

# TOWARD A CLASSIFICATION OF KILLING VECTOR FIELDS OF CONSTANT LENGTH ON PSEUDO-RIEMANNIAN NORMAL HOMOGENEOUS SPACES

JOSEPH A. WOLF, FABIO PODESTÀ & MING XU

## Abstract

In this paper we develop the basic tools for a classification of Killing vector fields of constant length on pseudo-Riemannian homogeneous spaces. This extends a recent paper of M. Xu and J. A. Wolf, which classified the pairs  $(M, \xi)$  where  $M = G/H$  is a Riemannian normal homogeneous space,  $G$  is a compact simple Lie group, and  $\xi \in \mathfrak{g}$  defines a nonzero Killing vector field of constant length on  $M$ . The method there was direct computation. Here we make use of the moment map  $M \rightarrow \mathfrak{g}^*$  and the flag manifold structure of  $\text{Ad}(G)\xi$  to give a shorter, more geometric proof which does not require compactness and which is valid in the pseudo-Riemannian setting. In that context we break the classification problem into three parts. The first is easily settled. The second concerns the cases where  $\xi$  is elliptic and  $G$  is simple (but not necessarily compact); that case is our main result here. The third, which remains open, is a more combinatorial problem involving elements of the first two.

## 1. Introduction

We consider a connected real reductive Lie group  $G$ , a nondegenerate invariant bilinear form  $b$  on  $\mathfrak{g}$ , and a closed reductive subgroup  $H$  in  $G$  such that  $b$  is nondegenerate on  $\mathfrak{h}$ . Decompose  $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$  where  $\mathfrak{m}$  is the  $b$ -orthocomplement of  $\mathfrak{h}$ . Then  $b$  is nondegenerate on  $\mathfrak{m}$  and induces a pseudo-Riemannian metric  $ds^2$  on  $M = G/H$ . Those are our *normal* pseudo-Riemannian metrics. This includes the Riemannian case, where  $ds^2$  is either positive definite (as usual) or negative definite (so that  $b$  can be the Killing form when  $G$  is a compact semisimple Lie group).

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Note the dependence on the pair  $(G, b)$ . If  $G'$  is another transitive group of isometries of  $(M, ds^2)$  then  $ds^2$  need not be normal as a homogeneous space of  $G'$ .

Let  $\xi \in \mathfrak{g}$ . It induces a Killing vector field on  $M$  which we denote  $\xi^M$ . If  $x \in M$  then  $\xi_x^M$  is the corresponding tangent vector at  $x$ . We say that  $\xi^M$  has *constant length* (perhaps pseudo-length would be a better term) if the function  $x \mapsto ds^2(\xi_x^M, \xi_x^M)$  is constant on  $M$ . The goal of this paper is the classification of triples  $(G, H, \xi)$  where  $\xi \in \mathfrak{g}$  is nonzero and elliptic, and where  $\xi^M$  has constant length.

In the setting of pseudo-Riemannian manifolds, constant length Killing vector fields (also called Clifford–Killing or CK vector fields; see [11]) are the appropriate replacement for isometries of constant displacement (Clifford–Wolf or CW isometries).

In Section 2, we discuss a flag manifold  $G_{\mathbb{C}}/Q$  that connects the moment map for conjugation orbits in  $\mathfrak{g}$  with the length function for  $\xi^M$ . Then in Section 3, we develop a method of passage through the complex domain that carries this connection to flag domains and the pseudo-Riemannian setting. In Section 4, we use these tools to carry out the classification for the cases where  $G_{\mathbb{C}}$  is simple; the main result is Theorem 4.4. Those tools don't apply directly to the case where  $G$  is simple but  $G_{\mathbb{C}}$  is not, but in Section 5, we use other methods to carry out the classification; there the main result is Theorem 5.2. Section 6 summarizes these classifications to give one of the two main results of this paper, Theorem 6.1. As a consequence of these classifications, Corollary 6.2 indicates the pseudo-Riemannian analog of the correspondence between homogeneity for quotient manifolds and isometries of constant displacement.

The other principal result is Theorem 7.6, which in effect describes current progress toward a classification where  $G$  need not be simple.

Let  $pr_{\mathfrak{h}}$  and  $pr_{\mathfrak{m}}$  denote the respective orthogonal projections of  $\mathfrak{g}$  to  $\mathfrak{h}$  and  $\mathfrak{m}$ . Then  $ds^2(\xi_x^M, \xi_x^M) = b(pr_{\mathfrak{m}}(\text{Ad}(g)\xi), pr_{\mathfrak{m}}(\text{Ad}(g)\xi))$  where  $x = gH$ . Since  $b(\text{Ad}(g)\xi, \text{Ad}(g)\xi)$  is independent of  $g \in G$ , and

$$b(\text{Ad}(g)\xi, \text{Ad}(g)\xi) = b(pr_{\mathfrak{h}}(\text{Ad}(g)\xi), pr_{\mathfrak{h}}(\text{Ad}(g)\xi)) + b(pr_{\mathfrak{m}}(\text{Ad}(g)\xi), pr_{\mathfrak{m}}(\text{Ad}(g)\xi)),$$

**Lemma 1.1.** *Let  $\xi \in \mathfrak{g}$ . Then  $\xi^M$  has constant length if and only if*

$$f_{\xi}(g) := b(pr_{\mathfrak{h}}(\text{Ad}(g)\xi), pr_{\mathfrak{h}}(\text{Ad}(g)\xi))$$

*is independent of  $g \in G$ .*

In view of Lemma 1.1 and our assumption that  $G$  is connected, the constant length property for  $\xi^M$  depends only on the pair  $(\mathfrak{g}, \mathfrak{h})$ . Thus we can (and will) be casual about passing to and from covering groups of  $G$  and about connectivity of  $H$ . In practice this will be only a matter of whether it is more convenient to write *Spin* or *SO*.

### 2. The flag domain

Recall that  $G$  is a real reductive Lie group and  $b$  is a nondegenerate  $\text{Ad}(G)$ -invariant symmetric bilinear form on its Lie algebra  $\mathfrak{g}$ . We use  $b$  to identify adjoint orbits of  $G$  on  $\mathfrak{g}$  and coadjoint orbits of  $G$  on  $\mathfrak{g}^*$ .

**Proposition 2.1.** *Suppose that  $\xi \in \mathfrak{g}$  is elliptic, in other words that  $\text{ad}(\xi)$  is semisimple (diagonalizable over  $\mathbb{C}$ ) with pure imaginary eigenvalues. Let  $L$  denote the centralizer of  $\xi$  in  $G$ . Then  $G_{\mathbb{C}}$  has a parabolic subgroup  $Q$  with the properties*

- $L$  is the isotropy subgroup of  $G$  at the base point  $z_0 = 1Q$  for the action of  $G$  (as a subgroup of  $G_{\mathbb{C}}$ ) on the complex flag manifold  $Z = G_{\mathbb{C}}/Q$ ,
- $L_{\mathbb{C}}$  is the reductive part of  $Q$ ,
- the orbit  $G(z_0) \subset Z$  is open and carries a  $G$ -invariant pseudo-Kähler metric, which can be normalized so that
- $\text{Ad}(g)\xi \mapsto gQ$  is a symplectomorphism of  $\mathcal{O}_{\xi} := \text{Ad}(G)\xi$  onto  $G(z_0)$  where the symplectic form on  $\mathcal{O}_{\xi}$  is the Kostant–Souriau form  $\omega(\eta, \zeta) = b(\xi, [\eta, \zeta])$  and the symplectic form on  $G(z_0)$  is the imaginary part of the invariant pseudo-Kähler metric.

In particular,  $\mathcal{O}_{\xi}$  has a  $G$ -invariant pseudo-Kähler structure.

*Proof.* By construction  $L$  is reductive. In fact,  $\xi$  is contained in a fundamental (maximally compact) Cartan subalgebra  $\mathfrak{t}$  of  $\mathfrak{g}$ , and  $\mathfrak{l}_{\mathbb{C}}$  is  $\mathfrak{t}_{\mathbb{C}}$  plus all the  $\mathfrak{t}_{\mathbb{C}}$ -root spaces  $\mathfrak{g}_{\alpha}$  for roots  $\alpha$  that vanish on  $\xi$ . Define  $\mathfrak{q} = \mathfrak{l}_{\mathbb{C}} + \sum_{\alpha(i\xi) < 0} \mathfrak{g}_{\alpha}$ . It is a parabolic subalgebra of  $\mathfrak{g}_{\mathbb{C}}$  with reductive part  $\mathfrak{l}_{\mathbb{C}}$ .

Let  $\tau$  denote complex conjugation of  $\mathfrak{g}_{\mathbb{C}}$  over  $\mathfrak{g}$ . Then  $\tau(i\xi) = -i\xi$  so  $\mathfrak{q} + \tau\mathfrak{q} = \mathfrak{g}_{\mathbb{C}}$ , and also  $\mathfrak{q} \cap \tau\mathfrak{q} = \mathfrak{l}_{\mathbb{C}}$ . There are two immediate consequences: (i)  $G(z_0)$  is open in  $Z = G_{\mathbb{C}}/Q$  and (ii)  $\text{Ad}(g)\xi \mapsto gQ$  is a diffeomorphism of  $\mathcal{O}_{\xi}$  onto  $G(z_0)$ . Note that (ii) uses simple connectivity of both  $Z$  and  $\mathcal{O}_{\xi}$ .

Since  $\mathfrak{q} \cap \tau\mathfrak{q} = \mathfrak{l}_{\mathbb{C}}$ , which is reductive,  $G(z_0)$  carries a  $G$ -invariant measure. Any such measure comes from the volume form of an invariant indefinite-Kähler metric; see [9], or see the exposition of flag domains in [2]. This metric is constructed in [9] using an invariant bilinear form; as is the Kostant–Souriau form, and by the construction a proper normalization of the metric has imaginary part equal to the Kostant–Souriau form. q.e.d.

**Remark 2.2.** In our flag domain cases, Proposition 2.1 extends the structural result of [1, Theorem 1.3(4)] from symplectic to pseudo-Kähler.

**Remark 2.3.** The parabolic  $\mathfrak{q}$  is the sum of the non-positive eigenspaces of  $\text{ad}(i\xi)$  on  $\mathfrak{g}_{\mathbb{C}}$ . If  $g \in G_{\mathbb{C}}$  now  $\text{Ad}(g)\mathfrak{q}$  is the sum of the non-positive eigenspaces of  $\text{ad}(\text{Ad}(g)\xi)$  on  $\mathfrak{g}_{\mathbb{C}}$ . As  $Q$  is its own nor-

malizer in  $G_{\mathbb{C}}$  we can identify  $Z = G_{\mathbb{C}}/Q$  with the space of  $\text{Ad}(G_{\mathbb{C}})$ -conjugates of  $\mathfrak{q}$ . Thus, if  $S$  is any subgroup of  $G_{\mathbb{C}}$ , we see exactly how  $\text{Ad}(S)\xi \subset Z$ .

We are using  $b$  to identify  $\mathfrak{g}$  with  $\mathfrak{g}^*$ ; similarly use  $b|_{\mathfrak{h}}$  to identify  $\mathfrak{h}$  with  $\mathfrak{h}^*$ . The inclusion

$$(2.4) \quad \mu_G : \mathcal{O}_{\xi} \hookrightarrow \mathfrak{g}$$

coincides with the moment map for the (necessarily Hamiltonian) action of  $G$  on  $\mathcal{O}_{\xi}$ . Now consider the action of  $H$  on  $\mathcal{O}_{\xi}$ . The corresponding moment map is

$$(2.5) \quad \mu_H := pr_{\mathfrak{h}} \circ \mu_G : \mathcal{O}_{\xi} \rightarrow \mathfrak{h}.$$

Thus Lemma 1.1 can be reformulated as

**Lemma 2.6.** *Let  $\xi \in \mathfrak{g}$ . Then  $\xi^M$  has constant length if and only if  $\zeta \mapsto b(\mu_H(\zeta), \mu_H(\zeta))$  is constant on  $\mathcal{O}_{\xi}$ .*

### 3. Holomorphic considerations

The group  $H$  is reductive in  $G$  because  $b$  is nondegenerate on  $\mathfrak{h}$ . Thus [4] there is a Cartan involution  $\theta$  of  $G$  such that  $\theta|_H$  is a Cartan involution on  $H$ . That gives us the decompositions

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} \text{ and } \mathfrak{h} = (\mathfrak{h} \cap \mathfrak{k}) + (\mathfrak{h} \cap \mathfrak{p}),$$

into  $\pm 1$  eigenspaces of  $d\theta$ .

From now on we suppose that  $G$  is semisimple and that  $b$  is a positive linear combination of the Killing forms of the simple ideals of  $\mathfrak{g}$ . Thus  $b$  is negative definite on  $\mathfrak{k}$  and positive definite on  $\mathfrak{p}$ . {The reader can extend many of our results to the case of reductive  $G$  by stipulating  $b(\mathfrak{k}, \mathfrak{p}) = 0$ ,  $b$  negative definite on  $\mathfrak{k}$ , and  $b$  positive definite on  $\mathfrak{p}$ .} The decompositions of  $\mathfrak{g}$  and  $\mathfrak{h}$  give us compact real forms

$$\begin{aligned} \mathfrak{g}_u &= \mathfrak{k} + i\mathfrak{p} = \mathfrak{h}_u + \mathfrak{m}_u, \\ \mathfrak{h}_u &= (\mathfrak{h} \cap \mathfrak{k}) + i(\mathfrak{h} \cap \mathfrak{p}) \text{ and } \mathfrak{m}_u = (\mathfrak{m} \cap \mathfrak{k}) + i(\mathfrak{m} \cap \mathfrak{p}), \end{aligned}$$

of  $\mathfrak{g}_{\mathbb{C}}$ ,  $\mathfrak{h}_{\mathbb{C}}$  and  $\mathfrak{m}_{\mathbb{C}}$ . Let  $G_u$  and  $H_u$  denote the compact real forms of  $G_{\mathbb{C}}$  and  $H_{\mathbb{C}}$  corresponding to  $\mathfrak{g}_u$  and  $\mathfrak{h}_u$ .

Extend  $b$  to a  $\mathbb{C}$ -bilinear form  $b_{\mathbb{C}}$  on  $\mathfrak{g}_{\mathbb{C}}$ . Then  $b_u := b_{\mathbb{C}}|_{\mathfrak{g}_u}$  is negative definite. As  $b(\mathfrak{h}, \mathfrak{m}) = 0$  we have  $b_{\mathbb{C}}(\mathfrak{h}_{\mathbb{C}}, \mathfrak{m}_{\mathbb{C}}) = 0$  and thus  $b_u(\mathfrak{h}_u, \mathfrak{m}_u) = 0$ . The orthogonal projection  $pr_{\mathfrak{h}_{\mathbb{C}}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{h}_{\mathbb{C}}$  restricts to orthogonal projection  $pr_{\mathfrak{h}_u} : \mathfrak{g}_u \rightarrow \mathfrak{h}_u$ .

**Lemma 3.1.** *Define  $f_{\xi} : G_{\mathbb{C}} \rightarrow \mathbb{C}$  by*

$$f_{\xi}(g) = b_{\mathbb{C}}(pr_{\mathfrak{h}_{\mathbb{C}}}(\text{Ad}(g)\xi), pr_{\mathfrak{h}_{\mathbb{C}}}(\text{Ad}(g)\xi)).$$

*Then  $f_{\xi}$  is holomorphic.*

*Proof.* The map  $g \mapsto \text{Ad}(g)\xi$  is holomorphic on  $G_{\mathbb{C}}$ , the projection  $pr_{\mathfrak{h}_{\mathbb{C}}} : \mathfrak{g}_{\mathbb{C}} \rightarrow \mathfrak{h}_{\mathbb{C}}$  is holomorphic, and  $b_{\mathbb{C}}$  is complex bilinear. q.e.d.

**Lemma 3.2.**  $\xi^M$  has constant length if and only if  $f_\xi : G_{\mathbb{C}} \rightarrow \mathbb{C}$  is constant.

*Proof.* If  $\xi^M$  has constant length then  $f_\xi$  is constant on  $G$ . Since  $G$  is a real form of  $G_{\mathbb{C}}$  and  $f_\xi$  is holomorphic, it follows that  $f_\xi$  is constant. q.e.d.

Denote  $M_u = G_u/H_u$  where  $M_u$  carries the normal homogeneous Riemannian metric defined by  $b_u|_{\mathfrak{m}_u}$ . In effect it is the natural compact real form of the affine algebraic variety  $M_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ . Extending the notion of Cartan duality from Riemannian symmetric space theory,  $M_u = G_u/H_u$  is the compact dual to  $M = G/H$ . If  $\xi \in \mathfrak{k}$ , in particular if  $\xi \in \mathfrak{g}_u$ , we write  $\xi^{M_u}$  for the corresponding vector field on  $M_u$ . Now Lemmas 2.6 and 3.2 give us

**Proposition 3.3.** *If  $\xi \in \mathfrak{k}$  and  $\xi^M$  has constant length on  $M$  if, and only if,  $\xi^{M_u}$  has constant length on the Riemannian normal homogeneous space  $M_u := G_u/H_u$ .*

#### 4. Classification for $G_{\mathbb{C}}$ simple

In this section, we carry out the classification of constant length Killing vector fields  $\xi^M$ , on reductive normal homogeneous pseudo-Riemannian manifolds  $M = G/H$  when the group  $G_{\mathbb{C}}$  is simple. The compact version of this classification was done by direct computation in [11], but here we have a less computational approach that starts with classification ([5], or see [6]) of Onishchik for irreducible complex flag manifolds  $Z = G_u/L_u$ , on which a proper closed subgroup  $H_u$  of  $G_u$  acts transitively. On the other hand we need the classification where  $H$  need not be compact. For that we use methods from [10]. In Section 5, we give a separate argument to deal with the case where  $G$  is simple but  $G_{\mathbb{C}}$  is not. Then in Section 6, we translate those results to the classification of constant length Killing vector fields  $\xi^M$  on reductive normal homogeneous pseudo-Riemannian manifolds  $M = G/H$ , with  $G$  simple and  $\xi$  nonzero and elliptic.

For clarity of exposition we always assume that  $G_{\mathbb{C}}$  is connected and simply connected, that the real forms  $G$  and  $G_u$  are analytic subgroups of  $G_{\mathbb{C}}$ , and that  $H, H_{\mathbb{C}}$  and  $H_u$  are analytic subgroups of  $G, G_{\mathbb{C}}$  and  $G_u$ .

**Proposition 4.1** ([5]). *Consider a complex flag manifold  $Z = G_{\mathbb{C}}/Q$ . Suppose that  $Z$  is irreducible, i.e., that  $G_{\mathbb{C}}$  is simple. Then the closed connected subgroups  $H_u \subset G_u$  transitive on  $Z$ ,  $\{1\} \neq H_u \subsetneq G_u$ , are precisely those given as follows:*

1.  $Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C})$ , complex projective  $(2n - 1)$ -space; there  $G_{\mathbb{C}} = SL(2n; \mathbb{C})$  and  $H_u = Sp(n)$ .
2.  $Z = SO(2n + 2)/U(n + 1)$ , unitary structures on  $\mathbb{R}^{2n+2}$ ; there  $G_{\mathbb{C}} = SO(2n + 2; \mathbb{C})$  and  $H_u = SO(2n + 1)$ .

3.  $Z = Spin(7)/(Spin(5) \cdot Spin(2))$ , nonsingular complex quadric; there  $G_{\mathbb{C}} = Spin(7; \mathbb{C})$  and  $H_u$  is the compact exceptional group  $G_2$ .

Here is the noncompact version of Proposition 4.1.

**Proposition 4.2.** *Consider a complex flag manifold  $Z = G_{\mathbb{C}}/Q$  with  $G_{\mathbb{C}}$  simple. Here is a complete list of the connected subgroups  $H \subset G$  with  $H \neq \{1\}$  and  $H_u$  transitive on  $Z$ .*

1.  $Z = SU(2n)/U(2n-1) = P^{2n-1}(\mathbb{C})$  and  $H_u = Sp(n)$ . Then  $(G, H)$  is one of

- (i)  $(SU(2p, 2q), Sp(p, q))$  with  $p + q = n$  or
- (ii)  $(SL(2n; \mathbb{R}), Sp(n; \mathbb{R}))$ .

2.  $Z = SO(2n+2)/U(n+1)$  and  $H_u = SO(2n+1)$ . Then  $(G, H)$  must be

- (i)  $(SO(2p+1, 2q+1), SO(2p+1, 2q))$  with  $p + q = n$  or
- (ii)  $(SO(2p+2, 2q), SO(2p+1, 2q))$  with  $p + q = n$ .

3.  $Z = Spin(7)/(Spin(5) \cdot Spin(2))$  and  $H_u = G_2$ . Then the pair  $(G, H)$  must be

- (i)  $(Spin(7), G_2)$  or
- (ii)  $(Spin(3, 4), (G_2)_{\mathbb{R}})$ . (Here  $(G_2)_{\mathbb{R}}$  is the split real form of  $(G_2)_{\mathbb{C}}$ ).

*Proof.* Suppose  $Z = SU(2n)/U(2n-1; \mathbb{C}) = P^{2n-1}(\mathbb{C})$  and  $H_u = Sp(n)$ . The real forms of  $(H_u)_{\mathbb{C}} = Sp(n; \mathbb{C})$  are the  $Sp(p, q)$ ,  $p + q = n$ , and  $Sp(n; \mathbb{R})$ , and the real forms of  $(G_u)_{\mathbb{C}} = SL(2n; \mathbb{C})$  are the  $SU(r, s)$ ,  $r + s = 2n$  and the special linear groups  $SL(2n; \mathbb{R})$  and  $SL(n; \mathbb{H})$ .

If  $G = SU(r, s)$  and  $J = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}$  then  $G = \{g \in SL(2n; \mathbb{C}) \mid g \cdot J \cdot {}^t \bar{g} = J\}$ . Thus  $H \neq Sp(n; \mathbb{R})$ , for that group cannot have both a symmetric and an antisymmetric bilinear invariant on  $\mathbb{R}^{2n}$ . Now  $G = SU(r, s)$  implies  $H = Sp(p, q)$ , which in turn implies  $G = SU(2p, 2q)$ . Also, if  $G = SL(2n; \mathbb{R})$  then  $H \not\cong Sp(p, q)$  so  $H = Sp(n; \mathbb{R})$ .

Next suppose  $Z = SO(2n+2)/U(n+1)$  and  $H_u = SO(2n+1)$ . The real forms of  $(H_u)_{\mathbb{C}} = SO(2n+1; \mathbb{C})$  are the  $SO(r, s)$  with  $r + s = 2n+1$ , and the real forms of  $(G_u)_{\mathbb{C}} = SO(2n+2; \mathbb{C})$  are the  $SO(k, \ell)$  with  $k + \ell = 2n+2$  and  $SO^*(2n+2)$ . The maximal compact subgroup of  $SO^*(2n+2)$  is  $U(n+1)$ , which does not contain any  $SO(r) \times SO(s)$  with  $r + s = 2n+1$ ; so  $G \neq SO^*(2n+2)$ . Thus  $G = SO(k, \ell)$  and  $H = SO(r, s)$  with  $r \leq k$ ,  $s \leq \ell$  and  $k + \ell = r + s + 1$ , as asserted.

Finally, suppose  $Z = Spin(7)/(Spin(5) \cdot Spin(2))$  and  $H_u = G_2$ . The real forms of  $(G_u)_{\mathbb{C}} = Spin(7; \mathbb{C})$  are the  $Spin(a, b)$  with  $a + b = 7$ , and the real forms of  $(H_u)_{\mathbb{C}} = (G_2)_{\mathbb{C}}$  are the compact form  $G_2$  and the split form  $(G_2)_{\mathbb{R}}$ . Now [8, Theorem 3.1] completes the argument that  $G/H$  is  $Spin(7)/G_2$  or  $Spin(3, 4)/(G_2)_{\mathbb{R}}$ . q.e.d.

To continue we need

**Theorem 4.3** (Gori & Podestà [3]). *Suppose  $M$  is a compact Kähler  $K$ -Hamiltonian manifold, where  $K$  is a compact connected Lie group*

acting effectively and isometrically on  $M$ . If  $\mu$  denotes the moment map, then the squared moment map  $f = |\mu|^2$  is constant if and only if  $K$  is semisimple and the manifold  $M$  is biholomorphically and  $K$ -equivariantly isometric to the product of a flag manifold and a compact Kähler manifold which is acted on trivially by  $K$ .

Now we summarize, including the case where  $H_u$  acts trivially on  $Z$ . In the proof, one case is eliminated by the requirement that  $\xi \in \mathfrak{g}$ . The main ingredients in the proof of Theorem 4.4 are Proposition 4.2 and the theorem of Gori and Podestà just quoted.

**Theorem 4.4.** *Suppose that  $G$  is absolutely simple, i.e., that  $G_{\mathbb{C}}$  is simple. Then there is a nonzero elliptic element  $\xi \in \mathfrak{g}$  such that the Killing vector field  $\xi^M$  on the normal homogeneous space  $M = G/H$  has constant length, if and only if, up to finite covering,  $(G, H)$  is one of the following pairs:*

1.  $Z = SU(2n)/U(2n - 1) = P^{2n-1}(\mathbb{C})$  and  $H_u = Sp(n)$ . Then  $(G, H)$  is one of the  $(SU(2p, 2q), Sp(p, q))$  with  $p + q = n$ , or is  $(SL(2n; \mathbb{R}), Sp(n; \mathbb{R}))$ .
2.  $Z = SO(2n)/U(n)$  and  $H_u = SO(2n - 1)$ . Then  $(G, H)$  is one of the  $(SO(2p, 2q), SO(2p - 1, 2q))$  with  $p + q = n$ .
3.  $Z = Spin(7)/(Spin(5) \cdot Spin(2))$  and  $H_u = G_2$ . Then  $(G, H)$  is  $(Spin(7), G_2)$  or  $(Spin(3, 4), (G_2)_{\mathbb{R}})$ .
4.  $\mathfrak{h} = 0$  and  $(G, H)$  is the group manifold pair  $(G, \{1\})$ .

*Proof.* Retain the notation of Section 3. We can suppose  $\xi \in \mathfrak{k} \subset \mathfrak{g}_u$ . By Proposition 3.3,  $\xi$  induces a Killing vector field  $\xi^{M_u}$  of constant length on the normal homogeneous Riemannian manifold  $M_u = G_u/H_u$ . The adjoint orbit  $Z := \text{Ad}(G_u)\xi \subset \mathfrak{g}_u$  is endowed with the  $G_u$ -invariant symplectic structure given by the Kostant–Souriau form. The  $b$ -orthogonal projection  $pr_{\mathfrak{h}} : Z \rightarrow \mathfrak{h}_u$  defines a moment map  $\mu$  for the Hamiltonian action of  $H_u$  on  $Z$ . By hypothesis  $\mu$  has constant length with respect to  $b|_{\mathfrak{h}_u}$ , and by [3] the flag manifold  $Z$  is a Kähler product  $Z_1 \times Z_2$  with  $H_u$  acting transitively on  $Z_1$  and trivially on  $Z_2$ . Since  $G_{\mathbb{C}}$  is simple, either  $Z = Z_1$  or  $Z = Z_2$ , and if  $H_u$  is not trivial then  $H_u$  acts transitively on  $Z$ . We have shown that  $(G, H)$  either is a group manifold or is one of the pairs listed in Propositions 4.1 and 4.2.

In the cases listed in Proposition 4.1, i.e., the cases where  $G$  is compact, we already have nonzero elliptic elements  $\xi \in \mathfrak{g}$  such that the centralizer of  $\xi$  in  $G$  is transitive on  $G/H$ . For  $G/H = SU(2n)/Sp(n)$  we use  $\xi_1 = \sqrt{-1} \text{diag}\{- (2n - 1), I_{2n-1}\}$ ; it has centralizer  $U(2n - 1)$  in  $G$ . For  $G/H = SO(2n)/SO(2n - 1)$  we use  $\xi_2 = \text{diag}\{J, \dots, J\}$  where  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ; it has centralizer  $U(n)$  in  $G$ . For  $G/H = Spin(7)/G_2$  we consider  $\mathfrak{spin}(5) \oplus \mathfrak{spin}(2) \subset \mathfrak{g}$  and take  $0 \neq \xi_3 \in \mathfrak{spin}(2)$ .

Now consider the noncompact cases listed in Proposition 4.2. Going case by case,  $\mathfrak{g}$  contains an appropriate multiple of the  $\xi_i \in \mathfrak{g}_u$  of

the previous paragraph, with the exception of the homogeneous spaces  $SO(2p + 1, 2q + 1)/SO(2p + 1, 2q)$ . That completes the proof. q.e.d.

**Remark 4.5.** In the case  $(G, H) = (G, \{1\})$ ,  $M$  is the group manifold, the metric is any nonzero multiple of the Killing form,  $G$  acts on itself by left translation, and  $\xi$  can be any element of  $\mathfrak{g}$  because it is centralized by all right translations. In this case  $\xi^M$  is of constant length without the requirement that  $\xi$  be elliptic.

### 5. Classification for $G$ complex simple

We now look at the case where  $G$  is simple but  $G_{\mathbb{C}}$  is not. That is when  $G$  is the underlying real structure of a complex simple Lie group  $E$ ; then  $G_{\mathbb{C}} = E \times \overline{E}$  where  $\overline{E}$  is the complex conjugate of  $E$  and  $G \hookrightarrow G_{\mathbb{C}}$  is the diagonal  $\delta E \hookrightarrow G_{\mathbb{C}}$ . It is convenient to use the following very general lemma, which is based on the infinitesimal version of [7, Théorème 1]:

**Lemma 5.1.** *Let  $(M, ds^2)$  be any connected pseudo-Riemannian homogeneous space. Let  $\xi \in \mathfrak{g}$ . If the centralizer*

$$L := \{g \in I(M, ds^2) \mid \text{Ad}(g)\xi = \xi\}$$

*of  $\xi$  in the isometry group  $I(M, ds^2)$  has an open orbit on  $M$  then  $\xi^M$  has constant length on  $M$ . In particular, if  $L$  is transitive on  $M$  then  $\xi^M$  has constant length on  $M$ .*

*Proof.* Let  $\mathcal{O}$  be an open  $L$ -orbit on  $M$ . If  $x, y \in \mathcal{O}$ , say  $gx = y$  with  $g \in L$ , then  $ds^2(\xi_y^M, \xi_y^M) = ds^2(dg(\xi_x^M), dg(\xi_x^M)) = ds^2(\xi_x^M, \xi_x^M)$ . Thus  $\|\xi^M\|^2$  is constant on  $\mathcal{O}$ . As  $\|\xi^M\|^2$  is real analytic on  $M$  it is constant. q.e.d.

**Theorem 5.2.** *Suppose that  $G$  is simple but  $G_{\mathbb{C}}$  is not. Then there is a nonzero elliptic element  $\xi \in \mathfrak{g}$  such that the Killing vector field  $\xi^M$  on the normal homogeneous space  $M = G/H$  has constant length, if and only if, up to finite covering,  $(G, H)$  is one of the pairs*

- (1)  $(SL(2n; \mathbb{C}), Sp(n; \mathbb{C}))$ ,
- (2)  $(SO(2n; \mathbb{C}), SO(2n - 1; \mathbb{C}))$ ,
- (3)  $(Spin(7; \mathbb{C}), (G_2)_{\mathbb{C}})$ , or
- (4) the group manifold pair  $(G, \{1\})$ .

**Remark 5.3.** In all cases of Theorem 5.2,  $G/H$  is a complex affine algebraic variety. Also, in the case  $(G, H) = (G, \{1\})$ ,  $M$  is the group manifold, the metric is any nonzero multiple of the Killing form,  $G$  acts on itself by left translation, and  $\xi$  can be any element of  $\mathfrak{g}$  because it is centralized by all right translations. In this case  $\xi^M$  is of constant length without the requirement that  $\xi$  be elliptic.



*Proof.* Let  $\xi \in \mathfrak{g}$  be nonzero and elliptic. We may assume that it is contained in the Lie algebra  $\mathfrak{k}$  of a maximal compact subgroup  $K$  of  $G$ . The point is that it is contained in a fundamental (maximally compact) Cartan subalgebra of  $\mathfrak{g}$ . All such Cartan subalgebras are  $\text{Ad}(G)$ -conjugate, and the  $\mathfrak{g}$ -centralizer of any Cartan subalgebra of  $\mathfrak{k}$  is one of them. Thus, for the proof, we may assume  $\xi \in \mathfrak{k}$ .

Note that  $K$  is a compact real form when  $G$  is regarded as a complex simple group. Passing to a conjugate,  $H$  is stable under the complex conjugation  $\tau$  of  $G$  with fixed point set  $K$ , for  $\tau$  is a Cartan involution. Now  $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$  under  $\tau$  and  $\mathfrak{h} = \mathfrak{h} \cap \mathfrak{k} + \mathfrak{h} \cap i\mathfrak{k}$ . These are orthogonal decompositions relative to the Killing form of  $G$ , and the invariant bilinear form  $b$  is a positive multiple of that Killing form.

Suppose that  $\xi^M$  has constant length, equivalently that

$$b(\text{pr}_{\mathfrak{h}}(\text{Ad}(g)\xi), \text{pr}_{\mathfrak{h}}(\text{Ad}(g)\xi))$$

is constant for  $g \in G$ . Then  $b(\text{pr}_{\mathfrak{h} \cap \mathfrak{k}}(\text{Ad}(g)\xi), \text{pr}_{\mathfrak{h} \cap \mathfrak{k}}(\text{Ad}(g)\xi))$  is constant for  $g \in K$ . In other words  $\xi$  defines a constant length Killing vector field on  $K/(K \cap H)$ .

If  $\mathfrak{k} \cap \mathfrak{h} = \mathfrak{k}$  then  $\mathfrak{k} \subset \mathfrak{h}$ . The adjoint action of  $\mathfrak{k}$  on  $\mathfrak{g}$  is the sum of two copies of the adjoint representation of  $\mathfrak{k}$ , which is irreducible, so  $\mathfrak{k} \subset \mathfrak{h}$  says that either  $\mathfrak{h} = \mathfrak{k}$  or  $\mathfrak{h} = \mathfrak{g}$ . If  $\mathfrak{h} = \mathfrak{k}$  there is no nonzero Killing vector field of constant length on  $G/H$ . If  $\mathfrak{h} = \mathfrak{g}$  then  $G/H$  is reduced to a point. So  $\mathfrak{k} \cap \mathfrak{h} \neq \mathfrak{k}$ .

If  $\mathfrak{k} \cap \mathfrak{h} = 0$  then  $b(\mathfrak{k}, \mathfrak{h}) = 0$  so  $\xi \in \mathfrak{k} \subset \mathfrak{m}$ . Then  $\text{pr}_{\mathfrak{m}}(\xi) = \xi$  and  $\text{pr}_{\mathfrak{h}}(\xi) = 0$ . In particular,  $b(\text{pr}_{\mathfrak{m}}(\text{Ad}(g)\xi), \text{pr}_{\mathfrak{m}}(\text{Ad}(g)\xi)) = b(\xi, \xi)$  and  $b(\text{pr}_{\mathfrak{h}}(\text{Ad}(g)\xi), \text{pr}_{\mathfrak{h}}(\text{Ad}(g)\xi)) = 0$  for all  $g \in G$ . Now  $\text{Ad}(G)\xi \subset \mathfrak{m}$ , so  $\mathfrak{m}$  contains a nonzero ideal of the simple Lie algebra  $\mathfrak{g}$ . In other words  $\mathfrak{m} = \mathfrak{g}$  and  $\mathfrak{h} = 0$ , so  $M$  is the group manifold  $G$ .

Now suppose  $\mathfrak{k} \cap \mathfrak{h} \neq 0$ . As  $\mathfrak{k} \cap \mathfrak{h} \subsetneq \mathfrak{k}$  and  $\xi$  defines a constant length Killing vector field on  $K/(K \cap H)$ , we know from [11] or from Theorem 4.4 that  $(\mathfrak{k}, \mathfrak{k} \cap \mathfrak{h})$  is one of

$$(5.4) \quad \text{(a) } (\mathfrak{su}(2n), \mathfrak{sp}(n)), \text{ (b) } (\mathfrak{so}(2n), \mathfrak{so}(2n - 1)), \text{ or (c) } (\mathfrak{so}(7), \mathfrak{g}_2).$$

Here  $(\mathfrak{h}, \mathfrak{k} \cap \mathfrak{h})$  is a symmetric pair,  $\mathfrak{h} \cap \mathfrak{k}$  is simple by (5.4), and of course  $\mathfrak{k} \neq \mathfrak{h} \subset \mathfrak{g}$ .

Decompose  $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$  where  $\mathfrak{h}_2 \cap \mathfrak{k} = 0$  and every ideal of  $\mathfrak{h}_1$  has nonzero intersection with  $\mathfrak{k}$ . Then  $\mathfrak{h}_2 \subset \mathfrak{p}$  and  $\mathfrak{h} \cap \mathfrak{k} = \mathfrak{h}_1 \cap \mathfrak{k}$ . Thus (5.4) limits the possibilities of  $\mathfrak{h}_1$  to

$$(5.5)$$

$$(5.4a) : \mathfrak{h}_1 = \text{(i) } \mathfrak{sp}(n; \mathbb{C}), \text{ (ii) } \mathfrak{sl}(n; \mathbb{H}), \text{ (iii) } \mathfrak{e}_{6,c_4}(n = 4),$$

$$(5.4b) : \mathfrak{h}_1 = \text{(iv) } \mathfrak{so}(2n - 1; \mathbb{C}), \text{ (v) } \mathfrak{sl}(2n - 1; \mathbb{R}), \text{ (vi) } \mathfrak{f}_{4,b_4}(n = 5),$$

$$(5.4c) : \mathfrak{h}_1 = \text{(vii) } \mathfrak{g}_{2,C}.$$

We eliminate case (iii) of (5.5) because  $\mathfrak{e}_6$  has no nontrivial representation of degree 8, and (vi) because  $\mathfrak{f}_4$  has no nontrivial representation

of degree 10. In case (v), passing to the complexification we would have  $\mathfrak{sl}(2n - 1; \mathbb{C}) \subset \mathfrak{so}(2n; \mathbb{C}) \oplus \mathfrak{so}(2n; \mathbb{C})$  while  $\mathfrak{sl}(2n - 1; \mathbb{C})$  has no nontrivial orthogonal representation of degree  $2n$ ; that eliminates case (v). At this point we notice that  $\mathfrak{h}_1$  is a maximal subalgebra of  $\mathfrak{g}$ , so  $\mathfrak{h} = \mathfrak{h}_1$ .

Case (ii) is more delicate. The analog of [11] reduces the existence of a Killing vector field  $\xi$  of constant length to the question of whether  $\xi' = i \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$  defines a Killing vector field of constant length on  $M' = SL(2; \mathbb{C})/SL(1; \mathbb{H})$ . Since  $M'$  is the noncompact Riemannian symmetric space  $SL(2; \mathbb{C})/SU(2)$ , the answer is negative. We have eliminated cases (ii), (iii), (v) and (vi) of (5.5), and we have shown  $\mathfrak{h} = \mathfrak{h}_1$ .

At this point we have shown that there is a nonzero elliptic  $\xi \in \mathfrak{g}$  such that  $\xi^M$  has constant length, if and only if  $(G, H)$  is one of the four pairs listed in Theorem 5.2. If  $(G, H) = (SL(2n; \mathbb{C}), Sp(n; \mathbb{C}))$  we can take  $\xi = \text{idia}g \{2n - 1; 1, \dots, 1\}$ ; it is centralized by  $GL(2n; \mathbb{C})$ . If  $(G, H) = (SO(2n; \mathbb{C}), SO(2n - 1; \mathbb{C}))$  we can take  $\xi = \text{diag} \{J, \dots, J\}$  where  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ ; it is centralized by  $GL(n; \mathbb{C})$ . If  $(G, H) = (Spin(7; \mathbb{C}), (G_2)_{\mathbb{C}})$  we can take  $\xi$  in a Cartan subalgebra dual to a short root. As noted in Remark 5.3, if  $(G, H) = (G, \{1\})$  we can take  $\xi$  to be any element of the Lie algebra  $\mathfrak{g}$  acting by right translations. Looking at the compact versions, in all cases one calculates  $\dim Z_G(\xi)/(Z_G(\xi) \cap H) = \dim G/H$ , so Lemma 5.1 ensures that the Killing vector field  $\xi^M$  has constant length. q.e.d.

**Remark 5.6.** Here is another argument to eliminate case (ii) of (5.5) in the proof of Theorem 5.2.  $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g} \oplus \bar{\mathfrak{g}}$  with  $\mathfrak{g}$  embedded diagonally, so  $\mathfrak{g}_{\mathbb{C}}$  has compact real form  $\mathfrak{g}_u \cong \mathfrak{k} \oplus \mathfrak{k}$  with  $\mathfrak{k}$  embedded diagonally. Now  $\xi \in \mathfrak{k}$  has form  $\xi = (\xi', \xi')$  inside  $\mathfrak{g}_{\mathbb{C}}$ , so it has nontrivial projections to each of the two simple summands of  $\mathfrak{g}_u$ . This is impossible here because  $H_u$  is the diagonal  $SU(2n)$  inside  $G_u \cong SU(2n) \times SU(2n)$ .

### 6. Summary for $G$ simple

Combining Theorems 4.4 and 5.2 we arrive at

**Theorem 6.1.** *Suppose that  $G$  is simple. Then there is a nonzero elliptic element  $\xi \in \mathfrak{g}$  such that the Killing vector field  $\xi^M$  on the normal homogeneous space  $M = G/H$  has constant length, if and only if, up to finite covering,  $(G, H)$  is one of the following:*

1.  $(SU(2p, 2q), Sp(p, q))$  with  $p + q = n$ ,  $(SL(2n; \mathbb{R}), Sp(n; \mathbb{R}))$  or  $(SL(2n; \mathbb{C}), Sp(n; \mathbb{C}))$ ;
2.  $(SO(2p + 2, 2q), SO(2p + 1, 2q))$  with  $p + q = n$  or  $(SO(2n + 2; \mathbb{C}), Sp(2n + 1; \mathbb{C}))$ ;
3.  $(Spin(7), G_2)$ ,  $(Spin(3, 4), (G_2)_{\mathbb{R}})$  or  $(Spin(7; \mathbb{C}), G_{2, \mathbb{C}})$ ;
4.  $(G, \{1\})$ .

Looking through this listing one sees

**Corollary 6.2.** *Suppose that  $G$  is simple, and that  $\xi \in \mathfrak{g}$  is nonzero and elliptic. Let  $L$  be the centralizer of  $\xi$  in  $G$ . Then the following are equivalent:*

1.  $\xi^M$  has constant length on  $M = G/H$ .
2.  $L$  has an open orbit on  $G/H$ .
3.  $H$  has an open orbit on the flag domain  $G/L$ .

### 7. The three cases

Retain the notation of Section 2. Note that  $G_u$  acts transitively on the complex flag manifold  $Z = G_{\mathbb{C}}/Q$ , so  $Z = G_u/L_u$  where  $L_u$  is a compact real form of  $L$ . This expresses  $Z$  as a compact simply connected homogeneous Kähler manifold.

By *coset space reduction* of  $G/H$  we mean a decomposition  $G = G' \times G''$  (locally) such that  $H = (H \cap G') \times (H \cap G'')$ , and, consequently,  $G/H = (G'/(H \cap G')) \times (G''/(H \cap G''))$ , with each factor of positive dimension. We will say that  $G/H$  is *coset space irreducible* if there is no such nontrivial reduction. The following is immediate from the definitions:

**Lemma 7.1.** *Suppose that  $G$  is semisimple and  $G/H = G'/H' \times G''/H''$  is a coset space reduction. If  $b$  is an invariant bilinear form on  $\mathfrak{g}$  then  $b = b' \oplus b''$  where  $b'$  (resp.  $b''$ ) is an invariant bilinear form on  $\mathfrak{g}'$  (resp.  $\mathfrak{g}''$ ). The corresponding decomposition  $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$  breaks up as  $\mathfrak{g}' = \mathfrak{h}' + \mathfrak{m}'$  and  $\mathfrak{g}'' = \mathfrak{h}'' + \mathfrak{m}''$  where  $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ ,  $\mathfrak{m} = \mathfrak{m}' \oplus \mathfrak{m}''$ ,  $\mathfrak{m}'$  is the  $b'$ -orthocomplement of  $\mathfrak{h}'$  in  $\mathfrak{g}'$ , and  $\mathfrak{m}''$  is the  $b''$ -orthocomplement of  $\mathfrak{h}''$  in  $\mathfrak{g}''$ . In particular, the corresponding factors of the pseudo-Riemannian product decomposition are normal homogeneous spaces.*

Let  $M = G/H$  with  $G$  reductive. Up to finite coverings we then have a decomposition

$$(7.2) \quad G = G_0 \times G_1 \times \cdots \times G_s \times G_{s+1} \times \cdots \times G_r,$$

where  $G_0$  is commutative,  $G_i$  is simple for  $i > 0$ .  $H$  is the (isomorphic) image of a reductive Lie group  $\tilde{H}$  under a homomorphism  $\varphi(x) = (\varphi_0(x), \dots, \varphi_r(x))$  where  $\varphi : \tilde{H} \rightarrow G_i$ . We are going to study constant length Killing vector fields on  $M = G/H$  defined by vectors

$$(7.3) \quad \xi = \xi_0 + \cdots + \xi_r \in \mathfrak{g}, \text{ where } \xi_i \in \mathfrak{g}_i, \xi_i \neq 0 \text{ for } 1 \leq i \leq s \text{ and } \xi_i = 0 \text{ for } s < i \leq r.$$

In view of Lemma 7.1 we need only consider the coset irreducible cases. There are three basic possibilities of reductive normal coset irreducible  $G/H$ :

- (7.4) (i) for some index  $i$  we have  $\varphi_i(\tilde{H}) = \{1\}$ ,  
 (ii) for every index  $i$  we have  $\{1\} \neq \varphi_i(\tilde{H}) \subsetneq G_i$ , and  
 (iii) for some index  $i$  we have  $\varphi_i(\tilde{H}) = G_i$ .

The first of these cases is somewhat trivial:

**Lemma 7.5.** *Let  $M = G/H$  be coset space irreducible with some  $\varphi_i(\tilde{H}) = \{1\}$  then  $G = G_i = M$  and every  $\xi \in \mathfrak{g}_i$  defines a constant length Killing vector field on  $M$ .*

*Proof.* The hypothesis says that  $G_i = G_i/\varphi_i(\tilde{H})$  is a factor in a coset space reduction of  $G/H$ , and coset space irreducibility says that  $G_i = G_i/\varphi_i(\tilde{H})$  must be all of  $G/H$ . As given,  $G_i$  acts isometrically on itself by left translations, and by normality the right translations also are isometries. If  $\xi \in \mathfrak{g}$  comes from the left action of  $G$  on itself, it is centralized by the right action, which is transitive, so the corresponding vector field  $\xi^M$  has constant length. q.e.d.

Now we may (and do) assume that each  $\dim \varphi_i(\tilde{H}) > 0$ . The second case is

**Theorem 7.6.** *Assume that  $M = G/H$  is a coset space irreducible normal homogeneous space with  $G$  semisimple and  $H$  reductive in  $G$ . In the notation of (7.2) suppose that  $\varphi_i(\tilde{H}) \subsetneq G_i$  and  $\dim \varphi_i(\tilde{H}) > 0$  for each  $i > 0$ . Let  $\xi = \xi_0 + \dots + \xi_r \in \mathfrak{g}$ , elliptic and decomposed as in (7.3). Consider the following conditions:*

- (1)  $\xi$  defines a constant length Killing vector field  $\xi^M$  on  $M = G/H$ .
- (2) For each  $i$ ,  $\xi_i$  defines a constant length Killing vector field  $\xi_i^{M_i}$  on  $M_i = G_i/\varphi_i(\tilde{H})$ .
- (3) For each  $i$ ,  $\xi_i$  defines a constant length Killing vector field  $\xi_i^M$  on  $M$ .
- (4) The  $\text{Ad}(G)$ -centralizer of  $\xi$  has an open orbit on  $M$ .

Then

- (a) (1) implies (2) but (2) does not imply (1);
- (b) (2) and (3) are equivalent; and
- (c) (1) and (4) are equivalent.

*Proof.* As in the first paragraph of the proof of Theorem 5.2 we may assume that  $\xi$  is contained in  $\mathfrak{k} = \mathfrak{g} \cap \mathfrak{g}_u$ .

We write  $L$  and  $L_u$  for the respective centralizers of  $\xi$  in  $G$  and  $G_u$  and  $Z$  for the complex flag manifold  $Z = G_u/L_u = G_{\mathbb{C}}/Q$ . We also write  $L_i$  and  $L_{i,u}$  for the respective centralizers of  $\xi_i$  in  $G_i$  and  $G_{i,u}$ , so  $Z$  is the product of the  $Z_i = G_{i,u}/L_i$ , where of course  $L_0 = G_0$  and  $L_i = G_i$  for  $i > s$ , so those  $Z_i$  are single points.

We first prove that (1) implies (2) and (4). As a subgroup of  $G$ ,  $H_u$  acts holomorphically and isometrically on  $Z$ . Here  $Z$  carries the  $G_u$ -invariant Kaehler metric whose associated symplectic form is the Kostant–Souriau form. The action is Hamiltonian. We are assuming (1), in other words that  $\xi^M$  has constant length on  $M = G/H$ , so Proposition 3.3 says that  $\xi^{M_u}$  has constant length on  $M_u = G_u/H_u$ . In other words the momentum map for the action of  $H_u$  on  $Z$  has constant square norm. Thus [3, Theorem 1]  $Z = Z' \times Z''$ , holomorphically and isometrically, where  $Z'$  and  $Z''$  are complex flag manifolds such that  $H_u$  is transitive on  $Z'$  and  $H_u$  acts trivially on  $Z''$ .

The group  $H_u$  acts nontrivially on  $Z_i$  for  $1 \leq i \leq s$ . For if the action were trivial then  $\varphi_i(\widetilde{H}_u)$  would be normal in  $G_{i,u}$ , while it is  $\neq \{1\}$ , forcing  $\varphi_i(\widetilde{H}_u) = G_{i,u}$ . This possibility was excluded by hypothesis. Thus  $Z' = Z_1 \times \dots \times Z_s$ . Now set  $G' = G_1 \times \dots \times G_s$ ,  $L' = L_1 \times \dots \times L_s$ ,  $\varphi' = \varphi_1 \times \dots \times \varphi_s$  and  $H' = \varphi'(\widetilde{H})$ . Then  $H'_u$  is transitive on  $Z'$ . It follows from [10, Proposition 2.1] that  $H'_{i,u} := \varphi_i(\widetilde{H}_u)$  is transitive on  $Z_i$  for  $1 \leq i \leq s$ . Equivalently  $G'_u = H'_u L'_u$ , which is the same (take inverses) as  $G'_u = L'_u H'_u$ , so  $L'_u$  is transitive on  $Z'$ . In particular,  $G_{u,i} = L'_{u,i} H'_{u,i}$ . Thus  $\xi_i^{M_{i,u}}$  has constant length on  $M_{i,u}$  for  $1 \leq i \leq s$ . Thus (1) implies (2) and (4), and (4) implies (1) by Lemma 5.1.

It is obvious that (3) implies (2). Given (2), the centralizer  $L_i$  of  $\xi_i$  in  $G_i$  is transitive on  $M_i$ , so the centralizer of  $\xi_i$  in  $G$  is transitive on  $M$ , and (3) follows.

It remains only to show that (2) does not imply (1). Consider the case  $G = SO(2n) \times SO(2n)$  with  $H = SO(2n - 1)$  embedded diagonally and  $\xi = \text{diag}\{J, \dots, J\}$  where  $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Then  $L = U(n) \times U(n)$  and  $L \cap H = U(n - 1)$  so  $\dim G/H = 2n^2 + n - 1 > 2n^2 = \dim L$ , so  $L$  cannot have an open orbit on  $G/H$  when  $n > 1$ . On the other hand the projections of  $\xi$  to the ideals of  $\mathfrak{g}$  define constant length Killing vector fields  $\xi_i^{M_i}$  on the  $M_i = G_i/\varphi_i(\widetilde{H})$  because  $L_i = U(n)$  is transitive on  $M_i = G_i/\varphi_i(\widetilde{H}) = SO(2n)/SO(2n - 1) = S^{2n-1}$ . Thus (2) does not imply (4). But (1) and (4) are equivalent, so (2) does not imply (1). q.e.d.

The third case includes the pseudo-Riemannian group manifolds  $(H \times H)/(\text{diag}\{H\})$  for real simple Lie groups  $H$ , but the following example shows that this case is more of a combinatorial problem than a geometric or Lie theoretic problem:

**Example 7.7.** Let  $G'$  and  $G''$  be reductive Lie groups. Let  $\widetilde{H}$  be reductive with homomorphisms  $\varphi' : \widetilde{H} \rightarrow G'$  and  $\varphi'' : \widetilde{H} \rightarrow G''$  such that  $h \mapsto (\varphi'(h), \varphi''(h))$  is an isomorphism of  $\widetilde{H}$  onto a reductive subgroup  $H$  of  $G := G' \times G''$ . Let  $M = G/H$  be the corresponding homogeneous space with any  $G$ -invariant pseudo-Riemannian metric. Suppose that

$\xi \in \mathfrak{g}'$  and that  $\varphi'(\tilde{H}) = G'$ . Then  $G''$  centralizes  $\xi$  and  $G = HG''$ , so the centralizer of  $\xi$  in  $G$  is transitive on  $M$ . Thus  $\xi^M$  has constant length on  $M$ . The most familiar case of this is a compact group manifold  $(H \times H)/(\text{diag}\{H\})$ .

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DEPARTMENT OF MATHEMATICS  
UNIVERSITY OF CALIFORNIA  
BERKELEY, CA 94720-3840, USA

*E-mail address:* jawolf@math.berkeley.edu

DIPARTIMENTO DI MATEMATICA E INFORMATICA “U.DINI”  
VIALE MORGAGNI 67/A  
I-50100-FIRENZE, ITALY

*E-mail address:* podesta@math.unifi.it

COLLEGE OF MATHEMATICS  
TIANJIN NORMAL UNIVERSITY  
TIANJIN 300387, P.R. CHINA

*E-mail address:* mgmgmgxu@163.com