(u)$ and $\gamma_{11} = \gamma_{11}(u)$ on $\mathbb{I}$ such that

$$\begin{cases} Y_{0u} = -\mu_1 Y_0 + P_1, \\ \hat{Y}_{0u} = \mu_1 \hat{Y}_0 + \rho_1 P_1 + \rho_2 P_2 + 4\gamma_{11}\psi_0, \\ P_{1u} = \mu_2 P_2 + 2k_1\psi_0 + \hat{Y}_0 + \rho_1 Y_0, \\ P_{2u} = -\mu_2 P_1 - 2k_2\psi_0 + \rho_2 Y_0, \\ \psi_{0u} = -2k_1 P_1 + 2k_2 P_2 + 4\gamma_{11}\hat{Y}_0 \end{cases} \tag{8.1}$$

holds. Set $\mu = \mu_1 + i\mu_2$, $k = k_1 + ik_2$, $\rho = \rho_1 + i\rho_2$ and $\gamma_1 = \gamma_{11} + i\gamma_{12}$. For a real analytic function $x(u)$ on $\mathbb{I}$, denote its analytic extension to a simply connected open subset containing $\mathbb{I}$ by $x(z)$. Consider the holomorphic potential

$$\Xi = \left(\lambda^{-1}\mathcal{A}_1 + \mathcal{A}_0 + \lambda\mathcal{A}_{-1}\right)\mathrm{d}z,$$

with

$$\mathcal{A}_0 = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},\ \mathcal{A}_1 = \begin{pmatrix} 0 & B_1 \\ -B_1^t I_{1,1} & 0 \end{pmatrix},\ \mathcal{A}_{-1}(z) = \overline{\mathcal{A}_1(\bar{z})},$$

$$A_1(z) = \begin{pmatrix} 0 & \mu_1(z) \\ \mu_1(z) & 0 \end{pmatrix},\ A_2(z) = \begin{pmatrix} 0 & -\mu_2(z) & -2k_1(z) \\ \mu_2(z) & 0 & 2k_2(z) \\ 2k_1(z) & -2k_2(z) & 0 \end{pmatrix},$$

$$B_1(z) = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\rho(z) & -i-i\rho(z) & 4\gamma_1 \\ 1-\rho(z) & -i+i\rho(z) & -4\gamma_1 \end{pmatrix}.$$

By DPW, Theorem 7.10*, the potential Ξ provides a half-isotropic harmonic map, together with a unique Willmore surface $y : \Sigma \to \mathbb{S}^3$, with conformal Gauss map ψ, Σ some simply connected open subset containing $\mathbb{I}$ and $z = u + iv$ a complex coordinate of Σ, such that the canonical lift Y of y satisfy $Y|_{\mathbb{I}} = Y_0$. Then $\psi|_{\mathbb{I}} = \psi_0$ and $\langle \psi_v|_{\mathbb{I}}, \hat{Y}_0\rangle = -\gamma_{12}$.*

Proof. The proof can be taken verbatim from the proof of Theorem 4.3, with the only difference being that here the function γ_1 in the matrix $B_1(z)$ is allowed to be non-zero. The real part of $\gamma_1(u)$ can be read off from (8.1). But the imaginary part of $\gamma_1(u)$ stays unknown, and we prescribe this as $\gamma_{12}(u)$. The rest is the same as the proof of Theorem 4.3. The equality $\langle \psi_v|_{\mathbb{I}}, \hat{Y}_0\rangle = -\gamma_{12}(u)$ follows from the fact that for a Willmore surface Y with a half-isotropic harmonic map $Y \wedge \hat{Y}$, $\gamma_1 = \frac{1}{2}\langle \hat{Y}_z, \psi\rangle$. q.e.d.

The potential Ξ defined in the above theorem is also called the *boundary potential* of the harmonic map.

Remark 8.3. (i) In contrast to the fully isotropic framework, here one can, for any Willmore surface y, locally choose a solution μ to the equation $\mu_z - \frac{\mu^2}{2} - s = 0$ with μ finite. Then one obtains a half-isotropic harmonic map $Y \wedge \hat{Y}$. Thus, the above theorem holds locally for any Willmore surface in S^3.

(ii) Choose $\hat{Y}_0$ to be an enveloping curve of ψ_0, pointwisely different from Y_0, and set $\gamma_{12} \equiv 0$. Then we re-obtain Theorem 4.3.

(iii) An extremal case is that $Y_0(\mathbb{I})$ is an umbilic curve of Y. For example, the Willmore tori constructed by Babich and Bobenko [**1**] contain an umbilic curve at the intersection of the upper and lower hemisphere models of $\mathbb{H}^3$. We can construct any Willmore surface with a line of umbilics with the following characterization (see Figure 15):

Corollary 8.4. *We retain the assumptions and notations of Theorems* 8.1 *and* 8.2. *Then* $Y_0(\mathbb{I})$ *is an umbilic curve of* Y *if and only if* $k_1 = k_2 \equiv 0$ *on* $\mathbb{I}$.

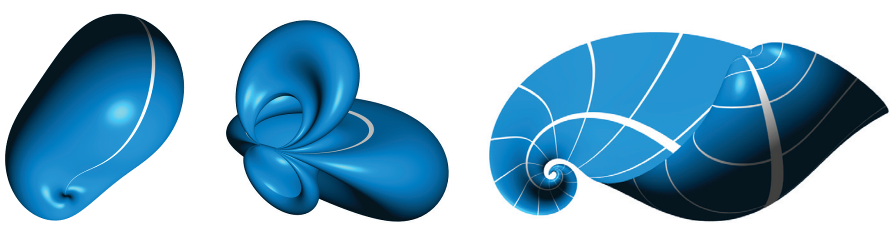


Figure 15. Willmore surfaces with umbilic lines (Example 8.5).

Example 8.5. Three examples with lines of umbilics are computed and displayed in Figure 15. From left to right, the Björling data are: $(\mu, k, \rho, \gamma_1) = (1+i, 0, 1+i, 1)$, $(\mu, k, \rho, \gamma_1) = (0, 0, 0, i)$ and $(\mu, k, \rho, \gamma_1) = (\sin u + e^{0.1u} + i(-1 + 0.5u + \sin u), 0, \cos 3u + i(1 + 0.3u), 1 + 0.2u + 2i(\sin u + 0.6u))$.

Example 8.6. Similar to Example 4.4, let us consider a Willmore surface in $\mathbb{S}^3$ containing the circle $(\cos u, \sin u, 0, 0)$, with a lift $Y = (1, \cos u, \sin u, 0, 0)$, $\hat{Y} = (1/2)(1, -\cos u, -\sin u, 0, 0)$ and a free function γ_{12}. Then similar to discussions in Example 4.4, we have

$$\begin{aligned} P_1 &= (0, -\sin u, \cos u, 0, 0), \quad P_2 = -E_3 \cos\theta - E_4 \sin\theta, \\ \psi &= -E_3 \sin\theta + E_4 \cos\theta, \end{aligned}$$

where θ is any real analytic map $\mathbb{R} \to \mathbb{R}$. We also have $\rho_1 = -1/2$, $k_2 = \theta'/2$ and $\rho_2 = \gamma_{11} = \mu_2 = k_1 = 0$. So we can say that all solutions corresponding to the pair Y and $\hat{Y}$ above are obtained from a choice of two functions θ and γ_{12} with the boundary potential given by the data:

$$(\mu, k, \rho, \gamma_1) = (0, i\theta'/2, -1/2, i\gamma_{12}).$$

Three examples are shown at Figure 16, the first with no umbilics on the circle, the second with two umbilics on the circle, and the last with a line of umbilics. The Björling data are, in order, $(\mu, k, \rho, \gamma_1) = (0, i/2, -1/2, i\sin 4u)$, $(\mu, k, \rho, \gamma_1) = (0, i\sin u, -1/2, i)$, $(\mu, k, \rho, \gamma_1) = (0, 0, -1/2, i\cos u)$.

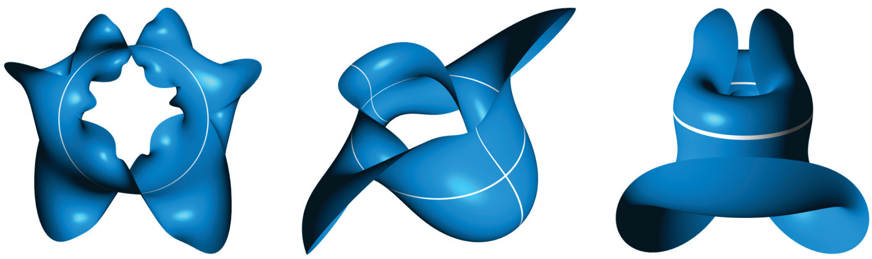


Figure 16. Willmore surfaces containing a circle (Example 8.6).

8.2. Generalized Björling problem for Willmore surfaces in $\mathbb{S}^{n+2}$. The above result can be generalized to Willmore surfaces in $\mathbb{S}^{n+2}$. We write down the solution to the generalized Björling problem for the half-isotropic harmonic maps associated to a Willmore surface in $\mathbb{S}^{n+2}$ as follows. In higher codimension, it will be convenient to use $Y_0 \wedge \hat{Y}_0 \wedge P_{01} \wedge P_{02}$ to represent sphere congruences. We refer to [**19**] for the representation of sphere congruences in $\mathbb{S}^{n+2}$ (See also [**8**], [**24**] for a discussion of mean curvature spheres).

Theorem 8.7. *Let $\Phi_0 = Y_0 \wedge \hat{Y}_0 \wedge P_1 \wedge P_2 : \mathbb{I} \to SO^+(1, n+3)/(SO^+(1,3) \times SO(n))$ denote a real analytic sphere congruence from $\mathbb{I}$ to $\mathbb{S}^{n+2}$ such that:*

(i) *$Y_0 : \mathbb{I} \to \mathcal{C}_+^{n+3} \subset \mathbb{R}_1^{n+4}$ is a real analytic curve with arc-parameter u and $[Y_0]$ is an enveloping curve of Φ_0;*
(ii) *The real analytic map $\hat{Y}_0 : \mathbb{I} \to \mathcal{C}_+^{n+3}$ satisfies $\langle Y_0, \hat{Y}_0 \rangle = -1$;*
(iii) *There is given a real analytic map $\zeta : \mathbb{I} \to \mathbb{R}_1^{n+4}$, perpendicular to $\{Y_0, \hat{Y}_0, P_1, P_2\}$.*

Then there exists a unique half-isotropic harmonic map $Y \wedge \hat{Y} : \Sigma \to SO^+(1, n+3)/(SO^+(1,1) \times SO(n+2))$ and a unique Willmore surface $y = [Y] : \Sigma \to \mathbb{S}^{n+2}$, with conformal Gauss map Φ, Σ some simply connected open subset containing $\mathbb{I}$ and $z = u + iv$ a complex coordinate of Σ, such that:

(i) *The canonical lift Y of y satisfies $Y|_{\mathbb{I}} = Y_0$;*
(ii) *The map $\hat{Y}$ satisfies $\hat{Y}|_{\mathbb{I}} = \hat{Y}_0, \quad \hat{Y}_v|_{\mathbb{I}} = \zeta \mod \{Y_0, \hat{Y}_0, P_1, P_2\}$;*
(iii) *The conformal Gauss map Φ of y satisfies $\Phi|_{\mathbb{I}} = \Phi_0$.*

Proof. Assume that the real analytic maps $P_1, P_2 : \mathbb{I} \rightarrow S^{n+3}_1$ satisfies

$$P_1 = Y_{0u} \mod Y_0,\ \{P_1, P_2\} \perp \{Y_0, \hat{Y}_0\} \text{ and } P_1 \perp P_2,$$

and $\{\psi_{01}, \dots \psi_{0n}\}$ is a real analytic orthonormal basis of the orthogonal complement of $\{P_1, P_2, Y_0, \hat{Y}_0\}$. The proof follows from the higher codimensional analogue of Theorem 8.2, the statement and proof of which generalize, replacing ψ_0 of Theorem 8.2 with $\psi_{01}, \dots \psi_{0n}$, substituting the equations

$$(8.2)\quad \begin{cases} Y_{0u} = -\mu_1 Y_0 + P_1, \\ \hat{Y}_{0u} = \mu_1 \hat{Y}_0 + \rho_1 P_1 + \rho_2 P_2 + 4\sum_j \gamma_{j1}\psi_{0j}, \\ P_{1u} = \mu_2 P_2 + 2\sum_{j=1}^{n} k_{j1}\psi_{0j} + \hat{Y}_0 + \rho_1 Y_0, \\ P_{2u} = -\mu_2 P_1 - 2\sum_{j=1}^{n} k_{j2}\psi_{0j} + \rho_2 Y_0, \\ \psi_{0ju} = \sum_{l=1}^{n} b_{jl1}\psi_{0l} - 2k_{j1}P_1 + 2k_{j2}P_2 + 4\gamma_{j1}Y,\ 1 \leq j \leq n, \end{cases}$$

for equations (8.1), and writing down the corresponding Maurer–Cartan form for the associated frame, which has the same form as (7.3). Note in this case γ_{j2} is given by ζ, i.e., $\gamma_{j2} = \frac{1}{4}\langle \zeta, \psi_{0j}\rangle$. We leave these details to the interested reader. q.e.d.

An interesting result is about the case $Y_0(\mathbb{I})$ being an umbilic curve.

Corollary 8.8. *We retain the assumptions and notations of Theorem 8.7. Then $Y_0(\mathbb{I})$ is an umbilic curve of Y if and only if $k_{j1} = k_{j2} \equiv 0$ on $\mathbb{I}$ for all $j = 1, \cdots, n$. Moreover, in this case Y must be an S-Willmore surface.*

Proof. It only remains to prove that Y is S-Willmore if $Y_0(\mathbb{I})$ is an umbilic curve. This comes from the discussion of the conformal Gauss map Φ of Y. First since $Y_0(\mathbb{I})$ is an umbilic curve, we have that $\kappa|_{\mathbb{I}} \equiv 0$, i.e., $k_{j1} = k_{j2} \equiv 0$ for all $j = 1, \cdots, n$ on $\mathbb{I}$. Then we note that from (8.2) one will also obtain a boundary potential of Φ which is of the form (See [**12**] for details)

$$\Xi_\Phi = \left(\lambda^{-1}\mathcal{A}_{1,\Phi} + \mathcal{A}_{0,\Phi} + \lambda\mathcal{A}_{-1,\Phi}\right) \mathrm{d}z,$$

with

$$\mathcal{A}_{1,\Phi} = \begin{pmatrix} 0 & B_{1,\Phi} \\ -B^t_{1,\Phi} I_{1,3} & 0 \end{pmatrix}.$$

In this case we obtain that (here $\gamma_j = \gamma_{j1} + i\gamma_{j2}$)

$$B_{1,\Phi} = \sqrt{2}\begin{pmatrix} \gamma_1 & \cdots & \gamma_n \\ -\gamma_1 & \cdots & -\gamma_n \\ 0 & \cdots & 0 \\ 0 & \cdots & 0 \end{pmatrix}.$$

So the rank of $B_{1,\Phi}$ is 1, which means that Y is S-Willmore by [**12**]. q.e.d.

Remark 8.9. In fact, s surface that is Willmore, but not S-Willmore, must have isolated umbilic points. The proof is straightforward. If Y is not S-Willmore, then by the Willmore equation $\kappa \wedge D_{\bar{z}}\kappa \mathrm{d}z^3$ is a global holomorphic 3-form (See also Theorem 1.2 of [**13**]). So it has only isolated zeros and, in particular, Y cannot admit an umbilic curve.

To adapt Theorem 8.7 to the isotropic case, we need only add the assumption that ζ has the same length as $\sum_j \gamma_{j1}\psi_{0j}$ in (8.2), which is to ensure that $\hat{Y}$ is also conformal in z. This is equivalent to prescribing the mean curvature sphere of $\hat{Y}$ (in addition to that of Y). See the proof of the following theorem for the details. Note that in general these two mean curvature spheres are different, which is also the geometric reason why two mean curvature spheres are needed to solve the Björling problem in the general case.

Theorem 8.10. *Let $\Phi_0 = Y_0 \wedge \hat{Y}_0 \wedge P_1 \wedge P_2$, $\hat{\Phi}_0 = \hat{Y}_0 \wedge Y_0 \wedge \hat{P}_1 \wedge \hat{P}_2 : \mathbb{I} \to SO^+(1,n+3)/(SO^+(1,3)\times SO(n))$ denote two real analytic sphere congruences from $\mathbb{I}$ to $\mathbb{S}^{n+2}$ such that:*

- (i) *$Y_0 : \mathbb{I} \to \mathcal{C}_+^{n+3} \subset \mathbb{R}_1^{n+4}$ is a real analytic curve with arc-parameter u and $[Y_0]$ is an enveloping curve of Φ_0;*
- (ii) *The real analytic map $\hat{Y}_0 : \mathbb{I} \to \mathcal{C}_+^{n+3}$ satisfies $\langle Y_0, \hat{Y}_0\rangle = -1$. And it is an enveloping curve of $\hat{\Phi}_0$ at the points it is immersed.*

Then there exists a unique isotropic harmonic map $Y \wedge \hat{Y} : \Sigma \to SO^+(1,n+3)/(SO^+(1,1)\times SO(n+2))$ and a unique Willmore surface $y = [Y] : \Sigma \to \mathbb{S}^{n+2}$, with an adjoint transform $\hat{y} = [\hat{Y}]$, Σ some simply connected open subset containing $\mathbb{I}$ and $z = u + iv$ a complex coordinate of Σ, such that:

- (i) *The canonical lift Y of y satisfies $Y|_{\mathbb{I}} = Y_0$;*
- (ii) *The map $\hat{Y}$ satisfies $\hat{Y}|_{\mathbb{I}} = \hat{Y}_0$;*
- (iii) *The conformal Gauss map Φ, $\hat{\Phi}$ of y and $\hat{y}$ satisfies $\Phi|_{\mathbb{I}} = \Phi_0$, $\hat{\Phi}|_{\mathbb{I}} = \hat{\Phi}_0$.*

Proof. Since $\hat{Y}_0$ is an enveloping curve of $\hat{\Phi}_0$,

$$\hat{Y}_{0u} \in \mathrm{Span}\{\hat{Y}_0, Y_0, \hat{P}_1, \hat{P}_2\}.$$

So we can assume that $\hat{Y}_{0u} = a\hat{P}_1$ and $\zeta = a\hat{P}_2 \mod \{Y_0, \hat{Y}_0, P_1, P_2\}$. Applying Theorem 8.7, we finish the proof. q.e.d.

Restricting to the case of a pair of dual S-Willmore surfaces in $\mathbb{S}^{n+2}$, we obtain the following:

Theorem 8.11. *Let $\Phi_0 : \mathbb{I} \to SO^+(1,n+3)/(SO^+(1,3)\times SO(n))$ denote a non-constant real analytic sphere congruence from $\mathbb{I}$ to $\mathbb{S}^{n+2}$, with enveloping curves $[Y_0]$ and $[\hat{Y}_0]$ such that $\langle Y_0, Y_0\rangle = \langle \hat{Y}_0, \hat{Y}_0\rangle = 0$, $\langle Y_0, \hat{Y}_0\rangle = -1$, and u is the arc-length parameter of Y_0. Then there exists* *a unique pair of dual (S-Willmore) Willmore surfaces* $y, \hat{y} : \Sigma \to \mathbb{S}^{n+2}$, *with* Σ *some open subset containing* $\mathbb{I}$, *such that:*

(i) *There exist lifts* Y, $\hat{Y}$ *of* $y, \hat{y}$ *such that* $Y|_{\mathbb{I}} = Y_0$, $\hat{Y}|_{\mathbb{I}} = \hat{Y}_0$;
(ii) *The conformal Gauss map* Φ *of* y *satisfies* $\Phi|_{\mathbb{I}} = \Phi_0$.

Note: applying this theorem to minimal surfaces in space forms gives a higher codimensional analogue of Corollary 4.2.

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