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STABILITY OF THE BERGMAN KERNEL ON A TOWER OF COVERINGS

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Dedicated to Takeo Ohsawa Sensei on the occasion of his 65^{th} *birthday*

Abstract

We obtain several results about stability of the Bergman kernel on a tower of coverings on complex manifolds. An effective estimate for stability of the Bergman kernel is given for a tower of coverings on a compact Riemann surface of genus ≥ 2. Stability of the Bergman kernel is established for towers of coverings on all hyperbolic Riemann surfaces and on complete Kähler manifolds that satisfy certain potential conditions. As a consequence, stability of the Bergman kernel is established for any tower of coverings of Riemann surfaces.

1. Introduction

The classical Bergman kernel – the reproducing kernel for L^2-holomorphic functions – has long played an important role in complex analysis. Its generalization to complex manifolds – in this case, the kernel for the projection onto the space of harmonic (p, q)-forms with L^2-coefficients – is encoded with information on the algebraic and geometric structures of the underlying manifolds. How the Bergman kernel behaves as the underlying structures change is a problem that has been extensively studied in a number of settings. Convergence of the Bergman kernel associated to tensor powers of a positive holomorphic line bundle over a compact complex manifold as the power goes to infinity was established in the celebrated work of Yau [**54**], Tian [**51**], Zelditch [**58**], and Catlin [**10**]. (See [**3**] and references therein for recent developments.)

In this paper, we study stability of the Bergman kernel on a quotient $\widetilde{M}/\Gamma$ of a complex manifold $\widetilde{M}$ by a free and properly discontinuous group Γ of automorphisms of $\widetilde{M}$ as Γ shrinks to the identity. We first recall the setup in Riemannian geometry. Let $\widetilde{M}$ be a Riemannian manifold and Γ a free and properly discontinuous group of isometries of $\widetilde{M}$. A *tower of subgroups* of Γ is a nested sequence of subgroups

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$\Gamma = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_j \supset \cdots \supset \cap \Gamma_j = \{\text{id}\}$ such that Γ_j is a normal subgroup of Γ of finite index $[\Gamma : \Gamma_j]$ for each j (see [**17**, p. 135]). The smooth manifolds $M_j = \widetilde{M}/\Gamma_j$ are equipped with the push-downs of the Riemannian metric on $\widetilde{M}$. The family $\{M_j\}$ is called a *tower of coverings* on the Riemannian manifold $M = \widetilde{M}/\Gamma$. We will refer to $\widetilde{M}$ as the *top manifold*, M_j the *covering manifolds*, and $M = \widetilde{M}/\Gamma$ the *base manifold* of the tower of coverings. (In applications, the top manifold is usually assumed to be the universal covering of the base manifold. However, we make no such assumption in this paper.) It is well known that every Riemannian manifold whose fundamental group is isomorphic to a finitely generated subgroup of $SL(n, \mathbb{C})$ admits a tower of coverings with the top manifold being the universal covering (cf. [**7**, Theorem B and Proposition 2.3]). This is the case, for instance, for an arithmetic quotient of a bounded symmetric domain. Limiting behavior of the spectrum of the Laplacian on M_j as $j \to \infty$ was studied for a tower of coverings on symmetric spaces of non-compact type by DeGeorge–Wallach [**17**].

In his work [**29**, **30**], Kazhdan employed the Bergman kernel to study arithmetic varieties and initiated the study of the Bergman kernel on a tower of coverings on a complex manifold. It follows from his work that for a tower of coverings $\{\widetilde{M}/\Gamma_j\}$ on a compact complex manifold, the Bergman kernel on the universal covering $\widetilde{M}$ is nontrivial provided $\limsup_{j\to\infty} h^{n,0}(M_j)/[\Gamma : \Gamma_j] > 0$, where $h^{n,0}(M_j)$ is the dimension of the space of global sections of the canonical line bundle on M_j (see [**30**, Theorem 1]). Kazhdan suggested that for a tower of coverings on a Riemann surface, the pull-back of the Bergman metric on M_j converges to that of the upper half plane $\widetilde{M} = \mathbb{H}$ (see [**35**, p. 12]). In [**54**, p. 139], Yau stated, as a result attributed to Kazhdan, that this also holds for a tower of coverings on any compact complex manifold. (See Section 4 below for a discussion on the link between Kazhdan's inequality and convergence of the Bergman kernels.)

For brevity, a tower of coverings $\{M_j\}$ on a complex manifold is said to be *Bergman stable* if the pull-back of the Bergman kernel on M_j converges locally uniformly to that of the top manifold $\widetilde{M}$ as $j \to \infty$. In 1993, Rhodes showed that a tower of covering on a compact Riemann surface of genus $g \geq 2$ is indeed Bergman stable [**43**]. Donnelly [**21**] proved analogous results for a tower of coverings on a Riemannian manifold M under the conditions that $\widetilde{M}$ has bounded sectional curvature and the smallest nonzero eigenvalue of the Laplacian on M_j is uniformly bounded from below by a positive constant. His method was based on Cheeger–Gromov–Taylor's estimates of the heat kernel [**14**] and Atiyah's L^2-index theorem [**2**]. Building on Donnelly's work, Yeung showed in [**56**, **57**] that the canonical line bundle of M_j is very ample when the injectivity radius of M_j is greater than a certain effective constant depending on the top manifold $\widetilde{M}$ and the base manifold M. More recently, using Donnelly–Fefferman's L^2-estimate for the $\bar{\partial}$-operator, Ohsawa [**39**] established Bergman stability for a tower of coverings on a complex manifold under certain assumptions on successive approximations of $\overline{\partial}$-closed $(n,0)$-forms on M_j by those on $\widetilde{M}$. He further gave an example of a tower of *non-normal* coverings on a compact Riemann surface that is not Bergman stable ([**40**]).

In this paper, we study stability of the Bergman kernel on a tower of coverings of complex manifolds, with an emphasis on the case when the base manifold is non-compact. For a tower of coverings on a compact Riemann surface, we establish the following effective version of Rhodes' theorem:

Theorem 1.1. *Let $M_j = \mathbb{D}/\Gamma_j$ be a tower of coverings on a compact Riemann surface of genus $g \geq 2$. Let τ_j be the injectivity radius of M_j and let $|\cdot|_{\rm hyp}$ be the pointwise length with respect to the hyperbolic metric. Then the Bergman kernel K_{M_j} of M_j satisfies*

$$|4\pi|K_{M_j}|_{\rm hyp} - 1| \leq \frac{12 \cdot 3^{2/3}}{\pi}(g-1)^{1/3}e^{-\tau_j/3}, \tag{1.1}$$

when $\tau_j \geq \log 3$. Furthermore, a similar estimate also holds for the Bergman metric.

Our proof of the above theorem is elementary; it uses only the Gauss–Bonnet formula and the reproducing property of the Bergman kernel. For a tower of coverings on a compact complex manifold, we exhibit a connection between the Bergman stability and the theory of L^2-Betti numbers, an area studied extensively in the literatures (cf. [**11**, **12**, **33**, **55**]; see Section 4 below).

The main focus in this paper, however, is on towers of coverings on *noncompact* complex manifolds. Recall that a Riemann surface M is *hyperbolic* if it carries a negative nonconstant subharmonic function. This is equivalent to existence of the Green function on M. (We refer the reader to [**24**, Chapter IV] for relevant background material.) For noncompact Riemann surfaces, as an application of the classical Myrberg's formula [**37**] and an idea from [**25**], we have:

Theorem 1.2. *Any tower of coverings on a hyperbolic Riemann surface is Bergman stable.*

Our main result on higher dimensional non-compact complex manifolds can be stated as follows:

Theorem 1.3. *Let M and $\widetilde{M}$ be complete Kähler manifolds with associated Kähler forms ω and $\widetilde{\omega}$ respectively. Let $M_j = \widetilde{M}/\Gamma_j$ be a tower of coverings on M. Then the tower is Bergman stable provided the following two conditions are satisfied:*

1) *There exist a compact set $K \subset M$, a C^2-smooth plurisubharmonic function ψ on $M \setminus K$, and a constant $C > 0$ such that $C^{-1}\omega \leq \partial\bar{\partial}\psi \leq C\omega$ and $\partial\bar{\partial}\psi \geq C^{-1}\partial\psi \wedge \bar{\partial}\psi$ on $M \setminus K$.*
2) *There exist a C^2-smooth plurisubharmonic function $\widetilde{\psi}$ on $\widetilde{M}$ and a constant $\widetilde{C} > 0$ such that $\widetilde{C}^{-1}\widetilde{\omega} \leq \partial\bar{\partial}\widetilde{\psi} \leq \widetilde{C}\widetilde{\omega}$ and $\partial\bar{\partial}\widetilde{\psi} \geq \widetilde{C}^{-1}\partial\widetilde{\psi} \wedge \bar{\partial}\widetilde{\psi}$ on $\widetilde{M}$.*

Recall that M is a hyperconvex complex manifold if it admits a C^2 strongly plurisubharmonic exhaustion function. It is easy to see that the conditions in the above theorem is satisfied if the base manifold M is hyperconvex (see Corollary 6.6 below). As a consequence of Theorem 1.2 and Theorem 1.3, we establish the following result, confirming a suggestion by Kazhdan (see [**35**, p. 12] and [**54**, p. 139]):

Theorem 1.4. *Any tower of coverings of Riemann surfaces with a simply-connected top manifold is Bergman stable.*

Our proof of Theorem 1.3 uses Donnelly–Fefferman type L^2-estimates [**22**] for the $\overline{\partial}$-Laplacian. It also uses spectral theory: the conditions on $\widetilde{M}$ and M ensure that the spectrum and essential spectrum of the respective complex Laplacian on $(n, 1)$-forms on $\widetilde{M}$ and M are positive. However, here instead of estimating the heat kernel as in [**21**], we study the spectral (Bergman) kernel. This enables us to streamline the arguments and replace curvature conditions by potential theoretic conditions on manifolds M and $\widetilde{M}$.

This paper is organized as follows. In Section 2, we review relevant definitions and properties for a tower of coverings and the Bergman kernel. Theorem 1.1 is proved in Section 3. In Section 4, we exhibit a connection among Kazhdan's inequality, the L^2-Betti numbers, and stability of the Bergman kernel on towers of coverings on compact complex manifolds. Theorem 1.2 is proved in Section 5 and Theorem 1.3 in Section 6. Applications of Theorem 1.3 to quotients of the polydisc and the ball are given in Section 7.

Throughout this paper, we will use C to denote a positive constant which may be different in various appearances. Also, the notation $f \gtrsim g$ means $f \geq Cg$ where C is a constant, its independence of certain parameters being clear from the context; and $A \approx B$ means $A \gtrsim B$ and $B \gtrsim A$.

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2. Preliminaries

We briefly recall the definition of the Bergman kernel and metric on a complex manifold (see [**31**]). Let X be a complex manifold and $A^2_{(n,0)}(X)$ space of square integrable holomorphic $(n,0)$-forms f equipped with the inner product

$$\langle f, g\rangle = i^{n^2} 2^{-n} \int_X f \wedge \overline{g}. \tag{2.1}$$

The Bergman kernel is an $2n$-form on $X \times X$ given by

$$K_X(z,w) = \sum_{j=1}^{\infty} b_j(z) \wedge \overline{b_j(w)}, \tag{2.2}$$

where $\{b_j\}$ is an orthonormal basis for $A^2_{(n,0)}(X)$. Write $K^z_X(\cdot) = K_X(z,\cdot)$. Then the Bergman kernel has the following reproducing property:

$$f(z) = (-1)^{n^2} \int_X K^z_X \wedge f, \qquad \forall f \in A^2_{(n,0)}(X). \tag{2.3}$$

The Bergman kernel on diagonal $K_X(z) = K_X(z,z)$ is a biholomorphically invariant (n,n)-form on X with the extremal property:

$$K_X(z) = \max\{f(z) \wedge \overline{f(z)} \mid f \in A^2_{(n,0)}(X), \|f\| = 1\}, \tag{2.4}$$

where $\|\cdot\|$ denotes the L^2-norm defined by (2.1). Furthermore, it satisfies the following decreasing property: $K_{X_2}(z) \le K_{X_1}(z)$ if X_1 and X_2 are subdomains of X with $X_1 \subset X_2$.

Given a local holomorphic coordinate chart $(z_1,\dots,z_n)$, write $\omega_n = \wedge_{j=1}^n (\frac{i}{2} dz_j \wedge d\bar{z}_j)$ and

$$K_X(z) = K^*(z)\omega_n.$$

When $K^*(z) > 0$, the Bergman (pseudo-)metric is given by

$$ds^2_X = \sum_{j,k=1}^n \frac{\partial^2 \log K^*(z)}{\partial z_j \partial \bar{z}_k} dz_j d\bar{z}_k.$$

The Bergman metric is a biholomorphically invariant metric and it can be regarded as the pull-back of the Fubini–Study metric of (possibly infinitely dimensional) complex projective spaces ([**31**]).

We review elements of the L^2-cohomology theory for the $\bar\partial$-operator. Let (M,ω) be a complex hermitian manifold of complex dimension n. Let $C^{p,q}_0(M)$ be the space of C^∞ (p,q)-forms with compact supports on M and let $L^{p,q}_{(2)}(M)$ be the completion of $C^{p,q}_0(M)$ with respect to the following L^2-norm:

$$\|u\| = \left(\int_M |u|^2 dV\right)^{1/2},$$

where $|\cdot|$ is the point-wise norm corresponding to ω and dV the volume form. The weak maximal extension $\bar\partial : L^{p,q}_{(2)}(M) \to L^{p,q+1}_{(2)}(M)$ is a densely defined closed operator. Let $\bar\partial^*$ be the adjoint of $\bar\partial$. Then the $\bar\partial$-Laplacian is $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ and the space of L^2-harmonic (p,q)-forms is given by

$$\mathcal{H}^{p,q}_{(2)}(M) = \left\{u \in L^{p,q}_{(2)}(M) : \Box u = 0\right\} = \left\{u \in L^{p,q}_{(2)}(M) : \bar\partial u = \bar\partial^* u = 0\right\}.$$

We now review relevant basic facts on covering spaces. Let $(\widetilde{M}, \widetilde{\omega})$ be a Riemannian manifold. Let Γ be a subgroup of the isometries that acts freely and properly discontinuously on $\widetilde{M}$. (Recall that Γ acts *freely* if the identify map is the only element in Γ that has a fixed point and *properly discontinuously* if for any compact set K, there is only finitely many $\gamma \in \Gamma$ such that $K \cap \gamma K \neq \emptyset$.) Let $M = \widetilde{M}/\Gamma$ be the quotient manifold and $\pi\colon \widetilde{M} \to M$ the covering map. We equip M with the push-down metric ω from $\widetilde{M}$ so that $\pi^*(\omega) = \widetilde{\omega}$. Denote by $d_{\widetilde{M}}$ and d_M the distances on $\widetilde{M}$ and M associated with $\widetilde{\omega}$ and ω respectively. For $x \in \widetilde{M}$, let

$$D(x) = \{y \in \widetilde{M} \mid d_{\widetilde{M}}(y, x) < d_{\widetilde{M}}(y, \gamma x), \forall \gamma \in \Gamma \setminus \{1\}\} \tag{2.5}$$

be the *Dirichlet fundamental domain* with center at x. It is easy to see that no pair of points in $D(x)$ are equivalent under Γ and every point in $\widetilde{M}$ has an equivalent point in $D(x)$ or its boundary. Let

$$\tau(x) = \frac{1}{2} \inf\left\{d_{\widetilde{M}}(x, \gamma x) : \gamma \in \Gamma \setminus \{1\}\right\}. \tag{2.6}$$

Evidently, the geodesic ball $B(x, \tau(x))$ is contained in $D(x)$. Moreover, when $\widetilde{M}$ has no conjugate points, (*i.e.*, any two points are joint uniquely – up to reparametrization – by a geodesic), $\tau(x)$ is the injectivity radius of $\pi(x)$ in M. In particular, this is the case when $\widetilde{M} = \mathbb{D}$, the unit disk.

Let $\{\Gamma_j\}$ be a tower of subgroups of Γ. Denote by $\tau_j(x)$ the quantity defined by (2.6) with Γ replaced by Γ_j. Since $\tau_j(\cdot)$ is invariant under Γ_j, it can be pushed down onto M_j. The following lemma is well known (compare, e.g., [**17**, Theorem 2.1] and [**20**, Lemma 2.1]):

Lemma 2.1. *$\tau_j(x)$ is an increasing sequence of positive continuous functions such that $\tau_j(x) \to \infty$ locally uniformly on $\widetilde{M}$ as $j \to \infty$.*

Proof. For the reader's convenience, we include a proof here. Since $\tau_j(x)$ is the infinum of a sequence of continuous functions, it is itself upper semi-continuous. To prove that $\tau_j(x)$ is continuous, it suffices to show that $A_\alpha = \{x \in \widetilde{M} \mid \tau_j(x) > \alpha\}$ is open for any $\alpha \geq 0$. Let $x_0 \in A_\alpha$. Choose $\varepsilon > 0$ sufficiently small so that $\tau_j(x_0) > \alpha + \varepsilon$. Then for any $x \in B(x_0, \varepsilon/2)$ and $\gamma \in \Gamma_j$,

$$\begin{aligned} d_{\widetilde{M}}(x, \gamma x) &\geq d_{\widetilde{M}}(x_0, \gamma x_0) - d_{\widetilde{M}}(x, x_0) - d_{\widetilde{M}}(\gamma x, \gamma x_0) \\ &= d_{\widetilde{M}}(x_0, \gamma x_0) - 2d_{\widetilde{M}}(x, x_0) \geq 2\alpha + \varepsilon. \end{aligned}$$

Thus $B(x_0, \varepsilon/2) \subset A_\alpha$. Therefore, A_α is open and hence $\tau_j(x)$ is continuous. It follows from the properly discontinuous property of Γ that $\tau_j(x) > 0$ and $\tau_j(x) \to \infty$ for any $x \in \widetilde{M}$. Since $\Gamma_j \supset \Gamma_{j+1}$, $\tau_j(x) \leq \tau_{j+1}(x)$. It then follows from Dini's theorem that $\tau_j(x) \to \infty$ locally uniformly on $\widetilde{M}$. q.e.d.

Hereafter, we assume that $(\widetilde{M}, \widetilde{\omega})$ is a complex Hermitian manifold and $\Gamma \subset \mathrm{Aut}(\widetilde{\mathrm{M}})$, the automorphism group of $\widetilde{M}$. Let $M_j = \widetilde{M}/\Gamma_j$ and $\widetilde{p}_j \colon \widetilde{M} \to M_j$ be the natural projection. Throughout the paper, when it is contextually clear, for the economy of notations, we will identify $K_{\widetilde{M}}$ with its push-down to M and likewise K_{M_j} with its pull-back on $\widetilde{M}$. The following proposition establishes the upper semi-continuity of the Bergman kernel K_{M_j} on a tower of coverings on complex manifolds (compare [**21**, Proposition 1.2]).

Proposition 2.2. *For each* $z \in \widetilde{M}$, $\limsup_{j\to\infty} \widetilde{p}_j^*(K_{M_j})(z) \leq K_{\widetilde{M}}(z)$.

Proof. Let $D_j(z) \subset \widetilde{M}$ be the Dirichlet fundamental domain of M_j as defined in (2.5) by replacing Γ by Γ_j. Then $\widetilde{p}_j$ maps $D_j(z)$ biholomorphically onto its image $\widetilde{p}_j(D_j(z))$. It follows from the decreasing property of the Bergman kernel that

$$K_{M_j}(\widetilde{p}_j(z)) \leq K_{\widetilde{p}_j(D_j(z))}(\widetilde{p}_j(z)).$$

Since $B(z, \tau_j(z)) \subset D_j(z)$, we have

$$\widetilde{p}_j^*(K_{M_j})(z) \leq \widetilde{p}_j^*(K_{\widetilde{p}_j(D_j(z))})(z) = K_{D_j(z)}(z) \leq K_{B(z,\tau_j(z))}(z). \tag{2.7}$$

We then conclude the proof by applying Lemma 2.1 and Ramadanov's theorem ([**42**]). q.e.d.

As an application of the above proposition, we provide a proof of the following version of Kazhdan's inequality (see [**30**, Theorem 1 and its proof]; also [**28**, pp. 13 and pp. 153]):

Proposition 2.3. *When* M *is compact,*

$$\limsup_{j\to\infty} \frac{h^{n,0}(M_j)}{[\Gamma : \Gamma_j]} \leq \int_M K_{\widetilde{M}}. \tag{2.8}$$

Proof. Since M is compact, there exists a compact set $A \subset \widetilde{M}$ such that $\widetilde{p}(A) = M$, where $\widetilde{p} = \widetilde{p}_1$ is as before the natural projection from $\widetilde{M}$ onto $M = M_1$. Let ε be the minimum of $\tau(x)$ on A. We cover A by finitely many geodesic balls $\{B(z_k, r_k/2)\}_{k=1}^m$ with $z_k \in A$ and $r_k < \varepsilon$

such that each $B(z_k, r_k)$ is contained in a normal neighborhood in $\widetilde{M}$. It follows from (2.7) that for $z \in B(z_k, r_k/2)$,

$$\widetilde{p}_j^*(K_{M_j})(z) \le K_{B(z,\tau(z))}(z) \le K_{B(z_k,r_k/2)}(z).$$

It then follows that K_{M_j} is uniformly bounded from above on M. (Here we identify K_{M_j} with its push-down onto M.) Inequality (2.8) is then obtained by integrating both sides of the inequality in Proposition 2.2 over M, the dominated convergence theorem, and the fact that

$$\int_M K_{M_j} = \frac{1}{[\Gamma : \Gamma_j]} \int_{M_j} K_{M_j} = \frac{h^{n,0}(M_j)}{[\Gamma : \Gamma_j]}. \qquad \text{q.e.d.}$$

The following proposition establishes the link between the convergence of the Bergman kernel and the Bergman metric (compare [**42**]).

Proposition 2.4. *Suppose the Bergman kernel $K_{\widetilde{M}}(z,z)$ is positive and the tower of coverings $\{M_j\}$ is Bergman stable. Then the pull-back of Bergman metric of M_j converges locally uniformly to the Bergman metric on $\widetilde{M}$.*

Proof. Let $z_0 \in \widetilde{M}$. Recall that $B_j = B(z_0, \tau_j(z_0)) \subset D_j(z_0)$, the Dirichlet fundamental domain of M_j with center at z_0. Denote by $\|\cdot\|_{B_j}$ the L^2-norm as defined by (2.1) over B_j. Let $w \in B_j$. It follows from the reproducing property of the Bergman kernel that

$$\begin{aligned}
&\left\|\widetilde{p}_j^*(K_{M_j})(\cdot,w) - K_{B_j}(\cdot,w)\right\|^2_{B_j} \\
&= \left\|\widetilde{p}_j^*(K_{M_j})(\cdot,w)\right\|^2_{B_j} - 2\mathrm{Re}\langle \widetilde{p}_j^*(K_{M_j})(\cdot,w), K_{B_j}(\cdot,w)\rangle + \left\|K_{B_j}(\cdot,w)\right\|^2_{B_j} \\
&= \left\|K_{M_j}(\cdot,w)\right\|^2_{\widetilde{p}_j(B_j)} - 2\widetilde{p}_j^*(K_{M_j})(w,w) + K_{B_j}(w,w) \\
&\le \left\|K_{M_j}(\cdot,w)\right\|^2_{M_j} - 2\widetilde{p}_j^*(K_{M_j})(w,w) + K_{B_j}(w,w) \\
&= K_{B_j}(w,w) - \widetilde{p}_j^*(K_{M_j})(w,w).
\end{aligned}$$

Similarly,

$$\|K_{\widetilde{M}}(\cdot,w) - K_{B_j}(\cdot,w)\|^2_{B_j} \le K_{B_j}(w,w) - K_{\widetilde{M}}(w,w).$$

Therefore,

$$\text{(2.9)} \quad \|K_{\widetilde{M}}(\cdot,w) - \widetilde{p}_j^*(K_{M_j})(\cdot,w)\|^2_{B_j} \le \\ 2\big(2K_{B_j}(w,w) - K_{\widetilde{M}}(w,w) - \widetilde{p}_j^*(K_{M_j})(w,w)\big).$$

It follows from Ramadanov's theorem [**42**] and the Bergman stability assumption that the right hand side above converges locally uniformly to zero for w near z_0. Since the Bergman metric is given locally by

$$\partial\bar\partial \log K^* = \frac{\partial\bar\partial K^*}{K^*} - \frac{\partial K^* \wedge \bar\partial K^*}{(K^*)^2},$$

we then conclude the proof by applying the Cauchy integral formula and the Cauchy–Schwarz inequality. q.e.d.

3. Effective estimates

Recall that the hyperbolic metric on the unit disk $\mathbb{D}$ is given by

$$ds^2_{\text{hyp}} = \frac{4\,|dz|^2}{(1-|z|^2)^2},$$

and the hyperbolic distance between z and 0 is

$$\text{dist}_{\text{hyp}}(0,z) = \log\frac{1+|z|}{1-|z|}.$$

It follows that the Euclidean and hyperbolic balls $B_{\text{eucl}}(0,r)$ and $B_{\text{hyp}}(0,\tau)$ are identical provided

$$r = \frac{e^\tau - 1}{e^\tau + 1}.$$

Furthermore,

$$K_{B_{\text{hyp}}(0,\tau)}(0) = \frac{1}{\pi}\frac{(e^\tau+1)^2}{(e^\tau-1)^2}dz\wedge d\bar{z},$$

where $K_{B_{\text{hyp}}(0,\tau)}$ denotes the Bergman kernel form on $B_{\text{hyp}}(0,\tau)$. Let $p\colon \mathbb{D}\to M$ be a covering map on a Riemann surface M. Then the hyperbolic metric $ds^2_{\text{hyp},M}$ satisfies

$$p^*(ds^2_{\text{hyp},M}) = ds^2_{\text{hyp}}.$$

Thus for a form K on M, we have $|p^*(K)(z)|_{\text{hyp}} = |K(p(z))|_{\text{hyp},M}$, where $|\cdot|_{\text{hyp}}$ denotes the pointwise norm with respect to the hyperbolic metric. (We will drop subscript M when doing so causes no confusion.)

We now prove Theorem 1.1. Let $\widetilde{p}_j\colon \mathbb{D}\to M_j = \mathbb{D}/\Gamma_j$ and $p_j\colon M_j\to M = M_j/(\Gamma/\Gamma_j)$ be the natural projections. Let $\tau_j(w)$ be the injectivity radius at $w\in M_j$ and let τ_j denote the injectivity radius of M_j. For any $z\in\mathbb{D}$, we have

$$\begin{aligned}|K_{M_j}(\widetilde{p}_j(z))|_{\text{hyp}} = |\widetilde{p}_j^*(K_{M_j})(z)|_{\text{hyp}} &\le |K_{B_{\text{hyp}}(z,\tau_j(z))}(z)|_{\text{hyp}}\\ &\le |K_{B_{\text{hyp}}(0,\tau_j)}(0)|_{\text{hyp}}.\end{aligned}$$

Hence

$$|K_{M_j}(\widetilde{p}_j(z))|_{\text{hyp}} - \frac{1}{4\pi} \le \frac{1}{\pi}\frac{e^{\tau_j}}{(e^{\tau_j}-1)^2}. \tag{3.1}$$

It follows from the Gauss–Bonnet theorem that

$$4\pi(g_j-1) = \text{vol}_{\text{hyp}}(M_j).$$

Since $\text{vol}_{\text{hyp}}(M_j) = [\Gamma:\Gamma_j]\text{vol}_{\text{hyp}}(M)$, we obtain

$$\frac{g_j}{[\Gamma:\Gamma_j]} = g - 1 + \frac{1}{[\Gamma:\Gamma_j]}.$$

Note that

$$g_j = \int_{M_j} |K_{M_j}|_{\text{hyp}} dV_{\text{hyp}},$$

and $|K_{M_j}|_{\text{hyp}}$ is invariant under the deck transformations of $p_j\colon M_j \to M$. Thus

$$\int_M |K_{M_j}|_{\text{hyp}} dV_{\text{hyp}} = \int_M \frac{1}{4\pi} dV_{\text{hyp}} + \frac{1}{[\Gamma : \Gamma_j]} \geq \int_M \frac{1}{4\pi}\, dV_{\text{hyp}}. \tag{3.2}$$

Here we identify K_{M_j} with its push-down to M.

It suffices to prove the theorem at $z = 0$; the general case is reduced to this case by applying a Möbius transformation. Let $r_j = (e^{\tau_j} - 1)/(e^{\tau_j} + 1)$. Then the Euclidean disk $B_{\text{eucl}}(0, r_j)$ is identical to the hyperbolic disk $B_{\text{hyp}}(0, \tau_j)$ and it is contained in the Dirichlet fundamental domain $D_j(0)$ of M_j. Write $\widetilde{p}_j^*(K_{M_j})(z) = K_{M_j}^*(z) dz \wedge d\bar{z}$ on $B_j = B_{\text{eucl}}(0, r_j)$. Let ϵ be a sufficiently small positive constant to be chosen. For $w \in B_j$, let f be a holomorphic 1-form on M_j with unit L^2-norm (as defined by (2.1)) such that $K_{M_j}(\widetilde{p}_j(w)) = f_j(\widetilde{p}_j(w)) \wedge \overline{f_j(\widetilde{p}_j(w))}$. Write $\widetilde{p}_j^*(f)(z) = f^*(z) dz$. Then

$$\int_{B_j} |f^*|^2 dV_{\text{eucl}} = \frac{1}{2} \left| \int_{B_j} \widetilde{p}_j^*(f) \wedge \overline{\widetilde{p}_j^*(f)} \right| \leq \frac{1}{2} \left| \int_{M_j} f \wedge \bar{f} \right| = 1.$$

Since

$$(f^*(z))^2 = \frac{r_j^2}{\pi} \int_{|\zeta| < r_j} \frac{(f^*(\zeta))^2}{(r_j^2 - z\bar{\zeta})^2} dV_{\text{eucl}}, \quad z \in B_j, \tag{3.3}$$

it follows that

$$|(f^*(z))^2 - (f^*(z'))^2| \leq \frac{48}{\pi r_j^3} |z - z'|,$$

for all $z, z' \in \frac{1}{2} B_j$. Now suppose $w \in \epsilon B_j$ where ϵ is a sufficiently small number to be chosen. Then

$$\begin{aligned} |\widetilde{p}_j^*(K_{M_j})(w)|_{\text{hyp}} &= \frac{(1 - |w|^2)^2}{4} |f^*(w)|^2 \leq \frac{1}{4} \left(|f^*(0)|^2 + \frac{48\epsilon}{\pi r_j^2} \right) \\ &\leq |K_{M_j}(0)|_{\text{hyp}} + \frac{12\epsilon}{\pi r_j^2}. \end{aligned} \tag{3.4}$$

From (3.2) and then (3.1), we have

$$\begin{aligned} \int_{\widetilde{p}_1(\epsilon B_j)} \left(\frac{1}{4\pi} - |K_{M_j}|_{\text{hyp}} \right) dV_{\text{hyp}} &\leq \int_{M \setminus \widetilde{p}_1(\epsilon B_j)} \left(|K_{M_j}|_{\text{hyp}} - \frac{1}{4\pi} \right) dV_{\text{hyp}} \\ &\leq \frac{4(g-1)e^{\tau_j}}{(e^{\tau_j} - 1)^2}. \end{aligned}$$

Combining this with (3.4), we obtain

$$\frac{1}{4\pi} - |K_{M_j}|_{\text{hyp}}(0) - \frac{12\epsilon}{\pi r_j^2} \leq \frac{4(g-1)e^{\tau_j}}{(e^{\tau_j} - 1)^2} \frac{1}{\text{vol}_{\text{hyp}}(\epsilon B_j)}.$$

Since $\mathrm{vol}_{\mathrm{hyp}}(\epsilon B_j) = 4\pi(\epsilon r_j)^2/(1-(\epsilon r_j)^2) \geq 4\pi\epsilon^2 r_j^2$,

$$\frac{1}{4\pi} - |K_{M_j}|_{\mathrm{hyp}}(0) \leq \frac{12\epsilon}{\pi r_j^2} + \frac{(g-1)e^{\tau_j}}{\pi(e^{\tau_j}-1)^2}\frac{1}{\epsilon^2 r_j^2}.$$

Choosing $\epsilon = ((g-1)e^{\tau_j}/6(e^{\tau_j}-1)^2)^{1/3}$, we then have

$$\frac{1}{4\pi} - |K_{M_j}|_{\mathrm{hyp}}(0) \leq \frac{18}{\pi r_j^2}\left(\frac{(g-1)e^{\tau_j}}{6(e^{\tau_j}-1)^2}\right)^{1/3}.$$

Note that the right hand side above is greater than that in (3.1). Since $\tau_j \geq \log 3$, we have

$$e^{\tau_j} - 1 \geq 2, \qquad r_j = \frac{e^{\tau_j}-1}{e^{\tau_j}+1} \geq \frac{1}{2}.$$

Hence

$$\left| |K_{M_j}(0)|_{\mathrm{hyp}} - \frac{1}{4\pi} \right| \leq \frac{C}{\pi}(g-1)^{1/3} e^{-\tau_j/3},$$

where

$$C = 18 \cdot 4 \cdot (6 \cdot 4)^{-1/3} = 12 \cdot 3^{2/3}.$$

This concludes the proof of Theorem 1.1 for the Bergman kernel.

We now show how to obtain effective estimates for the Bergman metric, without keeping track of the numerical constants. Let $K^*_{M_j}(z,w)$ denote the function on $\mathbb{D}$ representing the pull-back of the Bergman kernel form on M_j. Let $K^*_{\mathbb{D}}$ and $K^*_{B_j}$ be the Bergman kernel functions of $\mathbb{D}$ and $B_j = B_{\mathrm{eucl}}(0, r_j)$ respectively. Assume that $\tau_j \geq \log 3$. Then $r_j \geq 1/2$. From the first part of the theorem, we know that for $w \in \frac{1}{2}\mathbb{D}$,

$$|K^*_{M_j}(w,w) - K^*_{\mathbb{D}}(w,w)| \leq C(g-1)^{1/3} e^{-\tau_j/3}. \tag{3.5}$$

Furthermore, a simple calculation yields that

$$|K^*_{B_j}(w,w) - K^*_{\mathbb{D}}(w,w)| \leq C e^{-\tau_j}. \tag{3.6}$$

Following the same lines of argument as in the proof of (2.9), we have

$$\int_{B_j} |K^*_{M_j}(z,w) - K^*_{\mathbb{D}}(z,w)|^2 dV_{\mathrm{eucl}}(z) \leq 2\big(2K^*_{B_j}(w,w) - K^*_{\mathbb{D}}(w,w) - K^*_{M_j}(w,w)\big). \tag{3.7}$$

Combining (3.5)–(3.7), we then obtain

$$\int_{\frac{1}{2}\mathbb{D}} |K^*_{M_j}(z,w) - K^*_{\mathbb{D}}(z,w)|^2 dV_{\mathrm{eucl}}(z) \leq C(g-1)^{1/3} e^{-\tau_j/3}.$$

Using the reproducing property of the Bergman kernel on $\frac{1}{2}\mathbb{D}$ as in (3.3) and applying the Cauchy–Schwarz inequality, we have for any integers

$\alpha, \beta \geq 0$,

$$\left| \left(\frac{\partial^{\alpha+\beta} K^*_{M_j}}{\partial z^\alpha \partial \bar{z}^\beta} - \frac{\partial^{\alpha+\beta} K^*_{\mathbb{D}}}{\partial z^\alpha \partial \bar{z}^\beta} \right) (0,0) \right| \leq C(g-1)^{1/3} e^{-\tau_j/3}.$$

The above estimates then enable us to obtain an effective estimate for the Bergman metric. We leave the detail to the interested reader.

4. Compact complex manifolds

In this section, we study the Bergman stability for a tower of coverings on a compact complex manifold. In particular, we exhibit a connection between stability of the Bergman kernel and the theory of L^2-Betti numbers, a subject which has been studied extensively in literature (cf. [**11**, **12**, **33**, **55**]). The following proposition is implicit in [**30**][1]:

Proposition 4.1. *A tower of coverings $\{M_j\}$ on a compact complex manifold M is Bergman stable if and only if Kazhdan's inequality* (2.8) *becomes an equality:*

$$\lim_{j\to\infty} \frac{h^{n,0}(M_j)}{[\Gamma : \Gamma_j]} = \int_M K_{\widetilde{M}}. \tag{4.1}$$

This proposition is a consequence of Proposition 2.2. We present a proof below for the reader's convenience. We will need the following lemma:

Lemma 4.2. *Let $\{M_j\}$ be a tower of coverings on a complex manifold. Then the family $\{\widetilde{p}_j^*(K_{M_j})\}$ of the pull-backs of the Bergman kernels of the M_j's is locally equicontinuous on $\widetilde{M}$.*

Proof. Let $z_0 \in \widetilde{M}$. Let $U \subset\subset B_j = B(z_0, \tau_j(z_0))$ be a neighborhood of z_0 contained in a local coordinate chart. Let $K^*_{M_j}(z,w)$ be the Bergman kernel function, representing the pull-back to $\widetilde{M}$ of the Bergman kernel form $K_{M_j}(z,w)$ on M_j. Since

$$\begin{aligned} \left\|\widetilde{p}_j^*(K_{M_j})(\cdot,w)\right\|^2_U &= \left\|K_{M_j}(\cdot,w)\right\|^2_{\widetilde{p}_j(U)} \leq \left\|K_{M_j}(\cdot,w)\right\|^2_{M_j} \\ &= \widetilde{p}_j^*(K_{M_j})(w,w) \leq K_{B_j}(w,w), \end{aligned}$$

and $K_{B_j}(w,w)$ converges uniformly on U to $K_{\widetilde{M}}(w,w)$, the above expressions are uniformly bounded on U. As a consequence,

$$\int_U \int_U |K^*_{M_j}(z,w)|^2 \, dV(z) \, dV(w) \leq C < \infty.$$

The equicontinuity of $\widetilde{p}_j^*(K_{M_j})$ near z_0 then follows from the Cauchy estimate. q.e.d.

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[1]The authors thank Professor David Kazhdan for kindly sharing with us his proof of this proposition.

We now prove Proposition 4.1. The necessity is trivial, following from the uniform convergence theorem as in the proof of Proposition 2.3. To prove the sufficiency, we note that from Proposition 2.2 and (4.1), we have

$$\limsup_{j\to\infty} \widetilde{p}_j^*(K_{M_j})(z) = K_{\widetilde{M}}(z).$$

Thus it suffices to show $\liminf_{j\to\infty} \widetilde{p}_j^*(K_{M_j})(z) \geq K_{\widetilde{M}}(z)$. Proving by contradiction, we assume that there exist $z_0 \in \widetilde{M}$ and $\epsilon > 0$ such

$$K^*_{M_{j_k}}(z_0) < K^*_{\widetilde{M}}(z_0) - \epsilon,$$

for a subsequence $j_k \to \infty$. As before, K^* denotes the function representing the (pull-back) of the Bergman kernel form on a local coordinate chart U near z_0. By Lemma 4.2, after possible shrinking of U, we have

$$\widetilde{p}_{j_k}^*(K_{M_j})(z) < K_{\widetilde{M}}(z) - \frac{1}{2}\epsilon,$$

for $z \in U$. It then follows that

$$\begin{aligned}
\limsup_{j_k\to\infty} \frac{h^{n,0}(M_{j_k})}{[\Gamma : \Gamma_{j_k}]} &= \limsup_{j_k\to\infty} \int_M K_{M_{j_k}} \\
&\leq \limsup_{j_k\to\infty} \int_{M\setminus U} K_{M_{j_k}} + \int_U K_{\widetilde{M}} - \frac{1}{2}\epsilon \mathrm{vol}(U) \\
&\leq \int_M K_{\widetilde{M}} - \frac{1}{2}\epsilon \mathrm{vol}(U),
\end{aligned}$$

contradicting (4.1). We thus conclude the proof of Proposition 4.1.

We now recall relevant facts about the L^2-Betti numbers. (We refer the reader to [**2**], [**11**, **12**, **13**], and [**28**, Section 8] for extensive discussions on related topics.) Let $\widetilde{M}$ be a universal covering and let $M_j = \widetilde{M}/\Gamma_j$ be a tower of coverings on a complete Riemannian manifold M. Let $\mathcal{H}^s_{(2)}(\widetilde{M})$ be the space of L^2-harmonic s-forms on $\widetilde{M}$ corresponding to the d-Laplacian Δ. Let $K^s_{\widetilde{M}}$ be the Schwartz kernel of $\mathcal{H}^s_{(2)}(\widetilde{M})$. The L^2-*Betti number* of M is then given by

$$b^s_{(2)}(M) := \int_M |K^s_{\widetilde{M}}| dV.$$

When M has bounded geometry and finite volume, Cheeger and Gromov showed that

$$\lim_{j\to\infty} \frac{b^s(M_j)}{[\Gamma : \Gamma_j]} = b^s_{(2)}(M), \tag{4.2}$$

where $b^s(M_j)$ is the ordinary s-th Betti number of M_j ([**11**, **12**]). Similar result was obtained by Yeung [**55**] on compact Kähler manifolds with negative sectional curvatures. An analogous result was established for a finite connected CW-complex by Lück [**33**].

When M is a compact Kähler manifold, the L^2-Hodge number $h^{p,q}_{(2)}(M)$ of M is similarly given by

$$h^{p,q}_{(2)}(M) := \int_M |K^{p,q}_{\widetilde{M}}| dV,$$

where $K^{p,q}_{\widetilde{M}}$ is the Schwartz kernel for $\mathcal{H}^{p,q}_{(2)}(\widetilde{M})$, the space of L^2-harmonic (p,q)-forms corresponding to the $\bar\partial$-Laplacian $\Box$.

Proposition 4.3. *A tower of coverings $\{M_j\}$ on an n-dimensional compact Kähler manifold is Bergman stable if* (4.2) *holds for $s = n$.*

Proof. The Hodge–Kodaira decomposition

$$\mathcal{H}^s_{(2)}(\widetilde{M}) = \bigoplus_{p+q=s} \mathcal{H}^{p,q}_{(2)}(\widetilde{M})$$

implies that

$$b^s_{(2)}(M) = \sum_{p+q=s} h^{p,q}_{(2)}(M). \tag{4.3}$$

By Kazhdan's inequality,

$$\limsup_{j\to\infty} \frac{h^{p,q}(M_j)}{[\Gamma : \Gamma_j]} \le h^{p,q}_{(2)}(M) \tag{4.4}$$

([**30**, Theorem 1 and its proof], [**28**, pp. 153]; see also Proposition 2.3 above), where $h^{p,q}(M_j)$ denotes the ordinary Hodge numbers of M_j. Since $b^s(M_j) = \sum_{p+q=s} h^{p,q}(M_j)$, it follows from (4.2)–(4.4) that

$$\lim_{j\to\infty} \frac{h^{p,q}(M_j)}{[\Gamma : \Gamma_j]} = h^{p,q}_{(2)}(M).$$

In particular, we have (4.1) and thus the Bergman stability by Proposition 4.1. q.e.d.

5. Hyperbolic Riemann surfaces

Let Γ be a Fuchsian group, i.e., a properly discontinuous subgroup of $SL(2,\mathbb{R})$. Recall that Γ is of *convergence type* if $\sum_{\gamma\in\Gamma}(1-|\gamma(0)|) < \infty$. We refer the reader to [**52**, Chapter XI] for a treatment of the subject. Let $\widetilde{p}$ be the natural projection from $\mathbb{D}$ onto $\mathbb{D}/\Gamma$ (we will also use $\widetilde{p}$ to denote the natural projection from $\mathbb{D}\times\mathbb{D}$ onto $(\mathbb{D}/\Gamma)\times(\mathbb{D}/\Gamma)$.) A classical result of Myrberg states that Γ is of convergence type if and only if $\mathbb{D}/\Gamma$ is a hyperbolic Riemann surface, and in this case,

$$g_{\mathbb{D}/\Gamma}(\widetilde{p}(z), \widetilde{p}(w)) = \sum_{\gamma\in\Gamma} g_{\mathbb{D}}(z, \gamma(w)) = -\sum_{\gamma\in\Gamma} \log\left|\frac{z-\gamma(w)}{1-\overline{\gamma(w)}z}\right|, \tag{5.1}$$

where $g_{\mathbb{D}/\Gamma}$ and $g_{\mathbb{D}}$ are Green's functions of $\mathbb{D}/\Gamma$ and $\mathbb{D}$ respectively (see [**52**, Theorem XI. 13]). Note that since $g_{\mathbb{D}}(z,0) = g_{\mathbb{D}}(\gamma(z),\gamma(0))$ for any $\gamma \in \Gamma$,

$$1-|z|^2 = \frac{(1-|\gamma(0)|^2)(1-|\gamma(z)|^2)}{|1-\overline{\gamma(0)}\gamma(z)|^2} \le \frac{4(1-|\gamma(0)|)(1-|\gamma(z))|}{\max\{(1-|\gamma(z)|)^2,(1-|\gamma(0)|)^2\}}.$$

Therefore,

$$\frac{(1-|z|^2)(1-|\gamma(0)|)}{4} \le 1-|\gamma(z)| \le \frac{4(1-|\gamma(0)|)}{1-|z|^2},$$

from which it follows that when Γ is of convergent type, $\sum_{\gamma\in\Gamma}(1-|\gamma(z)|) < \infty$ for all $z \in \mathbb{D}$. Using

$$1-\left|\frac{z-\gamma(w)}{1-\overline{\gamma(w)}z}\right|^2 = \frac{(1-|z|^2)(1-|\gamma(w)|^2)}{|1-\overline{\gamma(w)}z|^2} \le 2\frac{1+|z|}{1-|z|}(1-|\gamma(w)|),$$

and the simple inequality $-\log x \le 2(1-x)$ when $x \ge 1/2$, we then have

$$-\frac{1}{2}\log\left|\frac{z-\gamma(w)}{1-\overline{\gamma(w)}z}\right|^2 \le 1-\left|\frac{z-\gamma(w)}{1-\overline{\gamma(w)}z}\right|^2 \le 2\frac{1+|z|}{1-|z|}(1-|\gamma(w)|),$$

provided $1-|\gamma(w)| \le (1-|z|)/4(1+|z|)$. Therefore, the series on the right hand side of (5.1) converges local uniformly in z and likewise in w.

We now establish a transformation formula of Bergman kernel for a normal covering map between hyperbolic Riemann surfaces. A related formula for Reinhardt domains in $\mathbb{C}^n$ was obtained in [**25**].

Proposition 5.1. *Let M and $\widetilde{M}$ be hyperbolic Riemann surfaces. Let $\widetilde{p}\colon \widetilde{M} \to M$ be a normal covering map. Then*

$$\big(\widetilde{p}^* K_M\big)(z,w) = \sum_{\gamma\in\Gamma}\big(\gamma^* K^z_{\widetilde{M}}\big)(w). \tag{5.2}$$

Proof. We first prove the case when $\widetilde{M} = \mathbb{D}$ and $M = \mathbb{D}/\Gamma$ where Γ is a Fuchsian group of convergence type. Write $K_{\mathbb{D}/\Gamma}(z,w) = K^*_{\mathbb{D}/\Gamma}(z,w)\times(\frac{i}{2}dz\wedge d\bar{w})$ where z and w are holomorphic coordinates induced by the covering map. Then (5.2) becomes

$$K^*_{\mathbb{D}/\Gamma}(\widetilde{p}(z),\widetilde{p}(w))\widetilde{p}(z)\overline{\widetilde{p}'(w)} = \sum_{\gamma\in\Gamma} K^*_{\mathbb{D}}(z,\gamma(w))\overline{\gamma'(w)}. \tag{5.3}$$

The above formula then follows from differentiating both sides of (5.1) with respect to z and $\bar{w}$ and by applying Schiffer's formula [**45**].

We now prove the general case. Let $\widetilde{M} = \mathbb{D}/\widetilde{\Gamma}$ and $M = \mathbb{D}/\Gamma$ with $\widetilde{\Gamma}$ a normal subgroup of Γ. Applying (5.3) to both $\mathbb{D}/\widetilde{\Gamma}$ and $\mathbb{D}/\Gamma$ and then combining the results, we then obtain formula (5.2). q.e.d.

We are now in position to prove Theorem 1.2. Let $M_j = \mathbb{D}/\Gamma_j$ be a tower of coverings on a hyperbolic Riemann surface $M = \mathbb{D}/\Gamma$ and $\widetilde{M} = \mathbb{D}/\widetilde{\Gamma}$ be the top manifold. Then Γ_j is a decreasing sequence of normal subgroups such that $\cap\Gamma_j = \widetilde{\Gamma}$. Let $\widehat{\Gamma}_j = \Gamma_j/\widetilde{\Gamma}$. Let $\widetilde{p}_j\colon \widetilde{M} \to M_j$ be the natural projection. Applying (5.3) to $\widetilde{p}_j$, we have

$$K^*_{M_j}(\widetilde{p}(z), \widetilde{p}(w))\widetilde{p_j}(z)\overline{\widetilde{p}'_j(w)} = \sum_{\widehat{\gamma}\in\widehat{\Gamma}_j} K^*_{\widetilde{M}}(z, \widehat{\gamma}(w))\overline{\widehat{\gamma}'(w)}. \tag{5.4}$$

For $\widehat{\gamma} \in \widehat{\Gamma}_j$, write $\widehat{\gamma} = [\gamma_j]$ with $\gamma_j \in \Gamma_j$. Let $\widehat{p}\colon \mathbb{D} \to \mathbb{D}/\widetilde{\Gamma}$ be the natural projection. Let $w \in \mathbb{D}$ and $\widetilde{\gamma} \in \widetilde{\Gamma}$. We have

$$\widetilde{\gamma}'(\gamma_j(w))\widehat{\gamma}'(\widehat{p}(w)) = \frac{\widehat{p}'(\gamma_j(w))\widehat{\gamma}'(\widehat{p}(w))}{\widehat{p}'(\widetilde{\gamma}(\gamma_j(w)))} = \frac{\widehat{p}'(\gamma_j(w))(\widetilde{\gamma}\circ\gamma_j)'(w)}{\widehat{p}'(w)}.$$

Therefore, for $z, w \in \mathbb{D}$,

$$\begin{aligned}
K^*_{\widetilde{M}}(\widehat{p}(z),&\widehat{\gamma}(\widehat{p}(w)))\overline{\widehat{\gamma}'(w)} = K^*_{\widetilde{M}}(\widehat{p}(z), \widehat{p}(\gamma_j(w)))\overline{\widehat{\gamma}'(\widehat{p}(w))}\\
&= \frac{1}{\widehat{p}'(z)}\frac{1}{\overline{\widehat{p}'(\gamma_j(w))}}\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}} K^*_{\mathbb{D}}(z, \widetilde{\gamma}(\gamma_j(w)))\overline{\widetilde{\gamma}'(\gamma_j(w))}\cdot\overline{\widehat{\gamma}'(\widehat{p}(w))}\\
&= \frac{1}{\widehat{p}'(z)}\frac{1}{\overline{\widehat{p}'(w)}}\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}} K^*_{\mathbb{D}}(z, \widetilde{\gamma}(\gamma_j(w)))\overline{(\widetilde{\gamma}\circ\gamma_j)'(w)}. \qquad (5.5)
\end{aligned}$$

Let

$$E_j = K^*_{M_j}(\widetilde{p}(\widehat{p}(z)), \widetilde{p}(\widehat{p}(w)))\widetilde{p}'_j(\widehat{p}(z))\overline{\widetilde{p}'_j(\widehat{p}(w))} - K^*_{\widetilde{M}}(\widehat{p}(z), \widehat{p}(w)).$$

Combining (5.4) and (5.5), we then have:

$$\begin{aligned}
E_j &= \sum_{\widehat{\gamma}\in\widehat{\Gamma}_j\setminus\{1\}} K^*_{\widetilde{M}}(\widehat{p}(z), \widehat{\gamma}(\widehat{p}(w)))\overline{\widehat{\gamma}'(\widehat{p}(w))}\\
&= \frac{1}{\widehat{p}'(z)\overline{\widehat{p}'(w)}}\sum_{[\gamma_j]\in\widehat{\Gamma}_j\setminus\{1\}}\sum_{\widetilde{\gamma}\in\widetilde{\Gamma}} K^*_{\mathbb{D}}(z, \widetilde{\gamma}\circ\gamma_j(w))\overline{(\widetilde{\gamma}\circ\gamma_j)'(w)}\\
&= \frac{1}{\widehat{p}'(z)\overline{\widehat{p}'(w)}}\sum_{\gamma\in\Gamma_j\setminus\widetilde{\Gamma}} K^*_{\mathbb{D}}(z, \gamma(w))\overline{\gamma'(w)}.
\end{aligned}$$

It follows from a simple computation that

$$|E_j| \le \frac{1}{\pi|\widehat{p}'(z)\widehat{p}'(w)|(1-|z|)^2(1-|w|)^2}\sum_{\gamma\in\Gamma_j\setminus\widetilde{\Gamma}}(1-|\gamma(0)|^2).$$

Since $\cap\Gamma_j = \widetilde{\Gamma}$, we have $|E_j| \to 0$ locally uniformly as $j \to \infty$. This concludes the proof of Theorem 1.2.

Remark. We have not used the condition $[\Gamma : \Gamma_j] < \infty$ in the above proof. Theorem 1.2 remains true even if the finiteness assumption on the indices $[\Gamma : \Gamma_j]$ is dropped from the definition of tower of coverings.

6. Complete Kähler manifolds

In this section, we study towers of coverings on complete Kähler manifolds, using tools from spectral theory of the complex Laplacian. Let (M,ω) be a complete Kähler manifold. Let $\square^M_{p,q}$ be the $\overline{\partial}$-Laplacian on $L^{p,q}_{(2)}(M)$. (We will drop the subscript or superscript from the notation $\square^M_{p,q}$ when doing so causes no confusions.) Denoted by $\sigma(\square)$ and $\sigma_e(\square)$ respectively the spectrum and essential spectrum of the operator $\square$. The following lemma is well-known:

Lemma 6.1. *Suppose there is a compact set $K \subset M$ and a constant $C > 0$ such that*

$$\|u\|^2 \leq C\left(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \int_K |u|^2 dV\right) \tag{6.1}$$

holds for all $u \in \mathrm{Dom}\,(\bar\partial) \cap \mathrm{Dom}\,(\bar\partial^*) \cap L^{p,q}_{(2)}(M)$. Then $\sigma_e(\square^M_{p,q}) \subset [\frac{1}{C},\ 0)$.

Proof. Let $\lambda \in \sigma_e(\square)$. Then by the Weyl criteria (cf. [**53**, Theorem 7.24]), there exists a sequence $u_j \in \mathrm{Dom}\,(\square)$ such that $\|u_j\| = 1$, $\|(\square - \lambda)u_j\| \to 0$, and $u_j \to 0$ weakly. Thus

$$\lim_{j\to\infty}\left(\|\overline{\partial} u_j\|^2 + \|\overline{\partial}^* u_j\|^2\right) = \lambda.$$

By the interior ellipticity of $\square$ and the Rellich compactness theorem, there exists a subsequence u_{j_k} converging in the L^2-norm on K. By the assumption, the limit must be 0. Plugging u_{j_k} into (6.1) and taking the limit, we then obtain $\lambda \geq 1/C$. q.e.d.

Note that if K is an empty set, then (6.1) is equivalent to $\sigma(\square) \subset [\frac{1}{C},\ \infty)$. Furthermore, if $\sigma_e(\square) \subset [\frac{1}{C},\ \infty)$, then $\sigma(\square) \cap [0,\ \frac{1}{C})$ is either empty or consists of eigenvalues of finite multiplicities (cf. [**16**, Theorem 4.5.2]).

Now let (M,ω) be an n-dimensional complex Kähler manifold as in the statement of Theorem 1.3. Then there exists a C^2-smooth plurisubharmonic function ψ on $M \setminus K$ satisfying

$$C_0^{-1}\omega \leq \partial\bar\partial\psi \leq C_0\omega, \quad |\bar\partial\psi|^2_\omega \leq C_0,$$

where K is a compact subset of M and $C_0 > 0$ is a constant. After a multiple of a cut-off function, we may assume that ψ is a C^2 real-valued function on M such that the above inequalities hold outside a geodesic ball $B(z_0,R) = \{z \in M;\ d_M(z_0,z) < R\}$ where $d_M(z_0,\cdot)$ is the distance to a fixed point $z_0 \in M$. Write $\psi_j = p_j^*(\psi)$, $\omega_j^* = p_j^*(\omega)$, and $d_j(\cdot) = p_j^*(d_M(z_0,\cdot))$ where $p_j : M_j \to M$ is the natural projection. Note that d_j need not be the distance function of M_j. Let $K_j = p_j^{-1}(\overline{B(z_0,2R)})$.

Lemma 6.2. *There exists a constant* $C_1 = C(n, C_0, R) > 0$ *such that*

$$\|u\|^2 \leq C_1 \left(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \int_{K_j} |u|^2 dV \right) \tag{6.2}$$

holds for all $u \in \mathrm{Dom}\,(\bar\partial) \cap \mathrm{Dom}\,(\bar\partial^*) \cap L^2_{n,1}(M_j)$. *As a consequence,* $\sigma_e(\Box^{M_j}_{n,1}) \subset [\frac{1}{C_1}, \infty)$.

Proof. This lemma is an easy consequence of an L^2-technique that goes back to Donnelly–Fefferman [**22**] and Gromov [**27**] (see also [**41**, **19**, **48**, **5**, **36**, **15**] for related results). We provide a proof for completeness. Let $v \in C^{n,1}_0(M_j)$ such that $v = 0$ on $p_j^{-1}(B(z_0, R))$. By the Bochner–Kodaira–Nakano formula, we have for any $\tau > 0$,

$$\begin{aligned} \|\bar\partial v\|^2_{\tau\psi_j} + \|\bar\partial^*_{\tau\psi_j} v\|^2_{\tau\psi_j} &\geq \int_{M_j} \langle [\sqrt{-1}\tau\partial\bar\partial\psi_j, \Lambda]v, v\rangle e^{-\tau\psi_j} dV \\ &\geq C_0^{-1}\tau\|v\|^2_{\tau\psi_j} \end{aligned} \tag{6.3}$$

where Λ is the adjoint of $Lu = \omega \wedge u$, $\|\cdot\|_{\tau\psi_j}$ the L^2-norm with weight $\tau\psi_j$, and $\bar\partial^*_{\tau\psi_j}$ the adjoint of $\bar\partial$ with respect to the inner product $\langle\cdot,\cdot\rangle_{\tau\psi_j}$ (see [**6**, p. 68]). Let $w = e^{-\tau\psi_j/2}v$. Note that

$$\bar\partial v = e^{\tau\psi_j/2}\left(\bar\partial w + \frac{\tau}{2}\bar\partial\psi_j \wedge w\right),$$

and

$$\bar\partial^*_{\tau\psi_j} v = e^{\tau\psi_j/2}\left(\bar\partial^* w - \frac{\tau}{2}\bar\partial\psi_j \lrcorner\, w\right),$$

where "$\lrcorner$" is the contraction operator. It follows from the Cauchy–Schwarz inequality that

$$\|\bar\partial v\|^2_{\tau\psi_j} + \|\bar\partial^*_{\tau\psi_j} v\|^2_{\tau\psi_j} \leq 2\|\bar\partial w\|^2 + 2\|\bar\partial^* w\|^2 + C_0\tau^2\|w\|^2.$$

Substituting this inequality into (6.3), we have

$$\|\bar\partial w\|^2 + \|\bar\partial^* w\|^2 \geq \frac{\tau}{2}(C_0^{-1} - C_0\tau)\|w\|^2 \geq C_1\|w\|^2, \tag{6.4}$$

provided $\tau < C_0^{-2}/2$. Now fix such a positive constant τ. Let $0 \leq \chi \leq 1$ be a C^∞ cut-off function such that $\chi = 0$ on $(-\infty, 1)$ and $\chi = 1$ on $(2, \infty)$. Let $u \in C^{n,1}_0(M_j)$ and $w = \chi(d_j/R)u$. Then

$$\begin{aligned} \bar\partial w &= \chi(d_j/R)\bar\partial u + \bar\partial\chi(d_j/R) \wedge u; \\ \bar\partial^* w &= \chi(d_j/R)\bar\partial^* u - \bar\partial\chi(d_j/R) \lrcorner\, u. \end{aligned}$$

From (6.4) and the Cauchy–Schwarz inequality, we have

$$\|u\|^2 \leq C_1 \left(\|\bar\partial u\|^2 + \|\bar\partial^* u\|^2 + \int_{K_j} |u|^2 dV \right).$$

By Andreotti–Vesentini's approximation theorem [**1**], the same inequality holds for all $u \in \mathrm{Dom}\,(\bar\partial) \cap \mathrm{Dom}\,(\bar\partial^*) \cap L^2_{n,1}(M_j)$. From Lemma 6.1, we have $\sigma_e(\Box^{M_j}_{n,1}) \subset [\frac{1}{C_1}, \infty)$. q.e.d.

Let C_1 be the constant in Lemma 6.2. For $0 < \delta < \frac{1}{C_1}$, let $\mathcal{H}^{n,1}_{(2)}(M_j, \delta)$ be the linear span of $(n,1)$-eigenforms of the $\bar\partial$-Laplacian $\Box^{M_j}_{n,1}$, with the corresponding eigenvalues smaller than or equal to δ. Note that $\mathcal{H}^{n,1}_{(2)}(M_j, \delta)$ is a finite dimensional complex vector space. Let $\{\phi_k\}$ be an orthonormal basis of eigenforms in $\mathcal{H}^{n,1}_{(2)}(M_j, \delta)$. We define the corresponding Bergman kernel function as

$$|K^1_{M_j,\delta}| = \sum |\phi_k|^2.$$

Then we have

$$\dim \mathcal{H}^{n,1}_{(2)}(M_j, \delta) = \int_{M_j} |K^1_{M_j,\delta}| dV.$$

It is easy to see that

$$|K^1_{M_j,\delta}|(z) \le n \sup \left\{ |f|^2(z) : f \in \mathcal{H}^{n,1}_{(2)}(M_j, \delta), \|f\| = 1 \right\}, \tag{6.5}$$

(e.g., [**4**, Lemma 4.1]).

Lemma 6.3. *For every* $\epsilon > 0$, *there exist* $0 < \delta_0 < \frac{1}{C_1}$ *and* $j_0 > 0$ *such that for* $\delta \le \delta_0$ *and* $j \ge j_0$,

$$|K^1_{M_j,\delta}|(z) < \epsilon, \quad \forall z \in K_j.$$

Proof. Let $z \in K_j$. Let $f \in \mathcal{H}^{n,1}(M_j, \delta)$ be the form that realizes the supremum on the right side of (6.5). Let κ be a C^∞-smooth cut-off function such that $0 \le \kappa \le 1$, $\kappa = 1$ on $(-\infty, 1/2)$, and $\kappa = 0$ on $(1, \infty)$. Let

$$\rho = \kappa(d_{M_j}(z, \cdot)/\tau_j(z)) f.$$

Here we use $\tau_j(\cdot)$ to denote also the push-down to M_j of $\tau_j(\cdot)$ from $\widetilde{M}$ (as defined by (2.6) with Γ replaced by Γ_j). Recall that $\widetilde{p}_j \colon \widetilde{M} \to M_j$ is the natural projection. Let $\widetilde{\rho} = \widetilde{p}_j^*(\rho)$. By an argument similar to that in the proof of Lemma 6.2, we have

$$\begin{aligned} \|\widetilde{\rho}\|^2_{\widetilde{M}} &\le C(\|\bar\partial \widetilde{\rho}\|^2_{\widetilde{M}} + \|\bar\partial^* \widetilde{\rho}\|^2_{\widetilde{M}}) \\ &\le C\left(\frac{\sup |\kappa'|^2}{\tau_j^2(z)} + \|\bar\partial f\|^2_{M_j} + \|\bar\partial^* f\|^2_{M_j} \right) \\ &\le C\left(\frac{\sup |\kappa'|^2}{\tau_j^2(z)} + \delta \right). \end{aligned}$$

Since $p_j(K_j) = \overline{B(z_0, 2R)}$ is compact, there is a constant $r = r(R) > 0$ such that $\tau_j(z) > 2r$ for all j. Thus $\rho = f$ on $B(z, r)$. Using Gårding's inequality together with Sobolev's estimates, we have that for sufficiently large j and $m > n$,

$$\begin{aligned} |f|^2(z) &\leq C\left(\int_{B(z,r)} |f|^2 dV + \int_{B(z,r)} |\Box^{(m)} f|^2 dV\right) \\ &\leq C\left(\|\widetilde{\rho}\|^2_{\widetilde{M}} + \|\Box^{(m)} f\|^2_{M_j}\right) \\ &\leq C\left(\frac{\sup|\kappa'|^2}{\tau_j^2(z)} + \delta + \delta^{2m}\right), \end{aligned}$$

where the constants depend only on m and r. Lemma 6.3 then follows from (6.5) and Lemma 2.1. q.e.d.

Lemma 6.4. *For every $\epsilon > 0$, there exist $0 < \delta_0 < \frac{1}{C_1}$ and $j_0 > 0$ such that for each $\delta \leq \delta_0$ and $j \geq j_0$,*

$$\dim \mathcal{H}^{n,1}_{(2)}(M_j, \delta) \leq \epsilon\, [\Gamma : \Gamma_j] \operatorname{vol}(B(z_0, 2R)).$$

Proof. Let $\{\phi_k\}$ be an orthonormal basis of $\mathcal{H}^{n,1}_{(2)}(M_j, \delta)$ as above. The estimate (6.2) implies

$$1 = \|\phi_k\|^2 \leq C\left(\delta + \int_{K_j} |\phi_k|^2 dV\right).$$

Summing up, we have

$$\begin{aligned} \dim \mathcal{H}^{n,1}_{(2)}(M_j, \delta) &\leq \frac{C}{1 - C\delta} \int_{K_j} |K^1_{M_j,\delta}|^2 dV < \frac{C\epsilon}{1 - C\delta} \operatorname{vol}(K_j) \\ &= \frac{C\epsilon}{1 - C\delta} [\Gamma : \Gamma_j] \operatorname{vol}(B(z_0, 2R)). \end{aligned}$$

q.e.d.

Observe that every positive eigenvalue λ of $\Box^{M_j}_{n,0}$ on $L^{n,0}_{(2)}(M_j)$ is also an eigenvalue of $\Box^{M_j}_{n,1}$ on $L^{n,1}_{(2)}(M_j)$: If $\{f\}$ is a normalized eigenform of $\Box^{M_j}_{n,0}$ on $L^{n,0}_{(2)}(M_j)$ associated with λ, then

$$\Box^{M_j}_{n,1}(\bar\partial f) = \lambda \bar\partial f \quad \text{and} \quad \|\bar\partial f\|^2_{M_j} = (\Box^{M_j}_{n,0} f, f) = \lambda.$$

Thus $\bar\partial$ induces a linear injection from $\mathcal{H}^{n,0}_{(2)}(M_j, \delta) \ominus \mathcal{H}^{n,0}_{(2)}(M_j)$ into $\mathcal{H}^{n,1}_{(2)}(M_j, \delta)$, and

$$\dim\left(\mathcal{H}^{n,0}_{(2)}(M_j, \delta) \ominus \mathcal{H}^{n,0}_{(2)}(M_j)\right) \leq \dim \mathcal{H}^{n,1}_{(2)}(M_j, \delta) < \infty, \tag{6.6}$$

where $\mathcal{H}^{n,0}_{(2)}(M_j, \delta)$ is the linear span of $(n, 0)$-eigenforms of $\Box^{M_j}_{n,0}$, associated with the eigenvalues smaller than or equal to δ. Let $|K_{M_j,\delta}|$ and $|K_{M_j}|$ be the Bergman kernel functions of $\mathcal{H}^{n,0}_{(2)}(M_j, \delta)$ and $\mathcal{H}^{n,0}_{(2)}(M_j)$ respectively.

Lemma 6.5. *For every* $\epsilon > 0$*, there exist* $0 < \delta_0 < \frac{1}{C_1}$ *and* $j_0 > 0$ *such that for* $\delta \leq \delta_0$ *and* $j \geq j_0$*,*

$$0 < |K_{M_j,\delta}|(z) - |K_{M_j}|(z) < \epsilon, \quad \forall\, z \in K_j.$$

Proof. By (6.6) and Lemma 6.4, we have for $\delta \leq \delta_0$ and $j \geq j_0$,

$$\int_{M_j} (|K_{M_j,\delta}| - |K_{M_j}|) dV \leq \epsilon\, [\Gamma : \Gamma_j]\, \mathrm{vol}(B(z_0, 2R)).$$

Since the function in the integral is invariant under Γ/Γ_j,

$$\int_{M} (|K_{M_j,\delta}| - |K_{M_j}|) dV < \epsilon\, \mathrm{vol}(B(z_0, 2R)). \tag{6.7}$$

Now take a normal coordinate ball $B(z, \epsilon_0)$ around z (lifted from M) where ϵ_0 depends only on M. Again it follows from Gårding's inequality and Sobolev's estimates that for $m > n$,

$$\begin{aligned} |f|^2(z) &\leq C_{m,\epsilon_0} \left(\int_{B(z,\epsilon_0)} |f|^2 dV + \int_{B(z,\epsilon_0)} |\Box^{(m)} f|^2 dV \right) \\ &\leq C_{m,\epsilon_0} (1 + \delta^{2m}) \int_M |f|^2 dV, \end{aligned} \tag{6.8}$$

for all $f \in \mathcal{H}^{n,0}_{(2)}(M_j, \delta) \ominus \mathcal{H}^{n,0}_{(2)}(M_j)$. By (6.7) and (6.8),

$$|K_{M_j,\delta}|(z) - |K_{M_j}|(z) \leq C_{m,\epsilon_0}(1 + \delta^{2m}) \epsilon\, \mathrm{vol}(B(z_0, 2R)),$$

completing the proof. q.e.d.

We now proceed to prove Theorem 1.3. In light of Proposition 2.2, it suffices to verify the lower semicontinuity of the Bergman kernels K_{M_j} as $j \to \infty$. Let $z \in \widetilde{M}$. Let R be sufficiently large so that $\widetilde{p}_1(z) \in B(z_0, R)$. Let δ_0 and j_0 be chosen as in Lemma 6.5. Let f be a candidate for the extremal property (2.4) of $K_{\widetilde{M}}(z)$ and let

$$\varrho_j = \kappa(d_{\widetilde{M}}(z, \cdot)/\tau_j(z)) f,$$

where κ is the cut-off function as in the proof of Lemma 6.3. Since ϱ_j is supported in a Dirichlet fundamental domain of M_j, we may push down ϱ_j onto M_j and regard it as a $(n, 0)$-form on M_j with $\|\varrho_j\|_{M_j} \leq 1$. Then we have the orthonormal decomposition

$$\varrho_j = u_j + v_j,$$

with $u_j \in \mathcal{H}^{n,0}_{(2)}(M_j, \delta_0)$ and

$$(\Box v_j, v_j) \geq \delta_0 \|v_j\|^2.$$

By Lemma 6.5, we have

$$|u_j|^2(z) \leq |K_{M_j,\delta_0}|(z) \|u_j\|^2 \leq |K_{M_j,\delta_0}|(z) < |K_{M_j}|(z) + \epsilon, \tag{6.9}$$

for $j \geq j_0$, whereas

$$\|v_j\|^2 \leq \delta_0^{-1}(\Box v_j, v_j) = \delta_0^{-1}\|\bar{\partial} v_j\|^2. \tag{6.10}$$

Since $\Box u_j \in \mathcal{H}^{n,0}_{(2)}(M_j, \delta_0)$, we have

$$(\bar{\partial} u_j, \bar{\partial} v_j) = (\Box u_j, v_j) = 0.$$

Hence

$$\|\bar{\partial} v_j\|^2 \leq \|\bar{\partial} \varrho_j\|^2 \leq C \sup |\kappa'|^2 \tau_j^{-2}(z). \tag{6.11}$$

By (6.10) and (6.11),

$$\|v_j\|^2 \leq C \frac{\sup |\kappa'|^2}{\delta_0} \tau_j^{-2}(z). \tag{6.12}$$

In order to obtain a pointwise estimate of v_j, we fix a coordinate unit ball $\mathbb{B}^n$ in $\widetilde{M}$, centered at z and lifted from M. It follows from Gårding's inequality together with Sobolev's estimates that for $m > n$,

$$\begin{aligned} |\bar{\partial} u_j|^2(z) &\leq C_m \left(\int_{\mathbb{B}^n} |\bar{\partial} u_j|^2 dV + \int_{\mathbb{B}^n} |\Box^{(m)}(\bar{\partial} u_j)|^2 dV \right) \\ &\leq C_m(1 + \delta_0^{2m})\|\bar{\partial} u_j\|^2 \leq C_m(1 + \delta_0^{2m})\|\bar{\partial} \varrho_j\|^2, \end{aligned}$$

for $z \in \mathbb{B}^n_{1/2}$. (Here we identify u_j and v_j with their pull-backs from M_j onto $\widetilde{M}$.) Since ϱ_j is holomorphic in $\mathbb{B}^n$ for large j, we conclude that

$$|\bar{\partial} v_j|^2 = |\bar{\partial} u_j|^2 \to 0, \tag{6.13}$$

uniformly on $\mathbb{B}^n_{1/2}$ as $j \to \infty$. Let K_{BM} be the Bochner–Martinelli kernel. Then

$$\begin{aligned} v_j(0) &= \int_{\mathbb{B}^n} \bar{\partial}(\kappa(2|z|) v_j) K_{\mathrm{BM}} \\ &= \int_{\mathbb{B}^n} v_j \bar{\partial} \kappa(2|z|) K_{\mathrm{BM}} \\ &\quad + \int_{\mathbb{B}^n} \kappa(2|z|) \bar{\partial} v_j K_{\mathrm{BM}} \end{aligned}$$

converges to 0 by (6.12) and (6.13); because K_{BM} is L^1 on $\mathbb{B}^n$. Combining this fact with (6.9), we conclude that

$$|K_{\widetilde{M}}|(z) \leq \liminf_{j \to \infty} |K_{M_j}|(z).$$

This concludes the proof of Theorem 1.3.

Corollary 6.6. *Any tower of coverings on a hyperconvex complex manifold is Bergman stable.*

Proof. Let $M_j = \widetilde{M}/\Gamma_j$ be a tower of covering on a hyperconvex complex manifold M. Let $\rho \colon M \to [-1,\ 0)$ be a smooth strictly plurisubharmonic proper map. Put $\psi = -\log(-\rho)$, $\omega = \partial\bar{\partial}\psi$, $\widetilde{\psi} = \widetilde{p}^*(\psi)$, and

$\widetilde{\omega} = \widetilde{p}^*(\omega)$, where $\widetilde{p}\colon \widetilde{M} \to M$ is the natural projection. Then the assumptions in Theorem 1.3 are satisfied. q.e.d.

Theorem 1.3 can be generalized to a tower of coverings on a complete Hermitian manifold. More specifically, we have the following:

Theorem 6.7. *A tower of coverings M_j on a complete Hermitian manifold (M, ω) is Bergman stable if the following conditions hold:*

1) *There exist a compact set $K \subset M$, a C^∞-smooth plurisubharmonic function on $M \backslash K$, and a constant $C > 0$ such that $\omega = \partial\bar{\partial}\psi$ and $\partial\bar{\partial}\psi \geq C^{-1}\partial\psi \wedge \bar{\partial}\psi$ on $M\backslash K$.*
2) *There exist a C^∞-smooth plurisubharmonic function $\tilde{\psi}$ on the top manifold $\widetilde{M}$ and a constant $\widetilde{C} > 0$ such that $\partial\bar{\partial}\tilde{\psi} \geq \widetilde{C}^{-1}\tilde{\omega}$ and $\partial\bar{\partial}\tilde{\psi} \geq \widetilde{C}^{-1}\partial\tilde{\psi} \wedge \bar{\partial}\tilde{\psi}$, where $\widetilde{\omega}$ is the lift of ω to $\widetilde{M}$.*

We indicate how the proof of Theorem 1.3 given above can be easily modified to prove Theorem 6.7. Since M is Kähler on $M \setminus K$, the conclusion of Lemma 6.2 is evidently valid under condition (1) above. The proof of Lemma 6.3 can be modified as follows: Since for each (n, q)-form on $\widetilde{M}$, the L^2-norm with respect to $\partial\bar{\partial}\tilde{\psi}$ is always dominated by the L^2-norm with respect to the metric $\tilde{\omega}$. Thus

$$\begin{aligned}\|\tilde{\rho}\|^2_{\widetilde{M},\partial\bar{\partial}\tilde{\psi}} &\leq C\left(\|\bar{\partial}\tilde{\rho}\|^2_{\widetilde{M},\partial\bar{\partial}\tilde{\psi}} + \|\bar{\partial}^*\tilde{\rho}\|^2_{\widetilde{M},\partial\bar{\partial}\tilde{\psi}}\right)\\ &\leq C\left(\|\bar{\partial}\rho\|^2_{M_j} + \|\bar{\partial}^*\rho\|^2_{M_j}\right) \leq C\left(\frac{\sup|\kappa'|^2}{\tau_j^2(z)} + \delta\right).\end{aligned}$$

The other arguments in the proof remain unchanged after replacing $\|\tilde{\rho}\|_{\widetilde{M}}$ by $\|\tilde{\rho}\|_{\widetilde{M},\partial\bar{\partial}\tilde{\psi}}$.

We are now in position to prove Theorem 1.4. Let M be a Riemann surface. When the universal covering space of M is $\mathbb{P}^1$, Theorem 1.4 is trivial; because in this case, M is also $\mathbb{P}^1$ ([**24**, Theorem IV.6.3, p. 193]). When the universal covering space of M is $\mathbb{C}$, then M and its normal covering spaces are conformally equivalent to either $\mathbb{C}$, the punctured complex plane $\mathbb{C}^*$, or a torus ([**24**, Theorem IV.6.4, p. 193]). Hence the Bergman kernels of the normal covering spaces of M vanish identically when M is non-compact and stability of the Bergman kernel is therefor established in this case. In the case when M is a torus, each covering space M_j in the tower is also a torus except the universal covering space $\mathbb{C}$. Since holomorphic 1-forms on a torus have the form of $c \cdot dz$ where c is a complex number, we have $|K_{M_j}| = 1/\mathrm{vol}\, M_j$, which tends to zero as $j \to \infty$. Thus we have Bergman stability in this case. It remains to deal with the case when the universal covering space of M is the unit disk $\mathbb{D}$. According to Rhodes' theorem and Theorem 1.2, it suffices to consider the case when M is parabolic. By a theorem of Nakai [**38**], there exists a harmonic function u outside of a compact subset such that

$u(z) \to +\infty$ as z tends to the ideal boundary ∂M of M. Let K be a compact subset of M such that $u > 1$ on $M\backslash K$. By Richberg's theorem, there is a C^∞-smooth strictly subharmonic function ϕ on $M\backslash K$ such that $|\phi - (-u)| \le 1$ holds on $M\backslash K$. Let $0 \le \chi \le 1$ be a function in $C_0^\infty(M)$ such that $\chi = 1$ on a neighborhood of K, and let $0 \le \kappa \le 1$ be a function in $C_0^\infty(M)$ such that $\kappa = 1$ in a neighborhood of $\operatorname{supp}\chi$. Let ω_0 be a Kähler metric on M. Put

$$\omega = C_1 \kappa\, \omega_0 + \partial\bar\partial\left((1-\chi)(-\log(-\phi))\right),$$

where $C_1 > 0$ is a constant. We see that ω is a complete Hermitian metric on M satisfying condition (1) in Theorem 6.7, provided C_1 is sufficiently large (we may take $\psi = -\log(-\phi)$). Let $\tilde p : \mathbb{D} \to M$ be the natural projection. We define

$$\widetilde{\psi}(z) = C_2\left(-\log(1-|z|^2)\right) + \tilde p^*\left((1-\chi)(-\log(-\phi))\right),$$

where $C_2 > 0$ is a constant. Since $\partial\bar\partial(-\log(1-|z|^2))$ descends to a complete Kähler metric on M, $\partial\bar\partial\tilde\psi$ dominates $\tilde\omega$ provided C_2 sufficiently large. It is also easy to verify $\partial\bar\partial\tilde\psi \ge C^{-1}\partial\tilde\psi \wedge \bar\partial\tilde\psi$. Applying Theorem 6.7, we then conclude the proof of Theorem 1.4.

7. Appendix: Applications

In this appendix, we apply Theorem 1.3 to study Bergman stability of towers of coverings on quotients of the polydisc and the ball. We first recall the setting for quotients of the polydisc.

Let $\mathbb{H}^n$ be the n-product of the upper half planes $\mathbb{H}$, equipped with the Bergman metric. The connected component G of the identity for the automorphism group of $\mathbb{H}^n$ contains all transformations in the form of $\sigma = (\sigma^{(1)}, \cdots, \sigma^{(n)})$, $\sigma^{(i)} \in PSL(2,\mathbb{R})$. An element σ of G is *parabolic* if each $\sigma^{(i)}$ has exactly one fixed point on $\overline{\mathbb{R}}$. Let Γ be a Hilbert modular group. (We refer the reader to [**50**, **47**] for the relevant material.) Then Γ has a fundamental domain F of the form

$$F = F_0 \cup V_1 \cup \cdots \cup V_t,$$

where F_0 is relatively compact in $\mathbb{H}^n$, the V_j's are disjoint, each $V_j \subset \sigma_j^{-1}(U_j)$ is the fundamental domain of the group Γ_j of all $\gamma \in \Gamma$ fixing some point x_j in $\overline{\mathbb{R}}^n$, and

$$U_j = \{z \in \mathbb{C}^n : \operatorname{Im} z_1 \times \cdots \times \operatorname{Im} z_n > d_j\},$$

with d_j being a suitably chosen positive number, σ_j an element in G such that $\sigma_j(x_j) = \infty$ (see [**47**, p. 48]). Since each nontrivial element of $\sigma_j\Gamma_j\sigma_j^{-1}$ fixes exactly one point ∞, it must be a translation. Let $\mathbb{D}^n$ be the unit polydisc in $\mathbb{C}^n$ and let Γ be a Hilbert modular group.

Proposition 7.1. *Any tower of coverings $M_j = \mathbb{D}^n/\Gamma_j$ on $M = \mathbb{D}^n/\Gamma$ is Bergman stable.*

Proof. Since $\mathbb{D}^n$ is biholomorphic to $\mathbb{H}^n$, it suffices to prove the proposition for $\mathbb{H}^n$. The Bergman kernel

$$K_{\mathbb{H}^n}(z) = \frac{1}{(4\pi)^n} \frac{1}{(\operatorname{Im} z_1 \times \cdots \times \operatorname{Im} z_n)^2},$$

of $\mathbb{H}^n$ is invariant under translations. In particular, it is $\sigma_j \Gamma_j \sigma_j^{-1}$-invariant. A direct computation shows that

$$\partial\bar{\partial} \log K_{\mathbb{H}^n} \gtrsim \partial \log K_{\mathbb{H}^n} \wedge \bar{\partial} \log K_{\mathbb{H}^n}.$$

By setting $\psi = \sigma_j^* \log K_{\mathbb{H}^n}$ on each parabolic end V_j/Γ_j, we see that conditions in Theorem 1.3 are satisfied. q.e.d.

Proposition 7.2. *Any tower of coverings on a complete Kähler manifold with pinched negative sectional curvature and finite volume is Bergman stable.*

Proof. Let (M, ω) be a complete Kähler manifold with pinched negative sectional curvature and finite volume. Let $(\widetilde{M}, \widetilde{\omega})$ be its universal covering. According to a result of Siu and Yau [**49**], each Busemann function ψ on $\widetilde{M}$ satisfies $C^{-1}\widetilde{\omega} \le \partial\bar{\partial}\psi \le C\widetilde{\omega}$ and $\partial\bar{\partial}\psi \ge C^{-1}\partial\psi \wedge \bar{\partial}\psi$. Furthermore, M is the union of a compact set and a finite number of cusp ends such that each end admits a function that is a push-down of some Busemann function on $\widetilde{M}$. Thus the conditions of Theorem 1.3 are satisfied. q.e.d.

Remark. Proposition 7.2 also holds when M is a ball quotient that is geometrically finite in the sense of Bowditch [**9**]. We leave the detail to the interested reader.

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