

NON-PROPERLY EMBEDDED H -PLANES IN \mathbb{H}^3 BARIS COSKUNUZER, WILLIAM H. MEEKS III
& GIUSEPPE TINAGLIA**Abstract**

For any $H \in [0, 1)$, we construct complete, non-proper, stable, simply-connected surfaces with constant mean curvature H embedded in hyperbolic three-space.

1. Introduction

In their ground breaking work [3], Colding and Minicozzi proved that complete minimal surfaces embedded in \mathbb{R}^3 with finite topology are proper. Based on the techniques in [3], Meeks and Rosenberg [8] then proved that complete minimal surfaces embedded in \mathbb{R}^3 are proper, if they have positive injectivity radius; since complete, immersed finite topology minimal surfaces in \mathbb{R}^3 have positive injectivity radius, their result generalized Colding and Minicozzi's work.

Recently Meeks and Tinaglia [9] proved that complete constant mean curvature surfaces embedded in \mathbb{R}^3 are proper if they have finite topology or have positive injectivity radius. With the convention that the mean curvature function of an oriented surface is the pointwise average of its principal curvatures, these results of Meeks and Tinaglia in \mathbb{R}^3 should generalize to show that a complete embedded surface Σ of constant mean curvature $H \in [1, \infty)$ in a complete hyperbolic three-manifold is proper if Σ has finite topology or it is connected and has positive injectivity radius; this is work in progress in [10].

In contrast to the above results, in this paper we prove the following existence theorem for non-proper, complete simply-connected surfaces embedded in \mathbb{H}^3 with constant mean curvature $H \in [0, 1)$. See the

The first author is partially supported by TUBITAK 2219 Grant, Fulbright Grant, BAGEP award of the Science Academy and by a Royal Society Newton Mobility Grant.

The second author was supported in part by NSF Grant DMS-1309236. Any opinions, findings, and conclusions or recommendations expressed in this publication are those of the authors and do not necessarily reflect the views of the NSF.

The third author was partially supported by EPSRC grant no. EP/M024512/1 and by a Royal Society Newton mobility Grant.

Received April 21, 2015.

Appendix where a description of the spherical catenoids appearing in the next theorem is given.

Theorem 1.1. *For any $H \in [0, 1)$ there exists a complete simply-connected surface Σ_H embedded in \mathbb{H}^3 with constant mean curvature H satisfying the following properties:*

- 1) *The closure of Σ_H is a lamination with three leaves, Σ_H , C_1 and C_2 , where C_1 and C_2 are stable spherical catenoids of constant mean curvature H in \mathbb{H}^3 with the same axis of revolution L . In particular, Σ_H is not properly embedded in \mathbb{H}^3 .*
- 2) *The asymptotic boundary of Σ_H is a pair of embedded curves in $\partial_\infty \mathbb{H}^3$ which spiral into the union of the round circles which are the asymptotic boundaries of C_1 and C_2 .*
- 3) *Let K_L denote the Killing field generated by rotations around L . Every integral curve of K_L that lies in the region between C_1 and C_2 intersects Σ_H transversely in a single point. In particular, the closed region between C_1 and C_2 is foliated by surfaces of constant mean curvature H , where the leaves are C_1 and C_2 and the rotated images $\Sigma_H(\theta)$ of Σ around L by angle $\theta \in [0, 2\pi)$.*

Previously Coskunuzer [5] constructed an example of a non-proper, stable, complete minimal plane in \mathbb{H}^3 that can be roughly described as a collection of “parallel” geodesic planes connected via “bridges at infinity”. However, his techniques do not generalize to construct non-proper, non-zero constant mean curvature planes in \mathbb{H}^3 .

There is a general conjecture related to Theorem 1.1 and the previously stated positive properness results. This conjecture states that if X is a simply-connected, homogeneous three-manifold with Cheeger constant $\text{Ch}(X)$, then for any $H \geq \frac{1}{2}\text{Ch}(X)$, every complete, connected H -surface embedded in X with positive injectivity radius or finite topology is proper. The Cheeger constant of \mathbb{H}^3 is 2.

In the case of the Riemannian product $X = \mathbb{H}^2 \times \mathbb{R}$, then $\text{Ch}(X) = 1$, and the validity of this conjecture would imply that every complete, connected H -surface embedded in X with positive injectivity radius or finite topology is properly embedded when $H \geq \frac{1}{2}$. In view of this conjecture and Theorem 1.1, it is natural to ask the question:

Given $H \in [0, \frac{1}{2})$, does there exist a complete, non-properly embedded H -surface of finite topology in $\mathbb{H}^2 \times \mathbb{R}$?

When $H = 0$, Rodríguez and Tinaglia [12] have constructed non-proper, complete minimal planes embedded in $\mathbb{H}^2 \times \mathbb{R}$. However, their construction does not generalize to produce complete, non-proper planes embedded in $\mathbb{H}^2 \times \mathbb{R}$ with non-zero constant mean curvature.

2. An outline of the construction

In this section, we outline the construction of the examples described in Theorem 1.1. Throughout the paper, we refer to an oriented surface embedded in \mathbb{H}^3 with constant mean curvature H as an H -surface, and call it an H -disk if it is simply-connected. After possibly reversing the orientation of an H -surface, we will always assume $H \geq 0$. Given a domain $\Omega \subset \mathbb{H}^3$ with smooth boundary $\partial\Omega$, we say that $\partial\Omega$ is H_0 -convex, $H_0 \geq 0$, if after orienting $\partial\Omega$ so that its unit normal is pointing into Ω , then $\inf_{\partial\Omega} H_{\partial\Omega} \geq H_0$, where $H_{\partial\Omega}$ denotes the mean curvature function of $\partial\Omega$.

We will work in \mathbb{H}^3 using the Poincaré ball model, that is we consider \mathbb{H}^3 as the unit ball in \mathbb{R}^3 and its ideal boundary $\partial_\infty\mathbb{H}^3$ at infinity corresponds to the boundary of the ball. Fix H in $[0, 1)$. Given $\lambda_1 > 0$ sufficiently large and $\lambda_2 > \lambda_1$, for any $\lambda \in [\lambda_1, \lambda_2]$, there exists a unique spherical H -catenoid \mathcal{C}^λ whose distance to its rotation axis, which we assume is the z -axis, is λ in the hyperbolic metric, is invariant under reflection in the (x, y) -plane and the mean curvature vectors of \mathcal{C}^λ point toward the z -axis; see Figure 1 and the discussion in the Appendix for further details. Throughout this paper if n denotes the unit normal, the mean curvature vector is the vector $\vec{H} = Hn$.

Let $W \subset \mathbb{H}^3$ denote the closed region between \mathcal{C}^{λ_1} and \mathcal{C}^{λ_2} . By Proposition 5.3 in the Appendix, the collection $\mathcal{F} = \{\mathcal{C}^\lambda \mid \lambda \in [\lambda_1, \lambda_2]\}$ is a foliation of W . We will identify W topologically with $[\lambda_1, \lambda_2] \times \mathbb{S}^1 \times \mathbb{R}$ and its boundary consists of the two H -surfaces \mathcal{C}^{λ_1} and \mathcal{C}^{λ_2} ; in the next section this identification is made explicit.

For λ_1 sufficiently large and for a fixed $\lambda_2 > \lambda_1$ that is sufficiently close to λ_1 , we will construct the surface $\Sigma_H \subset W$ described in Theorem 1.1 by creating a sequence of compact H -disks in W whose interiors converge to Σ_H on compact subsets of $\text{Int}(W)$. To do this, we will consider the universal cover $\widetilde{W} = [\lambda_1, \lambda_2] \times \mathbb{R} \times \mathbb{R}$ of W , which is an infinite slab with boundary H -planes $\widetilde{\mathcal{C}}^{\lambda_1}$ and $\widetilde{\mathcal{C}}^{\lambda_2}$. By construction, the mean curvature vectors of the surface $\widetilde{\mathcal{C}}^{\lambda_2} \subset \partial\widetilde{W}$ point into \widetilde{W} , while the mean curvature vectors of $\widetilde{\mathcal{C}}^{\lambda_1}$ point out of \widetilde{W} ; see Figure 1.

To create the compact sequence of H -disks we first exhaust \widetilde{W} by a certain increasing sequence of compact domains $\Omega_n \subset \Omega_{n+1}$ such that $\partial\Omega_n \setminus (\partial\Omega_n \cap \widetilde{\mathcal{C}}^{\lambda_1})$ is H -convex. Next we choose an appropriate sequence of simple closed curves Γ_n on $\partial\Omega_n$ so that each Γ_n is the boundary of an H -disk Σ_n embedded in Ω_n , with each such disk being a graph over its natural projection to $[\lambda_1, \lambda_2] \times \{0\} \times \mathbb{R}$; see Figure 4. A compactness argument then gives that a subsequence of the projected interiors of the surfaces $\Pi(\Sigma_n) \subset W$ converges to a complete H -disk Σ_H embedded in $\text{Int}(W)$, which we prove is a entire graph over $(\lambda_1, \lambda_2) \times \{0\} \times \mathbb{R}$. Finally, we will show that Σ_H satisfies the other conclusions of Theorem 1.1.

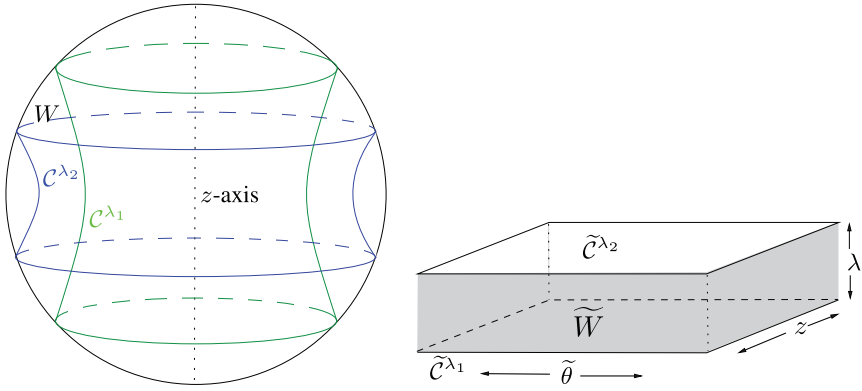


Figure 1. The induced coordinates $(\lambda, \tilde{\theta}, z)$ in \widetilde{W} .

3. The examples

3.1. The construction of the compact exhaustion of \widetilde{W} . We begin by explaining in detail the construction of the domains Ω_n briefly described in the previous section. For the remainder of the paper we fix a particular $H \in [0, 1)$.

Let $c_H > 0$ be the constant described in Lemma 5.2 and Proposition 5.3 in the Appendix. As described in the previous section, given $\lambda_2 > \lambda_1 \geq c_H$, W denotes the closed region between \mathcal{C}^{λ_1} and \mathcal{C}^{λ_2} . The number λ_2 will be fixed later. The region W is foliated by the collection $\mathcal{F} = \{\mathcal{C}^\lambda \mid \lambda \in [\lambda_1, \lambda_2]\}$ of spherical H -catenoids. By using the foliation \mathcal{F} of W , we introduce *cylindrical coordinates* (λ, θ, z) on W as follows. The coordinate λ indicates that the point is in \mathcal{C}^λ . The coordinate $\theta \in [0, 2\pi)$ parameterizes the core circles of the catenoids in \mathcal{F} and corresponds to θ in cylindrical coordinates in \mathbb{R}^3 . Finally, the z -coordinate of a point (λ, θ, z) represents the signed intrinsic distance on \mathcal{C}^λ to the core circle of the catenoid \mathcal{C}^λ which lies in the (x, y) -plane, where the sign is taken to be positive if the z -coordinate of the point in the ball model is positive, and otherwise it is negative. Note that this choice of z -coordinate is different from the one of the ball model. It is clear that for points of W , $\lambda \in [\lambda_1, \lambda_2]$, $\theta \in [0, 2\pi)$ and $z \in (-\infty, \infty)$ in these coordinates; see Figure 1.

Let \widetilde{W} be the universal cover of W , which is topologically an infinite slab $[\lambda_1, \lambda_2] \times \mathbb{R} \times \mathbb{R}$ with boundary H -planes $\widetilde{\mathcal{C}}^{\lambda_1}$ and $\widetilde{\mathcal{C}}^{\lambda_2}$. We will use the *induced* coordinates $(\lambda, \tilde{\theta}, z)$ in \widetilde{W} . Namely, if $\Pi : \widetilde{W} \rightarrow W$ is the covering map, then

$$\Pi(\lambda, \tilde{\theta}, z) = (\lambda, \tilde{\theta} \bmod 2\pi, z),$$

i.e., Π keeps fixed the λ and z coordinates, and sends $\tilde{\theta} \in (-\infty, \infty)$ to the point $(\tilde{\theta} \bmod 2\pi)$ corresponding to a point in the core circle of the

catenoid; see Figure 1. \widetilde{W} is endowed with the metric induced by W and in these coordinates, for any $\theta_0 \in (-\infty, \infty)$, the map

$$T_{\theta_0} : \widetilde{W} \rightarrow \widetilde{W}, \quad T_{\theta_0}(\lambda, \tilde{\theta}, z) = (\lambda, \tilde{\theta} + \theta_0, z)$$

is an isometry of \widetilde{W} as it is induced by the isometry of \mathbb{H}^3 which is a rotation by angle θ_0 about the z -axis. In particular, $T_{2\pi n}$ is a covering transformation for any n . We let ∂_θ denote the Killing field in W generated by the rotations about the z -axis and denote by $\partial_{\tilde{\theta}}$ the related Killing field in \widetilde{W} generated by the one-parameter group of isometries $\{T_\theta\}_{\theta \in \mathbb{R}}$.

Since when $H > 0$ the mean curvature vectors of the boundary of W point towards the rotation axis, the mean curvature vectors point into \widetilde{W} on $\widetilde{\mathcal{C}}^{\lambda_2}$ and out of \widetilde{W} on $\widetilde{\mathcal{C}}^{\lambda_1}$; thus, when considered to be a part of $\partial\widetilde{W}$, $\widetilde{\mathcal{C}}^{\lambda_2}$ is H -convex, while $\widetilde{\mathcal{C}}^{\lambda_1}$ is not.

As explained in Section 2, our next goal is to exhaust \widetilde{W} by an increasing sequence of compact domains $\Omega_n \subset \Omega_{n+1}$ such that $\partial\Omega_n \setminus (\partial\Omega_n \cap \mathcal{C}^{\lambda_1})$ is H -convex. Let $R_n \nearrow \infty$ as $n \nearrow \infty$ and let B_{R_n} be the closed geodesic ball in \mathbb{H}^3 with center the origin. Let $W_n = W \cap B_{R_n} \subset W$ and let \widetilde{W}_n be the universal cover of W_n ; see Figures 1 and 2. Assume R_1 is chosen sufficiently large so that every \widetilde{W}_n can be viewed as an infinite tube in \widetilde{W} which is bounded in the z -direction, but unbounded in $\tilde{\theta}$ -direction. Then there exists a sequence of bounded continuous positive functions

$$Z_n : [\lambda_1, \lambda_2] \rightarrow (0, \infty), \quad Z_{n+1} > Z_n,$$

such that

$$\widetilde{W}_n = \{(\lambda, \tilde{\theta}, z) \in \widetilde{W} \text{ with } z \in [-Z_n(\lambda), Z_n(\lambda)]\}.$$

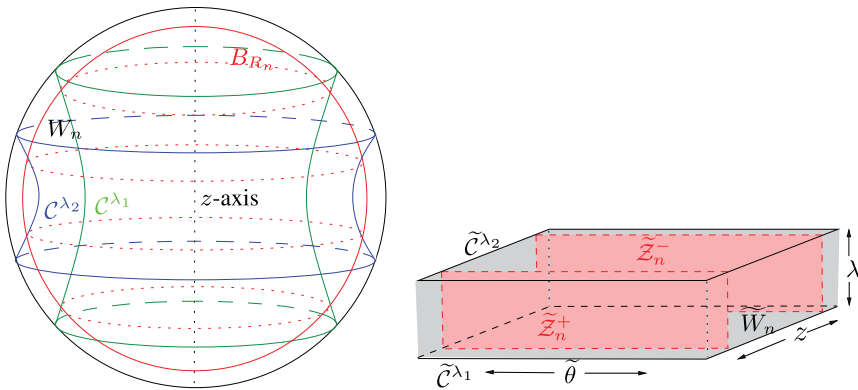


Figure 2. $W_n = W \cap B_{R_n}$ and \widetilde{W}_n denotes its universal cover. Note that $\partial\widetilde{W}_n \subset \widetilde{\mathcal{C}}^{\lambda_1} \cup \widetilde{\mathcal{C}}^{\lambda_2} \cup \mathcal{Z}_n^+ \cup \mathcal{Z}_n^-$.

Note that Z_n does not depend on $\tilde{\theta}$ because B_{R_n} is rotationally symmetric. Let Z_n^\pm be the two annular components of $\partial B_{R_n} \cap W$. The preimages of the surfaces Z_n^\pm in the universal cover \tilde{W} are

$$\tilde{Z}_n^\pm := \{(\lambda, \tilde{\theta}, \pm Z_n(\lambda)) \mid \lambda \in [\lambda_1, \lambda_2], \tilde{\theta} \in (-\infty, \infty)\}.$$

Since the mean curvature of ∂B_{R_n} with inner pointing unit normal is strictly greater than one, $\tilde{Z}_n^+ \cup \tilde{Z}_n^-$ is H -convex as part of the boundary of \tilde{W}_n . Note that these spheres would not be good barriers if $H \geq 1$ because their mean curvatures are converging to 1 as the radius goes to infinity. Therefore, if $H \geq 1$, this construction would not work.

The final and most difficult step in defining the piecewise-smooth compact domains Ω_n is to bound \tilde{W}_n in the $\tilde{\theta}$ -direction. In order to do this, we will again use spherical H -catenoids. Let $P^+ = (0, -\varepsilon, p_3)$ be a point on $\partial_\infty \mathbb{H}^3$ for some small $\varepsilon > 0$ and let $P^- = (0, -\varepsilon, -p_3) \in \partial_\infty \mathbb{H}^3$ be its symmetric point in $\partial_\infty \mathbb{H}^3$ with respect to the (x, y) -plane. Let γ be the geodesic in \mathbb{H}^3 connecting P^+ and P^- and let ϕ be the hyperbolic translation along the y -axis in the ball model for \mathbb{H}^3 and that maps the z -axis to γ .

For $\varepsilon > 0$ chosen sufficiently small, the asymptotic boundary circles of $\hat{\mathcal{C}}^{\lambda_1} = \phi(\mathcal{C}^{\lambda_1})$ intersect transversely the asymptotic boundary circles of \mathcal{C}^{λ_1} , and $\hat{\mathcal{C}}^{\lambda_1}$ intersects \mathcal{C}^{λ_1} transversely with $\hat{\mathcal{C}}^{\lambda_1} \cap \mathcal{C}^{\lambda_1} = l_1^+ \cup l_1^-$, where l_1^\pm is a pair of infinite ‘‘vertical’’ arcs in the intersecting catenoids with $\partial_\infty l_1^\pm \subset \partial_\infty \mathcal{C}^{\lambda_1}$. Similarly, by choosing $\lambda_2 - \lambda_1$ sufficiently small, we can make sure that $\hat{\mathcal{C}}^{\lambda_1}$ intersects \mathcal{C}^{λ_2} in a pair of infinite ‘‘vertical’’ arcs l_2^\pm , i.e., $\hat{\mathcal{C}}^{\lambda_1} \cap \mathcal{C}^{\lambda_2} = l_2^+ \cup l_2^-$. See the proof of Proposition 5.5 in the Appendix for the details on the existence of l_i^\pm and for some details on the next argument. Now, the intersection $\hat{\mathcal{C}}^{\lambda_1} \cap W$ consists of two thin infinite strips \mathcal{T}_+ and \mathcal{T}_- , where $\partial \mathcal{T}_\pm = l_1^\pm \cup l_2^\pm$ and \mathcal{T}_\pm looks like $l_i^\pm \times (\lambda_1, \lambda_2)$. The strips \mathcal{T}_+ and \mathcal{T}_- separate W into two components, say W^+ and W^- , and $\mathcal{T}_+ \cup \mathcal{T}_-$ is H -convex as boundary of one of these two components, say W^+ .

Notice that the strips \mathcal{T}_+ and \mathcal{T}_- have infinitely many lifts $\{\tilde{\mathcal{T}}_+^n\}_{n \in \mathbb{Z}}$ and $\{\tilde{\mathcal{T}}_-^n\}_{n \in \mathbb{Z}}$ in \tilde{W} . In particular, if we fix a lift $\tilde{\mathcal{T}}_+^0$, then $\tilde{\mathcal{T}}_+^n = T_{2\pi n}(\tilde{\mathcal{T}}_+^0)$ for any $n \in \mathbb{Z}$. Similarly, the same is true for $\tilde{\mathcal{T}}_-^n$. We fix the lifts $\tilde{\mathcal{T}}_+^0$ and $\tilde{\mathcal{T}}_-^0$ in \tilde{W} so that the mean curvature vectors are pointing into the region that they bound, and so that there is no other lift $\tilde{\mathcal{T}}_\pm^n$ between them. Then there exists a function $G: [\lambda_1, \lambda_2] \times (-\infty, \infty) \rightarrow (0, \pi)$ such that

$$\tilde{\mathcal{T}}_+^0 = \{(\lambda, G(\lambda, z), z)\} \text{ and } \tilde{\mathcal{T}}_-^0 = \{(\lambda, -G(\lambda, z), z)\}.$$

Moreover, let $G_n(\lambda, z) = G(\lambda, z) + 2\pi n$. Then, $\tilde{\mathcal{T}}_+^n = \{(\lambda, G_n(\lambda, z), z)\}$ and $\tilde{\mathcal{T}}_-^n = \{(\lambda, -G_n(\lambda, z), z)\}$.

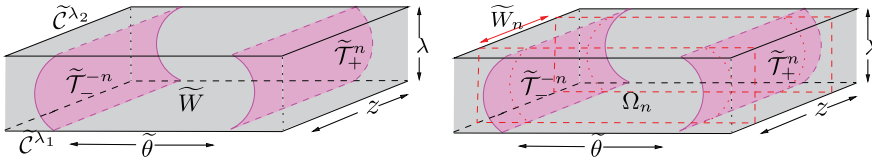


Figure 3. Ω_n is the region in W_n between \tilde{T}_+^n and \tilde{T}_-^n .

Finally, let Ω_n be the region in \tilde{W}_n between \tilde{T}_+^n and \tilde{T}_-^n . In particular,

$$\Omega_n = \{(\lambda, \tilde{\theta}, z) \in \tilde{W} \mid \tilde{\theta} \in [-G_n(\lambda, z), G_n(\lambda, z)] \text{ and } z \in [-Z_n(\lambda), Z_n(\lambda)]\}.$$

Hence, we have obtained an exhaustion of \tilde{W} by compact regions with the property that $\partial\Omega_n \setminus (\partial\Omega_n \cap \tilde{C}^{\lambda_1})$ is H -convex; see Figure 3.

3.2. The sequence Σ_n of graphical H -disks. Our aim in this section is to construct a sequence of compact H -disks $\Sigma_n \subset \Omega_n$ with $\partial\Sigma_n \subset \partial\Omega_n$, which are $\tilde{\theta}$ -graphs over their projections to $[\lambda_1, \lambda_2] \times \{0\} \times \mathbb{R}$.

Let

$$\partial_*\Omega_n := \partial\Omega_n \setminus (\partial\Omega_n \cap \{\tilde{C}^{\lambda_1} \cup \tilde{C}^{\lambda_2}\}),$$

and let γ be a piecewise smooth, embedded, simple closed curve in $\partial_*\Omega_n$ that does not bound a disk in $\partial_*\Omega_n$. Recall that $\partial_*\Omega_n$ is piecewise smooth and H -convex as part of the boundary of Ω_n , since the dihedral angles are less than π at the corners.

Consider the following variational problem. Let M be a compact surface embedded in Ω_n with $\partial M = \gamma \subset \partial_*\Omega_n$. Since Ω_n is simply-connected, M separates Ω_n into two regions, i.e., $\Omega_n - M = \Omega_M^+ \cup \Omega_M^-$ where Ω_M^+ denotes the region that contains $\tilde{C}^{\lambda_2} \cap \Omega_n$. Let $A(M)$ denote the area of M and let $V(M)$ denote the volume of the region Ω_M^+ . Then, let

$$(1) \quad I(M) = A(M) + 2HV(M).$$

By working with integral currents, it is known that for any simple closed essential curve γ_n in $\partial_*\Omega_n$ there exists a smooth (except at the 4 corners of γ_n), compact, embedded H -surface $\Sigma_n \subset W_n$ with $\text{Int}(\Sigma_n) \subset \text{Int}(W_n)$ and $\partial\Sigma_n = \gamma_n$. In fact, Σ_n can be chosen to be, and we will assume it is, a minimizer for this variational problem, i.e., $I(\Sigma_n) \leq I(M)$ for any $M \subset \Omega_n$ with $\partial M = \gamma_n$; see for instance [14, Theorem 2.1] [1, Theorem 1]. In particular, the fact that $\text{Int}(\Sigma_n) \subset \text{Int}(W_n)$ is proven in Lemma 3 of [7]. Moreover, Σ_n separates Ω_n into two regions and the mean curvature vectors of Σ_n points “down,” namely into $\Omega_{\Sigma_n}^-$.

If λ_{Σ_n} denotes the restriction of the λ -coordinate function to Σ_n , then the following holds. If P_+ and P_- are interior points of Σ_n where the function λ_{Σ_n} obtains, respectively, its maximum and minimum value,

then the mean curvature vector at P_+ and P_- points “down”, toward $\tilde{\mathcal{C}}^{\lambda_1} \cap \Omega_n$.

Lemma 3.1. *Let P^+ (respectively, P^-) be a point in Σ_n where the function λ_{Σ_n} attains its maximum (respectively, minimum) value. Then, P^+ and P^- cannot be in the interior of Σ_n unless $\Sigma_n = \tilde{\mathcal{C}}^\lambda \cap \Omega_n$ for a certain $\lambda \in (\lambda_1, \lambda_2)$.*

Proof. By applying the maximum principle for constant mean curvature surfaces, this lemma follows from the previous observation and the fact that the collection

$$\tilde{\mathcal{F}}_n = \{\tilde{\mathcal{C}}^\lambda \cap \Omega_n \mid \lambda \in [\lambda_1, \lambda_2]\}$$

foliates Ω_n .

q.e.d.

Note that with a suitable choice of γ_n , we will show later that Σ_n is a graphical disk, i.e., Lemma 4.1.

3.3. Choosing the right boundary curve Γ_n . Let \widetilde{W} be the hyperbolic compactification of \widetilde{W} , which is a covering of the related compactification of W when viewed to be a subset of $\overline{\mathbb{H}^3}$.

For each $n \in \mathbb{N}$ large, we will construct a simple closed curve Γ_n in $\partial_* \Omega_n$ such that the minimizer surface $\Sigma_n \subset \Omega_n$ for the functional I in (1) with $\partial \Sigma_n = \Gamma_n$ is a $\tilde{\theta}$ -graph over its projection to $[\lambda_1, \lambda_2] \times \{0\} \times \mathbb{R}$.

Let Γ_n be the union of four arcs in $\partial_* \Omega_n$,

$$\Gamma_n := \alpha_+^n \cup \beta_+^n \cup \alpha_-^n \cup \beta_-^n,$$

where $\alpha_+^n \subset \tilde{\mathcal{C}}^{\lambda_+^n} \cap \tilde{\mathcal{T}}_+^n$ with $\lambda_+^n \nearrow \lambda_2$ and with its endpoints on $\tilde{\mathcal{Z}}_n^\pm$, and $\alpha_-^n \subset \tilde{\mathcal{C}}^{\lambda_-^n} \cap \tilde{\mathcal{T}}_-^n$ with $\lambda_-^n \searrow \lambda_1$ and with its endpoints on $\tilde{\mathcal{Z}}_n^\pm$. The curve $\beta_+^n \subset \tilde{\mathcal{Z}}_+^n$ connects the endpoints of α_+^n and α_-^n that are contained in $\tilde{\mathcal{Z}}_+^n$ while the curve $\beta_-^n \subset \tilde{\mathcal{Z}}_-^n$ connects the endpoints of α_+^n and α_-^n that are contained in $\tilde{\mathcal{Z}}_-^n$; see Figure 4.

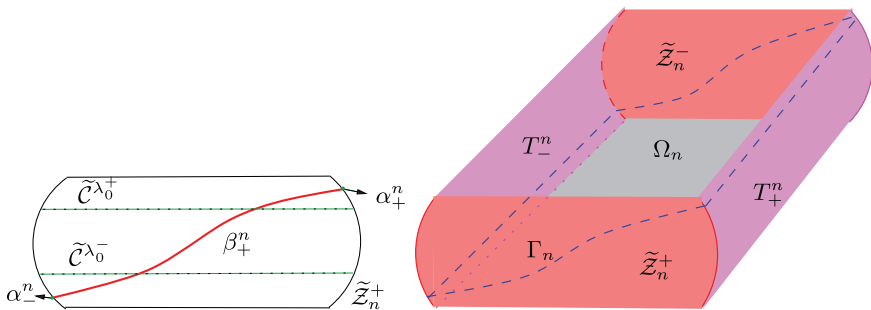


Figure 4. In the left, β_+^n is pictured in $\tilde{\mathcal{Z}}_+^n$. In the right, Γ_n curve is described in $\partial \Omega_n$.

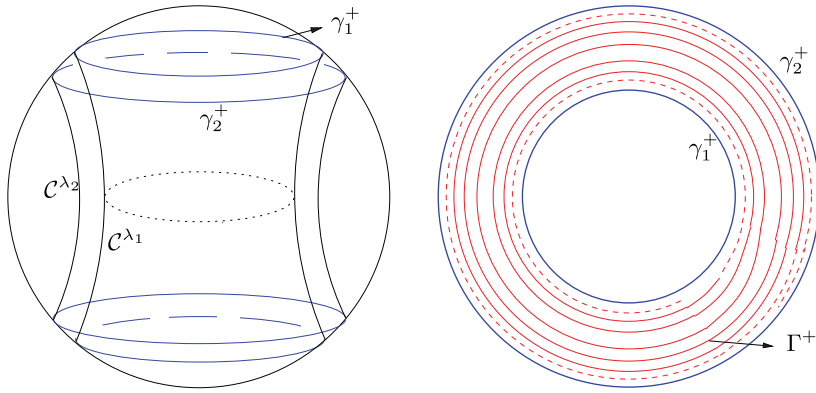


Figure 5. Let $\partial_\infty \mathcal{C}^{\lambda_i} = \gamma_i^+ \cup \gamma_i^-$, $i = 1, 2$, and $\Gamma = \Gamma^+ \cup \Gamma^-$. Then, Γ^+ is an infinite line in $\partial_\infty \mathbb{H}^3$ spiraling into γ_1^+ in one end, and spiraling into γ_2^+ in the other end.

Moreover, we chose β_+^n and β_-^n so that they are smooth graphs in the $\tilde{\theta}$ -variable with positive slope; see Figure 4. We will assume that the curves β_+^n and β_-^n converge, respectively, to a pair of curves β_+, β_- in $\partial_\infty \tilde{W}$. With these choices, if we denote by $\hat{\beta}_\pm^n$ the projections of β_\pm^n in $\mathbb{H}^3 \cap W_n$, the curves $\hat{\beta}_\pm^n$ are embedded curves contained in \mathcal{Z}_n^\pm . Finally, we require that $\hat{\beta}_\pm^n$ to converge to a pair of infinite smooth spiralling curves $\hat{\beta}_\pm$ in the pair of compact annuli A^+, A^- in $\partial_\infty \mathbb{H}^3 \cap \bar{W}$, each of which is a graph of some smooth function $\pm\lambda(\theta)$ with positive slope and the graphs converge to the asymptotic boundary curves of $\mathcal{C}^{\lambda_1}, \mathcal{C}^{\lambda_2}$. Here \bar{W} denotes the union of W with its limit points in $\partial_\infty \mathbb{H}^3$. We will also assume that $\hat{\beta}_\pm$ have positive bounded geodesic curvature and the reflection in the (x, y) -plane interchanges them; see Figure 5.

We will next make some further restrictions on the choices of β_+^n and β_-^n . For each $p \in \hat{\beta}_+$, let $C^1(p)$ and $C^2(p)$ be the two circles on opposite sides of $\hat{\beta}_+$ at p in $\partial_\infty W \cap \{z > 0\} \subset \partial_\infty \mathbb{H}^3$. Furthermore, they are maximal radius such that $C^1(p) \cap C^2(p) = p$ and the pairwise disjoint open disks that they bound in $\partial_\infty \mathbb{H}^3$ are disjoint from $\hat{\beta}_+$.

Definition 3.2. $\Delta^1(p)$ and $\Delta^2(p)$ the rotationally symmetric open H -disks in \mathbb{H}^3 with boundaries, respectively, $C^1(p)$ and $C^2(p)$, chosen so that the mean curvature vector of $\Delta^i(p)$ points into the component of $W \setminus \cup_{i=1}^2 \Delta^i(p)$ that contains $\mathcal{C}^{\lambda_1} \cup \mathcal{C}^{\lambda_2}$, $i = 1, 2$. Note that the mean curvature vectors of $\Delta^1(p)$ and $\Delta^2(p)$ also point into the component of $\mathbb{H}^3 - [\Delta^1(p) \cup \Delta^2(p)]$ that contains the origin.

Note that the disks $\Delta^1(p)$ and $\Delta^2(p)$ are disjoint because they are separated by the totally geodesic disk with boundary $C^1(p)$. By the

arguments in the proof of Lemma 3.4 below, these disks must be disjoint from $\mathcal{C}^{\lambda_1} \cup \mathcal{C}^{\lambda_2}$. The disks $\Delta^1(p)$ and $\Delta^2(p)$ can be defined in an analogous way when $p \in \widehat{\beta}_-$.

Definition 3.3. $\Gamma = \widehat{\beta}_+ \cup \widehat{\beta}_-$ and $\widetilde{\Gamma} = \beta_+ \cup \beta_-$.

Abusing the notation, for each $p \in \widetilde{\Gamma}$, we let $\Delta^1(p), \Delta^2(p)$ denote the lifts at p of the related disks $\Delta^1(\Pi(p)), \Delta^2(\Pi(p))$. The final condition on the convergence of β_+^n and β_-^n to β_+ and β_- is that

$$\beta_+^n \cap \bigcup_{p \in \beta_+} [\Delta^1(p) \cup \Delta^2(p)] = \emptyset, \quad \beta_-^n \cap \bigcup_{p \in \beta_-} [\Delta^1(p) \cup \Delta^2(p)] = \emptyset,$$

for all n sufficiently large.

Necessarily, if $\lambda_{\beta_{\pm}}$ denotes the restriction of the coordinate function λ to β_{\pm} then $\lambda_{\beta_{\pm}}(\theta) \rightarrow \lambda_2$ as $\theta \rightarrow +\infty$ and $\lambda_{\beta_{\pm}}(\theta) \rightarrow \lambda_1$ as $\theta \rightarrow -\infty$, which means that they are spiralling toward $[\partial\mathcal{C}^{\lambda_1} \cup \partial\mathcal{C}^{\lambda_2}] \subset \partial_{\infty}\mathbb{H}^3$, e.g.,

$$\lambda(\widetilde{\theta}) = \lambda_1 + (\lambda_2 - \lambda_1) \frac{2 \arctan \widetilde{\theta} + \pi}{2\pi}.$$

Lemma 3.4. *For all $p \in \Gamma$ and all n sufficiently large, the compact surfaces $\Pi(\Sigma_n)$ are disjoint from $\Delta^1(p) \cup \Delta^2(p)$. In particular, for n sufficiently large, the compact surfaces $\Pi(\Sigma_n)$ are disjoint from the two components of $\mathbb{H}^3 \setminus [\Delta^1(p) \cup \Delta^2(p)]$ that do not contain the origin. Furthermore, the limit set of the closed set $\overline{\cup_{n=1}^{\infty} \Pi(\Sigma_n)} \subset \mathbb{H}^3$ in $\partial_{\infty}\mathbb{H}^3$ must be contained in the closed set $\overline{\Gamma} \subset \partial_{\infty}\mathbb{H}^3$.*

Proof. The proof will follow from a simple application of the maximum principle and the mean curvature comparison principle. Recall that the mean curvature comparison principle implies that if two surfaces Λ_1, Λ_2 intersect at an interior point x , Λ_1 has a nonzero mean curvature vector $\vec{H}_{\Lambda_1}(x)$ at x and near x the surface Λ_2 lies on the side of Λ_1 where $\vec{H}_{\Lambda_1}(x)$ is pointing, then, for some $\lambda \geq 1$, $\vec{H}_{\Lambda_2}(x) = \lambda \vec{H}_{\Lambda_1}(x)$.

Fix $p \in \Gamma$. By the construction of the curves $\Gamma_n = \partial\Sigma_n$, for n sufficiently large, $\Pi(\partial\Sigma_n) \cap [\Delta^1(p) \cup \Delta^2(p)] = \emptyset$. If $\Pi(\Sigma_n) \cap [\Delta^1(p) \cup \Delta^2(p)] \neq \emptyset$, then one of the two disks, say $\Delta^1(p)$, intersects $\Pi(\Sigma_n)$ at some point. Note that the closure B_1 of the component of $\mathbb{H}^3 - \Delta^1(p)$ that is disjoint from the origin is foliated by rotationally symmetric open disks $D_1(t)$ of constant mean curvature H each of which is properly embedded in \mathbb{H}^3 , where $t \in [0, \infty)$ and $D_1(0) = \Delta^1(p)$. Also note that the mean curvature vectors of these disks are chosen to vary continuously and when considered to be the boundaries of associated domains $B_1(t)$, they point towards the complement of $B_1(t)$ in \mathbb{H}^3 ; in particular, $\mathbb{H}^3 - \text{Int}(B_1(t))$ is H -convex. The boundary circles of such disks $D_1(t)$ can be chosen to be disjoint and to converge to a point in $\partial_{\infty}\mathbb{H}^3$. Since for every $t \geq 0$ and n sufficiently large, $D_1(t)$ is disjoint from $\Pi(\partial\Sigma_n)$ and $\Pi(\Sigma_n)$ is compact, there exists a largest non-negative number t_1 such

that $D_1(t_1)$ intersects $\Pi(\Sigma_n)$. Therefore, $\Pi(\Sigma_n)$ locally lies on one side of $D_1(t_1)$ at an interior point of intersection. This contradicts the maximum principle in the case that the mean curvature vectors of the two surfaces agree at the point of intersection, a property that we now show must hold. Since $\Pi(\Sigma_n)$ lies in the H -convex region $\mathbb{H}^3 - \text{Int}(B_1(t_1))$, then the mean curvature comparison principle implies that the mean curvature vectors of the surfaces $D_1(t_1) = \partial B_1(t_1)$ and $\Pi(\Sigma_n)$ agree at the point of intersection. This completes the proof that $\Pi(\Sigma_n) \cap \Delta^1(p) = \emptyset$.

After viewing $\overline{\mathbb{H}^3}$ with the closed unit ball metric, let $q \in \partial_\infty \mathbb{H}^3 \setminus ((\partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2} \cup \Gamma) = \overline{\Gamma})$. By construction, the distance from q to $\cup_{i=1}^\infty \Pi(\partial \Sigma_n)$ is positive in the closed ball metric on $\overline{\mathbb{H}^3}$. The arguments in the first paragraph of this proof using the maximum principle show that there exists a disk $D^q \subset \mathbb{H}^3$ of revolution and constant mean curvature H , with boundary circle in $\partial_\infty \mathbb{H}^3$ centered at q , that is disjoint from $\cup_{n=1}^\infty \Sigma_n$. The existence of D^q implies that q is not in the closure of $\cup_{n=1}^\infty \Pi(\Sigma_n)$ in $\overline{\mathbb{H}^3}$. This completes the proof of the lemma. q.e.d.

4. Constructing the surface Σ_H

In this section, we construct Σ_H and finish the proof of Theorem 1.1.

Lemma 4.1. *Let Γ_n be as described in Section 3.3 and let $E_n = \Omega_n \cap ([\lambda_n^-, \lambda_n^+] \times \{0\} \times \mathbb{R})$. Then Σ_n is a $\tilde{\theta}$ -graph over the compact disk E_n . In particular, the related Jacobi function J_n on Σ_n induced by the inner product of the unit normal field to Σ_n with the Killing field $\partial_{\tilde{\theta}}$ is positive in the interior of Σ_n .*

Proof. Recall that T_α is an isometry of \widetilde{W} , which is translation by α , i.e., $T_\alpha(\lambda, \tilde{\theta}, z) = (\lambda, \tilde{\theta} + \alpha, z)$. Let $T_\alpha(\Sigma_n) = \Sigma_n^\alpha$ and $T_\alpha(\Gamma_n) = \Gamma_n^\alpha$. We claim that $\Sigma_n^\alpha \cap \Sigma_n = \emptyset$ for any $\alpha \in \mathbb{R} \setminus \{0\}$ which implies that Σ_n is a $\tilde{\theta}$ -graph.

Arguing by contradiction, suppose that $\Sigma_n^\alpha \cap \Sigma_n \neq \emptyset$ for a certain $\alpha \neq 0$. By compactness of Σ_n , there exists a largest positive number α' such that $\Sigma_n^{\alpha'} \cap \Sigma_n \neq \emptyset$. Let $p \in \Sigma_n^{\alpha'} \cap \Sigma_n$. Since $\partial \Sigma_n^{\alpha'} \cap \partial \Sigma_n = \emptyset$ and, by Lemma 3.1, the interiors of both $\Sigma_n^{\alpha'}$ and Σ_n lie in $(\lambda_n^-, \lambda_n^+) \times \mathbb{R} \times \mathbb{R}$, then $p \in \text{Int}(\Sigma_n^{\alpha'}) \cap \text{Int}(\Sigma_n)$. Since the surfaces $\text{Int}(\Sigma_n^{\alpha'})$, $\text{Int}(\Sigma_n)$ lie on one side of each other and intersect tangentially at the point p with the same mean curvature vector, then we obtain a contradiction to the maximum principle for constant mean curvature surfaces. This proves that Σ_n is graphical over its $\tilde{\theta}$ -projection to E_n .

Since by construction every integral curve, $(\bar{\lambda}, t, \bar{z})$ with $\bar{\lambda}, \bar{z}$ fixed and $(\bar{\lambda}, 0, \bar{z}) \in E_n$, of the Killing field $\partial_{\tilde{\theta}}$ has non-zero intersection number with any compact surface bounded by Γ_n , we conclude that every such integral curve intersects both the disk E_n and Σ_n in single points. This

means that Σ_n is a $\tilde{\theta}$ -graph over E_n and thus the related Jacobi function J_n on Σ_n induced by the inner product of the unit normal field to Σ_n with the Killing field $\partial_{\tilde{\theta}}$ is non-negative in the interior of Σ_n . Since J_n is a non-negative Jacobi function, then either $J_n \equiv 0$ or $J_n > 0$. Since J_n is positive somewhere in the interior, then J_n is positive everywhere in the interior. This finishes the proof of the lemma. q.e.d.

To summarize, with Γ_n as previously described, we have constructed a sequence of compact stable H -disks Σ_n with $\partial\Sigma_n = \Gamma_n \subset \partial\Omega_n$. By the curvature estimates for stable H -surfaces given in [13], the norms of the second fundamental forms of the $\Pi(\Sigma_n)$ are uniformly bounded from above at points at least $\varepsilon > 0$ intrinsically far from their boundaries, for any $\varepsilon > 0$. Since for any compact subset $X \subset \text{Int}(W)$ and for n sufficiently large, Γ_n is a positive distance from X , the norms of the second fundamental forms of the $\Pi(\Sigma_n)$ are uniformly bounded on compact sets of $\text{Int}(W)$.

A standard compactness argument, using the uniform curvature estimates for the surfaces Σ_n on compact subsets of $\text{Int}(W)$ and their graphical nature described in Lemma 4.1, implies that a subsequence $\Pi(\Sigma_{n(k)})$ of the surfaces $\Pi(\Sigma_n)$ converges to an H -lamination \mathcal{L} of $\text{Int}(W)$. By the nature of the convergence, each leaf of \mathcal{L} has bounded norm of its second fundamental form on compact sets of $\text{Int}(W)$ and this bound depends on the compact set but not on the leaf.

By similar arguments, there exists an H -lamination $\tilde{\mathcal{L}}$ of \tilde{W} that is a limit of some subsequence of the graphs $\Sigma_{n(k)}$; note that in this case we include the boundary of \tilde{W} in the domain, contrary to the previous case where we constructed the lamination \mathcal{L} in the interior of W . After a refinement of the original subsequence, we will assume that $\Pi(\Sigma_{n(k)})$ converges to \mathcal{L} and that $\Sigma_{n(k)}$ converges to $\tilde{\mathcal{L}}$. Since the boundaries of the $\Sigma_{n(k)}$ leave every compact subset of \tilde{W} , the leaves of $\tilde{\mathcal{L}}$ are complete. Note also that the leaves of $\tilde{\mathcal{L}}$ have uniformly bounded norms of their second fundamental forms. The fact that the laminations \mathcal{L} and $\tilde{\mathcal{L}}$ are not empty will follow from the next discussion.

Note that the region $\text{Int}(W)$ is foliated by the integral curves of the Killing field ∂_θ , which are circles and each such circle intersects $B = (\lambda_1, \lambda_2) \times \{0\} \times \mathbb{R}$ orthogonally in a unique point b ; let $S(b)$ denote this circle. We next study how the circles $S(b)$ intersect the leaves of \mathcal{L} and prove some properties of the laminations \mathcal{L} and $\tilde{\mathcal{L}}$. Let $\Theta: \text{Int}(W) \rightarrow B$ denote the natural projection.

Claim 4.2. *For every $b \in B$, $S(b)$ intersects at least one of the leaves of \mathcal{L} . Furthermore, if $S(b)$ intersects a leaf L of \mathcal{L} transversely at some point p , then L is the only leaf of \mathcal{L} that intersects $S(b)$ and $S(b) \cap L = \{p\}$.*

Proof. The first statement in this claim follows from the fact that for any $b \in B$, for n sufficiently large, $S(b)$ intersects the “graphical” surface $\Pi(\Sigma_{n(k)})$ in a single point $p_{n(k)}$, and since $S(b)$ is compact, some subsequence of these points converges to a point $p \in S(b) \cap \mathcal{L}$.

If $S(b)$ intersects a leaf L of \mathcal{L} transversely in a point q , then there exists an $\varepsilon(q) > 0$ such that for n sufficiently large, a small neighborhood $N(q, L) \subset L$ of q is a θ -graph of bounded gradient over the disk $D_B(b, \varepsilon(b)) \subset B$ centered at b of radius $\varepsilon(b)$, and $\{\Theta^{-1}(D_B(b, \varepsilon(b))) \cap \Sigma_{n(k)}\}_{n(k)}$ is a sequence of graphs converging smoothly to the graph $N(q, L) = \Theta^{-1}(D_B(b, \varepsilon(b))) \cap \Sigma_{n(k)}$. In particular, $S(b)$ intersects \mathcal{L} transversely in the single point q . q.e.d.

Claim 4.3. *The limit set $\partial_\infty \mathcal{L}$ contains Γ and it contains no other points in the two annuli in $\partial_\infty W \setminus [\partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}]$.*

Proof. Let $\text{Lim}(\mathcal{L})$ denote the limit set of \mathcal{L} that lies in $\partial_\infty W \setminus [\partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}]$. By Lemma 3.4, $\text{Lim}(\mathcal{L}) \subset \Gamma$. We next show that $\Gamma \subset \text{Lim}(\mathcal{L})$. Let $x \in \Gamma$ and choose a sequence of circles $\{S(b_k)\}_{k \in \mathbb{N}}$ that converges to the circle $C(x) \subset \partial_\infty \mathbb{H}^3$ passing through x . By Claim 4.2, there exist points $p_k \in S(b_k) \cap \mathcal{L}$, and by compactness of $\overline{\mathbb{H}^3}$, a subsequence of these points converges to a point $p \in C(x) \cap \text{Lim}(\mathcal{L})$. But since $C(x) \cap \Gamma = \{x\}$, then $x = p$, which completes the proof that $\text{Lim}(\mathcal{L}) = \Gamma$, and the claim holds. q.e.d.

Claim 4.4. *The limit set of $\tilde{\mathcal{L}}$ in $\partial_\infty \tilde{W}$ is equal to $\tilde{\Gamma}$.*

Proof. By arguing with barriers as in the proof of Lemma 3.4, the limit set of $\tilde{\mathcal{L}}$ must be contained in $\tilde{\Gamma}$. Let $p \in \beta^+ \subset \partial_\infty \tilde{W}$ and let $\Delta^1(p), \Delta^2(p)$ be the disks described at the end of the previous section. Consider a small arc $\alpha \subset \tilde{W}$ with end points in $\overline{\Delta^1(p)} \cup \overline{\Delta^2(p)}$ that links β^+ and such that $\Pi(\alpha)$ is the compactification of a geodesic in \tilde{W} . Then by previous arguments, α must intersect a leaf L of $\tilde{\mathcal{L}}$. Since a sequence of these arcs can be chosen to converge to p , then p is in the limit set of $\tilde{\mathcal{L}}$.

Using exactly the same arguments gives that the same is true of $p \in \beta^-$. This completes the proof of the claim. q.e.d.

Remark 4.5. Note that this claim implies that no leaves of $\tilde{\mathcal{L}}$ are invariant under the one-parameter group of translations T_θ^- , since such complete surfaces are the lifts of surfaces of revolution in \mathbb{H}^3 , and as such they have their limit sets that contain circles which are not contained in $\tilde{\Gamma}$. In particular, $\tilde{\mathcal{L}}$ does not contain $\tilde{\mathcal{C}}^{\lambda_1}$ nor $\tilde{\mathcal{C}}^{\lambda_2}$.

Claim 4.6. *Let $\alpha \subset \tilde{W} \subset [\mathbb{H}^3 \cup \partial_\infty \mathbb{H}^3]$ be a compact arc with $\text{Int}(\alpha) \subset W$, joining $\mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_1}$ to $\mathcal{C}^{\lambda_2} \cup \partial_\infty \mathcal{C}^{\lambda_2}$. Then there exists a leaf L of \mathcal{L} that intersects α and that is not invariant under rotations around the z -axis.*

Proof. Let $\tilde{\alpha} \subset \widetilde{W}$ be any fixed lift of α . By a linking argument, it follows that when $n(k)$ is sufficiently large, $\tilde{\alpha}$ intersect $\Sigma_{n(k)}$ at some interior point $p_{n(k)}$. Suppose that a subsequence of the points $p_{n(k)}$ converges to an end point of $\tilde{\alpha}$ that corresponds to an end point of α in $\mathcal{C}^{\lambda_1} \cup \mathcal{C}^{\lambda_2}$. This would imply that $\tilde{\mathcal{C}}^{\lambda_1}$ or $\tilde{\mathcal{C}}^{\lambda_2}$ is a leaf of the lamination $\tilde{\mathcal{L}}$, contradicting the previous remark. Next, suppose that a subsequence of the points $p_{n(k)}$ converges to an end point of $\tilde{\alpha}$ that corresponds to an end point of α in $\partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}$. This picture is ruled out by Claim 4.4. Therefore, a subsequence of the points $p_{n(k)}$ must converge to a point in the interior of $\tilde{\alpha}$, which is, therefore, a point on some leaf \tilde{L} of $\tilde{\mathcal{L}}$. Note that α intersects $L = \Pi(\tilde{L})$ and, since \tilde{L} is not invariant under the one-parameter group of translations $T_{\tilde{\theta}}$, L is not invariant under rotations around the z -axis. Thus the claim holds. q.e.d.

Claim 4.7. *Every complete leaf L of \mathcal{L} is the graph of a smooth function defined on its θ -projection $\Theta(L) \subset B$.*

Proof. Let L be a complete leaf of \mathcal{L} . Recall that the surfaces $\Pi(\Sigma_{n(k)})$ are θ -graphs and let $J_{n(k)}$ denote the related positive Jacobi functions induced by the inner product of the unit normal field to $\Pi(\Sigma_{n(k)})$ with the Killing field ∂_θ . Let J denote the limit Jacobi function on the leaves of \mathcal{L} and let J_L be the related Jacobi function on L . Since L is the limit of portions of the surfaces $\Pi(\Sigma_{n(k)})$, the previous observation implies that J_L is non-negative. Since J_L is a non-negative Jacobi function, then either $J_L \equiv 0$ or $J_L > 0$. If $J_L > 0$, then by Claim 4.2, L is the graph of a smooth function over its projection to $(\lambda_1, \lambda_2) \times \{0\} \times \mathbb{R}$. Therefore, to prove the claim, it suffices to show that $J_L > 0$.

Arguing by contradiction, assume that $J_L \equiv 0$, then L is a complete embedded surface in W that is invariant under rotations around the z -axis. In particular, there exists a complete arc β in $L \cap E$ and β is embedded since L is embedded; completeness of β follows from the completeness of L . Also this arc cannot be bounded in \mathbb{H}^3 since otherwise \bar{L} would be a bounded H -lamination in \mathbb{H}^3 , which is impossible since there would exist a leaf L' of \bar{L} that would be contained in a geodesic ball $B_{\mathbb{H}^3}$ centered at the origin and tangent to $\partial B_{\mathbb{H}^3}$ at some point q . But $\partial B_{\mathbb{H}^3}$ has mean curvature greater than one, which is a contradiction to the mean curvature comparison principle applied at the point q . Hence, since β is not bounded in \mathbb{H}^3 , then $\partial_\infty L \neq \emptyset$.

Since L is invariant under rotation around the z -axis and $\partial_\infty \mathcal{L} \subset \Gamma \cup \partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}$, then $\partial_\infty L \subset \partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}$. To obtain a contradiction we will consider separately each of the following three possible cases for the limiting behavior of L :

Case A: $\partial_\infty L \subset \partial_\infty \mathcal{C}^{\lambda_1}$ or $\partial_\infty L \subset \partial_\infty \mathcal{C}^{\lambda_2}$ and L is a spherical catenoid as described in the Appendix with its mean curvature vector pointing toward the z -axis.

Case B: $\partial_\infty L$ contains one component in $\partial_\infty \mathcal{C}^{\lambda_1}$ and another component in $\partial_\infty \mathcal{C}^{\lambda_2}$.

Case C: $\partial_\infty L \subset \partial_\infty \mathcal{C}^{\lambda_1}$ or $\partial_\infty L \subset \partial_\infty \mathcal{C}^{\lambda_2}$ and L is a surface of revolution as described in Gomes [6] with its mean curvature vector pointing away from the z -axis. Note that in this case it might hold that $\partial_\infty L$ is a single circle with multiplicity two.

First suppose that Case A holds. By the discussion in the Appendix and the description of stable catenoids, the only possible spherical catenoids are \mathcal{C}^{λ_1} or \mathcal{C}^{λ_2} but they do not intersect $\text{Int}(W)$, which gives a contradiction.

If Case B holds, then there is a compact arc $\alpha \subset L \cup \partial_\infty L$ satisfying the hypotheses of Claim 4.6. By the same claim, there must exist a non-rotational leaf L_1 of \mathcal{L} that intersects L , which is impossible since distinct leaves of \mathcal{L} are disjoint.

Finally, suppose that Case C holds. If $H = 0$, then using the maximum principle applied to the foliation of W by minimal catenoids gives an immediate contradiction; hence, assume that $H > 0$. As remarked in Gomes [6], the properly embedded surfaces of revolution in \mathbb{H}^3 extend to C^1 -immersed surfaces in $\overline{\mathbb{H}^3}$ and they make a particular oriented angle $\theta_H \neq \pi/2$ with $\partial_\infty \mathbb{H}^3$ according to their orientation. First consider the case where L contains a single circle S component in $\partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}$. This means that the surface L makes two different positive oriented angles along its single boundary circle S at infinity. Since L lies on the side of the spherical catenoid \mathcal{C}^{λ_2} that makes an acute angle with $\partial_\infty \mathbb{H}^3$, and \mathcal{C}^{λ_2} is disjoint from L , if $S \subset \partial_\infty \mathcal{C}^{\lambda_2}$, we obtain a contradiction. Hence, we may assume that S is a boundary curve of \mathcal{C}^{λ_1} . But in this case, there is a largest $\lambda \in [\lambda_2, \lambda_1)$ such that \mathcal{C}^λ intersects L at an interior point and L lies on the mean convex side of \mathcal{C}^λ . This contradicts the maximum principle. Therefore, if Case C holds, then either $\partial_\infty L = \partial_\infty \mathcal{C}^{\lambda_1}$ or $\partial_\infty L = \partial_\infty \mathcal{C}^{\lambda_2}$.

Reasoning as in the previous paragraph with the angles along the boundary, we find that the only possibility is $\partial_\infty L = \partial_\infty \mathcal{C}^{\lambda_1}$. But in this case, there is again a largest $\lambda \in [\lambda_2, \lambda_1)$ such that \mathcal{C}^λ intersects L at an interior point and L lies on the mean convex side of \mathcal{C}^λ . This contradicts the maximum principle. This final contradiction completes the proof of the claim. q.e.d.

Consider a complete leaf L of $\tilde{\mathcal{L}}$. Recall that L has bounded norm of the second fundamental form, is not invariant under the one-parameter group of translations $T_{\tilde{\theta}}$, and by the maximum principle $L \subset \text{Int}(\tilde{W})$. It follows from the construction of the H -laminations \mathcal{L} and $\tilde{\mathcal{L}}$ that $\Sigma_H = \Pi(L)$ is a complete leaf of \mathcal{L} . By the previous lemma, Σ_H is the graph of a smooth function defined on its θ -projection $\Theta(\Sigma_H) \subset B$.

We next prove that the leaf Σ_H is properly embedded in $\text{Int}(W)$. If not, then there exists a limit leaf $L \neq \Sigma_H$ of \mathcal{L} in the closure of Σ_H . Since this leaf is easily seen to be complete as well, by Claim 4.7 L is a graph of a smooth function over its projection $\Theta(L) \subset B$. Since the open sets $\Theta(\Sigma_H), \Theta(L)$ intersect near any point of $\Theta(L)$, this contradicts Claim 4.2 and so Σ_H is properly embedded in $\text{Int}(W)$. Since $\Theta: \text{Int}(W) \rightarrow B$ is a proper submersion, $\Theta(\Sigma_H) = B$ and thus, by Claim 4.2, Σ_H is the unique leaf of \mathcal{L} . Clearly the closure $\bar{\Sigma}_H$ of Σ_H in W is $\Sigma_H \cup \mathcal{C}^{\lambda_1} \cup \mathcal{C}^{\lambda_2}$.

To summarize, we have shown that Σ_H is a complete graph over B and is properly embedded in $\text{Int}(W)$. Moreover:

- 1) Σ_H has bounded norm of the second fundamental form.
- 2) The closure $\bar{\Sigma}_H$ of Σ_H in W is $\Sigma_H \cup \mathcal{C}^{\lambda_1} \cup \mathcal{C}^{\lambda_2}$.
- 3) $\partial_\infty \Sigma_H = \Gamma \cup \partial_\infty \mathcal{C}^{\lambda_1} \cup \partial_\infty \mathcal{C}^{\lambda_2}$; see Figure 5.

In particular, the surfaces $\{T_\theta(\Sigma_H) \mid \theta \in [0, 2\pi)\}$ together with the spherical catenoids $\mathcal{C}^{\lambda_1}, \mathcal{C}^{\lambda_2}$ form an H -foliation of W . This finishes the proof of the Theorem 1.1.

5. Appendix

In this appendix, we recall some facts about spherical H -catenoids in \mathbb{H}^3 that are used throughout the paper. As we have done in the previous sections, we will work in \mathbb{H}^3 using the Poincaré ball model, that is we consider \mathbb{H}^3 as the unit ball in \mathbb{R}^3 and its ideal boundary at infinity corresponds to the boundary of the ball. We will let $P_{x,y}, P_{x,z}$ and $P_{y,z}$ be the related totally geodesic coordinate planes in this ball model of \mathbb{H}^3 .

Definition 5.1. Let \mathcal{C} be a properly immersed annulus in \mathbb{H}^3 with constant mean curvature $H \in [0, 1]$ and $\partial_\infty \mathcal{C} = \alpha_1 \cup \alpha_2 \subset \partial_\infty \mathbb{H}^3$ where α_1 and α_2 are two disjoint round circles in $\partial_\infty \mathbb{H}^3$. Let γ be the unique geodesic in \mathbb{H}^3 from the center of α_1 to the center of α_2 . If \mathcal{C} is rotationally invariant with respect to γ , then we call \mathcal{C} a *spherical H -catenoid* with rotation axis γ .

In other words, spherical H -catenoids are obtained by rotating a certain curve, a catenary, around a geodesic. In [6], Gomes studied the spherical H -catenoids in \mathbb{H}^3 for $0 \leq H \leq 1$, and classified them in terms of the generating curve which is a solution to a certain ODE. In what follows, when $H \neq 0$, we will only focus on the spherical H -catenoids for which the mean curvature vector points towards the rotation axis.

After applying an isometry, we can assume that the x -axis is the rotation axis for the spherical H -catenoid and that the circles α_1 and α_2 are symmetric with respect to the (y, z) -plane. Gomes proved that for a fixed $H \in [0, 1)$ and $\lambda \in (0, \infty)$ there exists a unique spherical H -catenoid \mathcal{C}_H^λ with generating curve β_H^λ and satisfying the following properties:

- 1) β_H^λ is the graph in normal coordinates over the x -axis of a positive even function $g_H^\lambda(x)$ defined on an interval $(-d_H^\lambda, d_H^\lambda)$, namely

$$\beta_H^\lambda = \{(x, g_H^\lambda(x)) \mid x \in (-d_H^\lambda, d_H^\lambda)\},$$

and satisfying $\lambda = g_H^\lambda(0)$; see Figure 6-right;

- 2) Up to isometry, a spherical H -catenoid is isometric to a certain \mathcal{C}_H^λ .

In particular, \mathcal{C}_H^λ is embedded and symmetric with respect to the (y, z) -plane. Recall that if Σ is a properly embedded H -surface, $H < 1$ in \mathbb{H}^3 , with limit set at infinity a smooth compact embedded curve with multiplicity one, then the closure of Σ in the closed ball $\overline{\mathbb{H}^3}$ is a C^1 surface that makes a constant angle with the boundary sphere and this angle is the same as the one that a complete curve in \mathbb{H}^2 of constant geodesic curvature H makes with the circle $\partial_\infty \mathbb{H}^2$.

Gomes also analyzed the relation between the spherical H -catenoid \mathcal{C}_H^λ and its asymptotic boundary $\partial_\infty \mathcal{C}_H^\lambda = \tau_{H,\lambda}^+ \cup \tau_{H,\lambda}^-$ where each of the curves $\tau_{H,\lambda}^\pm$ is a circle in $\partial_\infty \mathbb{H}^3$. Define the asymptotic distance function $d_H(\lambda)$ as the distance between the geodesic planes $P_{H,\lambda}^+$ and $P_{H,\lambda}^-$ where $\partial_\infty P_{H,\lambda}^\pm = \tau_{H,\lambda}^\pm$, respectively. By an abuse of language, we will say that $d_H(\lambda)$ is the distance between the asymptotic circles of \mathcal{C}_H^λ . Recall that we use normal coordinates over the x -axis in the description of the function $g_H^\lambda(x)$. Let the asymptotic boundary of the generating curve β_H^λ be the points $\{w_{H,\lambda}^+, w_{H,\lambda}^-\} \subset \partial_\infty \mathbb{H}^3$, i.e., $\partial_\infty \beta_H^\lambda = \{w_{H,\lambda}^+, w_{H,\lambda}^-\} \subset \partial_\infty \mathbb{H}^3$. Then the geodesic projections of these points to the x -axis will be $\Pi(w_{H,\lambda}^\pm) = \pm d_H^\lambda$. Since $w_{H,\lambda}^\pm \subset \tau_{H,\lambda}^\pm$ and the construction is rotationally invariant, it is easy to see that $d_H(\lambda) = 2d_H^\lambda$.

Gomes showed that for a fixed $H \in [0, 1)$, the function $d_H(\lambda)$ increases from a non-negative value $\lim_{\lambda \rightarrow 0} d_H(\lambda)$, which is zero and not acquired when $H = 0$ and positive when $H > 0$, reaches a maximum at a certain $c_H \in (0, \infty)$, and then decreases to 0 as $\lambda \rightarrow \infty$; see Figure 6 and [6, Lemma 3.5].

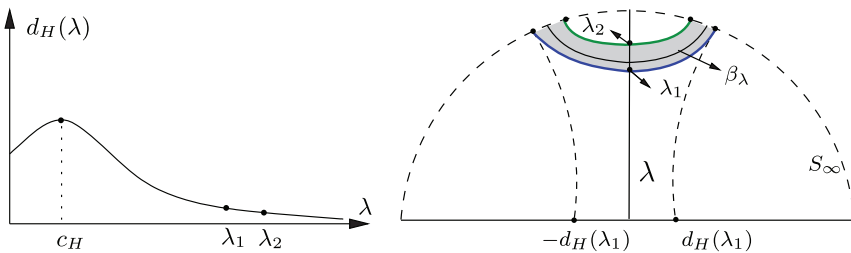


Figure 6. λ represents the distance from the rotation axis, and $d_H(\lambda)$ represents the asymptotic distance between the asymptotic circles of \mathcal{C}_H^λ .

In particular, this discussion implies that for a fixed $H \in [0, 1)$, the distance between the asymptotic boundaries of the spherical H -catenoids is bounded by $\max(d_H)$. He also showed that $\max(d_H)$ is monotone increasing in H , and $\max(d_H) \rightarrow \infty$ as $H \rightarrow 1$. The following lemma summarizes some of these results:

Lemma 5.2 ([6]). *For any $H \in [0, 1)$ and any $\lambda \in (0, \infty)$, there exists a unique spherical H -catenoid \mathcal{C}_H^λ in \mathbb{H}^3 such that the distance from its generating curve β_H^λ to the x -axis is λ . Moreover, there exists a number $c_H > 0$ such that $d_H(c_H) = \max_{(0, \infty)} d_H$, and $d_H(\lambda)$ decreases to 0 on the interval $[c_H, \infty)$ as λ goes to infinity; see Figure 6-left.*

In the next proposition we construct the foliations by spherical H -catenoids that are used in producing our examples.

Proposition 5.3. *For any $H \in [0, 1)$, the family of spherical H -catenoids $\mathcal{F}_H = \{\mathcal{C}_H^\lambda \mid \lambda \in [c_H, \infty)\}$ foliates the closure of the non-simply-connected component of $\mathbb{H}^3 \setminus \mathcal{C}_H^{c_H}$. In particular, each of the spherical catenoids in \mathcal{F}_H admits a positive Jacobi function, that is induced by the associated normal variational field.*

Proof. We first consider the case $H \in (0, 1)$. After adding the leaf $\mathcal{C}_H^{c_H}$, it suffices to show that the family of spherical H -catenoids $\mathcal{F}_H = \{\mathcal{C}_H^\lambda \mid \lambda \in (c_H, \infty)\}$ foliates the non-simply-connected component of $\mathbb{H}^3 \setminus \mathcal{C}_H^{c_H}$. Note that by construction, the elements in \mathcal{F}_H are embedded and form a continuous family with respect to λ . Therefore, if for $\lambda_0 \in (c_H, \infty)$ the spherical H -catenoids $\mathcal{C}_H^{c_H}$ and $\mathcal{C}_H^{\lambda_0}$ are disjoint, then an application of the maximum principle gives that $\{\mathcal{C}_H^\lambda \mid \lambda \in (c_H, \lambda_0)\}$ is a foliation of the region between $\mathcal{C}_H^{c_H}$ and $\mathcal{C}_H^{\lambda_0}$. Therefore, it suffices to show that for any λ sufficiently large, $\mathcal{C}_H^{c_H}$ and \mathcal{C}_H^λ are disjoint.

Since $\lim_{\lambda \rightarrow \infty} d_H(\lambda) = 0$, the end points of the generating curve β_H^λ of \mathcal{C}_H^λ converge to $(0, 0, 1)$ in $\partial_\infty \mathbb{H}^3$ as λ goes to infinity. Recall that the mean curvature vector of \mathcal{C}_H^λ is pointing toward the x -axis. Using barriers that are planes of constant mean curvature H and are rotationally invariant with respect to rotations around the x -axis, it can be shown that as $d_H(\lambda) \rightarrow 0$, the set \mathcal{C}_H^λ viewed in the unit ball converges to the unit circle in the (y, z) -plane. Thus, for any λ sufficiently large, $\mathcal{C}_H^{c_H}$ and \mathcal{C}_H^λ are disjoint and, by the previous discussion, the family of spherical H -catenoids \mathcal{F}_H foliates the non-simply-connected component of $\mathbb{H}^3 \setminus \mathcal{C}_H^{c_H}$ when $H \in (0, 1)$.

The foliation in the $H = 0$ case can be obtained as the limit as H goes to zero of the foliations \mathcal{F}_H . q.e.d.

Remark 5.4. Since, for any fixed $H \in [0, 1)$ and $\lambda < c_H$, the Jacobi function J_H^λ on \mathcal{C}_H^λ induced from the variational vector field of the Gomes family \mathcal{C}_H^λ is positive along the circle in \mathcal{C}_H^λ closest to its axis of revolution

but limits to $-\infty$ at its asymptotic boundary, then such a \mathcal{C}_H^λ is unstable. Thus, the pair of asymptotic boundary circles of \mathcal{C}_H^λ , $\lambda < c_H$, bounds two spherical H -catenoids, a stable one \mathcal{C}^{λ_2} , and an unstable one \mathcal{C}^{λ_1} , where $\lambda_1 < \lambda_2$.

We will need the next proposition in the proof of Theorem 1.1.

Proposition 5.5. *Fix $H \in [0, 1)$. For $t \in [0, \infty)$, let $S(t)$ be the parallel surface in \mathbb{H}^3 that lies “above” $P_{y,z}$ at distance t and for $t \in (-\infty, 0)$ let $S(t)$ denote the parallel surface that lies “below” $P_{y,z}$ at distance $-t$. Let V_x denote the related Killing field generated by the one-parameter group of isometries $\{\phi_\varepsilon: \mathbb{H}^3 \rightarrow \mathbb{H}^3 \mid \varepsilon \in \mathbb{R}\}$ where $\phi_\varepsilon: \mathbb{H}^3 \rightarrow \mathbb{H}^3$ is the hyperbolic translation along the y -axis by the signed distance ε . Then:*

1. *For each $t \in \mathbb{R}$, $S(t)$ intersects $\mathcal{C}_H^{c_H}$ transversely in a single circle (closed curve of constant geodesic curvature in $S(t)$) centered at the intersection of the x -axis with $S(t)$.*
2. *If we let $\mathcal{C}_H^{c_H}(+)$ denote the portion of $\mathcal{C}_H^{c_H}$ with non-negative y -coordinate, then $\mathcal{C}_H^{c_H}(+)$ is a V_y -Killing graph over its projection to $P_{x,z}$, and this projected domain has boundary $\mathcal{C}_H^{c_H} \cap P_{x,z}$.*
3. *Let $D(t) \subset S(t)$ be the open disk bounded by the circle $S(t) \cap \mathcal{C}_H^{c_H}$. For $\varepsilon < 0$ sufficiently close to zero, $\phi_\varepsilon(x\text{-axis})$ intersects each of the disks in a single point.*
4. *For ε satisfying the previous item and for $\lambda_2 \in (c_H, \infty)$ sufficiently close to c_H , $\phi_\varepsilon(\mathcal{C}_H^{c_H}) \cap \cup_{s \in [c_H, \lambda_2]} \mathcal{C}_H^s$ is a pair of infinite strips in $W = \cup_{s \in [c_H, \lambda_2]} \mathcal{C}_H^s$ that separate W into two regions, and for one of these two regions the portion of $\phi_\varepsilon(\mathcal{C}_H^{c_H})$ in its boundary has non-negative mean curvature with respect to the inward pointing to the boundary.*

Proof. Recall that the rotation axis of \mathcal{C}_H^λ is the x -axis. By definition, the planes $S(t)$ are equidistant planes to plane $P_{y,z}$ with distance t , and hence $\partial_\infty S(t) = \partial_\infty P_{y,z}$. Since the boundary circle of $P_{y,z}$ has linking number 1 with the arc $\beta_H^{c_H}$ and the x -axis (See Figure 6-right), each of the planes $S(t)$ intersect $\beta_H^{c_H}$ in some point, and so since these planes are also invariant under rotation around the x -axis, each point in $\beta_H^{c_H} \cap S(t)$ lies on a circle of intersection of $S(t)$ with $\mathcal{C}_H^{c_H}$. Also note that $S(0) = P_{y,z}$ intersects $\mathcal{C}_H^{c_H}$ orthogonally in a circle of radius c_H centered at $(0, 0, 0)$. If $S(t)$ fails to intersect $\mathcal{C}_H^{c_H}$ transversely at some point, then V_y is tangent to $\mathcal{C}_H^{c_H}$ along some circle S in $\mathcal{C}_H^{c_H}$ as well as to the related circle in $S' \subset \mathcal{C}_H^{c_H}$ obtained by reflection in the plane $P_{y,z}$. But in this case the compact subannulus of $\mathcal{C}_H^{c_H}$ bounded by $S \cup S'$ would represent a compact non-strictly stable subdomain of $\mathcal{C}_H^{c_H}$, which contradicts that $\mathcal{C}_H^{c_H}$ is stable. This contradiction shows that $S(t)$ always intersects $\mathcal{C}_H^{c_H}$ transversally. Since $S(0)$ and $\mathcal{C}_H^{c_H}$ intersect along a single circle, elementary arguments imply that every $S(t)$ intersects $\mathcal{C}_H^{c_H}$ transversely in a single circle centered at the point of intersection of $S(t)$ with the x -axis. This finishes the proof of item 1.

Items 2 and 3 follow from item 1 and the details will be left to the reader. Finally, item 4 follows from item 3 after observing that for λ_2 chosen sufficiently close to c_H , then for all $t \in \mathbb{R}$, $\phi_\varepsilon(\mathcal{C}_H^{c_H}) \cap S(t)$ is a translation of the circle $\mathcal{C}_H^{c_H} \cap S(t)$ and these two circles intersect transversely in two points; this is true since $S(t)$ is preserved under the isometry ϕ_ε . q.e.d.

References

- [1] H. Alencar and H. Rosenberg. Some remarks on the existence of hypersurfaces of constant mean curvature with a given boundary, or asymptotic boundary in hyperbolic space. *Bull. Sci. Math.*, 121(1):61–69, 1997. MR1431100, Zbl 0882.53044.
- [2] J. Bernstein and G. Tinaglia. Topological type of limit laminations of embedded minimal disks. *J. Differential Geom.*, 102(1):1–23, 2016. MR3447084.
- [3] T. H. Colding and W. P. Minicozzi II. The Calabi-Yau conjectures for embedded surfaces. *Ann. of Math.*, 167:211–243, 2008. MR2373154, Zbl 1142.53012.
- [4] B. Coskunuzer. Least area planes in hyperbolic 3-space are properly embedded. *Indiana Univ. Math. J.*, 58:381–392, 2009. MR2504417, Zbl 1161.53006.
- [5] B. Coskunuzer. Non-properly embedded minimal planes in hyperbolic 3-space. *Comm. Contemp. Math.*, 13:727–739, 2011. MR2847226, Zbl 1232.53010.
- [6] M. J. Gomes. Spherical surfaces with constant mean curvature in hyperbolic space. *Bol. Soc. Brasil. Mat.*, 18(2):49–73, 1987. MR1018445, Zbl 0752.53034.
- [7] R. Gulliver. The Plateau problem for surfaces of prescribed mean curvature in a Riemannian manifold. *J. Differential Geom.*, 8:317–330, 1973. MR0341260, Zbl 0275.53033.
- [8] W. H. Meeks III and H. Rosenberg. The minimal lamination closure theorem. *Duke Math. Journal*, 133(3):467–497, 2006. MR2228460, Zbl 1098.53007.
- [9] W. H. Meeks III and G. Tinaglia. Curvature estimates for constant mean curvature surfaces. Preprint available at <http://www.mth.kcl.ac.uk/~tinaglia/research.htm>.
- [10] W. H. Meeks III and G. Tinaglia. Embedded Calabi–Yau problem in hyperbolic 3-manifolds. Work in progress.
- [11] N. Nadirashvili. Hadamard’s and Calabi–Yau’s conjectures on negatively curved and minimal surfaces. *Invent. Math.*, 126(3):457–465, 1996. MR1419004, Zbl 0881.53053.
- [12] M. M. Rodríguez and G. Tinaglia. Non-proper complete minimal surfaces embedded in $\mathbb{H}^2 \times \mathbb{R}$. *IMRN*, (12):4322–4334, 2015. MR3356755, Zbl 1328.53079.
- [13] H. Rosenberg, R. Souam, and E. Toubiana. General curvature estimates for stable H -surfaces in 3-manifolds and applications. *J. Differential Geom.*, 84(3):623–648, 2010. MR2669367, Zbl 1198.53062.
- [14] Y. Tonegawa. Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space. *Math. Z.*, 221:591–615, 1996. MR0486522, Zbl 0353.53032.

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