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FOLIATED STRATIFIED SPACES AND A DE RHAM COMPLEX DESCRIBING INTERSECTION SPACE COHOMOLOGY

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Abstract

The method of intersection spaces associates cell-complexes depending on a perversity to certain types of stratified pseudomanifolds in such a way that Poincaré duality holds between the ordinary rational cohomology groups of the cell-complexes associated to complementary perversities. The cohomology of these intersection spaces defines a cohomology theory HI for singular spaces, which is not isomorphic to intersection cohomology IH. Mirror symmetry tends to interchange IH and HI. The theory IH can be tied to type IIA string theory, while HI can be tied to IIB theory. For pseudomanifolds with stratification depth 1 and flat link bundles, the present paper provides a de Rham-theoretic description of the theory HI by a complex of global smooth differential forms on the top stratum. We prove that the wedge product of forms introduces a perversity-internal cup product on HI, for every perversity. Flat link bundles arise for example in foliated stratified spaces and in reductive Borel-Serre compactifications of locally symmetric spaces. A precise topological definition of the notion of a stratified foliation is given.

1. Introduction

Let $\bar{p}$ be a perversity in the sense of intersection homology theory, [**25**], [**26**], [**29**], [**2**]. In [**3**], we introduced a general homotopy-theoretic framework that assigns to certain types of n-dimensional stratified pseudomanifolds X CW-complexes

$$I^{\bar{p}}X,$$

the *perversity-$\bar{p}$ intersection spaces* of X, such that for complementary perversities $\bar{p}$ and $\bar{q}$, there is a Poincaré duality isomorphism

$$\widetilde{H}^i(I^{\bar{p}}X;\mathbb{Q}) \cong \widetilde{H}_{n-i}(I^{\bar{q}}X;\mathbb{Q})$$

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when X is compact and oriented. In particular, this framework yields a new cohomology theory $HI^\bullet_{\bar{p},s}(X) = H^\bullet_s(I^{\bar{p}}X)$ for singular spaces, where $H^\bullet_s$ denotes ordinary singular cohomology. For the lower middle perversity $\bar{p} = \bar{m}$, we shall briefly write $IX = I^{\bar{m}}X$ and $HI^\bullet_s(X) = HI^\bullet_{\bar{m},s}(X)$. That this theory is indeed not isomorphic to intersection cohomology $IH^\bullet_{\bar{p}}(X)$ or to Cheeger's L^2-cohomology $H^\bullet_{(2)}(X)$ is apparent from the observation that, for every $\bar{p}$, $HI^\bullet_{\bar{p},s}(X)$ is an algebra under cup product, whereas it is well-known that $IH^\bullet_{\bar{p}}(X)$ and $H^\bullet_{(2)}(X)$ cannot generally be endowed with a $\bar{p}$-internal algebra structure compatible with the cup product.

The construction of intersection spaces is guided by the idea of mimicking spatially what intersection homology does algebraically. By Mayer-Vietoris sequences, the overall behavior of intersection homology is primarily controlled by its behavior on cones. If L is a closed l-dimensional manifold, $n > 0$, then

$$IH^{\bar{p}}_r(\overset{\circ}{\text{cone}}(L)) \cong \begin{cases} H_r(L), & r < l - \bar{p}(l+1), \\ 0, & \text{otherwise,} \end{cases}$$

where $\overset{\circ}{\text{cone}}(L)$ denotes the open cone on L and we are using intersection homology built from finite chains. Thus, intersection homology is a process of truncating the homology of singularity links algebraically above some cut-off degree given by the perversity and the dimension of the space. This is also apparent from Deligne's formula for the intersection chain sheaf. To implement homology truncation of the link spatially, we use the notion of Moore approximation of a space, which is Eckmann-Hilton dual to the notion of Postnikov approximation. Let k be a positive integer. A stage-k Moore approximation of a space L is a space $L_{<k}$ with $H_r(L_{<k}) = 0$ for $r \geq k$, together with a map $L_{<k} \to L$ which is a homology-isomorphism in degrees less than k. Such an approximation certainly exists when L is simply connected, but this sufficient condition is far from necessary. The intersection space is obtained roughly by replacing cones on links L by cones on their Moore approximations $L_{<k}$. For instance, the intersection space $I^{\bar{p}}X$ of an n-dimensional closed pseudomanifold X with one isolated singular point is by definition the homotopy cofiber of the composition

$$L_{<k} \longrightarrow L = \partial\overline{X} \hookrightarrow \overline{X},$$

where $k = n - 1 - \bar{p}(n)$ and $\overline{X}$, a compact manifold with boundary L, is the so-called blow-up of X obtained by replacing the singular point of X by L. In other words: we attach the cone on a spatial homology truncation of the link to the blow-up of X along the boundary of the blow-up. There is an obvious collapse map $I^{\bar{p}}X \to X$. (More background on intersection spaces is provided in Section 9.2. That section also lists the space classes for which $I^{\bar{p}}X$ has been presently constructed and Poincaré duality established.) If the singularities are not isolated, one attempts a process of fiberwise spatial homology truncation (i.e. equivariant Moore approximation) applied to the link bundle. Due to the lack of full functoriality of Moore approximations, this is generally obstructed and related to Steenrod's equivariant Moore space problem, [**16**].

The present paper serves a twofold purpose: It provides a de Rham-type description of $HI^{\bullet}_{\bar{p},s}(X;\mathbb{R})$ in terms of certain global differential forms on the top stratum of X. But by doing so, it simultaneously opens up a way of defining the theory $HI^{\bullet}_{\bar{p}}(X)$ on spaces X, for which the intersection space $I^{\bar{p}}X$ has not been constructed yet. Let X^n be a compact, oriented, stratified pseudomanifold of stratification depth 1 possessing Mather control data (see Definitions 2.1, 2.2 for details), in particular a link bundle for every component of the singular set Σ. Assume that all of these link bundles are flat and that each link can be endowed with a Riemannian metric such that the structure group of the bundle is contained in the isometries of the link. (Such a metric can always be found if the structure group is a compact Lie group. A fiber bundle is called flat if its transition functions are locally constant.) Do *not* assume that the links are simply connected — they may or may not be. For such X, we define a subcomplex $\Omega I^{\bullet}_{\bar{p}}(X-\Sigma)$ of the complex $\Omega^{\bullet}(X-\Sigma)$ of smooth differential forms on the top stratum $X-\Sigma$, set $HI^{\bullet}_{\bar{p}}(X) = H^{\bullet}(\Omega I^{\bullet}_{\bar{p}}(X-\Sigma))$, and show

Theorem 8.2. *(Generalized Poincaré Duality.) Let $\bar{p}$ and $\bar{q}$ be complementary perversities. Wedge product followed by integration induces a nondegenerate bilinear form*

$$\int : HI^{r}_{\bar{p}}(X) \times HI^{n-r}_{\bar{q}}(X) \longrightarrow \mathbb{R},\ ([\omega],[\eta]) \mapsto \int_{X-\Sigma} \omega \wedge \eta.$$

We prove our de Rham theorem for spaces with only isolated singularities. Since the intersection space $I^{\bar{p}}X$ is not smooth, but only a CW-complex, we introduce in Section 9.1 a *partial smoothing* tool that enables us to recover enough smoothness of singular simplices $\Delta \to I^{\bar{p}}X$ so that forms in $\Omega I^{\bullet}_{\bar{p}}(X-\Sigma)$ can be integrated over them and this induces an isomorphism:

Theorem 9.11. *(De Rham description of $HI^{\bullet}_{\bar{p},s}$.) Let X be a compact, oriented pseudomanifold with only isolated singularities and simply connected links. Then integrating a form in $\Omega I^{\bullet}_{\bar{p}}(X-\Sigma)$ over a smooth singular simplex in $X-\Sigma$ induces an isomorphism*

$$HI^{\bullet}_{\bar{p}}(X) \cong \widetilde{H}I^{\bullet}_{\bar{p},s}(X;\mathbb{R}).$$

Again, we will briefly put $HI^\bullet(X) = HI^\bullet_{\bar{m}}(X)$. An advantage of the differential form approach adopted in this paper is that it eliminates the simple connectivity assumption on links adopted in [**3**], where we use the Hurewicz theorem. As there is presently no general construction of $I^{\bar{p}}X$ available for X with flat link bundles, this paper extends the theory $HI^\bullet_{\bar{p}}$ to such spaces. Let us indicate some fields of application. If the link bundle is flat, then the total space of the bundle possesses a foliation so that the bundle becomes a transversely foliated fiber bundle. Conversely, flat link bundles arise in foliated stratified spaces. A precise definition of stratified foliations is given in Section 11 (Definitions 11.2, 11.3), at least for stratification depth 1. Such foliations play a role for instance in the work of Farrell and Jones on the topological rigidity of negatively curved manifolds, [**21**], [**22**]. Our definition of a stratified foliation is inspired by the conical foliations of Saralegi-Aranguren and Wolak, [**35**]. The orbits of an isometric Lie group action on a compact Riemannian manifold, for example, form a conical foliation. Theorem 11.7 of the present paper confirms that if a stratified foliation is zero-dimensional on the links, then the restrictions of the link bundle to the leaves of the singular stratum are flat bundles.

Reductive Borel-Serre compactifications of locally symmetric spaces may constitute another field of stratified spaces to which the theory $HI^\bullet$ can be applied. Let G be a connected reductive algebraic group defined over $\mathbb{Q}$ and $\Gamma \subset G(\mathbb{Q})$ an arithmetic subgroup. Let $K \subset G(\mathbb{R})$ be a maximal compact subgroup and A_G the connected component of the real points of the maximal $\mathbb{Q}$-split torus in the center of G. The associated symmetric space is $D = G(\mathbb{R})/KA_G$. The arithmetic quotient $X = \Gamma\backslash D$ is generally not compact and several compactifications of X have been studied. For simplicity, let us assume that Γ is neat, so that X is a manifold. (Otherwise, X may have mild singularities; it is in general a V-manifold. Any arithmetic group contains a neat subgroup of finite index.) The Borel-Serre compactification $\overline{X}$ ([**10**]) is a manifold with corners whose interior is X and whose faces Y_P are indexed by the Γ-conjugacy classes of parabolic $\mathbb{Q}$-subgroups P of G. Each Y_P fits into a *flat* bundle $Y_P \to X_P$, called the nilmanifold fibration because the fiber is a compact nilmanifold. The X_P are arithmetic quotients of the symmetric space associated to the Levi quotient of P. The reductive Borel-Serre compactification $\widehat{X}$, introduced by Zucker ([**37**]), is the quotient of $\overline{X}$ obtained by collapsing the fibers of the nilmanifold fibrations. The X_P are the strata of $\widehat{X}$ and their link bundles are the flat nilmanifold fibrations. A basic class of examples is given by Hilbert modular surfaces X associated to real quadratic fields $\mathbb{Q}(\sqrt{d})$. For these, the X_P are circles, the nilmanifold links are 2-tori and the flat link bundles are mapping tori, see [**7**].

Let us describe some of the features of $HI^\bullet_{\bar p}$. Since there is no general cup product $H^i(M) \otimes H^j(M) \to H^{i+j}(M, \partial M)$ for a manifold M with boundary ∂M, intersection cohomology $IH^\bullet_{\bar p}(X)$, for most $\bar p$, cannot be endowed with a $\bar p$-internal cup product. Similarly, the complex $\Omega^\bullet_{(2)}(X - \Sigma)$ of L^2-forms on the top stratum equipped with a conical metric in the sense of Cheeger ([**17**], [**18**], [**19**]) is not a differential graded algebra (DGA) under wedge product of forms — the product of two L^2-functions need not be L^2 anymore. We prove that for every perversity $\bar p$, the DGA-structure $(\Omega^\bullet(X - \Sigma), d, \wedge)$, where d denotes exterior derivation, restricts to a DGA-structure $(\Omega I^\bullet_{\bar p}(X - \Sigma), d, \wedge)$ (Theorem 10.1). Consequently, the wedge product induces a cup product

$$\cup : HI^i_{\bar p}(X) \otimes HI^j_{\bar p}(X) \longrightarrow HI^{i+j}_{\bar p}(X).$$

Contrary to $IH^\bullet_{\bar p}$ and $H^\bullet_{(2)}$, the theory $HI^\bullet_{\bar p}$ is quite stable under deformation of complex algebraic singularities. Consider for example the Calabi-Yau quintic

$$V_s = \left\{ z \in \mathbb{C}P^4 \mid z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 - 5(1+s) z_0 z_1 z_2 z_3 z_4 = 0 \right\},$$

depending on a complex parameter s. The variety V_s is smooth for small $s \neq 0$, while $V = V_0$ has 125 isolated singular points. Its ordinary cohomology has Betti numbers $\operatorname{rk} H^2(V) = 1$, $\operatorname{rk} H^3(V) = 103$, $\operatorname{rk} H^4(V) = 25$ and its middle perversity intersection cohomology has ranks $\operatorname{rk} IH^2(V) = 25$, $\operatorname{rk} IH^3(V) = 2$, $\operatorname{rk} IH^4(V) = 25$. Both of these sets of Betti numbers differ considerably from the Betti numbers of the nearby smooth deformation V_s ($s \neq 0$): $\operatorname{rk} H^2(V_s) = 1$, $\operatorname{rk} H^3(V_s) = 204$, $\operatorname{rk} H^4(V_s) = 1$. Now the calculations of [**3**, Section 3.9], together with our de Rham theorem, show that

$$\operatorname{rk} HI^2(V) = 1, \ \operatorname{rk} HI^3(V) = 204, \ \operatorname{rk} HI^4(V) = 1,$$

in perfect agreement with the Betti numbers of V_s, $s \neq 0$. Indeed, jointly with L. Maxim, we have established the following Stability Theorem, see [**8**]: Let V be a complex n-dimensional projective hypersurface with one isolated singularity and let V_s be a nearby smooth deformation of V. Then for all $i < 2n$, and $i \neq n$, $\widetilde{HI}^i_s(V; \mathbb{Q}) \cong \widetilde{H}^i(V_s; \mathbb{Q})$. For the middle dimension $HI^n_s(V; \mathbb{Q}) \cong H^n(V_s; \mathbb{Q})$ if, and only if, the monodromy operator acting on the cohomology of the Milnor fiber of the singularity is trivial. At least if $H_{n-1}(L; \mathbb{Z})$ is torsionfree, where L is the link of the singularity, the isomorphism is induced by a continuous map $IV \to V_s$ and is thus a ring isomorphism. We use this in [**8**] to endow $HI^\bullet_s(V; \mathbb{Q})$ with a mixed Hodge structure so that the canonical map $IV \to V$ induces homomorphisms of mixed Hodge structures in cohomology. Even if the monodromy is not trivial, $IV \to V_s$ induces a monomorphism on homology. This statement for $HI^\bullet$ may be viewed as a "mirror image" of the well-known fact that the intersection homology of a complex variety V is a linear subspace of the ordinary homology of any resolution

$\widetilde{V} \to V$, as follows from the Beilinson-Bernstein-Deligne-Gabber decomposition theorem, [**9**]. If the resolution is small, then $IH^i(V) \cong H^i(\widetilde{V})$. Thus the monodromy condition for deformations may be viewed as a "mirror image" of the smallness condition for resolutions.

The relationship between $IH^\bullet$ and $HI^\bullet$ is indeed illuminated well by mirror symmetry, which tends to exchange resolutions and deformations. In [**34**] for example, it is conjectured that the mirror of a conifold transition, which consists of a degeneration $s \to 0$ followed by a small resolution, is again a conifold transition, but performed in the reverse direction. The results of Section 3.8 in [**3**] together with the de Rham theorem of this paper imply that if V° is the mirror of a conifold V, both sitting in mirror symmetric conifold transitions, then

$$\begin{array}{rcl} \operatorname{rk} IH^3(V) & = & \operatorname{rk} HI^2(V^\circ) + \operatorname{rk} HI^4(V^\circ) + 2, \\ \operatorname{rk} IH^3(V^\circ) & = & \operatorname{rk} HI^2(V) + \operatorname{rk} HI^4(V) + 2, \\ \operatorname{rk} HI^3(V) & = & \operatorname{rk} IH^2(V^\circ) + \operatorname{rk} IH^4(V^\circ) + 2, \text{ and} \\ \operatorname{rk} HI^3(V^\circ) & = & \operatorname{rk} IH^2(V) + \operatorname{rk} IH^4(V) + 2. \end{array}$$

Since mirror symmetry is a phenomenon that arose originally in string theory, it is not surprising that the theories $IH^\bullet$, $HI^\bullet$ have a specific relevance for type IIA, IIB string theories, respectively. While $IH^\bullet$ yields the correct count of massless 2-branes on a conifold in type IIA theory, the theory $HI^\bullet$ yields the correct count of massless 3-branes on a conifold in type IIB theory, see [**3**]. The author hopes that the de Rham description of $HI^\bullet$ by differential forms offered here is closer to physicists' intuition of cohomology than the homotopy theory of [**3**]. The present paper makes it possible, for example, to obtain differential form representatives for the above mentioned massless 3-branes in IIB string theory.

Let us briefly indicate the construction of $\Omega I^\bullet_{\bar{p}}$. Our approach requires a Riemannian metric on a particular copy L of the link, but no metric on the entire link bundle or on the entire top stratum. To obtain a de Rham description of intersection cohomology, one uses a truncation $\tau_{<k}\Omega^\bullet(L)$, where $k = \bar{p}(m+1)+1$ and $m = \dim(L)$, of the forms on the link, as is well-known. To pass from this local normal truncation to a global complex, one must perform fiberwise normal truncation. This is technically easy to accomplish, since an automorphism of L induces an automorphism of $\Omega^\bullet(L)$, which restricts to an automorphism of $\tau_{<k}\Omega^\bullet(L)$. Ultimately, the result will indeed be a subcomplex of $\Omega^\bullet(X-\Sigma)$, since there is a canonical monomorphism $\tau_{<k}\Omega^\bullet(L) \to \Omega^\bullet(L)$. By contrast, a de Rham model for $HI^\bullet_s$ requires the use of *cotruncation* $\tau_{\geq k}\Omega^\bullet(L)$ (with $k = m - \bar{p}(m+1)$). If one uses standard cotruncation of a complex, one runs into two problems: standard cotruncation comes with a canonical

epimorphism $\Omega^\bullet(L) \to \tau_{\geq k}\Omega^\bullet(L)$, so one will not obtain a *sub*complex of $\Omega^\bullet(X - \Sigma)$. Furthermore, one must implement normal cotruncation as a subcomplex in such a way that it can be carried out in a fiberwise fashion. This paper solves these problems as follows: In Section 4, we use Riemannian Hodge theory to define cotruncation as a subcomplex $\tau_{\geq k}\Omega^\bullet(L) \subset \Omega^\bullet(L)$ (Definition 4.2). This is the reason for requiring a metric on L. By Proposition 4.4, $\tau_{\geq k}\Omega^\bullet(L)$ is independent (up to isomorphism) of the metric on L. A key advantage of cotruncation over truncation is that $\tau_{\geq k}\Omega^\bullet(L)$ is a subalgebra of $\Omega^\bullet(L)$ (Proposition 4.3), whereas $\tau_{<k}\Omega^\bullet(L)$ is not. This property of cotruncation will entail that the cohomology theory $HI^\bullet_{\bar{p}}(X)$ has a $\bar{p}$-internal cup product for all $\bar{p}$, while intersection cohomology does not. An isometry $L \to L$ induces an automorphism of $\tau_{\geq k}\Omega^\bullet(L)$, a property that is important for fiberwise cotruncation and explains why we assume the structure group of the link bundle to lie in the isometries of L. Examples of flat sphere bundles with nonzero real Euler class constructed by Milnor [**33**] show that our isometry assumption cannot be dropped without substitute; flatness alone is certainly not sufficient for the existence of a fiberwise cotruncation yielding a subcomplex. In order to implement fiberwise cotruncation, we develop a model $\Omega^\bullet_{\mathcal{MS}}(\Sigma)$, called the *multiplicatively structured forms*, for the forms on the total space E of the link bundle $E \to \Sigma$, which is structured enough so that fiberwise cotruncation is fairly straightforward, yielding a subcomplex $\operatorname{ft}_{\geq k}\Omega^\bullet_{\mathcal{MS}}(\Sigma)$, but at the same time rich enough so that it computes the ordinary cohomology of E (Theorem 3.9). The multiplicative structuring of forms uses the flatness assumption on the bundle in an essential way. These techniques then allow us to construct the subcomplex $\Omega I^\bullet_{\bar{p}}(X - \Sigma) \subset \Omega^\bullet(X - \Sigma)$ as follows: Let $\overline{X}$ be the blow-up (defined precisely in Section 2) of X with boundary $\partial\overline{X} = E$. Let π be the retraction from a collar neighborhood of $\partial\overline{X}$ in $\overline{X}$ to $\partial\overline{X}$. Then

$$\begin{aligned}\Omega I^\bullet_{\bar{p}}(X - \Sigma) \quad = \quad & \{\omega \in \Omega^\bullet(X - \Sigma) \mid \exists \text{ open neighborhood } U \text{ of } \partial\overline{X} : \\ & \quad \omega|_{U-\Sigma} = \pi^*\eta, \text{ some } \eta \in \operatorname{ft}_{\geq m-\bar{p}(m+1)}\Omega^\bullet_{\mathcal{MS}}(\Sigma)\}.\end{aligned}$$

Comparing this to the definition of $I^{\bar{p}}X$, we see that coning off the truncated Moore approximation of the link has the homological effect of cotruncating the homology of the link, which explains why we must use cotruncation, rather than truncation, of forms on the links. This leads one to expect a de Rham isomorphism between the cohomology of the two constructions. There is a parallel between the above definition of $\Omega I^\bullet_{\bar{p}}$ and the special differential forms introduced by Goresky, Harder and MacPherson in [**24**, Section 13]. Following this parallel, we sheafify the complex $\Omega I^\bullet_{\bar{p}}(X - \Sigma)$ and construct a complex $\mathbf{\Omega I}^\bullet_{\bar{p}}$ of soft sheaves on X, whose global hypercohomology is $HI^\bullet_{\bar{p}}(X)$. It is perhaps worthwhile to point out that no completely general sheaf-theoretic formula, analogous to the well-known Deligne-Goresky-MacPherson formula for the intersection chain sheaf, expressed in terms of the standard functors operating on sheaf complexes on an arbitrary stratified pseudomanifold, can exist to describe $HI^\bullet$. For such a formula would then enable fiberwise cotruncation in arbitrary degrees for arbitrary link bundles, which implies the collapse of the Leray-Serre spectral sequence (with real coefficients) of the link bundle at E_2, as we have shown in [**5**]. Thus nonvanishing differentials in that spectral sequence are obstructions to fiberwise cotruncation. These obstructions may be interpreted as the price one must pay for the richer internal algebraic structure discussed above. For complex projective hypersurfaces with an isolated singularity, we show in [**6**] that the cohomology of intersection spaces is the hypercohomology of a perverse sheaf on the hypersurface. Moreover, this perverse sheaf underlies a mixed Hodge module. For general X, there cannot exist a perverse sheaf $\mathbf{P}^\bullet$ on X such that $HI^\bullet(X)$ is computed by the hypercohomology groups $\mathcal{H}^\bullet(X;\mathbf{P}^\bullet)$, as follows from the stalk vanishing conditions that such a $\mathbf{P}^\bullet$ satisfies. To construct *some* sheaf complex $\mathbf{S}^\bullet$ on X with $\mathcal{H}^\bullet(X;\mathbf{S}^\bullet) \cong HI^\bullet_{\bar{p},s}(X;\mathbb{R})$, simply take the pushforward $\mathbf{S}^\bullet = Rf_*\mathbb{R}_{I^{\bar{p}}X}$ of the constant sheaf on $I^{\bar{p}}X$ under the collapse map $f : I^{\bar{p}}X \to X$.

The methods introduced in the present paper radiate out into fields that are not (directly) linked to singularities. For example, let $\pi : E \to B$ be a flat fiber bundle of closed, smooth manifolds with oriented fiber and compact Lie structure group. Then our method of fiberwise cotruncation and multiplicatively structured forms can, as mentioned above, be used to show that the cohomological Leray-Serre spectral sequence of π for real coefficients collapses at the E_2-term. We can furthermore show that if M is an oriented, closed, Riemannian manifold and G a discrete group, whose Eilenberg-MacLane space $K(G,1)$ may be taken to be a closed, smooth manifold (e.g. $G = \mathbb{Z}^n$), and which acts isometrically on M, then the equivariant cohomology $H^\bullet_G(M;\mathbb{R})$ of this action can be computed as

$$H^k_G(M;\mathbb{R}) \cong \bigoplus_{p+q=k} H^p(G;\mathbf{H}^q(M;\mathbb{R})),$$

where the $\mathbf{H}^q(M;\mathbb{R})$ are the cohomology G-modules determined by the action. (We do not assume that G is closed in the isometry group of M.) These consequences are detailed in [**5**]. In a similar vein, the fiberwise spatial homology truncation methods used to construct intersection spaces yield, for simply connected singular sets where nontrivial link bundles are not flat, information on cases of the Halperin conjecture, [**27**], [**20**].

An analytic description of the cohomology theory $HI^\bullet$ remains to be found. A partial result in this direction is the following. Let M be a smooth, compact manifold with boundary ∂M. Let x be a boundary-defining function, i.e. on ∂M we have $x \equiv 0$, and $dx \neq 0$. A Riemannian metric g on the interior N of M is called a *scattering metric* if near ∂M it has the form

$$g = \frac{dx^2}{x^4} + \frac{h}{x^2},$$

where h is a metric on ∂M. Let $L^2\mathcal{H}^\bullet(N,g)$ denote the Hodge cohomology space of L^2-harmonic forms on N. From Melrose [**32**], the work of Hausel, Hunsicker and Mazzeo [**28**], and the results of [**3**], one can readily derive:

Proposition 1.1. *Suppose that X^n is an even-dimensional pseudomanifold with only one isolated singularity so that $X = M \cup \operatorname{cone}(\partial M)$, where M is a compact manifold with boundary. If the complement N of the singular point is endowed with a scattering metric g and the restriction map $H^{n/2}(M) \to H^{n/2}(\partial M)$ is zero (a "Witt-type" condition), then*

$$HI^\bullet(X) \cong L^2\mathcal{H}^\bullet(N,g).$$

General Notation. For a real vector space V, we denote the linear dual $\operatorname{Hom}(V,\mathbb{R})$ by $V^\dagger$. The tangent space of a smooth manifold M at a point $x \in M$ is written as T_xM. For a smooth manifold M, $H^\bullet(M)$ will always denote the de Rham cohomology of M, whereas $H^\bullet_s(X)$ denotes the singular cohomology with real coefficients of a topological space X. Singular homology with real coefficients will be written as $H_\bullet(X)$. Reduced cohomology and homology are indicated by $\widetilde{H}^\bullet, \widetilde{H}^\bullet_s, \widetilde{H}_\bullet$. The ring of smooth real-valued functions on a smooth manifold M is written as $C^\infty(M)$. Boldface letters denote sheaves.

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2. Preparatory Material on Stratified Spaces and Differential Forms

We shall work with stratified spaces that possess Mather-type control data, see for example [**31**] or [**1**]. Since the present paper only considers stratifications of depth 1, we limit the following definition to this depth although it is of course available in full generality.

Definition 2.1. A 2-*strata space* is a pair (X, Σ) such that

(1) X is a locally compact, Hausdorff, second-countable topological space, $\Sigma \subset X$ is a closed subspace and a closed, connected, smooth manifold, $X - \Sigma$ is a smooth manifold dense in X;

(2) Σ possesses control data (T, p, ρ), where
(2.1) $T \subset X$ is an open neighborhood of Σ,
(2.2) $p : T \to \Sigma$ is a continuous retraction,
(2.3) $\rho : T \to [0, 2)$ is a continuous radial function such that $\rho^{-1}(0) = \Sigma$, and
(2.4) the restrictions of p and ρ to $T - \Sigma$ are smooth;

(3) $p : T \to \Sigma$ is a locally trivial fiber bundle with fiber the open cone $cL = (L \times [0, 2))/(L \times 0)$ over some closed smooth manifold L (the *link* of Σ) and structure group given by homeomorphisms $cL \to cL$ of the form $c(\phi)$, where $\phi : L \to L$ is a diffeomorphism. These ϕ are to vary smoothly with points in charts of Σ;

(4) Locally, the radius ρ is the cone-line coordinate: If $U \subset \Sigma$ is an open set and

$$\begin{tikzcd} U \times cL \arrow[rr, "\psi", "\cong"'] \arrow[dr, "\mathrm{proj}_1"'] & & p^{-1}(U) \arrow[dl, "p|"] \\ & U & \end{tikzcd}$$

a local trivialization with ψ the identity on $U \times \{c\}$ (where c is the cone vertex), then

(1)
$$\begin{tikzcd} U \times cL \arrow[r, "\psi"] \arrow[d, "\mathrm{proj}_2"'] & p^{-1}(U) \arrow[d, "\rho|"] \\ cL \arrow[r, "\tau"] & {[0,2)} \end{tikzcd}$$

commutes, where $\tau(l, t) = t$, $l \in L$, $t \in [0, 2)$.

The triple (T, p, ρ) is called *control data* at Σ. For $E = \rho^{-1}(1)$, the above axioms imply that the restriction $p : E \to \Sigma$ is a smooth fiber bundle with fiber L. We call this bundle the *link bundle* of Σ. Note that a space X satisfying (1) is metrizable by Urysohn's metrization theorem. The dense open subspace $X - \Sigma$ is called the *top stratum* or *regular part* of X. The set Σ is often called the *singular set* or *singular stratum*, though it need not actually contain singular points.

Definition 2.2. A *stratified space of depth 1* (or *depth-1 space* for short) is a tuple $(X, \Sigma_1, \ldots, \Sigma_r)$ such that X is a locally compact, Hausdorff, second-countable topological space and the Σ_i are mutually disjoint, closed subspaces of X such that $(X - \bigcup_{j \neq i} \Sigma_j, \Sigma_i)$ is a 2-strata space for every $i = 1, \ldots, r$.

(A locally compact, Hausdorff, second-countable space is normal — thus every Σ_i has an open neighborhood T_i in X such that $T_i \cap T_j = \varnothing$ for $i \neq j$.) A stratified space X in the sense of Definition 2.1 has a blow-up (sometimes also called resolution) $\overline{X}$, see e.g. [**12**], [**1**], which is obtained, roughly, by replacing points of the singular stratum by their links. The precise construction to be used in this paper is as follows. Let (T, p, ρ) be control data at Σ. The (open) mapping cylinder of the link bundle projection is stratified isomorphic to T by an isomorphism which is the identity on Σ, see [**12**, A.I.6.3]. Its restriction to $E \times (0,2)$ is a diffeomorphism $F : E \times (0,2) \to T - \Sigma$ such that

$$\begin{tikzcd} E \times (0,2) \arrow[rr, "\cong", "F"'] \arrow[dr, "\mathrm{proj}_2"'] & & T - \Sigma \arrow[dl, "\rho"] \\ & (0,2) & \end{tikzcd}$$

commutes. We set

$$\overline{X} = ((X - \Sigma) \sqcup (E \times [0,2)))/\sim,$$

where $x \sim F(x)$ for all $x \in E \times (0,2)$. Then $\overline{X}$ is a smooth manifold with boundary $\partial\overline{X} = E \times \{0\} = E$ and comes equipped with a continuous map $\overline{X} \to X$, which is the identity on the interior $N := \overline{X} - \partial\overline{X} = X - \Sigma$ and p on $\partial\overline{X}$.

Definition 2.3. The map $\overline{X} \to X$ (and sometimes $\overline{X}$ itself) is called the *blow-up* of X.

Throughout the paper, $\pi : E \times [0,2) \to E$ denotes the projection to the first factor. More generally, we will also use π to denote the restriction of this projection to any subset $U \subset \overline{X}$ which is contained in the collar $E \times [0,2)$. The formula for F given in [**12**, A.I.6.3] shows that the composition

$$E = \rho^{-1}(1) \hookrightarrow T - \Sigma \xrightarrow{F^{-1}} E \times (0,2) \xrightarrow{\pi} E$$

is the identity map and the diagram

$$\begin{tikzcd} E \times (0,2) \arrow[r, "F"] \arrow[d, "\pi"'] & T - \Sigma \arrow[d, "p"] \\ E \arrow[r, "p"] & \Sigma \end{tikzcd}$$

commutes.

Let X be a stratified compact n-dimensional pseudomanifold in the sense of Definition 2.1, with b-dimensional singular stratum Σ. The main geometric assumption throughout this paper is that the link bundle be flat and isometrically structured. Thus $\partial\overline{X} = E$ is the total space of a flat fiber bundle $p : E \to \Sigma$ with fiber the link L^m, a closed Riemannian $(m = n - 1 - b)$-dimensional manifold. The structure group of p is the isometry group of L. We shall often write $B = \Sigma$ when we think of the singular stratum as the base space of its link bundle. For any smooth manifold M, let $\Omega^\bullet(M)$ denote the de Rham complex of smooth differential forms on M and let $\Omega^\bullet_c(M) \subset \Omega^\bullet(M)$ denote the subcomplex of forms with compact support. The exterior differential will be denoted by d. The differential forms that compute the cohomology of intersection spaces will be constant in the collar direction near the boundary of $\overline{X}$. We shall show that the complex

$$\begin{aligned}\Omega^\bullet_{\partial\mathcal{C}}(N) = \{\omega \in \Omega^\bullet(N) \mid \exists \text{ open neighborhood } U \subset E \times [0,2) \subset \overline{X} \\ \text{of } E = \partial\overline{X} : \omega|_{U\cap N} = \pi^*\eta, \quad \text{some } \eta \in \Omega^\bullet(\partial\overline{X})\}\end{aligned}$$

computes the cohomology of N.

Proposition 2.4. *The inclusion $\Omega^\bullet_{\partial\mathcal{C}}(N) \subset \Omega^\bullet(N)$ induces a cohomology isomorphism.*

Proof. We are grateful to the anonymous referee for pointing out the following argument, which is substantially shorter than the author's original argument. Let $\iota : \Omega^\bullet_{\partial\mathcal{C}}(N) \hookrightarrow \Omega^\bullet(N)$ be the subcomplex inclusion. Choose a smooth cutoff function $\chi : \overline{X} \to \mathbb{R}$ which is identically 1 on $E \times [0,1]$ and vanishes on the complement of $E \times [0, \frac{3}{2})$. Choose an $a \in (0,1)$. Let $\omega \in \Omega^\bullet(N)$ be any form. Its restriction to $E \times (0,2)$ can be written as $\omega_0 + dt \wedge \omega_1$, with $\omega_0(t), \omega_1(t) \in \Omega^\bullet(\partial\overline{X})$ for every $t \in (0,2)$. Setting

$$\rho(\omega) = (1-\chi)\omega + \chi\pi^*\omega_0(a) - d\chi \wedge \int_a^t \omega_1 dt,$$

we observe that $\rho(\omega) \in \Omega^\bullet_{\partial\mathcal{C}}(N)$ (take $U = E \times (0,1)$). Furthermore, ρ commutes with the de Rham differential d and thus defines a map $\rho : \Omega^\bullet(N) \to \Omega^\bullet_{\partial\mathcal{C}}(N)$ of complexes. Defining a homotopy operator K by $K(\omega) = \chi \int_a^t \omega_1 dt$, the equation $dK(\omega) + K(d\omega) = \omega - \rho(\omega)$ holds. This shows that $\iota \circ \rho$ is chain homotopic to the identity on $\Omega^\bullet(N)$. Now if $\omega \in \Omega^\bullet_{\partial\mathcal{C}}(N)$ then $K(\omega) \in \Omega^\bullet_{\partial\mathcal{C}}(N)$ as well. Hence $\rho \circ \iota$ is homotopic to the identity on $\Omega^\bullet_{\partial\mathcal{C}}(N)$. In particular, ι (and ρ) is a chain homotopy equivalence. q.e.d.

A map $\Omega^\bullet_{\partial\mathcal{C}}(N) \to \Omega^\bullet(\partial\overline{X})$ is given by $\omega \mapsto \eta$, since the equation $\omega|_{U\cap N} = \pi^*\eta$ determines η uniquely.

3. A Complex of Multiplicatively Structured Forms on Flat Bundles

Let F be a closed, oriented, smooth manifold and $p : E \to B$ a flat, smooth fiber bundle over the closed smooth base manifold B^b with fiber F. An open cover of a b-manifold is called *good*, if all nonempty finite intersections of sets in the cover are diffeomorphic to $\mathbb{R}^b$. Every smooth manifold has a good cover and if the manifold is compact, then the cover can be chosen to be finite. Let $\mathfrak{U} = \{U_\alpha\}$ be a finite good open cover of the base B such that p trivializes with respect to $\mathfrak{U}$. Let $\{\phi_\alpha\}$ be a system of local trivializations, that is, the ϕ_α are diffeomorphisms such that

$$\begin{tikzcd} p^{-1}(U_\alpha) \arrow[rr, "\phi_\alpha"] \arrow[dr, "p|"'] & & U_\alpha \times F \arrow[dl, "\pi_1"] \\ & U_\alpha & \end{tikzcd}$$

commutes for every α. Flatness means that the transition functions

$$\rho_{\beta\alpha} = \phi_\beta| \circ \phi_\alpha|^{-1} : (U_\alpha \cap U_\beta) \times F \longrightarrow p^{-1}(U_\alpha \cap U_\beta) \longrightarrow (U_\alpha \cap U_\beta) \times F$$

are of the form $\rho_{\beta\alpha}(t, x) = (t, g_{\beta\alpha}(x))$. If X is a topological space, let $\pi_2 : X \times F \to F$ denote the second-factor projection. Let $V \subset B$ be a $\mathfrak{U}$-small open subset and suppose that $V \subset U_\alpha$.

Definition 3.1. A differential form $\omega \in \Omega^q(p^{-1}(V))$ is called α-*multiplicatively structured*, if it has the form

$$\omega = \phi_\alpha^* \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j, \;\; \eta_j \in \Omega^\bullet(V), \;\; \gamma_j \in \Omega^\bullet(F)$$

(finite sums).

Flatness is crucial for the following basic lemma.

Lemma 3.2. *Suppose $V \subset U_\alpha \cap U_\beta$. Then ω is α-multiplicatively structured if, and only if, ω is β-multiplicatively structured.*

Proof. The flatness allows us to construct a commutative diagram

$$\begin{tikzcd} (U_\alpha \cap U_\beta) \times F \arrow[r, "\rho_{\alpha\beta}"] \arrow[d, "\pi_2"'] & (U_\alpha \cap U_\beta) \times F \arrow[d, "\pi_2"'] \\ F \arrow[r, "g_{\alpha\beta}"] & F. \end{tikzcd}$$

If the form is α-multiplicatively structured, then, using the equations $\pi_1 \rho_{\alpha\beta} = \pi_1$, $\pi_2 \rho_{\alpha\beta} = g_{\alpha\beta} \pi_2$, we derive the *transformation law*

$$\begin{aligned} \omega &= \phi_\alpha^* \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j = \phi_\beta^* (\phi_\beta^{-1})^* \phi_\alpha^* \sum_j \pi_1^* \eta_j \wedge \pi_2^* \gamma_j \\ &= \phi_\beta^* \sum_j \rho_{\alpha\beta}^* \pi_1^* \eta_j \wedge \rho_{\alpha\beta}^* \pi_2^* \gamma_j = \phi_\beta^* \sum_j \pi_1^* \eta_j \wedge \pi_2^* (g_{\alpha\beta}^* \gamma_j). \end{aligned}$$

Thus ω is β-multiplicatively structured. The converse implication follows from symmetry. q.e.d.

The lemma shows that the property of being multiplicatively structured over V is invariantly defined, independent of the choice of α such that $V \subset U_\alpha$. Let $U \subset B$ be an open subset. A linear subspace, the subspace of *multiplicatively structured forms*, of $\Omega^q(p^{-1}U)$ is obtained by setting

$$\Omega^q_{\mathcal{MS}}(U) = \{\omega \in \Omega^q(p^{-1}U) \mid \omega|_{p^{-1}(U\cap U_\alpha)} \text{ is } \alpha\text{-mult. structured, all } \alpha\}.$$

The Leibniz rule applied to a term of the form $\pi_1^*\eta \wedge \pi_2^*\gamma$ shows that the de Rham differential $d : \Omega^q(p^{-1}U) \to \Omega^{q+1}(p^{-1}U)$ restricts to a differential $d : \Omega^q_{\mathcal{MS}}(U) \to \Omega^{q+1}_{\mathcal{MS}}(U)$. This shows that $\Omega^\bullet_{\mathcal{MS}}(U) \subset \Omega^\bullet(p^{-1}U)$ is a subcomplex. Since there are well-defined restriction maps, the assignment $U \mapsto \Omega^\bullet_{\mathcal{MS}}(U)$ is a presheaf on B. As this presheaf satisfies the unique gluing property for sections, it is actually a sheaf $\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ on B, whose sections $\Gamma(U;\mathbf{\Omega}^\bullet_{\mathcal{MS}})$ are given by $\Gamma(U;\mathbf{\Omega}^\bullet_{\mathcal{MS}}) = \Omega^\bullet_{\mathcal{MS}}(U)$. Sending an open set $U \subset B$ to

$$C^\infty_p(U) = \{f \in C^\infty(p^{-1}U) \mid f = g \circ p,\ g \in C^\infty(U)\}$$

defines a presheaf of rings with unit on B, which satisfies the unique gluing property for sections and thus is a sheaf $\mathbf{C}^\infty_p$ of rings with unit on B. The sections of $\mathbf{C}^\infty_p$ over U are $\Gamma(U;\mathbf{C}^\infty_p) = C^\infty_p(U)$. Using pullbacks of suitable bump functions on B one verifies easily:

Lemma 3.3. *The sheaf $\mathbf{C}^\infty_p$ is soft.*

Since $\mathbf{C}^\infty_p$ is a sheaf of rings with unit, the above lemma implies that $\mathbf{C}^\infty_p$ is in fact a fine sheaf, see [**13**, Theorem II.9.16].

Lemma 3.4. *The vector space $\Omega^q_{\mathcal{MS}}(U)$ is a module over $C^\infty_p(U)$ for every q.*

Proof. Let $\omega \in \Omega^q_{\mathcal{MS}}(U)$ and $f \in C^\infty_p(U)$. The restriction of ω to $p^{-1}U_\alpha$ has a representation $\omega|_{p^{-1}(U\cap U_\alpha)} = \phi_\alpha^* \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j$, $\eta_j \in \Omega^\bullet(U\cap U_\alpha)$, $\gamma_j \in \Omega^\bullet(F)$. Over U_α, the equation $p = \pi_1 \circ \phi_\alpha$ holds and thus $f = g\circ p = \phi_\alpha^*\pi_1^*(g)$. Hence

$$(f\omega)|_{p^{-1}(U\cap U_\alpha)} = \phi_\alpha^*\pi_1^*(g)\cdot \phi_\alpha^* \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j = \phi_\alpha^* \sum_j \pi_1^*(g\eta_j)\wedge \pi_2^*\gamma_j$$

with $g\eta_j \in \Omega^\bullet(U\cap U_\alpha)$. This shows that $f\omega$ is multiplicatively structured. q.e.d.

We conclude that the sheaf $\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ is a module over the fine sheaf $\mathbf{C}^\infty_p$. By [**13**, Theorem II.9.16], $\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ is a fine sheaf. Let $\mathbf{\Omega}^\bullet_p$ be the fine sheaf on B with sections $\Gamma(U;\mathbf{\Omega}^\bullet_p) = \Omega^\bullet(p^{-1}U)$. The inclusion maps $\Omega^\bullet_{\mathcal{MS}}(U) \hookrightarrow \Omega^\bullet(p^{-1}U)$ define a morphism $\mathbf{\Omega}^\bullet_{\mathcal{MS}} \to \mathbf{\Omega}^\bullet_p$. We shall show that this morphism is a quasi-isomorphism.

Regarding $\mathbb{R}^b \times F$ as a trivial fiber bundle over $\mathbb{R}^b$ with projection π_1, the multiplicatively structured complex $\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ is defined as

$$\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) = \{\omega \in \Omega^\bullet(\mathbb{R}^b \times F) \mid \omega = \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j, \\ \eta_j \in \Omega^\bullet(\mathbb{R}^b),\ \gamma_j \in \Omega^\bullet(F)\}.$$

Let $s : \mathbb{R}^{b-1} \hookrightarrow \mathbb{R} \times \mathbb{R}^{b-1} = \mathbb{R}^b$ be the standard inclusion $s(u) = (0, u)$, $u \in \mathbb{R}^{b-1}$. Let $q : \mathbb{R}^b = \mathbb{R} \times \mathbb{R}^{b-1} \to \mathbb{R}^{b-1}$ be the standard projection $q(t, u) = u$, so that $qs = \mathrm{id}_{\mathbb{R}^{b-1}}$. Set

$$S = s \times \mathrm{id}_F : \mathbb{R}^{b-1} \times F \hookrightarrow \mathbb{R}^b \times F,\ Q = q \times \mathrm{id}_F : \mathbb{R}^b \times F \to \mathbb{R}^{b-1} \times F$$

so that $QS = \mathrm{id}_{\mathbb{R}^{b-1}\times F}$. The induced map $S^* : \Omega^\bullet(\mathbb{R}^b \times F) \to \Omega^\bullet(\mathbb{R}^{b-1} \times F)$ restricts to a map $S^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \to \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1})$, and the induced map $Q^* : \Omega^\bullet(\mathbb{R}^{b-1} \times F) \to \Omega^\bullet(\mathbb{R}^b \times F)$ restricts to a map $Q^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}) \to \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$.

Proposition 3.5. *The maps $S^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \rightleftarrows \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}) : Q^*$ are chain homotopy inverses of each other and thus induce mutually inverse isomorphisms*

$$H^\bullet(\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \underset{Q^*}{\overset{S^*}{\rightleftarrows}} H^\bullet(\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}))$$

on cohomology.

Proof. We define a homotopy operator $K : \Omega^\bullet(\mathbb{R}^b \times F) \to \Omega^{\bullet-1}(\mathbb{R}^b \times F)$ satisfying

$$dK + Kd = \mathrm{id} - Q^*S^*. \tag{2}$$

Think of $\mathbb{R}^b \times F$ as $\mathbb{R} \times M$, with $M = \mathbb{R}^{b-1} \times F$. Let $(t, t_2, \ldots, t_b)$ be coordinates on $\mathbb{R}^b = \mathbb{R} \times \mathbb{R}^{b-1}$ and let y denote (local) coordinates on F. Then $x = (t_2, \ldots, t_b, y)$ are coordinates on M. Every form on $\mathbb{R} \times M$ can be uniquely written as a linear combination of forms that do not contain dt, that is, forms $f(t, x)Q^*\alpha$, where $\alpha \in \Omega^\bullet(M)$, and forms that do contain dt, that is, forms $f(t, x)dt \wedge Q^*\alpha$. We define K by $K(f(t, x)Q^*\alpha) = 0$ and

$$K(f(t, x)dt \wedge Q^*\alpha) = g(t, x)Q^*\alpha, \text{ with } g(t, x) = \int_0^t f(\tau, x)d\tau.$$

Equation (2) is verified by a standard calculation. The operator K restricts to a homotopy operator $K_{\mathcal{MS}} : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \to \Omega^{\bullet-1}_{\mathcal{MS}}(\mathbb{R}^b)$. Since $S^*Q^* = \mathrm{id}$, S^* and Q^* are thus chain homotopy inverse chain homotopy equivalences through multiplicatively structured forms. q.e.d.

Let $S_0 : F = \{0\} \times F \hookrightarrow \mathbb{R}^b \times F$ be the inclusion at 0. The equations $\pi_1 \circ S_0 = c_0$, $\pi_2 \circ S_0 = \mathrm{id}_F$ hold, where $c_0 : F \to \mathbb{R}^b$ is the constant map $c_0(y) = 0$ for all $y \in F$. Thus, if $\eta \in \Omega^\bullet(\mathbb{R}^b)$ and $\gamma \in \Omega^\bullet(F)$, then

$$S_0^*(\pi_1^*\eta \wedge \pi_2^*\gamma) = c_0^*\eta \wedge \gamma = \begin{cases} \eta(0)\gamma, & \text{if } \deg \eta = 0 \\ 0, & \text{if } \deg \eta > 0. \end{cases}$$

The inclusion S_0 induces a map $S_0^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \longrightarrow \Omega^\bullet(F)$. The map $\pi_2^* : \Omega^\bullet(F) \to \Omega^\bullet(\mathbb{R}^b \times F)$ restricts to a map $\pi_2^* : \Omega^\bullet(F) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$.

Proposition 3.6. *The maps $S_0^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \rightleftarrows \Omega^\bullet(F) : \pi_2^*$ are chain homotopy inverses of each other and thus induce mutually inverse isomorphisms*

$$H^\bullet(\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \underset{\pi_2^*}{\overset{S_0^*}{\rightleftarrows}} H^\bullet(F)$$

on cohomology.

Proof. The statement holds for $b = 0$, since then $S_0 : \{0\} \times F \to \mathbb{R}^0 \times F$ is the identity map, $\pi_2 : \mathbb{R}^0 \times F \to F$ is the identity map, and $\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^0) = \Omega^\bullet(F)$. The statement then follows from Proposition 3.5 by an induction on b. q.e.d.

Proposition 3.6 together with homotopy invariance of classical de Rham cohomology implies:

Proposition 3.7. *The inclusion $\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \subset \Omega^\bullet(\mathbb{R}^b \times F)$ induces an isomorphism $H^\bullet(\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \cong H^\bullet(\mathbb{R}^b \times F)$ on cohomology.*

From this proposition, we deduce:

Proposition 3.8. *The map $\mathbf{\Omega}^\bullet_{\mathcal{MS}} \to \mathbf{\Omega}^\bullet_p$ is a quasi-isomorphism.*

Proof. Let $\mathbf{H}^q(-)$ denote the q-th derived sheaf of a complex of sheaves. We must show that the induced map $\mathbf{H}^q(\mathbf{\Omega}^\bullet_{\mathcal{MS}}) \to \mathbf{H}^q(\mathbf{\Omega}^\bullet_p)$ is an isomorphism in every degree q. This can be established stalkwise. The stalk $\mathbf{H}^q(\mathbf{\Omega}^\bullet_{\mathcal{MS}})_x$ at a point $x \in B$ can be expressed as

$$\mathbf{H}^q(\mathbf{\Omega}^\bullet_{\mathcal{MS}})_x = H^q(\mathbf{\Omega}^\bullet_{\mathcal{MS},x}) = H^q(\lim_j \Omega^\bullet_{\mathcal{MS}}(U_{x,j})) = \lim_j H^q(\Omega^\bullet_{\mathcal{MS}}(U_{x,j})),$$

with $\{U_{x,j}\}_j$ a neighborhood basis of x; similarly we have $\mathbf{H}^q(\mathbf{\Omega}^\bullet_p)_x = \lim_j H^q(p^{-1}U_{x,j})$. Now by Proposition 3.7 and the local triviality of the bundle, x has a neighborhood basis $\{U_{x,j}\}_j$ such that for every j, the inclusion $\Omega^\bullet_{\mathcal{MS}}(U_{x,j}) \hookrightarrow \Omega^\bullet(p^{-1}U_{x,j})$ induces an isomorphism on cohomology. q.e.d.

Theorem 3.9. *The inclusion $\Omega^\bullet_{\mathcal{MS}}(B) \hookrightarrow \Omega^\bullet(E)$ induces an isomorphism*

$$H^\bullet(\Omega^\bullet_{\mathcal{MS}}(B)) \xrightarrow{\cong} H^\bullet(E)$$

on cohomology.

Proof. By Proposition 3.8, $\mathbf{\Omega}^\bullet_{\mathcal{MS}} \to \mathbf{\Omega}^\bullet_p$ is a quasi-isomorphism and thus induces an isomorphism $H^\bullet(B;\mathbf{\Omega}^\bullet_{\mathcal{MS}}) \cong H^\bullet(B;\mathbf{\Omega}^\bullet_p)$ of global hypercohomology groups. Since $\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ is fine,

$$H^\bullet(B;\mathbf{\Omega}^\bullet_{\mathcal{MS}}) = H^\bullet\Gamma(B;\mathbf{\Omega}^\bullet_{\mathcal{MS}}) = H^\bullet(\Omega^\bullet_{\mathcal{MS}}(B))$$

and since $\mathbf{\Omega}^\bullet_p$ is fine,

$$H^\bullet(B;\mathbf{\Omega}^\bullet_p) = H^\bullet\Gamma(B;\mathbf{\Omega}^\bullet_p) = H^\bullet(\Omega^\bullet(p^{-1}B)) = H^\bullet(E).$$

q.e.d.

4. Truncation and Cotruncation Over a Point

Standard truncation of a differential complex yields a subcomplex, while standard cotruncation yields a quotient complex. Let F be a closed, oriented, m-dimensional Riemannian manifold as in Section 3. We shall here use the Riemannian metric and Hodge theory to define *both* truncation $\tau_{<k}$ and cotruncation $\tau_{\geq k}$ of the complex $\Omega^\bullet(F)$ as subcomplexes. A key advantage of cotruncation over truncation is that $\tau_{\geq k}\Omega^\bullet(F)$ is a subalgebra of $\Omega^\bullet(F)$, whereas $\tau_{<k}\Omega^\bullet(F)$ is not. This property of cotruncation will entail that the cohomology theory $HI^\bullet_{\bar{p}}(X)$ has a $\bar{p}$-internal cup product for all $\bar{p}$, while intersection cohomology does not.

The bilinear form $(\cdot,\cdot) : \Omega^r(F) \times \Omega^r(F) \to \mathbb{R}$, $(\omega,\eta) \mapsto \int_F \omega \wedge *\eta$, where $*$ is the Hodge star, is symmetric and positive definite, thus defines an inner product on $\Omega^\bullet(F)$. The Hodge star acts as an isometry with respect to this inner product, $(*\omega, *\eta) = (\omega,\eta)$, and the codifferential

$$d^* = (-1)^{m(r+1)+1} * d* : \Omega^r(F) \longrightarrow \Omega^{r-1}(F)$$

is the adjoint of the differential d, $(d\omega,\eta) = (\omega, d^*\eta)$. The classical Hodge decomposition theorem provides orthogonal splittings

$$\Omega^r(F) = \operatorname{im} d^* \oplus \operatorname{Harm}^r(F) \oplus \operatorname{im} d,$$

$$\ker d = \operatorname{Harm}^r(F) \oplus \operatorname{im} d,\ \ker d^* = \operatorname{im} d^* \oplus \operatorname{Harm}^r(F),$$

where $\operatorname{Harm}^r(F) = \ker d \cap \ker d^*$ are the closed and coclosed, i.e. harmonic, forms on F. In particular, $\Omega^r(F) = \operatorname{im} d^* \oplus \ker d = \ker d^* \oplus \operatorname{im} d$. Let k be a nonnegative integer.

Definition 4.1. The *truncation* $\tau_{<k}\Omega^\bullet(F)$ of $\Omega^\bullet(F)$ is the complex

$$\tau_{<k}\Omega^\bullet(F) = \cdots \to \Omega^{k-2}(F) \to \Omega^{k-1}(F) \xrightarrow{d^{k-1}} \operatorname{im} d^{k-1} \to 0 \to 0 \to \cdots,$$

where $\operatorname{im} d^{k-1} \subset \Omega^k(F)$ is placed in degree k.

The inclusion $\tau_{<k}\Omega^\bullet(F) \subset \Omega^\bullet(F)$ is a morphism of complexes. The induced map on cohomology, $H^r(\tau_{<k}\Omega^\bullet F) \to H^r(F)$, is an isomorphism

for $r < k$, while $H^r(\tau_{<k}\Omega^\bullet F) = 0$ for $r \geq k$. Using the orthogonal projection

$$\mathrm{proj} : \Omega^k(F) = \ker d^* \oplus \operatorname{im} d \twoheadrightarrow \operatorname{im} d,$$

we define a surjective morphism of complexes

$$\begin{tikzcd}
\Omega^\bullet(F) = \cdots \arrow[r] \arrow[d, "\mathrm{proj}"'] & \Omega^{k-2}(F) \arrow[r] \arrow[d, equal] & \Omega^{k-1}(F) \arrow[r, "d^{k-1}"] \arrow[d, equal] & \Omega^k(F) \arrow[r] \arrow[d, "\mathrm{proj}"'] & \Omega^{k+1}(F) \arrow[r] \arrow[d] & {} \\
\tau_{<k}\Omega^\bullet(F) = \cdots \arrow[r] & \Omega^{k-2}(F) \arrow[r] & \Omega^{k-1}(F) \arrow[r, "d^{k-1}"] & \operatorname{im} d^{k-1} \arrow[r] & 0 \arrow[r] & .
\end{tikzcd}$$

(Note that $\mathrm{proj} \circ d^{k-1} = d^{k-1}$.) The composition

$$\tau_{<k}\Omega^\bullet(F) \hookrightarrow \Omega^\bullet(F) \overset{\mathrm{proj}}{\twoheadrightarrow} \tau_{<k}\Omega^\bullet(F)$$

is the identity. Taking cohomology, this implies in particular that $\mathrm{proj}^* : H^r(F) \to H^r(\tau_{<k}\Omega^\bullet F)$ is an isomorphism for $r < k$. We move on to cotruncation.

Definition 4.2. The *cotruncation* $\tau_{\geq k}\Omega^\bullet(F)$ of $\Omega^\bullet(F)$ is the complex

$$\tau_{\geq k}\Omega^\bullet(F) = \cdots \to 0 \to 0 \to \ker d^* \xrightarrow{d^k|} \Omega^{k+1}(F) \xrightarrow{d^{k+1}} \Omega^{k+2}(F) \to \cdots,$$

where $\ker d^* \subset \Omega^k(F)$ is placed in degree k.

The inclusion $\tau_{\geq k}\Omega^\bullet(F) \subset \Omega^\bullet(F)$ is a morphism of complexes. By construction, $H^r(\tau_{\geq k}\Omega^\bullet F) = 0$ for $r < k$. There are several ways to see that $\tau_{\geq k}\Omega^\bullet(F) \hookrightarrow \Omega^\bullet(F)$ induces an isomorphism $H^r(\tau_{\geq k}\Omega^\bullet F) \xrightarrow{\cong} H^r(F)$ in the range $r \geq k$. One way is to compare $\tau_{\geq k}\Omega^\bullet(F)$ to the standard cotruncation

$$\widetilde{\tau}_{\geq k}\Omega^\bullet(F) = \cdots \to 0 \to \operatorname{coker} d^{k-1} \xrightarrow{d^k} \Omega^{k+1}(F) \xrightarrow{d^{k+1}} \Omega^{k+2}(F) \to \cdots$$

via the isomorphism

$$\ker d^* \xrightarrow{\cong} \frac{\ker d^* \oplus \operatorname{im} d}{\operatorname{im} d} = \frac{\Omega^k F}{\operatorname{im} d} = \operatorname{coker} d^{k-1}.$$

Alternatively, one observes that

$$H^k(\tau_{\geq k}\Omega^\bullet F) = \ker d \cap \ker d^* = \mathrm{Harm}^k(F) \cong H^k(F)$$

and

$$\begin{aligned} H^{k+1}(\tau_{\geq k}\Omega^\bullet F) &= \frac{\ker d^{k+1}}{d^k(\ker d^*)} = \frac{\ker d^{k+1}}{d^k(\ker d^* \oplus \operatorname{im} d^{k-1})} \\ &= \frac{\ker d^{k+1}}{\operatorname{im} d^k} = H^{k+1}(F). \end{aligned}$$

The kernel of $\mathrm{proj} : \Omega^\bullet(F) \twoheadrightarrow \tau_{<k}\Omega^\bullet F$ is precisely $\tau_{\geq k}\Omega^\bullet(F)$. Thus there is an exact sequence

$$0 \to \tau_{\geq k}\Omega^\bullet F \longrightarrow \Omega^\bullet F \longrightarrow \tau_{<k}\Omega^\bullet F \to 0. \tag{3}$$

(The associated long exact cohomology sequence gives a third way to see that $\tau_{\geq k}\Omega^\bullet F \hookrightarrow \Omega^\bullet F$ is a cohomology isomorphism in degrees $r \geq k$.)

We turn to the multiplicative properties of cotruncation.

Proposition 4.3. *The complex $\tau_{\geq k}\Omega^\bullet F$ is a sub-DGA of $(\Omega^\bullet(F), \wedge)$.*

Proof. It remains to be shown that if $\omega, \eta \in \tau_{\geq k}\Omega^\bullet F$, then $\omega \wedge \eta \in \tau_{\geq k}\Omega^\bullet F$. Let $p \geq 0$ be the degree of ω and $q \geq 0$ the degree of η. If $p+q > k$, then $(\tau_{\geq k}\Omega^\bullet F)^{p+q} = \Omega^{p+q}(F)$ and there is nothing to prove. If $p+q < k$, then both p and q are less than k. In this case, $(\tau_{\geq k}\Omega^\bullet F)^p = 0 = (\tau_{\geq k}\Omega^\bullet F)^q$ and $\omega\wedge\eta = 0 \in \tau_{\geq k}\Omega^\bullet F$. Suppose $p+q = k$. If one of p, q is less than k, then $\omega\wedge\eta = 0\wedge\eta = 0$ or $\omega\wedge\eta = \omega\wedge 0 = 0$ and the assertion follows as before. If $p, q \geq k$, then $k = p+q \geq 2k$ implies $k = 0 = p = q$. But for $k = 0$, $d^* = 0 : \Omega^0 F \to \Omega^{-1} F = 0$ so that $\ker d^* = \Omega^0 F$. Thus for functions $\omega, \eta \in \Omega^0 F$, we have $\omega \wedge \eta \in \Omega^0(F) = \ker d^* = (\tau_{\geq k}\Omega^\bullet F)^{p+q}$. q.e.d.

Proposition 4.4. *The isomorphism type of $\tau_{\geq k}\Omega^\bullet F$ in the category of cochain complexes is independent of the Riemannian metric on F.*

Proof. Let g and g' be two Riemannian metrics on F, determining codifferentials $d^*_g, d^*_{g'}$, harmonic forms $\mathrm{Harm}^\bullet_g(F), \mathrm{Harm}^\bullet_{g'}(F)$, and cotruncations $\tau^g_{\geq k}\Omega^\bullet F, \tau^{g'}_{\geq k}\Omega^\bullet F$. We observe first that $D := d^k(\ker d^*_g) = d^k(\ker d^*_{g'})$, as follows from

$$\begin{aligned} d^k(\ker d^*_g) &= d^k(\operatorname{im} d^{k-1} \oplus \ker d^*_g) = d^k(\Omega^k F) \\ &= d^k(\operatorname{im} d^{k-1} \oplus \ker d^*_{g'}) = d^k(\ker d^*_{g'}). \end{aligned}$$

Furthermore, as harmonic forms are closed,

$$\begin{aligned} d^k(\operatorname{im} d^*_g) &= d^k(\operatorname{im} d^*_g \oplus \mathrm{Harm}^k_g(F)) = d^k(\ker d^*_g) \\ &= d^k(\ker d^*_{g'}) = d^k(\operatorname{im} d^*_{g'} \oplus \mathrm{Harm}^k_{g'}(F)) = d^k(\operatorname{im} d^*_{g'}). \end{aligned}$$

Let $d_g : \operatorname{im} d^*_g \longrightarrow D$, $d_{g'} : \operatorname{im} d^*_{g'} \longrightarrow D$ be the restrictions of $d^k : \Omega^k F \to \Omega^{k+1} F$ to $\operatorname{im} d^*_g$ and $\operatorname{im} d^*_{g'}$, respectively. By the above observations, d_g and $d_{g'}$ are surjective. Since the decomposition $\Omega^k F = \operatorname{im} d^*_g \oplus \ker d^k$ is direct, d_g and $d_{g'}$ are injective, thus both isomorphisms. Since F is closed, the inclusions $\mathrm{Harm}^\bullet_g(F), \mathrm{Harm}^\bullet_{g'}(F) \subset \Omega^\bullet(F)$ induce isomorphisms

$$h_g : \mathrm{Harm}^k_g(F) \xrightarrow{\cong} H^k(F),\ \ h_{g'} : \mathrm{Harm}^k_{g'}(F) \xrightarrow{\cong} H^k(F).$$

Define an isomorphism $\kappa : \ker d^*_g \longrightarrow \ker d^*_{g'}$ by

$$\kappa : \ker d^*_g = \operatorname{im} d^*_g \oplus \mathrm{Harm}^k_g(F) \xrightarrow{d^{-1}_{g'} d_g \oplus h^{-1}_{g'} h_g} \operatorname{im} d^*_{g'} \oplus \mathrm{Harm}^k_{g'}(F) = \ker d^*_{g'}.$$

For $\alpha \in \operatorname{im} d_g^*$, $\beta \in \operatorname{Harm}_g^k(F)$, we have

$$d^k \kappa(\alpha+\beta) = d^k d_{g'}^{-1} d_g(\alpha) + d^k h_{g'}^{-1} h_g(\beta) = d_g(\alpha) = d^k(\alpha+\beta),$$

since harmonic forms are closed, which verifies that

$$\begin{tikzcd} \ker d_g^* \arrow[r, "d^k"] \arrow[d, "\kappa"', "\cong"] & \Omega^{k+1}F \arrow[d, equal] \\ \ker d_{g'}^* \arrow[r, "d^k"] & \Omega^{k+1}F \end{tikzcd}$$

commutes. This square can be embedded in an isomorphism of complexes

$$\begin{tikzcd} \tau_{\geq k}^{g}\Omega^\bullet F = \cdots \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] \arrow[d, equal] & \ker d_g^* \arrow[r] \arrow[d, "\kappa"', "\cong"] & \Omega^{k+1}F \arrow[r] \arrow[d, equal] & \Omega^{k+2}F \arrow[r] \arrow[d, equal] & \cdots \\ \tau_{\geq k}^{g'}\Omega^\bullet F = \cdots \arrow[r] & 0 \arrow[r] & \ker d_{g'}^* \arrow[r] & \Omega^{k+1}F \arrow[r] & \Omega^{k+2}F \arrow[r] & \cdots . \end{tikzcd}$$

q.e.d.

Lemma 4.5. *Let $f : F \to F$ be a smooth self-map.*
(1) f induces an endomorphism f^* of $\tau_{<k}\Omega^\bullet F$.
(2) If f is an isometry, then f induces an automorphism f^* of $\tau_{\geq k}\Omega^\bullet F$.

Proof. (1) Since $f^* : \Omega^\bullet F \to \Omega^\bullet F$ commutes with d, f^* restricts to a map $f^*| : \operatorname{im} d^{k-1} \to \operatorname{im} d^{k-1}$.

(2) If f is an isometry, then it preserves the orthogonal splitting $\Omega^k F = \operatorname{im} d^{k-1} \oplus \ker d^*$: For an isometry, one has $f^* \circ * = \epsilon \cdot * \circ f^*$ with $\epsilon = 1$ if f is orientation preserving and $\epsilon = -1$ if f is orientation reversing. Thus $d^* \circ f^* = f^* \circ d^*$, which implies $f^*(\ker d^*) \subset \ker d^*$. The preservation of $\operatorname{im} d^{k-1}$ was discussed in (1). The restriction $f^*| : \ker d^* \to \ker d^*$ continues to be injective, and is also onto: Given $\omega \in \ker d^*$, there exist $\alpha \in \operatorname{im} d$, $\beta \in \ker d^*$ such that $f^*(\alpha+\beta) = \omega$, since $f^* : \Omega^k F \to \Omega^k F$ is onto. Then $f^*\alpha = \omega - f^*\beta \in \ker d^*$ and $f^*\alpha \in \operatorname{im} d$ so that $f^*\alpha \in \ker d^* \cap \operatorname{im} d = 0$. Therefore, $f^*\beta = \omega$ and $f^*| : \ker d^* \to \ker d^*$ is surjective. q.e.d.

5. Fiberwise Truncation and Poincaré Duality

5.1. Local Fiberwise Truncation and Cotruncation. Let F be a closed, oriented, m-dimensional Riemannian manifold as in Section 3. Regarding $\mathbb{R}^b \times F$ as a trivial fiber bundle over $\mathbb{R}^b$ with projection π_1 and fiber F, a subcomplex $\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \subset \Omega^\bullet(\mathbb{R}^b \times F)$ of multiplicatively structured forms was defined in Section 3 as

$$\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) = \Big\{ \omega \in \Omega^\bullet(\mathbb{R}^b \times F) \mid \omega = \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j, \\ \eta_j \in \Omega^\bullet(\mathbb{R}^b),\ \gamma_j \in \Omega^\bullet(F) \Big\}.$$

We shall here define the *fiberwise truncation* $\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \subset \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ and the *fiberwise cotruncation* $\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \subset \Omega_{\mathcal{MS}}(\mathbb{R}^b)$, depending on an integer k. Analogous concepts for forms with compact supports will be introduced as well. In Section 4, a truncation $\tau_{<k}\Omega^\bullet(F)$ and a cotruncation $\tau_{\geq k}\Omega^\bullet(F)$ were defined using the Riemannian metric on F. Define

$$\begin{aligned}\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \quad = \quad & \Big\{\omega \in \Omega^\bullet(\mathbb{R}^b \times F) \mid \omega = \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j, \\ & \qquad\qquad \eta_j \in \Omega^\bullet(\mathbb{R}^b),\ \gamma_j \in \tau_{<k}\Omega^\bullet(F)\Big\}.\end{aligned}$$

The Leibniz rule shows that $\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ is a subcomplex of $\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$. Define $\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ by replacing $\tau_{<k}\Omega^\bullet(F)$ by $\tau_{\geq k}\Omega^\bullet(F)$ in the above definition of fiberwise truncation. This is a subcomplex of $\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$. Similar complexes can be defined using compact supports. We define the subcomplex $\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \subset \Omega^\bullet_c(\mathbb{R}^b \times F)$ of *multiplicatively structured forms with compact supports* on $\mathbb{R}^b \times F$ to be

$$\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) = \Omega^\bullet_c(\mathbb{R}^b \times F) \cap \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b).$$

Note that we have the alternative description

$$\begin{aligned}\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) = & \Big\{\omega \in \Omega^\bullet(\mathbb{R}^b \times F) \mid \omega = \sum_j \pi_1^*\eta_j \wedge \pi_2^*\gamma_j, \\ & \qquad\qquad \eta_j \in \Omega^\bullet_c(\mathbb{R}^b),\ \gamma_j \in \Omega^\bullet(F)\Big\}\end{aligned}$$

because a form $\omega \in \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$ can be written as $\omega = \pi_1^*(\rho)\omega$, where $\rho : \mathbb{R}^b \to \mathbb{R}$ is a bump function such that $\rho|_{\pi_1(\operatorname{supp}\omega)} \equiv 1$ and $\operatorname{supp}(\rho)$ is compact. Absorbing ρ into the η_j shows that we may as well assume that each η_j has compact support. (The converse inclusion is obvious as the sum over the j is finite.) As above, fiberwise truncations and cotruncations

$$\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \subset \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \supset \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$$

are defined as

$$\begin{aligned}\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) &= \Omega^\bullet_c(\mathbb{R}^b \times F) \cap \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b), \\ \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) &= \Omega^\bullet_c(\mathbb{R}^b \times F) \cap \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b),\end{aligned}$$

or in other words, by requiring the γ_j to lie in $\tau_{<k}\Omega^\bullet(F)$ and $\tau_{\geq k}\Omega^\bullet(F)$, respectively.

5.2. Poincaré Lemmas for Fiberwise Truncations. Let s, S, q, Q be the standard inclusion and projection maps used in Section 3. The formula $S^*(\pi_1^*\eta \wedge \pi_2^*\gamma) = \pi_1^*(s^*\eta) \wedge \pi_2^*\gamma$, $\gamma \in \tau_{<k}\Omega^\bullet(F)$, shows that $S^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \to \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1})$ restricts to a map $S^* : \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \longrightarrow \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1})$. The formula $Q^*(\pi_1^*\eta \wedge \pi_2^*\gamma) = \pi_1^*(q^*\eta) \wedge \pi_2^*\gamma$ shows

that the map $Q^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}) \to \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ restricts to a map $Q^* : \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}) \longrightarrow \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$.

Lemma 5.1. *The maps* $S^* : \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \rightleftarrows \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}) : Q^*$ *are chain homotopy inverses of each other and thus induce mutually inverse isomorphisms*

$$H^\bullet(\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \underset{Q^*}{\overset{S^*}{\rightleftarrows}} H^\bullet(\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^{b-1}))$$

on cohomology.

Proof. The homotopy operator $K_{\mathcal{MS}} : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \to \Omega^{\bullet-1}_{\mathcal{MS}}(\mathbb{R}^b)$, defined in the proof of Proposition 3.5, applied to a form $\omega = \pi_1^*\eta \wedge \pi_2^*\gamma$ yields a result that can be written as $\pi_1^*\eta' \wedge \pi_2^*\gamma$ for some η'. Thus $K_{\mathcal{MS}}$ does not transform γ and if $\gamma \in \tau_{<k}\Omega^\bullet F$, then $\pi_1^*\eta' \wedge \pi_2^*\gamma = K_{\mathcal{MS}}(\omega)$ again lies in $\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$. Thus $K_{\mathcal{MS}}$ restricts to a homotopy operator

$$K_{\mathcal{MS}} : \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \longrightarrow (\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b))^{\bullet-1}$$

satisfying $K_{\mathcal{MS}}d + dK_{\mathcal{MS}} = \mathrm{id} - Q^*S^*$. Consequently, Q^*S^* is chain homotopic to the identity on $\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$. Since $S^*Q^* = \mathrm{id}$, S^* and Q^* are thus chain homotopy inverse chain homotopy equivalences through fiberwise truncated, multiplicatively structured forms. q.e.d.

As in Section 3, let $S_0 : F = \{0\} \times F \hookrightarrow \mathbb{R}^b \times F$ be the inclusion at 0. If $\gamma \in \tau_{<k}\Omega^\bullet(F)$, then

$$S_0^*(\pi_1^*\eta \wedge \pi_2^*\gamma) = \begin{cases} \eta(0)\gamma, & \text{if } \deg \eta = 0 \\ 0, & \text{if } \deg \eta > 0 \end{cases}$$

lies in $\tau_{<k}\Omega^\bullet(F)$ for any $\eta \in \Omega^\bullet(\mathbb{R}^b)$. Thus $S_0^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \to \Omega^\bullet(F)$ restricts to a map $S_0^* : \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \longrightarrow \tau_{<k}\Omega^\bullet(F)$. The map $\pi_2^* : \Omega^\bullet(F) \to \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ restricts to a map $\pi_2^* : \tau_{<k}\Omega^\bullet(F) \to \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$ by the definition of $\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)$.

Lemma 5.2. *(Poincaré Lemma, truncation version.)*
The maps $S_0^* : \mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \rightleftarrows \tau_{<k}\Omega^\bullet(F) : \pi_2^*$ *are chain homotopy inverses of each other and thus induce mutually inverse isomorphisms*

$$H^r(\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \underset{\pi_2^*}{\overset{S_0^*}{\rightleftarrows}} H^r(\tau_{<k}\Omega^\bullet(F)) \cong \begin{cases} H^r(F), & r < k \\ 0, & r \geq k \end{cases}$$

on cohomology.

Proof. The statement holds for $b = 0$, since then S_0 and π_2 are both the identity map and $\mathrm{ft}_{<k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^0) = \tau_{<k}\Omega^\bullet(F)$. For positive b, the statement follows, as in the proof of Proposition 3.6, from an induction on b, using Lemma 5.1. q.e.d.

An analogous argument, replacing $\tau_{<k}\Omega^\bullet(F)$ by $\tau_{\geq k}\Omega^\bullet(F)$, proves a version for fiberwise cotruncation:

Lemma 5.3. *(Poincaré Lemma, cotruncation version.)*
The maps $S_0^* : \mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \rightleftarrows \tau_{\geq k}\Omega^\bullet(F) : \pi_2^*$ *are chain homotopy inverses of each other and thus induce mutually inverse isomorphisms*

$$H^r(\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \underset{\pi_2^*}{\overset{S_0^*}{\rightleftarrows}} H^r(\tau_{\geq k}\Omega^\bullet(F)) \cong \begin{cases} H^r(F), & r \geq k \\ 0, & r < k. \end{cases}$$

on cohomology.

In order to set up a Poincaré lemma for fiberwise cotruncation of multiplicatively structured compactly supported forms, we need to discuss integration along the fiber. Let Y be a smooth manifold and $\pi_2 : \mathbb{R}^k \times Y \to Y$ the second-factor projection. Integration along the fiber $\mathbb{R}^k$ of π_2 is a map $\pi_{2*} : \Omega^\bullet_c(\mathbb{R}^k \times Y) \to \Omega^{\bullet - k}_c(Y)$ of degree $-k$, given as follows. Let $t = (t_1, \ldots, t_k)$ be the standard coordinates on $\mathbb{R}^k$ and let dt denote the k-form $dt = dt_1 \wedge \cdots \wedge dt_k$. A compactly supported form on $\mathbb{R}^k \times Y$ is a linear combination of two types of forms: those which do not contain dt as a factor and those which do. The former can be written as $f(t,y)dt_{i_1} \wedge \cdots \wedge dt_{i_r} \wedge \pi_2^*\gamma$, $r < k$, and the latter as $g(t,y)dt \wedge \pi_2^*\gamma$, where $\gamma \in \Omega^\bullet_c(Y)$, y is a (local) coordinate on Y, and f,g have compact support. Define π_{2*} by

$$\begin{aligned} \pi_{2*}(f(t,y)dt_{i_1} \wedge \cdots \wedge dt_{i_r} \wedge \pi_2^*\gamma) &= 0 \qquad\qquad (r < k), \\ \pi_{2*}(g(t,y)dt \wedge \pi_2^*\gamma) &= \Big(\int_{\mathbb{R}^k} g dt_1 \cdots dt_k \Big) \cdot \gamma. \end{aligned}$$

This is a chain map $\pi_{2*} : \Omega^\bullet_c(\mathbb{R}^k \times Y) \to \Omega^{\bullet - k}_c(Y)$, provided the shifted complex $\Omega^{\bullet-k}_c(Y)$ is given the differential $d_{-k} = (-1)^k d$. For $\omega \in \Omega^\bullet_c(\mathbb{R}^k \times Y)$, one has the *projection formula*

$$\pi_{2*}(\omega \wedge \pi_2^*\gamma) = (\pi_{2*}\omega) \wedge \gamma.$$

In particular, for a multiplicatively structured form involving the pullback of $\eta \in \Omega^\bullet_c(\mathbb{R}^k)$, we obtain $\pi_{2*}(\pi_1^*\eta \wedge \pi_2^*\gamma) = \pi_{2*}(\pi_1^*\eta) \wedge \gamma$. Applying this concept to our $\pi_2 : \mathbb{R}^b \times F \to F$, we receive a map $\pi_{2*} : \Omega^\bullet_c(\mathbb{R}^b \times F) \to \Omega^{\bullet - b}(F)$, and, by restriction, $\pi_{2*} : \Omega_{\mathcal{MS},c}(\mathbb{R}^b) \to \Omega^{\bullet-b}(F)$.

Lemma 5.4. *For* $\omega \in \Omega^r_{\mathcal{MS},c}(\mathbb{R}^b)$ *and* $\gamma \in \Omega^{b+m-r}(F)$, *the integration formula*

$$\int_{\mathbb{R}^b \times F} \omega \wedge \pi_2^*\gamma = \int_F (\pi_{2*}\omega) \wedge \gamma$$

holds.

Now suppose that $\gamma \in \tau_{\geq k}\Omega^\bullet(F)$ and $\deg \eta = b$, so that $\pi_1^*\eta \wedge \pi_2^*\gamma$ lies in the cotruncation $\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$. Then $\pi_{2*}(\pi_1^*\eta \wedge \pi_2^*\gamma) = \pm(\int_{\mathbb{R}^b} \eta) \cdot \gamma$

lies in $\tau_{\geq k}\Omega^\bullet(F)$ as well. Thus integration along the fiber restricts to a map

$$\pi_{2*} : \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \longrightarrow (\tau_{\geq k}\Omega^\bullet(F))^{\bullet-b}.$$

Choose any compactly supported 1-form $e_1 = \varepsilon(t)dt \in \Omega^1_c(\mathbb{R}^1)$ with $\int_{-\infty}^{+\infty}\varepsilon(t)dt = 1$. Then $e = e_1 \wedge e_1 \wedge \cdots \wedge e_1 = \prod_{i=1}^b \varepsilon(t_i)dt_1 \wedge \cdots \wedge dt_b$ is a compactly supported b-form on $\mathbb{R}^b$ with $\int_{\mathbb{R}^b} e = 1$. A chain map $e_* : \Omega^{\bullet-b}(F) \longrightarrow \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$ is given by $e_*(\gamma) = \pi_1^* e \wedge \pi_2^*\gamma$. By definition of $\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$, e_* restricts to a map

$$e_* : (\tau_{\geq k}\Omega^\bullet(F))^{\bullet-b} \longrightarrow \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b).$$

Lemma 5.5. *(Poincaré Lemma for Cotruncation with Compact Supports.) The maps $\pi_{2*} : \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \rightleftarrows (\tau_{\geq k}\Omega^\bullet(F))^{\bullet-b} : e_*$ are chain homotopy inverses of each other and thus induce mutually inverse isomorphisms*

$$H^r(\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)) \underset{e_*}{\overset{\pi_{2*}}{\rightleftarrows}} H^r((\tau_{\geq k}\Omega^\bullet(F))^{\bullet-b}) \cong \begin{cases} H^{r-b}(F), & r-b \geq k \\ 0, & r-b < k. \end{cases}$$

on cohomology.

Proof. The plan is to factor π_{2*} and e_* by peeling off one $\mathbb{R}^1$-factor at a time. Each map in the factorization will be shown to be a homotopy equivalence. Let M be the manifold $M = \mathbb{R}^{b-1} \times F$ so that $\mathbb{R}^b \times F = \mathbb{R}^1 \times \mathbb{R}^{b-1} \times F = \mathbb{R}^1 \times M$. The coordinate on the $\mathbb{R}^1$-factor is t_1, coordinates on the $\mathbb{R}^{b-1}$-factor will be $u = (t_2, \ldots, t_b)$ and coordinates on F will be y. We shall also write $x = (u, y)$ for points in M. Let $\pi : \mathbb{R}^1 \times M \to M$ be the projection given by $\pi(t_1, x) = x$.

Step 1. We shall show that integration along the fiber of π, $\pi_* : \Omega^\bullet_c(\mathbb{R}^1 \times M) \to \Omega^{\bullet-1}_c(M)$, restricts to the complex of fiberwise cotruncated multiplicatively structured forms. Let $\pi_1^*\eta \wedge \pi_2^*\gamma \in \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$ be a multiplicatively structured form, $\eta \in \Omega^p_c(\mathbb{R}^b)$, $\gamma \in \tau_{\geq k}\Omega^\bullet(F)$. The p-form η can be uniquely decomposed as

$$\eta = \sum_I f_I(t_1, u)du_I + \sum_J g_J(t_1, u)dt_1 \wedge du_J,$$

$$du_I = dt_{i_1} \wedge \cdots \wedge dt_{i_p},\ du_J = dt_{j_1} \wedge \cdots \wedge dt_{j_{p-1}},$$

where I ranges over all strictly increasing multi-indices $2 \leq i_1 < i_2 < \ldots < i_p \leq b$ and J over $2 \leq j_1 < j_2 < \ldots < j_{p-1} \leq b$. The functions f_I and g_J have compact support. As the terms $\pi_1^*(f_I(t_1, u)du_I) \wedge \pi_2^*\gamma$ do not contain dt_1, they are sent to 0 by π_*. Let

$$\mathbb{R}^{b-1} \xleftarrow{\widehat{\pi}_1} \mathbb{R}^{b-1} \times F \xrightarrow{\widehat{\pi}_2} F$$

be the standard projections $\widehat{\pi}_1(u,y) = u$, $\widehat{\pi}_2(u,y) = y$, and set $G_J(u) = \int_{-\infty}^{+\infty} g_J(t_1,u)dt_1$. The map π_* sends the term

$$\pi_1^*(g_J(t_1,u)dt_1 \wedge du_J) \wedge \pi_2^*\gamma = g_J(t_1,u)dt_1 \wedge \pi^*(\widehat{\pi}_1^* du_J \wedge \widehat{\pi}_2^*\gamma)$$

to

$$G_J(u) \cdot (\widehat{\pi}_1^* du_J \wedge \widehat{\pi}_2^*\gamma) = \widehat{\pi}_1^*(G_J(u)du_J) \wedge \widehat{\pi}_2^*\gamma,$$

which lies in $(\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^{b-1}))^{\bullet-1}$. Thus π_* restricts to a map

$$\pi_* : \mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \longrightarrow (\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^{b-1}))^{\bullet-1}.$$

Step 2. We shall construct a candidate e_{1*} for a homotopy inverse for π_* and show that it, too, restricts to the complex of fiberwise cotruncated multiplicatively structured forms. We define a chain map $e_{1*} : \Omega_c^{\bullet-1}(M) \longrightarrow \Omega_c^\bullet(\mathbb{R}^1 \times M)$, that is,

$$e_{1*} : \Omega_c^{\bullet-1}(\mathbb{R}^{b-1} \times F) \longrightarrow \Omega_c^\bullet(\mathbb{R}^b \times F),$$

by $e_{1*}(\omega) = e_1 \wedge \pi^*\omega$. By construction, $\pi_* \circ e_{1*} = \mathrm{id}$. (Recall that $\int_{\mathbb{R}^1} e_1 = 1$.) The equations $\hat{\pi} \circ \pi_1 = \widehat{\pi}_1 \circ \pi$, $\widehat{\pi}_2 \circ \pi = \pi_2$ hold, where $\hat{\pi} : \mathbb{R} \times \mathbb{R}^{b-1} \to \mathbb{R}^{b-1}$ is the standard projection $\hat{\pi}(t,u) = u$. The image of a form $\widehat{\pi}_1^*\eta \wedge \widehat{\pi}_2^*\gamma \in (\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^{b-1}))^{\bullet-1}$, $\eta \in \Omega_c^\bullet(\mathbb{R}^{b-1})$, $\gamma \in \tau_{\geq k}\Omega^\bullet(F)$, under e_{1*} is

$$e_{1*}(\widehat{\pi}_1^*\eta \wedge \widehat{\pi}_2^*\gamma) = \pi_1^*(e_1 \wedge \hat{\pi}^*\eta) \wedge \pi_2^*\gamma,$$

which lies in $\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$. Thus e_{1*} restricts to a map

$$e_{1*} : (\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^{b-1}))^{\bullet-1} \longrightarrow \mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b).$$

Step 3. We shall show that $e_{1*}\pi_*$ is homotopic to the identity by exhibiting a homotopy operator $K : \mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \to (\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b))^{\bullet-1}$ such that

$$\mathrm{id} - e_{1*}\pi_* = dK + Kd \tag{4}$$

on $\mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$. First, define $K : \Omega_c^\bullet(\mathbb{R}^1 \times M) \longrightarrow \Omega_c^{\bullet-1}(\mathbb{R}^1 \times M)$ by

$$\begin{aligned} K(f(t_1,x) \cdot \pi^*\mu) &= 0, \\ K(g(t_1,x)dt_1 \wedge \pi^*\mu) &= (G(t_1,x) - E_1(t_1)G(\infty,x)) \cdot \pi^*\mu, \end{aligned}$$

where $G(t_1,x) = \int_{-\infty}^{t_1} g(\tau,x)d\tau$, $E_1(t_1) = \int_{-\infty}^{t_1} e_1$. Equation (4) holds on $\Omega_c^\bullet(\mathbb{R}^1 \times M)$. Let $\pi_1^*\eta \wedge \pi_2^*\gamma \in \mathrm{ft}_{\geq k}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$ be a multiplicatively structured form, $\eta \in \Omega_c^p(\mathbb{R}^b)$, $\gamma \in \tau_{\geq k}\Omega^\bullet(F)$. The basic form η is again decomposed as in Step 1. As the terms $\pi_1^*(f_I(t_1,u)du_I) \wedge \pi_2^*\gamma$ do not contain dt_1, they are sent to 0 by K. With $H_J(t_1,u) = G_J(t_1,u) - E_1(t_1)G_J(\infty,u)$, which has compact support, K maps the terms

$$\pi_1^*(g_J(t_1,u)dt_1 \wedge du_J) \wedge \pi_2^*\gamma = g_J(t_1,u)dt_1 \wedge \pi^*(\widehat{\pi}_1^* du_J \wedge \widehat{\pi}_2^*\gamma)$$

to

$$\begin{aligned} H_J(t_1,u)\cdot \pi^*(\widehat{\pi}_1^* du_J \wedge \widehat{\pi}_2^*\gamma) &= H_J(t_1,u)\cdot \pi_1^* du_J \wedge \pi_2^*\gamma \\ &= \pi_1^*(H_J(t_1,u)du_J)\wedge \pi_2^*\gamma, \end{aligned}$$

which lie in $(\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b))^{\bullet-1}$. Consequently, K restricts to a map

$$K : \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \longrightarrow (\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b))^{\bullet-1}.$$

By equation (4), it is a homotopy operator between

$$e_{1*}\pi_* : \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \longrightarrow \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$$

and the identity.

Step 4. By Step 3 and $\pi_* e_{1*} = \mathrm{id}$, the maps

$$\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b) \underset{e_{1*}}{\overset{\pi_*}{\rightleftarrows}} (\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^{b-1}))^{\bullet-1}$$

are mutually chain homotopy inverse chain homotopy equivalences. As b was arbitrary, we may iterate the application of these maps and obtain homotopy equivalences π_*^b (the b-fold iteration of π_*) and e_{1*}^b (the b-fold iteration of e_{1*}). Since $\pi_*^b = \pi_{2*}$ and $e_{1*}^b = e_*$, the lemma is proved.

q.e.d.

5.3. Local Poincaré Duality for Truncated Structured Forms. The Poincaré Lemmas of the previous section, together with the integration formula of Lemma 5.4 imply local Poincaré duality between fiberwise truncated multiplicatively structured forms and fiberwise cotruncated compactly supported multiplicatively structured forms, as we will demonstrate in this section. Given complementary perversities $\bar{p}$ and $\bar{q}$, and the dimension m of F, we define truncation values

$$K = m - \bar{p}(m+1),\ \ K^* = m - \bar{q}(m+1).$$

The bilinear form $\Omega^r(\mathbb{R}^b \times F) \times \Omega_c^{b+m-r}(\mathbb{R}^b \times F) \to \mathbb{R}$, $(\omega,\omega') \mapsto \int_{\mathbb{R}^b\times F}\omega\wedge\omega'$, restricts to $\int : \Omega^r_{\mathcal{MS}}(\mathbb{R}^b) \times \Omega^{b+m-r}_{\mathcal{MS},c}(\mathbb{R}^b) \longrightarrow \mathbb{R}$ and further to

$$\int : (\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b))^r \times (\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b))^{b+m-r} \longrightarrow \mathbb{R}. \tag{5}$$

Stokes' theorem implies:

Lemma 5.6. *The bilinear forms* (5) *induce bilinear forms*

$$\int : H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \times H^{b+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)) \longrightarrow \mathbb{R}$$

on cohomology.

Lemma 5.7. *Integration induces a nondegenerate bilinear form*

$$H^r(\tau_{<K}\Omega^\bullet(F)) \times H^{m-r}(\tau_{\geq K^*}\Omega^\bullet(F)) \longrightarrow \mathbb{R}.$$

Proof. If $r \geq K$, then $H^r(\tau_{<K}\Omega^\bullet(F)) = 0$. The inequality $r \geq K$ implies the inequality $m - r < K^*$. Thus $H^{m-r}(\tau_{\geq K^*}\Omega^\bullet(F)) = 0$ as well and the lemma is proved for $r \geq K$. When $r < K$, then $H^r(\tau_{<K}\Omega^\bullet(F)) = H^r(F)$. The inequality $r < K$ implies $m - r \geq K^*$. Hence $H^{m-r}(\tau_{\geq K^*}\Omega^\bullet(F)) = H^{m-r}(F)$. Classical Poincaré duality for the closed, oriented m-manifold F asserts that the bilinear form $H^r(F) \times H^{m-r}(F) \to \mathbb{R}$, $([\omega], [\eta]) \mapsto \int_F \omega \wedge \eta$, is nondegenerate. q.e.d.

Lemma 5.8. *(Local Poincaré Duality.) The bilinear form*

$$\int : H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \times H^{b+m-r}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)) \longrightarrow \mathbb{R}$$

is nondegenerate.

Proof. By Lemma 5.7, the map

$$H^r(\tau_{<K}\Omega^\bullet(F)) \longrightarrow H^{m-r}(\tau_{\geq K^*}\Omega^\bullet(F))^\dagger,\ [\omega] \mapsto \int_F - \wedge \omega,$$

is an isomorphism. We have to show that the map

$$H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \longrightarrow H^{b+m-r}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b))^\dagger,\ [\omega] \mapsto \int_{\mathbb{R}^b \times F} - \wedge \omega,$$

is an isomorphism. By the Poincaré Lemma 5.2,

$$\pi_2^* : H^r(\tau_{<K}\Omega^\bullet(F)) \longrightarrow H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b))$$

is an isomorphism. According to the Poincaré lemma for cotruncation with compact supports, Lemma 5.5,

$$\pi_{2*} : H^{b+m-r}(\mathrm{ft}_{\geq K^*}\, \Omega_{\mathcal{MS},c}(\mathbb{R}^b)) \longrightarrow H^{m-r}(\tau_{\geq K^*}\Omega^\bullet(F))$$

is an isomorphism. The desired conclusion will follow once we have verified that the diagram

$$\begin{tikzcd}
H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \arrow[d, "\int"'] & H^r(\tau_{<K}\Omega^\bullet(F)) \arrow[l, "\cong"', "\pi_2^*"] \arrow[d, "\int", "\cong"'] \\
H^{b+m-r}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b))^\dagger & H^{m-r}(\tau_{\geq K^*}\Omega^\bullet(F))^\dagger \arrow[l, "\cong"', "\pi_{2*}^\dagger"]
\end{tikzcd}$$

commutes. Commutativity means that for $\gamma \in \tau_{<K}\Omega^\bullet(F)$ and $\omega \in \mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$, the identity $\int_{\mathbb{R}^b \times F} \omega \wedge \pi_2^*\gamma = \int_F \pi_{2*}\omega \wedge \gamma$ holds. This is precisely the integration formula of Lemma 5.4. q.e.d.

5.4. Global Poincaré Duality for Truncated Structured Forms. Let $F \to E \xrightarrow{p} B$ be a flat fiber bundle as in Section 3. The manifold F is Riemannian and we now assume that the structure group of the bundle are the isometries of F. The smooth, compact base B is covered by a finite good open cover $\mathfrak{U} = \{U_\alpha\}$ with respect to which the bundle trivializes. The local trivializations are denoted by $\phi_\alpha : p^{-1}(U_\alpha) \xrightarrow{\cong} U_\alpha \times F$, as before. For $U \subset B$ open, a compactly supported version

$\Omega^\bullet_{\mathcal{MS},c}(U)$ is obtained by setting $\Omega^\bullet_{\mathcal{MS},c}(U) = \Omega^\bullet_c(p^{-1}U) \cap \Omega^\bullet_{\mathcal{MS}}(U)$. Note that this is consistent with our earlier definition of $\Omega^\bullet_{\mathcal{MS},c}(\mathbb{R}^b)$ for $U = \mathbb{R}^b$. For any integer k, a subcomplex

$$\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U) \subset \Omega^\bullet_{\mathcal{MS}}(U)$$

of *fiberwise truncated* multiplicatively structured forms on $p^{-1}(U)$ is given by requiring, for all α, every γ_j (see Definition 3.1) to lie in $\tau_{<k}\Omega^\bullet(F)$. This is well-defined by the transformation law of Lemma 3.2 together with Lemma 4.5(1). A subcomplex

$$\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(U) \subset \Omega^\bullet_{\mathcal{MS}}(U)$$

of *fiberwise cotruncated* multiplicatively structured forms on $p^{-1}(U)$ is given by requiring, for all α, every γ_j to lie in $\tau_{\geq k}\Omega^\bullet(F)$. This is well-defined by the transformation law and Lemma 4.5(2). (At this point it is used that the transition functions of the bundle are isometries.) Since there are well-defined restriction maps, the assignment $U \mapsto \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(U)$ is a presheaf on B. As this presheaf satisfies the unique gluing property for sections, it is actually a sheaf $\mathrm{ft}_{\geq k}\,\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ on B, whose sections are given by $\Gamma(U; \mathrm{ft}_{\geq k}\,\mathbf{\Omega}^\bullet_{\mathcal{MS}}) = \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(U)$. The proof of Lemma 3.4 also shows:

Lemma 5.9. *The vector space* $(\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}})^q(U)$ *is a* $C^\infty_p(U)$*-module for every* q*.*

Hence, the sheaf $\mathrm{ft}_{\geq k}\,\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ is a module over the fine sheaf $\mathbf{C}^\infty_p$. In particular, $\mathrm{ft}_{\geq k}\,\mathbf{\Omega}^\bullet_{\mathcal{MS}}$ is a fine sheaf. A subcomplex $\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(U) \subset \Omega^\bullet_{\mathcal{MS},c}(U)$ of *fiberwise cotruncated* multiplicatively structured compactly supported forms on $p^{-1}(U)$ is given by setting

$$\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(U) = \Omega^\bullet_c(p^{-1}U) \cap \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS}}(U).$$

Let $K = m - \bar{p}(m+1)$, $K^* = m - \bar{q}(m+1)$ be the truncation values defined in Section 5.3. The bilinear form $\Omega^r(p^{-1}U) \times \Omega^{b+m-r}_c(p^{-1}U) \to \mathbb{R}$, $(\omega, \omega') \mapsto \int_{p^{-1}U} \omega \wedge \omega'$, restricts to $\int : \Omega^r_{\mathcal{MS}}(U) \times \Omega^{b+m-r}_{\mathcal{MS},c}(U) \to \mathbb{R}$ and further to

$$\int : (\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(U))^r \times (\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(U))^{b+m-r} \longrightarrow \mathbb{R}. \tag{6}$$

Replacing $\mathbb{R}^b$ by U and $\mathbb{R}^b \times F$ by $p^{-1}U$ in the proof of Lemma 5.6, we obtain a globalized version of that lemma:

Lemma 5.10. *The bilinear forms* (6) *induce bilinear forms*

$$\int : H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(U)) \times H^{b+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(U)) \longrightarrow \mathbb{R}$$

on cohomology.

Lemma 5.11. *(Bootstrap.) Let $U, V \subset B$ be open subsets such that*

$$\int : H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(W)) \times H^{b+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(W)) \longrightarrow \mathbb{R} \tag{7}$$

is nondegenerate for $W = U, V, U \cap V$. Then (7) is nondegenerate for $W = U \cup V$.

Proof. Using pullbacks of a partition of unity subordinate to $\{U, V\}$ and Lemma 3.4, one shows that for any $k \in \mathbb{Z}$ the map

$$\begin{array}{rcl} \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U) \oplus \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(V) & \longrightarrow & \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U \cap V) \\ (\omega, \tau) & \mapsto & \tau|_{p^{-1}(U \cap V)} - \omega|_{p^{-1}(U \cap V)} \end{array}$$

is surjective. We demonstrate the exactness of the sequence

$$0 \to \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(W_\cup) \to \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U) \oplus \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(V) \to \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(W_\cap) \to 0 \tag{8}$$

at the middle group, where $W_\cup = U \cup V$, $W_\cap = U \cap V$. Given forms $\omega \in \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U)$ and $\tau \in \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(V)$ such that $\omega|_{p^{-1}(U \cap V)} = \tau|_{p^{-1}(U \cap V)}$, there exists a unique differential form $\delta \in \Omega^\bullet(p^{-1}(U \cup V))$ with $\delta|_{p^{-1}U} = \omega$, $\delta|_{p^{-1}V} = \tau$. Again, using the above partition of unity and Lemma 3.4, δ lies in $\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U \cup V) \subset \Omega^\bullet(p^{-1}(U \cup V))$. Since

$$\mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U \cup V) \longrightarrow \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(U) \oplus \mathrm{ft}_{<k}\,\Omega^\bullet_{\mathcal{MS}}(V)$$

is clearly injective, the sequence (8) is exact.

Our next immediate objective is to create a similar sequence for cotruncated multiplicatively structured forms with compact supports. Using extension by zero, the sum of two forms defines a map $\Omega^\bullet_c(p^{-1}U) \oplus \Omega^\bullet_c(p^{-1}V) \longrightarrow \Omega^\bullet_c(p^{-1}(U \cup V))$. We claim that this map restricts to a map

$$\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(U) \oplus \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(V) \longrightarrow \mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(U \cup V). \tag{9}$$

To prove the claim, let ω, ω' be forms in $\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(U)$, $\mathrm{ft}_{\geq k}\,\Omega^\bullet_{\mathcal{MS},c}(V)$, respectively. The image $K = p(\mathrm{supp}(\omega))$ is compact. Let $\rho_U : B \to \mathbb{R}$ be a bump function such that $\rho_U|_K \equiv 1$, $\mathrm{supp}(\rho_U) \subset U$ and $\mathrm{supp}(\rho_U)$ compact. Then $\widetilde{\rho}_U\omega = \omega$, where $\widetilde{\rho}_U = p^*(\rho_U)$. Since ω is multiplicatively structured and fiberwise cotruncated, its restriction to $p^{-1}(U \cap U_\alpha)$ can be written as

$$\omega|_{p^{-1}(U \cap U_\alpha)} = \phi^*_\alpha \sum_i \pi^*_1\eta_i \wedge \pi^*_2\gamma_i,\ \eta_i \in \Omega^\bullet(U \cap U_\alpha),\ \gamma_i \in \tau_{\geq k}\Omega^\bullet(F).$$

Multiplying by $\widetilde{\rho}_U$, we obtain $\omega|_{p^{-1}(U \cap U_\alpha)} = \phi^*_\alpha \sum_i \pi^*_1(\rho_U\eta_i) \wedge \pi^*_2\gamma_i$, where the support of $\rho_U\eta_i$ is contained in the compact space $\mathrm{supp}(\rho_U)$. Every point $x \in (V \cap U_\alpha) - U$ has an open neighborhood $U_x \subset (U \cup V) \cap U_\alpha$ such that the restriction of $\rho_U\eta_i$ to $U_x \cap U \cap U_\alpha$ vanishes. (If no such U_x existed, there would be a sequence of points $x_n \in U \cap U_\alpha$, converging to x as $n \to \infty$, with $(\rho_U\eta_i)(x_n) \neq 0$. Then $x_n \in \mathrm{supp}(\rho_U\eta_i) \subset \mathrm{supp}(\rho_U)$. As $\mathrm{supp}(\rho_U)$ is compact, $x \in \mathrm{supp}(\rho) \subset U$, which would contradict $x \notin U$.) Therefore, extending $\rho_U \eta_i$ by zero to $(U \cup V) \cap U_\alpha$ yields a smooth form. Similarly, we introduce $\rho_V : B \to \mathbb{R}$ such that $\widetilde{\rho}_V \omega' = \omega'$ and the extension by zero of $\rho_V \eta'_j$ to $(U \cup V) \cap U_\alpha$ is smooth, where

$$\omega'|_{p^{-1}(V \cap U_\alpha)} = \phi_\alpha^* \sum_j \pi_1^* \eta'_j \wedge \pi_2^* \gamma'_j, \;\; \eta'_j \in \Omega^\bullet(V \cap U_\alpha), \;\; \gamma'_j \in \tau_{\geq k} \Omega^\bullet(F).$$

Then the sum $\omega + \omega'$ on $U \cup V$ is multiplicatively structured and fiberwise cotruncated, as

$$(\omega+\omega')|_{p^{-1}((U\cup V)\cap U_\alpha)} = \phi_\alpha^* \Big(\sum_i \pi_1^*(\rho_U \eta_i) \wedge \pi_2^* \gamma_i + \sum_j \pi_1^*(\rho_V \eta'_j) \wedge \pi_2^* \gamma'_j \Big),$$

$\rho_U \eta_i \in \Omega^\bullet((U \cup V) \cap U_\alpha)$, $\rho_V \eta'_j \in \Omega^\bullet((U \cup V) \cap U_\alpha)$. This proves the claim. The map (9) is onto: Let $\{\rho_U, \rho_V\}$ be a partition of unity (not necessarily with compact supports) subordinate to $\{U, V\}$. Given a form $\omega \in \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U \cup V)$, consider the forms $p^*(\rho_U)\omega \in \Omega^\bullet(p^{-1}U)$ and $p^*(\rho_V)\omega \in \Omega^\bullet(p^{-1}V)$. Their supports are closed subsets of compact sets and thus themselves compact. They are multiplicatively structured and fiberwise cotruncated by Lemma 3.4. The summation map sends the pair $(p^*(\rho_U)\omega, p^*(\rho_V)\omega)$ to $(p^*\rho_U + p^*\rho_V)\omega = \omega$, establishing surjectivity.

Given a form $\omega \in \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U \cap V)$, extension by zero $\iota_* : \Omega^\bullet_c(p^{-1}(U \cap V)) \to \Omega^\bullet_c(p^{-1}U)$ allows us to regard ω as a form $\iota_* \omega \in \Omega^\bullet_c(p^{-1}U)$. We claim that this form lies in fact in $\mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U)$. This can be seen as above by writing $\omega = p^*(\rho)\omega$, where $\rho_U : B \to \mathbb{R}$ is a bump function such that $\rho_U|_{p(\operatorname{supp}\omega)} \equiv 1$, $\operatorname{supp}(\rho_U) \subset U$ and $\operatorname{supp}(\rho_U)$ compact. Extension by zero thus defines a map

$$\mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U \cap V) \to \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U) \oplus \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(V), \;\; \omega \mapsto (-\iota_*\omega, \iota_*\omega),$$

which is clearly injective. We obtain a sequence

$$\begin{aligned} 0 \to \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U \cap V) \to \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U) \oplus \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(V) \\ \to \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U \cup V) \to 0. \end{aligned} \tag{10}$$

Exactness in the middle follows from the exactness of the standard sequence

$$0 \to \Omega^\bullet_c(p^{-1}(U \cap V)) \to \Omega^\bullet_c(p^{-1}U) \oplus \Omega^\bullet_c(p^{-1}V) \to \Omega^\bullet_c(p^{-1}(U \cup V)) \to 0,$$

since the unique form $\tau = \omega|_{p^{-1}(U \cap V)} \in \Omega^\bullet_c(p^{-1}(U \cap V))$ which hits a given pair of the form $(-\omega, \omega)$ in $\mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U) \oplus \mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(V)$ must actually lie in $\mathrm{ft}_{\geq k} \Omega^\bullet_{\mathcal{MS},c}(U \cap V)$, as multiplicatively forms form a presheaf (i.e. the restriction of a multiplicatively structured form is again multiplicatively structured). We have shown that the sequence (10) is exact. The long exact cohomology sequences induced by (8) and (10) are dually paired by the bilinear forms of Lemma 5.10 ($W_\cup = U \cup V$, $W_\cap = U \cap V$):

$$\begin{tikzcd}
H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(W_\cup)) \arrow[d] & \otimes & H^{n+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(W_\cup)) \arrow[r, "\int_{p^{-1}(W_\cup)}"] & \mathbb{R} \\
\begin{array}{c} H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(U)) \\ \oplus H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(V)) \end{array} \arrow[d] & \otimes & \begin{array}{c} H^{n+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(U)) \\ \oplus H^{n+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(V)) \end{array} \arrow[u] \arrow[r, "\int_{p^{-1}(U)} + \int_{p^{-1}(V)}"] & \mathbb{R} \\
H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(W_\cap)) \arrow[d, "d^*"'] & \otimes & H^{n+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(W_\cap)) \arrow[u] \arrow[r, "\int_{p^{-1}(W_\cap)}"] & \mathbb{R} \\
H^{r+1}(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(W_\cup)) & \otimes & H^{n+m-r-1}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(W_\cup)) \arrow[u, "d_*"] \arrow[r, "\int_{p^{-1}(W_\cup)}"] & \mathbb{R}
\end{tikzcd}$$

The proof of Lemma 5.6 on page 45 of [**11**] shows that this diagram commutes up to sign. Since Poincaré duality holds over U, V and $U \cap V$ by assumption, the 5-lemma implies that it holds over $U \cup V$ as well.

q.e.d.

Proposition 5.12. *(Global Poincaré Duality for Truncated Multiplicatively Structured Forms.) Wedge product followed by integration induces a nondegenerate form*

$$H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(B)) \times H^{b+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS}}(B)) \longrightarrow \mathbb{R},$$

where $b = \dim B$, $m = \dim F$, $K = m - \bar{p}(m+1)$, $K^* = m - \bar{q}(m+1)$, *and* $\bar{p}, \bar{q}$ *are complementary perversities.*

Proof. We will prove that

$$H^r(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(U)) \times H^{b+m-r}(\mathrm{ft}_{\geq K^*}\,\Omega^\bullet_{\mathcal{MS},c}(U)) \to \mathbb{R}$$

is nondegenerate for all open subsets $U \subset B$ that are of the form $U = \bigcup_{i=1}^s U_{\alpha_0^i \ldots \alpha_{p_i}^i}$ by an induction on s. For $s = 1$, so that $U = U_{\alpha_0 \ldots \alpha_p} \cong \mathbb{R}^b$, the statement holds by Local Poincaré Duality, Lemma 5.8. Suppose the bilinear form is nondegenerate for all U of the form $U = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \ldots \alpha_{p_i}^i}$. Let V be a set $V = U_{\alpha_0^s \ldots \alpha_{p_s}^s}$. By induction hypothesis, the form is nondegenerate for U and for

$$U \cap V = \Big(\bigcup_{i=1}^{s-1} U_{\alpha_0^i \ldots \alpha_{p_i}^i}\Big) \cap U_{\alpha_0^s \ldots \alpha_{p_s}^s} = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \ldots \alpha_{p_i}^i \alpha_0^s \ldots \alpha_{p_s}^s}.$$

Since it also holds for V by the induction basis, it follows from the Bootstrap Lemma 5.11 that the form is nondegenerate for $U \cup V = \bigcup_{i=1}^s U_{\alpha_0^i \ldots \alpha_{p_i}^i}$. The statement for $U = B$ follows as B is compact and equals the finite union $B = \bigcup_\alpha U_\alpha$.

q.e.d.

6. The Complex $\Omega I^\bullet_{\bar p}$

Let X^n be a stratified, compact pseudomanifold as in Section 2. We continue to use the notation $(\overline{X}, \partial\overline{X})$ for the blow-up, $p : \partial\overline{X} = E \to B = \Sigma$ for the link bundle, $L = F$ for the link and $N = \overline{X} - \partial\overline{X}$ for the interior as introduced in that section. The link bundle p is assumed to be flat and has structure group the isometries of L. Let $b = \dim B$ and $\pi : E \times [0, 2) \to E$ be the first-factor projection. To the bundle p one can associate a complex $\Omega^\bullet_{\mathcal{MS}}(B) \subset \Omega^\bullet(\partial\overline{X})$ of multiplicatively structured forms as shown in Section 3. We define forms on N that are multiplicatively structured near the end of N (i.e. near the boundary of $\overline{X}$) as

$$\Omega^r_{\partial\mathcal{MS}}(N) = \{\omega \in \Omega^r(N) \mid \exists \text{ open neighborhood } U \subset E \times [0,2) \subset \overline{X}$$
$$\text{of } E = \partial\overline{X} : \omega|_{U \cap N} = \pi^*\eta, \text{ some } \eta \in \Omega^r_{\mathcal{MS}}(B)\}.$$

Then $\Omega^\bullet_{\partial\mathcal{MS}}(N) \subset \Omega^\bullet(N)$ is a subcomplex and we shall show below that this inclusion is a quasi-isomorphism. Cutoff values K and K^* are defined by $K = m - \bar p(m+1)$, $K^* = m - \bar q(m+1)$, with $\bar p, \bar q$ complementary perversities and $m = \dim L$. In Section 5, we defined and investigated a fiberwise cotruncation $\mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)$. Using this complex, we now define the complex $\Omega I^\bullet_{\bar p}(N)$ by

$$\Omega I^\bullet_{\bar p}(N) = \{\omega \in \Omega^\bullet(N) \mid \exists \text{ open neighborhood } U \subset E \times [0,2) \subset \overline{X}$$
$$\text{of } E = \partial\overline{X} : \omega|_{U \cap N} = \pi^*\eta, \text{ some } \eta \in \mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)\}.$$

It is obviously a subcomplex of $\Omega^\bullet_{\partial\mathcal{MS}}(N)$.

Definition 6.1. The cohomology groups $HI^\bullet_{\bar p}(X)$ are defined to be

$$HI^r_{\bar p}(X) = H^r(\Omega I^\bullet_{\bar p}(N)).$$

It follows from Proposition 4.4 that the groups $HI^\bullet_{\bar p}(X)$ are independent of the Riemannian metric on the link, where the metric is allowed to vary within all metrics such that the transition functions of the link bundle are isometries.

We shall construct a complex $\mathbf{\Omega I}^\bullet_{\bar p}$ of soft sheaves on X, whose global sections are given by $\Omega I^\bullet_{\bar p}(N)$. Guided by the definition of the presheaf of special differential forms given in [**24**], we set

$$\Omega I^\bullet_{\bar p}(U) = \{\omega \in \Omega^\bullet(U) \mid \exists \widetilde{\omega} \in \Omega I^\bullet_{\bar p}(N) : \widetilde{\omega}|_U = \omega\}$$

for open subsets $U \subset N$. Then the assignment $U \mapsto \Omega I^\bullet_{\bar p}(U)$ is a presheaf complex on N, but usually not a sheaf complex. A presheaf on X is obtained by assigning $\Omega I^\bullet_{\bar p}(V \cap N)$ to an open subset $V \subset X$. Let $\mathbf{\Omega I}^\bullet_{\bar p}$ be the sheafification of this presheaf. The next two facts are verified using standard sheaf theoretic methods, using cut-off functions that are sections of $\mathbf{C}^\infty_p$ (Section 3).

Lemma 6.2.
1. *The global sections of* $\mathbf{\Omega I}^\bullet_{\bar{p}}$ *are given by* $\Gamma(X; \mathbf{\Omega I}^\bullet_{\bar{p}}) = \Omega I^\bullet_{\bar{p}}(N)$.
2. *The complex of sheaves* $\mathbf{\Omega I}^\bullet_{\bar{p}}$ *is soft.*

This implies that the global hypercohomology $\mathcal{H}^\bullet(X; \mathbf{\Omega I}^\bullet_{\bar{p}})$ is

$$\mathcal{H}^\bullet(X; \mathbf{\Omega I}^\bullet_{\bar{p}}) = H^\bullet\Gamma(X; \mathbf{\Omega I}^\bullet_{\bar{p}}) = H^\bullet \Omega I^\bullet_{\bar{p}}(N) = HI^\bullet_{\bar{p}}(X).$$

The proof of the following proposition will use the complex $\Omega^\bullet_{\partial\mathcal{C}}(N)$ of forms constant in the collar direction, defined in Section 2. Mapping a form $\omega \in \Omega^\bullet_{\partial\mathcal{MS}}(N)$ to η with $\omega|_{U\cap N} = \pi^*\eta$ defines a map $j^* : \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B)$.

Proposition 6.3. *The inclusion* $\Omega^\bullet_{\partial\mathcal{MS}}(N) \subset \Omega^\bullet(N)$ *induces an isomorphism*

$$H^\bullet(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \cong H^\bullet(N)$$

on cohomology.

Proof. The map $j^* : \Omega^\bullet_{\partial\mathcal{MS}}(N) \to \Omega^\bullet_{\mathcal{MS}}(B)$ is onto: Given a form $\eta \in \Omega^\bullet_{\mathcal{MS}}(B)$, multiply the pullback $\pi^*\eta$, $\pi : E \times (0,2) \to E$, by a cutoff function which is identically 1 on $E \times (0,1)$ and zero on the complement in N of $E \times (0, \frac{3}{2})$. Since the kernel of j^* is $\Omega^\bullet_c(N)$, we have an exact sequence

$$0 \to \Omega^\bullet_c(N) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B) \to 0.$$

Similarly, the map $\Omega^\bullet_{\partial\mathcal{C}}(N) \to \Omega^\bullet(E)$ is onto. Its kernel is also $\Omega^\bullet_c(N)$, and we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \Omega^\bullet_c(N) \arrow[r] & \Omega^\bullet_{\partial\mathcal{C}}(N) \arrow[r] & \Omega^\bullet(E) \arrow[r] & 0 \\
0 \arrow[r] & \Omega^\bullet_c(N) \arrow[r] \arrow[u, equal] & \Omega^\bullet_{\partial\mathcal{MS}}(N) \arrow[r] \arrow[u, hook] & \Omega^\bullet_{\mathcal{MS}}(B) \arrow[r] \arrow[u, hook] & 0.
\end{tikzcd}$$

On cohomology, we arrive at a commutative diagram with long exact rows,

$$\begin{tikzcd}
H^\bullet_c(N) \arrow[r] & H^\bullet_{\partial\mathcal{C}}(N) \arrow[r] & H^\bullet(\partial\overline{X}) \arrow[r] & H^{\bullet+1}_c(N) \\
H^\bullet_c(N) \arrow[r] \arrow[u, equal] & H^\bullet_{\partial\mathcal{MS}}(N) \arrow[r] \arrow[u] & H^\bullet(\Omega^\bullet_{\mathcal{MS}}(B)) \arrow[r] \arrow[u, "\cong"'] & H^{\bullet+1}_c(N). \arrow[u, equal]
\end{tikzcd}$$

The vertical arrow $H^\bullet(\Omega^\bullet_{\mathcal{MS}}(B)) \to H^\bullet(\partial\overline{X})$ is an isomorphism by Theorem 3.9. By the 5-lemma, $H^\bullet_{\partial\mathcal{MS}}(N) \to H^\bullet_{\partial\mathcal{C}}(N)$ is an isomorphism. The inclusion $\Omega^\bullet_{\partial\mathcal{C}}(N) \subset \Omega^\bullet(N)$ induces an isomorphism $H^\bullet_{\partial\mathcal{C}}(N) \to H^\bullet(N)$ by Proposition 2.4. Thus the composition $H^\bullet_{\partial\mathcal{MS}}(N) \to H^\bullet(N)$ is an isomorphism as well. q.e.d.

A form $\omega \in \Omega^\bullet_{\partial\mathcal{MS}}(N)$ has a unique extension $\widetilde{\omega} \in \Omega^\bullet(\overline{X})$, given by setting $\widetilde{\omega}(x,0) = \eta(x)$ for $(x,0) \in E \times [0,2) \subset \overline{X}$, where $\eta = j^*(\omega)$. In this way, $\Omega^\bullet_{\partial\mathcal{MS}}(N)$ becomes a subcomplex of $\Omega^\bullet(\overline{X})$, since $\widetilde{d\omega} =$

$d(\widetilde{\omega})$. The obvious inclusions and restrictions induce on cohomology the diagram

$$\begin{tikzcd} & H^\bullet(N) \\ H^\bullet_{\partial \mathcal{MS}}(N) \arrow[ur, "\cong"] \arrow[r] & H^\bullet(\overline{X}). \arrow[u, "\cong"'] \end{tikzcd}$$

The diagonal arrow is an isomorphism by Proposition 6.3, while the vertical arrow is an isomorphism by homotopy invariance. This shows:

Proposition 6.4. *The inclusion $\Omega^\bullet_{\partial \mathcal{MS}}(N) \subset \Omega^\bullet(\overline{X})$ induces on cohomology an isomorphism $H^\bullet_{\partial \mathcal{MS}}(N) \cong H^\bullet(\overline{X})$.*

The surjection $j^* : \Omega^\bullet_{\partial \mathcal{MS}}(N) \to \Omega^\bullet_{\mathcal{MS}}(B)$ can now be thought of as restricting a form to the boundary $\partial \overline{X}$, since the diagram

$$\begin{tikzcd} \Omega^\bullet_{\partial \mathcal{MS}}(N) \arrow[r, "j^*"] \arrow[d, hook] & \Omega^\bullet_{\mathcal{MS}}(B) \arrow[d, hook] \\ \Omega^\bullet(\overline{X}) \arrow[r, "\mathrm{restr}"'] & \Omega^\bullet(\partial \overline{X}) \end{tikzcd}$$

commutes. For an open subset $U \subset B$, we set

$$Q^\bullet(U) = \Omega^\bullet_{\mathcal{MS}}(U) / \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(U).$$

Lemma 6.5. *Given open subsets $U, V \subset B$, there is a Mayer-Vietoris long exact sequence*

$$\cdots \xrightarrow{\delta^*} H^r Q^\bullet(U \cup V) \to H^r Q^\bullet(U) \oplus H^r Q^\bullet(V) \to H^r Q^\bullet(U \cap V) \xrightarrow{\delta^*} \cdots .$$

Proof. Use the exact fiberwise cotruncation sequence

$$0 \to \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(U \cup V) \to \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(U) \oplus \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(V) \\ \to \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(U \cap V) \to 0$$

and standard 3×3-diagram arguments. q.e.d.

For every open subset $U \subset B$, we define a canonical map

$$\gamma_U : \operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(U) \to Q^\bullet(U)$$

by composing

$$\operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(U) \overset{\mathrm{incl}}{\hookrightarrow} \Omega^\bullet_{\mathcal{MS}}(U) \xrightarrow{\mathrm{quot}} Q^\bullet(U).$$

Our next goal is to show that γ_B is a quasi-isomorphism. To prove this, we will use the following bootstrap principle:

Lemma 6.6. *Let $U, V \subset B$ be open subsets. If γ_U, γ_V and $\gamma_{U \cap V}$ are quasi-isomorphisms, then $\gamma_{U \cup V}$ is a quasi-isomorphism as well.*

Proof. Map the Mayer-Vietoris sequence developed in the proof of Lemma 5.11 to the Mayer-Vietoris sequence of Lemma 6.5 via γ, and use the 5-lemma. q.e.d.

Lemma 6.7. *The map $\gamma_B : \mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(B) \to Q^\bullet(B)$ induces an isomorphism*

$$H^\bullet(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(B)) \longrightarrow H^\bullet Q^\bullet(B)$$

on cohomology.

Proof. We shall show that γ_U is a quasi-isomorphism for all open U of the form $U = \bigcup_{i=1}^s U_{\alpha_0^i\ldots\alpha_{p_i}^i}$ by an induction on s, where $\{U_\alpha\}$ is a finite good cover of B with respect to which the link bundle trivializes. Let $s=1$ so that $U = U_{\alpha_0\ldots\alpha_p} \cong \mathbb{R}^b$. The inclusion $\operatorname{im} d^{K-1} \subset \Omega^K F$ induces an isomorphism

$$\operatorname{im} d^{K-1} \xrightarrow{\cong} \frac{\ker d^* \oplus \operatorname{im} d^{K-1}}{\ker d^*} = \frac{\Omega^K F}{(\tau_{\geq K}\Omega^\bullet F)^K},$$

which can be extended to an isomorphism of complexes

$$\begin{tikzcd}
\tau_{<K}\Omega^\bullet(F) = \cdots \arrow[r] \arrow[d, "\gamma"', "\cong"] & \Omega^{K-2}(F) \arrow[r] \arrow[d, equal] & \Omega^{K-1}(F) \arrow[r] \arrow[d, equal] & \operatorname{im} d^{K-1} \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] \arrow[d] & {} \\
\Omega^\bullet F/\tau_{\geq K}\Omega^\bullet F = \cdots \arrow[r] & \Omega^{K-2}(F) \arrow[r] & \Omega^{K-1}(F) \arrow[r] & \frac{\Omega^K F}{(\tau_{\geq K}\Omega^\bullet F)^K} \arrow[r] & 0 \arrow[r] & {}.
\end{tikzcd}$$

This isomorphism factors as

$$\gamma : \tau_{<K}\Omega^\bullet(F) \overset{\text{incl}}{\hookrightarrow} \Omega^\bullet(F) \xrightarrow{\text{quot}} \frac{\Omega^\bullet(F)}{\tau_{\geq K}\Omega^\bullet(F)}.$$

According to the Poincaré Lemmas 5.2 and 5.3, the restriction S_0^* of a form on $\mathbb{R}^b \times F$ to $\{0\} \times F = F$ provides a homotopy equivalence $S_0^* : \mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \xrightarrow{\simeq} \tau_{<K}\Omega^\bullet(F)$ and a homotopy equivalence $S_0^* : \mathrm{ft}_{\geq K}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \xrightarrow{\simeq} \tau_{\geq K}\Omega^\bullet(F)$. Taking K negative in the latter homotopy equivalence (or K larger than m in the former), we get in particular a homotopy equivalence $S_0^* : \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \xrightarrow{\simeq} \Omega^\bullet(F)$. The map S_0^* induces a unique map $Q^\bullet(\mathbb{R}^b) \to \Omega^\bullet(F)/\tau_{\geq K}\Omega^\bullet(F)$ such that

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{ft}_{\geq K}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \arrow[r] \arrow[d, "\simeq"', "S_0^*"] & \Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b) \arrow[r] \arrow[d, "\simeq"', "S_0^*"] & Q^\bullet(\mathbb{R}^b) \arrow[r] \arrow[d, dotted] & 0 \\
0 \arrow[r] & \tau_{\geq K}\Omega^\bullet(F) \arrow[r] & \Omega^\bullet(F) \arrow[r] & \frac{\Omega^\bullet(F)}{\tau_{\geq K}\Omega^\bullet(F)} \arrow[r] & 0
\end{tikzcd}$$

commutes. This map is a quasi-isomorphism by the 5-lemma. By the commutativity of

$$\begin{tikzcd}
H^\bullet(\mathrm{ft}_{<K}\,\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \arrow[r, "\cong", "S_0^*"'] \arrow[d, "\text{incl}^*"] \arrow[dd, bend right=60, "\gamma^*_{\mathbb{R}^b}"'] & H^\bullet(\tau_{<K}\Omega^\bullet(F)) \arrow[d, "\text{incl}^*"'] \arrow[dd, bend left=60, "{\gamma^*,\ \cong}"] \\
H^\bullet(\Omega^\bullet_{\mathcal{MS}}(\mathbb{R}^b)) \arrow[r, "\cong", "S_0^*"'] \arrow[d, "\text{quot}^*"] & H^\bullet(\Omega^\bullet(F)) \arrow[d, "\text{quot}^*"'] \\
H^\bullet Q^\bullet(\mathbb{R}^b) \arrow[r, "\cong", "S_0^*"'] & H^\bullet(\Omega^\bullet(F)/\tau_{\geq K}\Omega^\bullet(F)),
\end{tikzcd}$$

the map $\gamma_{\mathbb{R}^b}$ is a quasi-isomorphism. This furnishes the induction basis. Suppose γ_U is a quasi-isomorphism for all U of the form $U = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \cdots \alpha_{p_i}^i}$. Let V be a set $V = U_{\alpha_0^s \cdots \alpha_{p_s}^s}$. By the induction hypothesis, γ_U is a quasi-isomorphism and $\gamma_{U\cap V}$ is a quasi-isomorphism, as $U \cap V = \bigcup_{i=1}^{s-1} U_{\alpha_0^i \cdots \alpha_{p_i}^i \alpha_0^s \cdots \alpha_{p_s}^s}$. Since γ_V is a quasi-isomorphism as well ($s = 1$), the bootstrap Lemma 6.6 implies that $\gamma_{U\cup V}$ is a quasi-isomorphism, $U \cup V = \bigcup_{i=1}^{s} U_{\alpha_0^i \cdots \alpha_{p_i}^i}$. The statement for $U = B$ follows as B is the finite union $B = \bigcup_\alpha U_\alpha$. q.e.d.

Let $\mathcal{D}(\mathbb{R})$ denote the derived category of complexes of real vector spaces. The exact sequence

$$0 \longrightarrow \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow Q^\bullet(B) \longrightarrow 0$$

induces a distinguished triangle

$$\operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow Q^\bullet(B) \longrightarrow \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B)[1]$$

in $\mathcal{D}(\mathbb{R})$. Using the quasi-isomorphism γ_B of Lemma 6.7, we may replace $Q^\bullet(B)$ in the triangle by $\operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B)$ and thus arrive at a distinguished triangle

(11)
$$\operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B)[1].$$

On the basis of this triangle, we shall next construct a distinguished triangle

$$\Omega I^\bullet_{\bar p}(N) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \Omega I^\bullet_{\bar p}(N)[1]. \tag{12}$$

Since $\Omega I^\bullet_{\bar p}(N)$ is a subcomplex of $\Omega^\bullet_{\partial\mathcal{MS}}(N)$, there is an exact sequence

$$0 \longrightarrow \Omega I^\bullet_{\bar p}(N) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \frac{\Omega^\bullet_{\partial\mathcal{MS}}(N)}{\Omega I^\bullet_{\bar p}(N)} \longrightarrow 0.$$

The surjection $j^* : \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \Omega^\bullet_{\mathcal{MS}}(B)$ restricts further to a map $j^*_{\bar p} : \Omega I^\bullet_{\bar p}(N) \longrightarrow \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B)$, which is also surjective. Moreover, j^* induces a unique surjective map

$$\bar{j}^* : \frac{\Omega^\bullet_{\partial\mathcal{MS}}(N)}{\Omega I^\bullet_{\bar p}(N)} \twoheadrightarrow \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{\operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B)} = Q^\bullet(B)$$

such that

$$\begin{tikzcd}
0 \arrow[r] & \Omega I^\bullet_{\bar p}(N) \arrow[r] \arrow[d, two heads, "j^*_{\bar p}"'] & \Omega^\bullet_{\partial\mathcal{MS}}(N) \arrow[r] \arrow[d, two heads, "j^*"'] & \frac{\Omega^\bullet_{\partial\mathcal{MS}}(N)}{\Omega I^\bullet_{\bar p}(N)} \arrow[r] \arrow[d, two heads, dashed, "\bar{j}^*"'] & 0 \\
0 \arrow[r] & \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B) \arrow[r] & \Omega^\bullet_{\mathcal{MS}}(B) \arrow[r] & Q^\bullet(B) \arrow[r] & 0
\end{tikzcd}$$

commutes. The kernel of both j^* and $j^*_{\bar{p}}$ is $\Omega^\bullet_c(N)$. Thus, by standard arguments in homological algebra, $\overline{j}^*$ is an isomorphism. According to Lemma 6.7, the map $\gamma_B : \operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B) \to Q^\bullet(B)$ is a quasi-isomorphism. Using the isomorphism

$$\gamma_B^{-1} \circ \overline{j}^* : \frac{\Omega^\bullet_{\partial\mathcal{MS}}(N)}{\Omega I^\bullet_{\bar{p}}(N)} \xrightarrow{\cong} \operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B)$$

in $\mathcal{D}(\mathbb{R})$ to replace the quotient in the distinguished triangle

$$\Omega I^\bullet_{\bar{p}}(N) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N)/\Omega I^\bullet_{\bar{p}}(N) \longrightarrow \Omega I^\bullet_{\bar{p}}(N)[1]$$

by $\operatorname{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B)$, we arrive at the desired triangle (12). As the kernel of the surjective map $j^*_{\bar{p}} : \Omega I^\bullet_{\bar{p}}(N) \twoheadrightarrow \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B)$ is $\Omega^\bullet_c(N)$, there is also a distinguished triangle

$$\Omega^\bullet_c(N) \xrightarrow{\text{incl}} \Omega I^\bullet_{\bar{p}}(N) \xrightarrow{j^*_{\bar{p}}} \operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \Omega^\bullet_c(N)[1]. \tag{13}$$

These triangles will be used in proving Poincaré duality for $HI^\bullet(X)$.

7. Integration on $\Omega I^\bullet_{\bar{p}}$

Integration defines bilinear forms

$$\int : \Omega^r_{\partial\mathcal{MS}}(N) \times \Omega^{n-r}_{\partial\mathcal{MS}}(N) \longrightarrow \mathbb{R},\ (\omega, \eta) \mapsto \int_N \omega \wedge \eta.$$

Since $\Omega I^\bullet_{\bar{p}}(N)$ is a subcomplex of $\Omega^\bullet_{\partial\mathcal{MS}}(N)$, we obtain in particular:

Lemma 7.1. *Integration defines bilinear forms*

$$\int : \Omega I^r_{\bar{p}}(N) \times \Omega I^{n-r}_{\bar{q}}(N) \longrightarrow \mathbb{R}.$$

Lemma 7.2. *For differential forms $\nu_0 \in (\operatorname{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B))^{r-1}$ and $\eta_0 \in (\operatorname{ft}_{\geq K^*} \Omega^\bullet_{\mathcal{MS}}(B))^{n-r}$, the vanishing result $\int_{\partial\overline{X}} \nu_0 \wedge \eta_0 = 0$ holds.*

Proof. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\mathfrak{U} = \{U_\alpha\}$, $\operatorname{supp}(\rho_\alpha) \subset U_\alpha$ compact. Then $\{\widetilde{\rho}_\alpha\}$, $\widetilde{\rho}_\alpha = \rho_\alpha \circ p$, is a partition of unity subordinate to $p^{-1}\mathfrak{U} = \{p^{-1}U_\alpha\}$. Since

$$\begin{aligned}\int_{\partial\overline{X}} \nu_0 \wedge \eta_0 &= \int_{\partial\overline{X}} (\sum \widetilde{\rho}_\alpha) \cdot \nu_0 \wedge \eta_0 = \sum \int_{\partial\overline{X}} \widetilde{\rho}_\alpha \nu_0 \wedge \eta_0 \\ &= \sum \int_{p^{-1}U_\alpha} \widetilde{\rho}_\alpha \nu_0 \wedge \eta_0,\end{aligned}$$

it suffices to show that $\int_{p^{-1}U_\alpha} \widetilde{\rho}_\alpha \nu_0 \wedge \eta_0 = 0$ for all α. Let $\phi_\alpha : p^{-1}U_\alpha \xrightarrow{\cong} U_\alpha \times F$ be the trivialization over U_α. Over U_α, ν_0 has the form $\nu_0|_{p^{-1}U_\alpha} = \phi^*_\alpha \sum_{i=1}^k \pi_1^* \nu_i \wedge \pi_2^* \gamma_i$, with $\nu_i \in \Omega^\bullet(U_\alpha)$, $\gamma_i \in \tau_{\geq K}\Omega^\bullet(F)$, for $1 \leq i \leq k$, $\deg \nu_i + \deg \gamma_i = r - 1$, and η_0 has the local form

$\eta_0|_{p^{-1}U_\alpha} = \phi_\alpha^* \sum_{j=1}^l \pi_1^* \eta_j \wedge \pi_2^* \widetilde{\gamma}_j$, with $\eta_j \in \Omega^\bullet(U_\alpha)$, $\widetilde{\gamma}_j \in \tau_{\geq K^*}\Omega^\bullet(F)$, $\deg \eta_j + \deg \widetilde{\gamma}_j = n - r$, for $1 \leq j \leq l$. We have

$$(\widetilde{\rho}_\alpha \nu_0)|_{p^{-1}U_\alpha} = \phi_\alpha^* \sum_i \pi_1^*(\rho_\alpha \nu_i) \wedge \pi_2^* \gamma_i,$$

where $\rho_\alpha \nu_i \in \Omega_c^\bullet(U_\alpha)$ has compact support in U_α. Thus

$$\begin{aligned} \int_{p^{-1}U_\alpha} \widetilde{\rho}_\alpha \nu_0 \wedge \eta_0 &= \sum_{i,j} (\pm) \int_{U_\alpha \times F} \pi_1^*(\rho_\alpha \nu_i \wedge \eta_j) \wedge \pi_2^*(\gamma_i \wedge \widetilde{\gamma}_j) \\ &= \sum_{i,j} (\pm) \int_{U_\alpha} \rho_\alpha \nu_i \wedge \eta_j \cdot \int_F \gamma_i \wedge \widetilde{\gamma}_j. \end{aligned}$$

We claim that $\int_F \gamma_i \wedge \widetilde{\gamma}_j = 0$, which will finish the proof. Let D denote the degree of γ_i; we may assume that $\deg \widetilde{\gamma}_j = m - D$ ($m = \dim F$). If $D < K$, then $\gamma_i = 0$, so the claim is verified for this case. Suppose that $D \geq K$. Since $K = m - \bar{p}(m+1)$, $K^* = m - \bar{q}(m+1)$, and $\bar{p}(m+1) + \bar{q}(m+1) = m - 1$, the inequality $D \geq K$ implies that $m - D < K^*$. Hence $\widetilde{\gamma}_j = 0$ and the claim is correct in the case $D \geq K$ as well. q.e.d.

The next result then follows from Stokes' theorem and the previous lemma:

Lemma 7.3. *If $\nu \in \Omega I_{\bar{p}}^{r-1}(N)$ and $\eta \in \Omega I_{\bar{q}}^{n-r}(N)$, then $\int_N d(\nu \wedge \eta) = 0$.*

8. Poincaré Duality for $HI_{\bar{p}}^\bullet$

Lemma 7.3 implies readily:

Proposition 8.1. *The bilinear form of Lemma 7.1 induces a bilinear form*

$$\int : HI_{\bar{p}}^r(X) \times HI_{\bar{q}}^{n-r}(X) \longrightarrow \mathbb{R}, \ ([\omega], [\eta]) \mapsto \int_N \omega \wedge \eta,$$

on cohomology.

Theorem 8.2. *(Generalized Poincaré Duality.) The bilinear form*

$$\int : HI_{\bar{p}}^r(X) \times HI_{\bar{q}}^{n-r}(X) \longrightarrow \mathbb{R}$$

of Proposition 8.1 is nondegenerate.

Proof. By Proposition 6.3, the inclusion $\Omega_{\partial \mathcal{MS}}^\bullet(N) \subset \Omega^\bullet(N)$ induces an isomorphism $H_{\partial \mathcal{MS}}^r(N) \xrightarrow{\cong} H^r(N)$. Classical Poincaré duality asserts that

$$H^r(N) \longrightarrow H_c^{n-r}(N)^\dagger, \ [\omega] \mapsto \int_N \omega \wedge -$$

is an isomorphism. Composing these two isomorphisms, we obtain an isomorphism

$$H^r_{\partial \mathcal{MS}}(N) \xrightarrow{\cong} H^{n-r}_c(N)^\dagger, \ [\omega] \mapsto \int_N \omega \wedge -. \tag{14}$$

The nondegenerate form of Proposition 5.12 can be rewritten as an isomorphism

$$H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B)) \xrightarrow{\cong} H^{n-r-1}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS}}(B))^\dagger, \tag{15}$$

while the bilinear form of Proposition 8.1 can be rewritten as a map

$$H^r(\Omega I^\bullet_{\bar p}(N)) \longrightarrow H^{n-r}(\Omega I^\bullet_{\bar q}(N))^\dagger. \tag{16}$$

The distinguished triangle (12) induces a long exact cohomology sequence

$$\begin{aligned} \cdots \to H^{r-1}(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B)) \to H^r(\Omega I^\bullet_{\bar p}(N)) \to H^r(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \\ \to H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B)) \to \cdots . \end{aligned}$$

The distinguished triangle (13) induces a long exact cohomology sequence

$$\begin{aligned} \cdots \to H^{n-r}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS}}(B))^\dagger \xrightarrow{(j^*_{\bar p})^\dagger} H^{n-r}(\Omega I^\bullet_{\bar q}(N))^\dagger \xrightarrow{\mathrm{incl}^{*\dagger}} H^{n-r}(\Omega^\bullet_c(N))^\dagger \\ \longrightarrow H^{n-r-1}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS}}(B))^\dagger \longrightarrow \cdots . \end{aligned}$$

Using the maps (14), (15) and (16), we map the former sequence to the latter:

$$\begin{tikzcd} H^{r-1}(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B)) \arrow[r, "\cong"] \arrow[d] & H^{n-r}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS}}(B))^\dagger \arrow[d, "(j^*_{\bar p})^\dagger"] \\ H^r(\Omega I^\bullet_{\bar p}(N)) \arrow[r] \arrow[d] & H^{n-r}(\Omega I^\bullet_{\bar q}(N))^\dagger \arrow[d, "\mathrm{incl}^{*\dagger}"] \\ H^r(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \arrow[r, "\cong"] \arrow[d] & H^{n-r}(\Omega^\bullet_c(N))^\dagger \arrow[d] \\ H^r(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B)) \arrow[r, "\cong"] & H^{n-r-1}(\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS}}(B))^\dagger \end{tikzcd} \tag{17}$$

Let us denote the top square, middle square and bottom square of this diagram by (TS), (MS), (BS), respectively. We shall verify that all three squares commute up to sign. Let us start with (TS). We begin by describing the map

$$\delta : H^{r-1}(\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B)) \longrightarrow H^r(\Omega I^\bullet_{\bar p}(N)).$$

Let $\iota : \Omega I^\bullet_{\bar p}(N) \hookrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N)$ denote the subcomplex inclusion and $C^\bullet(\iota)$ the algebraic mapping cone of ι, that is, $C^r(\iota) = \Omega I^{r+1}_{\bar p}(N) \oplus \Omega^r_{\partial\mathcal{MS}}(N)$ and $d : C^r(\iota) \to C^{r+1}(\iota)$ is given by $d(\tau, \sigma) = (-d\tau, \tau + d\sigma)$. Let $P : C^\bullet(\iota) \to \Omega I^{\bullet+1}_{\bar p}(N)$, $P(\tau, \sigma) = \tau$, be the standard projection and $f : C^\bullet(\iota) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N)/\Omega I^\bullet_{\bar p}(N)$ be the map given by $f(\tau, \sigma) = q(\sigma)$, where

$q : \Omega^\bullet_{\partial \mathcal{MS}}(N) \longrightarrow \Omega^\bullet_{\partial \mathcal{MS}}(N)/\Omega I^\bullet_{\bar{p}}(N)$ is the canonical quotient map. The map f is a quasi-isomorphism. Recall that the isomorphism

$$\overline{j}^* : \frac{\Omega^\bullet_{\partial \mathcal{MS}}(N)}{\Omega I^\bullet_{\bar{p}}(N)} \xrightarrow{\cong} \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{\mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)}$$

can be thought of as first uniquely extending a form from N to $\overline{X}$ and then restricting to $\partial \overline{X}$. The quasi-isomorphism

$$\gamma_B : \mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B) \longrightarrow \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{\mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)}$$

was defined to be the composition

$$\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B) \overset{\mathrm{incl}}{\hookrightarrow} \Omega^\bullet_{\mathcal{MS}}(B) \xrightarrow{\mathrm{quot}} \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{\mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)}.$$

Let $\omega \in (\mathrm{ft}_{<K}\, \Omega^\bullet_{\mathcal{MS}}(B))^{r-1}$ be a closed form. Then $d(\gamma_B \omega) = 0$ as well. As $\overline{j}^*$ is an isomorphism, there exists a unique element $w \in \Omega^\bullet_{\partial \mathcal{MS}}(N)/\Omega I^\bullet_{\bar{p}}(N)$ such that $\overline{j}^*(w) = \gamma_B(\omega)$ and $\overline{j}^*(dw) = d(\overline{j}^* w) = d\gamma_B(\omega) = 0$. The injectivity of $\overline{j}^*$ implies that

$$dw = 0 \in \Omega^\bullet_{\partial \mathcal{MS}}(N)/\Omega I^\bullet_{\bar{p}}(N).$$

Let $\overline{\omega} \in \Omega^{r-1}_{\partial \mathcal{MS}}(N)$ be a representative for w so that $q(\overline{\omega}) = w$. From $q(d\overline{\omega}) = dq(\overline{\omega}) = dw = 0$ we conclude that $d\overline{\omega} \in \Omega I^r_{\bar{p}}(N)$. The element

$$c = (-d\overline{\omega}, \overline{\omega}) \in C^{r-1}(\iota) = \Omega I^r_{\bar{p}}(N) \oplus \Omega^{r-1}_{\partial \mathcal{MS}}(N)$$

is a cocycle, since $dc = (d^2 \overline{\omega}, -d\overline{\omega} + d\overline{\omega}) = (0,0)$. Furthermore, $f(c) = q(\overline{\omega}) = w$ and hence $\overline{j}^* f(c) = \overline{j}^* w = \gamma_B(\omega)$, i.e. c is a lift of $\gamma_B(\omega)$ to a cocycle in the mapping cone. Since $P(c) = -d\overline{\omega} \in \Omega I^r_{\bar{p}}(N)$, the element $\delta(\omega)$ can be described as $\delta(\omega) = -d\overline{\omega}$. (Note that this does of course not mean that $\delta(\omega)$ represents the zero class in cohomology, since only $d\overline{\omega}$ is known to lie in $\Omega I^\bullet_{\bar{p}}(N)$, but $\overline{\omega}$ itself lies only in $\Omega^\bullet_{\partial \mathcal{MS}}(N)$, not necessarily in $\Omega I^\bullet_{\bar{p}}(N)$.) Since $j^*(\overline{\omega})$ satisfies

$$[j^* \overline{\omega}] = \overline{j}^* q(\overline{\omega}) = \gamma_B(\omega) \in \frac{\Omega^\bullet_{\mathcal{MS}}(B)}{\mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)},$$

we have $\alpha := j^*(\overline{\omega}) - \omega \in \mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)$. Thus $j^*(\overline{\omega})$ equals ω up to an element in $\mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)$.

Let $\overline{\omega}_0 = j^* \overline{\omega} \in \Omega^{r-1}_{\mathcal{MS}}(B) \subset \Omega^{r-1}(\partial \overline{X})$. Thus there exists an open neighborhood $U \subset \overline{X}$ of $\partial \overline{X}$ such that $\overline{\omega}|_{U \cap N} = \pi^* \overline{\omega}_0$. Let $\eta \in \Omega I^{n-r}_{\bar{q}}(N)$ be a closed form and set $\eta_0 = j^*_{\bar{q}} \eta \in (\mathrm{ft}_{\geq K^*}\, \Omega^\bullet_{\mathcal{MS}}(B))^{n-r} \subset \Omega^{n-r}(\partial \overline{X})$. Thus there exists an open neighborhood $V \subset \overline{X}$ of $\partial \overline{X}$ such that $\eta|_{V \cap N} = \pi^* \eta_0$. In order to verify the commutativity of (TS), we must show that

$$\int_N \delta(\omega) \wedge \eta = \pm \int_{\partial \overline{X}} \omega \wedge j^*_{\bar{q}}(\eta).$$

Since η is closed, $(d\overline{\omega}) \wedge \eta = d(\overline{\omega} \wedge \eta)$. The integral of this form over a sufficiently small open strip (contained in $U \cap V$) near the boundary vanishes, since the form is zero there. On the compact complement C of this strip we have by Stokes' theorem

$$\int_C d(\overline{\omega} \wedge \eta) = \int_{\partial \overline{X}} \omega \wedge j_{\bar{q}}^* \eta + \int_{\partial \overline{X}} \alpha \wedge j_{\bar{q}}^* \eta.$$

From $\alpha \in (\mathrm{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B))^{r-1}$, $j_{\bar{q}}^* \eta = \eta_0 \in (\mathrm{ft}_{\geq K^*} \Omega^\bullet_{\mathcal{MS}}(B))^{n-r}$ and Lemma 7.2 it follows that $\int_{\partial \overline{X}} \alpha \wedge j_{\bar{q}}^* \eta = 0$. Thus (TS) commutes.

Let us move on to (BS). We begin by describing the map

$$D : H^{n-r-1}(\mathrm{ft}_{\geq K^*} \Omega^\bullet_{\mathcal{MS}}(B)) \longrightarrow H^{n-r}(\Omega^\bullet_c(N)).$$

Let $\rho : \Omega^\bullet_c(N) \hookrightarrow \Omega I^\bullet_{\bar{q}}(N)$ be the subcomplex inclusion and $C^\bullet(\rho)$ its algebraic mapping cone. Let $P : C^\bullet(\rho) \longrightarrow \Omega^{\bullet+1}_c(N)$, $P(\tau, \sigma) = \tau$, be the projection and let $f : C^\bullet(\rho) \to \mathrm{ft}_{\geq K^*} \Omega^\bullet_{\mathcal{MS}}(B)$ be the quasi-isomorphism given by $f(\tau, \sigma) = j_{\bar{q}}^*(\sigma)$. The kernel of

$$j_{\bar{q}}^* : \Omega I^\bullet_{\bar{q}}(N) \twoheadrightarrow \mathrm{ft}_{\geq K^*} \Omega^\bullet_{\mathcal{MS}}(B)$$

is $\operatorname{im} \rho = \Omega^\bullet_c(N)$. Let $\eta \in (\mathrm{ft}_{\geq K^*} \Omega^\bullet_{\mathcal{MS}}(B))^{n-r-1}$ be a closed form. Since $j_{\bar{q}}^*$ is surjective, there exists an $\overline{\eta} \in \Omega I^{n-r-1}_{\bar{q}}(N)$ such that $j_{\bar{q}}^*(\overline{\eta}) = \eta$. We have $j_{\bar{q}}^*(d\overline{\eta}) = dj_{\bar{q}}^*(\overline{\eta}) = d\eta = 0$. Thus $d\overline{\eta} \in \ker j_{\bar{q}}^* = \Omega^{n-r}_c(N)$. The element

$$c = (-d\overline{\eta}, \overline{\eta}) \in \Omega^{n-r}_c(N) \oplus \Omega I^{n-r-1}_{\bar{q}}(N) = C^{n-r-1}(\rho)$$

is a cocycle, for $dc = (d^2\overline{\eta}, -d\overline{\eta} + d\overline{\eta}) = (0,0)$. Moreover, $f(c) = j_{\bar{q}}^*(\overline{\eta}) = \eta$ and $P(c) = -d\overline{\eta}$. We conclude that the image $D(\eta)$ can be described as $D(\eta) = -d\overline{\eta}$. We shall next describe the map

$$Q : H^r(\Omega^\bullet_{\partial \mathcal{MS}}(N)) \longrightarrow H^r(\mathrm{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B)).$$

Let $\omega \in \Omega^r_{\partial \mathcal{MS}}(N)$ be a closed form. Its image under

$$\Omega^r_{\partial \mathcal{MS}}(N) \xrightarrow{q} \!\!\!\!\rightarrow \frac{\Omega^r_{\partial \mathcal{MS}}(N)}{\Omega I^r_{\bar{p}}(N)} \xrightarrow[\overline{j}^*]{\cong} \frac{\Omega^r_{\mathcal{MS}}(B)}{(\mathrm{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B))^r}$$

is represented by $\omega_0 = j^*(\omega)$,

$$\overline{j}^* q(\omega) = [\omega_0] \in \frac{\Omega^r_{\mathcal{MS}}(B)}{\mathrm{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B))^r}.$$

Let $[\![\overline{j}^* q(\omega)]\!] \in H^r(Q^\bullet(B))$ denote the cohomology class determined by $\overline{j}^* q(\omega)$. Since γ_B is a quasi-isomorphism, there exists a unique class $[\![\overline{\omega}]\!] \in H^r(\mathrm{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B))$, which is represented by a closed form $\overline{\omega} \in (\mathrm{ft}_{<K} \Omega^\bullet_{\mathcal{MS}}(B))^r$, with $\gamma_B^* [\![\overline{\omega}]\!] = [\![\overline{j}^* q(\omega)]\!]$. Consequently, there exists a form $\xi \in \Omega^{r-1}_{\mathcal{MS}}(B)$, representing an element $[\xi] \in Q^{r-1}(B)$ such that $\gamma_B(\overline{\omega}) - \overline{j}^* q(\omega) = d[\xi]$. We deduce that $\alpha = \overline{\omega} - \omega_0 - d\xi \in \mathrm{ft}_{\geq K} \Omega^\bullet_{\mathcal{MS}}(B)$.

The map Q is described by $Q(\omega) = \overline{\omega}$. In order to verify the commutativity of (BS), we must show that

$$\int_N \omega \wedge D(\eta) = \pm \int_{\partial \overline{X}} Q(\omega) \wedge \eta.$$

We split the left integral $-\int_N \omega \wedge d\overline{\eta}$ into an integral over a sufficiently small open strip at the boundary and an integral over the compact complement C. The integral over the small strip vanishes as $d\overline{\eta} \in \Omega_c^{n-r}(N)$, so that its restriction to a sufficiently small strip at the boundary is zero. By Stokes' theorem on C, we are reduced to showing $\int_{\partial \overline{X}} \omega \wedge \overline{\eta} = \pm \int_{\partial \overline{X}} \overline{\omega} \wedge \eta$. Rewriting the integrand on the left-hand side as

$$\omega_0 \wedge j_{\bar{q}}^*(\overline{\eta}) = (\overline{\omega} - \alpha - d\xi) \wedge \eta = \overline{\omega} \wedge \eta - \alpha \wedge \eta - (d\xi) \wedge \eta,$$

it remains to show that $\int_{\partial M} \alpha \wedge \eta = 0$ and $\int_{\partial M} d\xi \wedge \eta = 0$. The former statement is implied by Lemma 7.2 and the latter follows from Stokes' theorem.

Finally (MS) commutes, since the map $H^r(\Omega I_{\bar{p}}^\bullet(N)) \to H^r(\Omega^\bullet_{\partial \mathcal{MS}}(N))$ is induced by the subcomplex inclusion $\Omega I_{\bar{p}}^\bullet(N) \subset \Omega^\bullet_{\partial \mathcal{MS}}(N)$, and the map $H^{n-r}(\Omega_c^\bullet(N)) \to H^{n-r}(\Omega I_{\bar{q}}^\bullet(N))$ is induced by the subcomplex inclusion $\Omega_c^\bullet(N) \subset \Omega I_{\bar{q}}^\bullet(N)$, whence the two integrals whose equality has to be demonstrated are both just $\int_N \omega \wedge \eta$, $\omega \in \Omega I_{\bar{p}}^r(N)$, $\eta \in \Omega_c^{n-r}(N)$. Since the diagram (17) is now known to commute (up to sign), the statement of the theorem is implied by the 5-lemma. q.e.d.

9. The de Rham Theorem to the Cohomology of Intersection Spaces

9.1. Partial Smoothing. Our method to establish the de Rham isomorphism between $HI_{\bar{p}}^\bullet$ and the cohomology of the corresponding intersection space requires building an interface between smooth objects and techniques, such as smooth differential forms and smooth singular chains in a smooth manifold, and nonsmooth objects, such as the intersection space, which arises from a homotopy-theoretic construction and is a CW-complex, not generally a (pseudo)manifold. The interface will be provided by a certain *partial smoothing* technique that we shall now develop.

For a topological space X, let $S_\bullet(X)$ denote its singular chain complex with real coefficients. Homology $H_\bullet(X)$ will mean singular homology, $H_\bullet(S_\bullet(X))$. For a smooth manifold V (which is allowed to have a boundary), let $S_\bullet^\infty(V)$ denote its smooth singular chain complex with real coefficients, generated by smooth singular simplices $\Delta^k \to V$. For a continuous map $g : X \to V$, we shall define the *partially smooth chain* *complex* $S^{\pitchfork}_\bullet(g)$. In degree k, we set

$$S^{\pitchfork}_k(g) = H_{k-1}(X) \oplus S^\infty_k(V).$$

Let $\iota : S^\infty_\bullet(V) \hookrightarrow S_\bullet(V)$ be the inclusion and $s : S_\bullet(V) \longrightarrow S^\infty_\bullet(V)$ Lee's smoothing operator, [**30**], pp. 416 – 424. The map s is a chain map such that $s \circ \iota$ is the identity and $\iota \circ s$ is chain homotopic to the identity. Thus s and ι induce mutually inverse isomorphisms on homology. If V has a nonempty boundary ∂V and $J : \partial V \hookrightarrow V$ is the inclusion, then a continuous singular simplex that lies in the boundary can be smoothed within the boundary. Thus, we can assume that s has been arranged so that the square

$$\begin{tikzcd} S_\bullet(\partial V) \arrow[r, "s"] \arrow[d, "J_*"'] & S^\infty_\bullet(\partial V) \arrow[d, "J_*"'] \\ S_\bullet(V) \arrow[r, "s"] & S^\infty_\bullet(V) \end{tikzcd} \tag{18}$$

commutes. Let Z_k denote the subspace of k-cycles in $S_k(X)$ and $B_k = \partial S_{k+1}(X)$ the subspace of k-boundaries. Choosing direct sum decompositions $Z_k = B_k \oplus H'_k$, we obtain a quasi-isomorphism $q : H_\bullet(X) = H_\bullet(S_\bullet(X)) \longrightarrow S_\bullet(X)$, which is given in degree k by the composition

$$H_k(X) = \frac{Z_k}{B_k} = \frac{B_k \oplus H'_k}{B_k} \xrightarrow{\cong} H'_k \hookrightarrow Z_k \hookrightarrow S_k(X).$$

Here, we regard $H_\bullet(X)$ as a chain complex with zero boundary operators. By construction, the formula

$$[q(x)] = x \tag{19}$$

holds for a homology class $x \in H_k(X)$, that is, $q(x)$ is a cycle representative for x. Let $x \in H_{k-1}(X)$ be a homology class in X and $v : \Delta^k \to V$ be a smooth singular simplex $v \in S^\infty_k(V)$. We define the boundary operator $\partial : S^{\pitchfork}_k(g) \longrightarrow S^{\pitchfork}_{k-1}(g)$ by

$$\partial(x, v) = (0, \partial v + s g_* q(x)),$$

where $g_* : S_{k-1}(X) \to S_{k-1}(V)$ is the chain map induced by g. The algebraic mapping cone $C_\bullet(g_*)$ of g_* is given by

$$C_k(g_*) = S_{k-1}(X) \oplus S_k(V),\ \ \partial(x, v) = (-\partial x, \partial v + g_*(x)).$$

The homology $H_\bullet(g)$ of the map g is $H_\bullet(g) = H_\bullet(C_\bullet(g_*))$. We wish to show that the partially smooth chain complex $S^{\pitchfork}_\bullet(g)$ computes $H_\bullet(g)$. To do this, we construct an intermediate complex $U_\bullet(g)$, which underlies both complexes,

$$\begin{tikzcd} C_\bullet(g_*) \arrow[dr] & & S^{\pitchfork}_\bullet(g) \arrow[dl] \\ & U_\bullet(g) & \end{tikzcd}$$

such that the two maps are quasi-isomorphisms. Set

$$U_k(g) = S_{k-1}(X) \oplus S_k^{\infty}(V),\ \partial(x,v) = (-\partial x, \partial v + s g_*(x));$$

then $U_\bullet(g)$ is a chain complex. The maps $\mathrm{id} \oplus s : C_\bullet(g_*) \longrightarrow U_\bullet(g)$ and $q \oplus \mathrm{id} : S_\bullet^{\propto}(g) \longrightarrow U_\bullet(g)$ are both chain maps. We leave the proof of the next lemma as an exercise.

Lemma 9.1. *The maps* $\mathrm{id} \oplus s$ *and* $q \oplus \mathrm{id}$ *are both quasi-isomorphisms.*

Hence:

Proposition 9.2. *(Partial Smoothing.) The maps* $\mathrm{id} \oplus s$ *and* $q \oplus \mathrm{id}$ *induce an isomorphism* $H_\bullet(S_\bullet^{\propto}(g)) \cong H_\bullet(g)$.

This concludes the construction of the partially smooth model to compute the homology of the map g.

9.2. Background on Intersection Spaces. We provide a quick review of the construction of intersection spaces. For more details, we ask the reader to consult [**3**]. Let k be an integer and let $C_\bullet(K)$ denote the integral cellular chain complex of a CW-complex K.

Definition 9.3. The category $\mathbf{CW}_{k\supset\partial}$ of *k-boundary-split CW- complexes* consists of the following objects and morphisms: Objects are pairs (K, Y), where K is a simply connected CW-complex and $Y \subset C_k(K)$ is a subgroup that arises as the image $Y = s(\mathrm{im}\,\partial)$ of some splitting $s : \mathrm{im}\,\partial \to C_k(K)$ of the boundary map $\partial : C_k(K) \to \mathrm{im}\,\partial (\subset C_{k-1}(K))$. (Given K, such a splitting always exists, since $\mathrm{im}\,\partial$ is free abelian.) A morphism $(K, Y_K) \to (L, Y_L)$ is a cellular map $h : K \to L$ such that $h_*(Y_K) \subset Y_L$.

Let $\mathbf{HoCW}_{k-1}$ denote the category whose objects are CW-complexes and whose morphisms are rel $(k-1)$-skeleton homotopy classes of cellular maps. Let

$$t_{<\infty} : \mathbf{CW}_{k\supset\partial} \longrightarrow \mathbf{HoCW}_{k-1}$$

be the natural projection functor, that is, $t_{<\infty}(K, Y_K) = K$ for an object (K, Y_K) in $\mathbf{CW}_{k\supset\partial}$, and $t_{<\infty}(h) = [h]$ for a morphism $h : (K, Y_K) \to (L, Y_L)$ in $\mathbf{CW}_{k\supset\partial}$. The following theorem is proved in [**3**].

Theorem 9.4. *Let $k \geq 3$ be an integer. There is a covariant assignment $t_{<k} : \mathbf{CW}_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$ of objects and morphisms together with a natural transformation $\mathrm{emb}_k : t_{<k} \to t_{<\infty}$ such that for an object (K, Y) of $\mathbf{CW}_{k\supset\partial}$, $H_r(t_{<k}(K, Y); \mathbb{Z}) = 0$ for $r \geq k$, and $\mathrm{emb}_k(K, Y)_* : H_r(t_{<k}(K, Y); \mathbb{Z}) \xrightarrow{\cong} H_r(K; \mathbb{Z})$ is an isomorphism for $r < k$.*

This means in particular that given a morphism h, one has squares

$$\begin{tikzcd} t_{<k}(K,Y_K) \arrow[r, "\mathrm{emb}_k(K,Y_K)"] \arrow[d, "t_{<k}(h)"'] & t_{<\infty}(K,Y_K) \arrow[d, "t_{<\infty}(h)"] \\ t_{<k}(L,Y_L) \arrow[r, "\mathrm{emb}_k(L,Y_L)"] & t_{<\infty}(L,Y_L) \end{tikzcd}$$

that commute in $\mathbf{HoCW}_{k-1}$. If $k \leq 2$ (and the CW-complexes are simply connected), then it is of course a trivial matter to construct such truncations.

Let X be an n-dimensional pseudomanifold with one isolated singularity. For a given perversity $\bar{p}$, set $k = n-1-\bar{p}(n)$. As usual, $\overline{X}$ denotes the blow-up of X and N continues to denote the interior of $\overline{X}$. To be able to apply the general spatial homology truncation Theorem 9.4, we require the link $L = \partial\overline{X}$ to be simply connected. This assumption is not always necessary, as in many non-simply connected situations, ad hoc truncation constructions can be used. If $k \geq 3$, we can and do fix a completion (L, Y) of L so that (L, Y) is an object in $\mathbf{CW}_{k\supset\partial}$. If $k \leq 2$, no group Y has to be chosen. Applying the truncation $t_{<k} : \mathbf{CW}_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$, we obtain a CW-complex $t_{<k}(L, Y) \in Ob\,\mathbf{HoCW}_{k-1}$. The natural transformation $\mathrm{emb}_k : t_{<k} \to t_{<\infty}$ of Theorem 9.4 gives a homotopy class $\mathrm{emb}_k(L, Y)$ represented by a map $f : t_{<k}(L, Y) \to L$ such that for $r < k$, $f_* : H_r(t_{<k}(L, Y)) \cong H_r(L)$, while $H_r(t_{<k}(L, Y)) = 0$ for $r \geq k$. The intersection space $I^{\bar{p}}X$ is defined to be

$$I^{\bar{p}}X = \mathrm{cone}(g),$$

where g is the composition

$$\begin{tikzcd} t_{<k}(L,Y) \arrow[r, "f"] \arrow[dr, "g"'] & L = \partial\overline{X} \arrow[d, hook, "J"] \\ & \overline{X}. \end{tikzcd} \tag{20}$$

(This notation will be retained in the rest of Section 9.) Thus, to form the intersection space, we attach the cone on a suitable spatial homology truncation of the link to the blow-up of the singularity along the boundary of the blow-up. Let us briefly write $t_{<k}L$ for $t_{<k}(L, Y)$. More generally, $I^{\bar{p}}X$ has at present been constructed, and Poincaré duality established, for the following classes of X, where all links are generally assumed to be simply connected:

• X has stratification depth 1 and every connected component of the singular set Σ has trivializable link bundle ([**3**]). This includes all X with only isolated singularities (and simply connected links). Under the name *framified sets*, stratified spaces with trivial link bundles play a role in the work of Buoncristiano, Rourke and Sanderson, [**14**].

• X has depth 1 and Σ is a simply connected sphere, whose link either has no odd-degree homology or has a cellular chain complex all of whose boundary operators vanish ([**23**], the link bundle may be twisted here).

• In [**4**], we took first steps in higher stratification depth: X has depth 2 with one-dimensional Σ such that the links of the components of the pure one-dimensional stratum satisfy a condition similar to Weinberger's *antisimplicity* condition [**36**], which itself is an algebraic version of a somewhat stronger geometric condition due to Hausmann, requiring a manifold to have a handlebody without middle-dimensional handles.

9.3. $\Omega I^\bullet_{\bar p}$ in the Isolated Singularity Case. In the isolated singularity case,

$$\Omega^r_{\partial\mathcal{MS}}(N) = \{\omega \in \Omega^r(N) \mid \exists \text{ open neighborhood } U \subset E \times [0,2) \subset \overline{X} \text{ of } E = \partial\overline{X} : \omega|_{U\cap N} = \pi^*\eta, \quad \text{some } \eta \in \Omega^r(\partial\overline{X})\}$$

and

$$\Omega I^r_{\bar p}(N) = \{\omega \in \Omega^r(N) \mid \exists \text{ open neighborhood } U \subset E \times [0,2) \subset \overline{X} \text{ of } E = \partial\overline{X} : \omega|_{U\cap N} = \pi^*\eta, \text{ some } \eta \in (\tau_{\geq k}\Omega^\bullet(\partial\overline{X}))^r\}.$$

In Section 4, an orthogonal projection $\mathrm{proj} : \Omega^\bullet(\partial\overline{X}) \to \tau_{<k}\Omega^\bullet(\partial\overline{X})$ was defined. Composing with $j^* : \Omega^\bullet_{\partial\mathcal{MS}}(N) \twoheadrightarrow \Omega^\bullet_{\mathcal{MS}}(\mathrm{pt}) = \Omega^\bullet(\partial\overline{X})$, we obtain an epimorphism $\mathrm{proj}\circ j^* : \Omega^\bullet_{\partial\mathcal{MS}}(N) \twoheadrightarrow \tau_{<k}\Omega^\bullet(\partial\overline{X})$. Using the exact sequence (3) in Section 4, one verifies:

Lemma 9.5. *The kernel of* $\mathrm{proj}\circ j^* : \Omega^\bullet_{\partial\mathcal{MS}}(N) \twoheadrightarrow \tau_{<k}\Omega^\bullet(\partial\overline{X})$ *is* $\Omega I^\bullet_{\bar p}(N)$*.*

Thus we have an exact sequence

$$0 \longrightarrow \Omega I^\bullet_{\bar p}(N) \longrightarrow \Omega^\bullet_{\partial\mathcal{MS}}(N) \longrightarrow \tau_{<k}\Omega^\bullet(\partial\overline{X}) \longrightarrow 0. \tag{21}$$

9.4. The de Rham Theorem. Let us define a map

$$\Psi_L : H^{r-1}(\tau_{<k}\Omega^\bullet(L)) \longrightarrow H_{r-1}(t_{<k}L)^\dagger.$$

For $r-1 \geq k$, $\Psi_L = 0$, since both $H^{r-1}(\tau_{<k}\Omega^\bullet(L))$ and $H_{r-1}(t_{<k}L)$ are zero in this case. Suppose $r-1 < k$. Then $H^{r-1}(\tau_{<k}\Omega^\bullet(L)) = H^{r-1}(L)$ and we define

$$\widetilde{\Psi}_L : H^{r-1}(L) \longrightarrow H_{r-1}(S^\infty_\bullet(L))^\dagger$$

by $\widetilde{\Psi}_L[\omega][b] = \int_b \omega$ for a smooth singular cycle $b \in S^\infty_{r-1}(L)$. Standard de Rham theory shows that $\widetilde{\Psi}_L$ is well-defined and an isomorphism. The smoothing operator s induces on homology an isomorphism $s_* : H_\bullet(L) \xrightarrow{\cong} H_\bullet(S^\infty_\bullet(L))$. The structural map f from diagram (20) induces an isomorphism $f_* : H_{r-1}(t_{<k}L) \overset{\cong}{\to} H_{r-1}(L)$ since $r-1<k$. The map Ψ_L is defined to be the composition

$$H^{r-1}(L) \xrightarrow[\widetilde{\Psi}_L]{\cong} H_{r-1}(S^\infty_\bullet(L))^\dagger \xrightarrow[s_*^\dagger]{\cong} H_{r-1}(L)^\dagger \xrightarrow[f_*^\dagger]{\cong} H_{r-1}(t_{<k}L)^\dagger$$

for $r-1<k$. By construction:

Lemma 9.6. *The map $\Psi_L : H^{r-1}(\tau_{<k}\Omega^\bullet(L)) \longrightarrow H_{r-1}(t_{<k}L)^\dagger$ is an isomorphism for all r.*

Next, we shall define an isomorphism

$$\Psi_N : H^\bullet(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \xrightarrow{\cong} H_\bullet(S^\infty_\bullet(\overline{X}))^\dagger.$$

By Proposition 6.4, the inclusion $\Omega^\bullet_{\partial\mathcal{MS}}(N) \subset \Omega^\bullet(\overline{X})$ induces an isomorphism

$$H^\bullet(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \xrightarrow{\cong} H^\bullet(\overline{X}).$$

The classical de Rham isomorphism $\Psi_{\overline{X}} : H^\bullet(\Omega^\bullet(\overline{X})) \xrightarrow{\cong} H_\bullet(S^\infty_\bullet(\overline{X}))^\dagger$ is given by $\Psi_{\overline{X}}[\omega][a] = \int_a \omega$. The isomorphism Ψ_N is defined by the composition

$$H^\bullet(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \xrightarrow{\cong} H^\bullet(\Omega^\bullet(\overline{X})) \xrightarrow[\Psi_{\overline{X}}]{\cong} H_\bullet(S^\infty_\bullet(\overline{X}))^\dagger.$$

Lemma 9.7. *The diagram*

$$\begin{tikzcd}
H^r(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \arrow[r, "\mathrm{j}^*"] \arrow[d, "\Psi_N"', "\cong"] & H^r(\Omega^\bullet(L)) \arrow[r, "\mathrm{proj}"] & H^r(\tau_{<k}\Omega^\bullet(L)) \arrow[d, "\cong"', "\Psi_L"] \\
H_r(S^\infty_\bullet(\overline{X}))^\dagger \arrow[r, "\cong", "s_*^\dagger"'] & H_r(\overline{X})^\dagger \arrow[r, "g_*^\dagger"] & H_r(t_{<k}L)^\dagger
\end{tikzcd}$$

commutes.

Proof. The statement holds trivially for $r \geq k$, since then $H_r(t_{<k}L) = 0$. Assume that $r<k$. Let $\omega \in \Omega^r_{\partial\mathcal{MS}}(N)$ be a closed r-form and $[a] \in H_r(t_{<k}L)$ a class represented by a cycle $a \in S_r(t_{<k}L)$. Let $U \subset \overline{X}$ be an open neighborhood of $\partial\overline{X}$ such that $\omega|_{U\cap N} = \pi^*\eta$, i.e. $j^*(\omega) = \eta \in \Omega^r(L)$. We must prove that the equation

$$\Psi_L(\mathrm{proj}(\eta))[a] = \Psi_N(\omega)(sg_*(a))$$

holds. The following computation verifies this, observing that in degrees $r<k$, proj is the identity. Recall from diagram (20) that $g = J \circ f$, where $J : L = \partial\overline{X} \hookrightarrow \overline{X}$ is the inclusion of the boundary. Also, as $\Omega^\bullet_{\partial\mathcal{MS}}(N) \subset \Omega^\bullet(\overline{X})$, we can and will view ω as a form on $\overline{X}$. Then

$J^*(\omega) = \eta$ and we compute

$$\begin{aligned} \Psi_L(\mathrm{proj}(\eta))[a] &= f_*^\dagger s_*^\dagger \widetilde{\Psi}_L(\eta)[a] = \widetilde{\Psi}_L(\eta)[sf_*(a)] \\ &= \int_{sf_*(a)} \eta = \int_{sf_*(a)} J^*\omega = \int_{J_* sf_*(a)} \omega \\ &= \int_{sJ_*f_*(a)} \omega \qquad\qquad \text{by (18)} \\ &= \Psi_{\overline{X}}(\omega)(sJ_*f_*(a)) = \Psi_N(\omega)(sg_*(a)). \end{aligned}$$

q.e.d.

Let us define a map $\Psi_{\bar{p}} : H^r(\Omega I^\bullet_{\bar{p}}(N)) \to H_r(S^\propto_\bullet(g))^\dagger$. Given a closed form $\omega \in \Omega I^r_{\bar{p}}(N)$ and a cycle $(x, v) \in S^\propto_r(g) = H_{r-1}(t_{<k}L) \oplus S^\infty_r(\overline{X})$, we set $\Psi_{\bar{p}}[\omega][(x,v)] = \int_v \omega$.

Proposition 9.8. *The map $\Psi_{\bar{p}}$ is well-defined.*

Proof. Let $\omega \in \Omega I^{r-1}_{\bar{p}}(N)$ be any form and $(x, v) \in S^\propto_r(g)$ a cycle. Suppose $r - 1 < k$. This implies by definition of $\Omega I^\bullet_{\bar{p}}(N)$ that $j^*\omega = 0$, that is, $J^*\omega = 0$, thinking of ω as a form on the compactification $\overline{X}$. Furthermore, $0 = \partial(x, v) = (0, \partial v + sg_*q(x))$ so that $\partial v = -sg_*q(x) = -J_*sf_*q(x)$. Hence,

$$\Psi_{\bar{p}}(d\omega)(x, v) = \int_v d\omega = \int_{\partial v} \omega = -\int_{J_*sf_*q(x)} \omega = -\int_{sf_*q(x)} J^*\omega = 0,$$

using Stokes' theorem for chains. Suppose that $r - 1 \geq k$. Then $x \in H_{r-1}(t_{<k}L) = 0$ and $\Psi_{\bar{p}}(d\omega)(x, v) = -\int_{sf_*q(x)} J^*\omega = 0$.

Let $\omega \in \Omega I^{r-1}_{\bar{p}}(N)$ be a closed form and $(x, v) \in S^\propto_r(g)$ any chain. If $r - 1 \geq k$, then $x \in H_{r-1}(t_{<k}L) = 0$ is zero and

$$\Psi_{\bar{p}}(\omega)(\partial(x, v)) = \Psi_{\bar{p}}(\omega)(0, \partial v) = \int_{\partial v} \omega = \int_v d\omega = 0,$$

as ω is closed. If $r - 1 < k$, then $j^*\omega = 0 = J^*\omega$ and

$$\begin{aligned} \Psi_{\bar{p}}(\omega)(\partial(x, v)) &= \Psi_{\bar{p}}(\omega)(0, \partial v + sg_*q(x)) = \int_{\partial v} \omega + \int_{sg_*q(x)} \omega \\ &= \int_v d\omega + \int_{sf_*q(x)} J^*\omega = 0. \end{aligned}$$

q.e.d.

The inclusion $\Omega I^\bullet_{\bar{p}}(N) \subset \Omega^\bullet_{\partial\mathcal{MS}}(N)$ induces a map

$$HI^\bullet_{\bar{p}}(X) \longrightarrow H^\bullet(\Omega^\bullet_{\partial\mathcal{MS}}(N)).$$

The standard inclusions $S^\infty_r(\overline{X}) \hookrightarrow H_{r-1}(t_{<k}L) \oplus S^\infty_r(\overline{X}) = S^\propto_r(g)$, $v \mapsto (0, v)$, form a chain map $\mathrm{inc} : S^\infty_\bullet(\overline{X}) \hookrightarrow S^\propto_\bullet(g)$, which induces a map $\mathrm{inc}_* : H_\bullet(S^\infty_\bullet(\overline{X})) \to H_\bullet(S^\propto_\bullet(g))$ on homology.

Lemma 9.9. *The square*

$$\begin{tikzcd} HI^r_{\bar{p}}(X) \arrow[r] \arrow[d, "\Psi_{\bar{p}}"'] & H^r(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \arrow[d, "\cong"', "\Psi_N"] \\ H_r(S^\propto_\bullet(g))^\dagger \arrow[r, "\mathrm{inc}^\dagger_*"] & H_r(S^\infty_\bullet(\overline{X}))^\dagger \end{tikzcd}$$

commutes.

Proof. For a closed form $\omega \in \Omega I^r_{\bar{p}}(N)$ and a cycle $v \in S^\infty_r(\overline{X})$, we calculate

$$\begin{aligned} \mathrm{inc}^\dagger_* \Psi_{\bar{p}}[\omega][v] &= \Psi_{\bar{p}}[\omega][\mathrm{inc}(v)] = \Psi_{\bar{p}}[\omega][(0,v)] = \int_v \omega \\ &= \Psi_{\overline{X}}[\omega][v] = \Psi_N[\omega][v]. \end{aligned}$$

q.e.d.

The short exact sequence (21) induces a long exact sequence on cohomology, which contains the connecting homomorphism

$$\delta^* : H^{r-1}(\tau_{<k}\Omega^\bullet(L)) \longrightarrow H^r(\Omega I^\bullet_{\bar{p}}(N)).$$

The standard projections $\mathrm{pro} : S^\propto_r(g) = H_{r-1}(t_{<k}L) \oplus S^\infty_r(\overline{X}) \to H_{r-1}(t_{<k}L)$, $(x,v) \mapsto x$, form a chain map $\mathrm{pro} : S^\propto_\bullet(g) \to H_{\bullet-1}(t_{<k}L)$, which induces on homology $\mathrm{pro}_* : H_\bullet(S^\propto_\bullet(g)) \to H_{\bullet-1}(t_{<k}L)$.

Lemma 9.10. *The square*

$$\begin{tikzcd} H^{r-1}(\tau_{<k}\Omega^\bullet(L)) \arrow[r, "\delta^*"] \arrow[d, "\Psi_L"', "\cong"] & H^r(\Omega I^\bullet_{\bar{p}}(N)) \arrow[d, "\Psi_{\bar{p}}"] \\ H_{r-1}(t_{<k}L)^\dagger \arrow[r, "\mathrm{pro}^\dagger_*"] & H_r(S^\propto_\bullet(g))^\dagger \end{tikzcd}$$

commutes up to sign.

Proof. If $r-1 \geq k$, then $H^{r-1}(\tau_{<k}\Omega^\bullet(L)) = 0$ and the statement of the lemma is correct. Assume that $r-1 < k$. Let $\omega \in (\tau_{<k}\Omega^\bullet(L))^{r-1} = \Omega^{r-1}(L)$ be a closed form on $L = \partial\overline{X}$. We shall first describe $\delta^*(\omega)$. The form $\pi^*\omega$ can be smoothly extended to a form $\overline{\omega} \in \Omega^{r-1}_{\partial\mathcal{MS}}(N)$. Its differential $d\overline{\omega}$ lies in $\Omega I^r_{\bar{p}}(N) \subset \Omega^r_{\partial\mathcal{MS}}(N)$, since $(d\overline{\omega})|_{L\times(0,1)} = d(\overline{\omega}|_{L\times(0,1)}) = d\pi^*\omega = \pi^* d\omega = 0$. The connecting homomorphism is then described as $\delta^*(\omega) = d\overline{\omega}$. Let $(x,v) \in S^\propto_r(g)$ be a cycle, i.e.

$0 = \partial(x,v) = (0, \partial v + sg_*q(x))$. The required commutativity is verified as follows:

$$\begin{aligned}
\Psi_{\bar{p}}[\delta^*\omega](x,v) &= \Psi_{\bar{p}}[d\overline{\omega}](x,v) = \int_v d\overline{\omega} = \int_{\partial v} \overline{\omega} \\
&= -\int_{sg_*q(x)} \overline{\omega} = -\int_{sf_*q(x)} J^*\overline{\omega} = -\int_{sf_*q(x)} \omega \\
&= -\widetilde{\Psi}_L(\omega)(s_*f_*[q(x)]) = -\widetilde{\Psi}_L(\omega)(s_*f_*x) \qquad \text{by (19)} \\
&= -f_*^\dagger s_*^\dagger \widetilde{\Psi}_L(\omega)(x) = -\Psi_L(\omega)(x) = -\Psi_L(\omega)(\mathrm{pro}(x,v)) \\
&= -\,\mathrm{pro}_*^\dagger\, \Psi_L(\omega)(x,v).
\end{aligned}$$

q.e.d.

Theorem 9.11. *(De Rham Description of $HI^\bullet_{\bar{p}}$.) The map $\Psi_{\bar{p}}$, induced by integrating a form in $\Omega I^\bullet_{\bar{p}}(N)$ over a smooth singular simplex in N, defines an isomorphism*

$$HI^\bullet_{\bar{p}}(X) \xrightarrow{\cong} H_\bullet(S^\infty_\bullet(g))^\dagger \cong \widetilde{H}_\bullet(I^{\bar{p}}X)^\dagger \cong \widetilde{H}^\bullet_s(I^{\bar{p}}X).$$

Proof. The short exact sequence (21) induces a long exact cohomology sequence

$$H^{r-1}(\tau_{<k}\Omega^\bullet(L)) \to H^r(\Omega I^\bullet_{\bar{p}}(N)) \to H^r(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \to H^r(\tau_{<k}\Omega^\bullet(L)).$$

The short exact sequence

$$0 \longrightarrow S^\infty_\bullet(\overline{X}) \xrightarrow{\mathrm{inc}} S^\infty_\bullet(g) \xrightarrow{\mathrm{pro}} H_{\bullet-1}(t_{<k}L) \longrightarrow 0$$

induces a long exact sequence

$$H_{r-1}(t_{<k}L)^\dagger \xrightarrow{\mathrm{pro}_*^\dagger} H_r(S^\infty_\bullet(g))^\dagger \xrightarrow{\mathrm{inc}_*^\dagger} H_r(S^\infty_\bullet(\overline{X}))^\dagger \xrightarrow{g_*^\dagger s_*^\dagger} H_r(t_{<k}L)^\dagger.$$

By Lemmas 9.7, 9.9 and 9.10, the diagram

$$\begin{tikzcd}
H^{r-1}(\tau_{<k}\Omega^\bullet L) \arrow[r] \arrow[d, "\Psi_L"', "\cong"] & H^r(\Omega I^\bullet_{\bar{p}}(N)) \arrow[r] \arrow[d, "\Psi_{\bar{p}}"'] & H^r(\Omega^\bullet_{\partial\mathcal{MS}}(N)) \arrow[r] \arrow[d, "\Psi_N"', "\cong"] & H^r(\tau_{<k}\Omega^\bullet L) \arrow[d, "\Psi_L"', "\cong"] \\
H_{r-1}(t_{<k}L)^\dagger \arrow[r] & H_r(S^\infty_\bullet(g))^\dagger \arrow[r] & H_r(S^\infty_\bullet(\overline{X}))^\dagger \arrow[r] & H_r(t_{<k}L)^\dagger
\end{tikzcd}$$

commutes (up to sign). The maps Ψ_L are isomorphisms by Lemma 9.6. The maps Ψ_N are isomorphisms by construction. By the 5-lemma, $\Psi_{\bar{p}}$ is an isomorphism. The identification $H_\bullet(S^\infty_\bullet(g))^\dagger \cong \widetilde{H}_\bullet(I^{\bar{p}}X)^\dagger$ follows from Proposition 9.2 (Partial Smoothing). q.e.d.

10. The Differential Graded Algebra Structure

The theory $HI^\bullet_{\bar{p}}$ possesses a perversity-internal cup product structure, as we shall now show. The theorem applies to any depth-1 stratified space with flat, isometrically structured link bundles.

Theorem 10.1. *For every perversity $\bar{p}$, the DGA structure $(d, \wedge)$ on $\Omega^\bullet(N)$ restricts to a DGA structure $(\Omega I^\bullet_{\bar{p}}(N), d, \wedge)$. In particular, the wedge product of forms induces a cup product $\cup : HI^r_{\bar{p}}(X) \otimes HI^s_{\bar{p}}(X) \longrightarrow HI^{r+s}_{\bar{p}}(X)$.*

Proof. Given $\omega, \omega' \in \Omega I^\bullet_{\bar{p}}(N)$, there are $\eta, \eta' \in \mathrm{ft}_{\geq K}\, \Omega^\bullet_{\mathcal{MS}}(B)$ and open neighborhoods $U, U' \subset \overline{X}$ of $\partial \overline{X}$ so that $\omega|_{U \cap N} = \pi^* \eta$ and $\omega'|_{U' \cap N} = \pi^* \eta'$. Over $p^{-1}(U_\alpha)$, η and η' have expressions

$$\eta|_{p^{-1}U_\alpha} = \phi^*_\alpha \sum_i \pi^*_1 \eta_i \wedge \pi^*_2 \gamma_i, \;\; \eta'|_{p^{-1}U_\alpha} = \phi^*_\alpha \sum_j \pi^*_1 \eta'_j \wedge \pi^*_2 \gamma'_j,$$

with $\gamma_i, \gamma'_j \in \tau_{\geq K} \Omega^\bullet(F)$. Then the product $\gamma_i \wedge \gamma'_j$ again lies in $\tau_{\geq K} \Omega^\bullet(F)$ by Proposition 4.3. (Note that the direction in which we truncate enters crucially here — if we had used $\tau_{<K}$, the product would not usually lie in the truncated complex.) The proof is completed by observing $(\omega \wedge \omega')|_{U \cap U' \cap N} = \pi^*(\eta \wedge \eta')$ and

$$\begin{aligned} (\eta \wedge \eta')|_{p^{-1}U_\alpha} &= \phi^*_\alpha \sum_{i,j} \pi^*_1 \eta_i \wedge \pi^*_2 \gamma_i \wedge \pi^*_1 \eta'_j \wedge \pi^*_2 \gamma'_j \\ &= \phi^*_\alpha \sum_{i,j} (-1)^{\deg \gamma_i \deg \eta'_j} \pi^*_1(\eta_i \wedge \eta'_j) \wedge \pi^*_2(\gamma_i \wedge \gamma'_j) \end{aligned}$$

with $\gamma_i \wedge \gamma'_j \in \tau_{\geq K} \Omega^\bullet(F)$. q.e.d.

11. Foliated Stratified Spaces

We shall here give a precise definition of what we mean by a stratified foliation. Since this paper is mostly concerned with depth-1 spaces, we shall restrict our discussion of foliations to the depth-1 case as well, though the definition can easily be recursively extended to arbitrary stratified spaces. We will compare our definition to the one given by Farrell and Jones in [**21**] and to the conical foliations of [**35**]. The main formal difference is that our definition is purely topological, whereas the definition of Farrell and Jones requires a system of metrics on the strata satisfying a number of conditions with respect to Mather-type control data of the stratification. The main result of this section (Theorem 11.7) explains how flat link bundles arise in foliated stratified spaces.

Recall that a (smooth) k-dimensional foliation $\mathcal{F}$ of a manifold M^m without boundary is a decomposition $\mathcal{F} = \{F_j\}_{j \in J}$ of M into connected immersed smooth submanifolds of dimension k (called *leaves*) so that the following local triviality condition is satisfied: each point in M has an open neighborhood $U \cong \mathbb{R}^m$ such that the partition of U into the connected components of the $U \cap F_j$, $j \in J$, corresponds under the diffeomorphism $\phi : U \cong \mathbb{R}^m$ to the decomposition of $\mathbb{R}^m = \mathbb{R}^k \times \mathbb{R}^{m-k}$ into the parallel affine subspaces $\mathbb{R}^k \times \mathrm{pt}$. Such a (U, ϕ) is called a *foliation chart* and the connected components of the $U \cap F_j$ are called *plaques*.

The plaques contained in a leaf constitute a basis for the topology of the leaf. This topology does not, in general, coincide with the topology induced on the leaf by the topology on M. Thus F_j is not generally an embedded submanifold. The foliation $\mathcal{F}$ induces a foliation $\mathcal{F}_V$ on any open subset $V \subset M$ by taking $\mathcal{F}_V$ to consist of the connected components of all the $V \cap F_j$.

Definition 11.1. The *cone on a foliation* $(M, \mathcal{F})$ is the pair $(cM, c\mathcal{F})$, where cM is the cone on M with cone vertex c and $c\mathcal{F}$ is the decomposition of cM given by

$$c\mathcal{F} = \{F \times \{t\} \mid F \in \mathcal{F}, t \in (0,2)\} \cup \{c\}.$$

Note that $c\mathcal{F}$ is a "singular foliation" of cM, since it contains leaves of different dimensions. The collection $c\mathcal{F} - \{c\}$ is a smooth foliation of the manifold $cM - \{c\} = M \times (0,2)$.

Definition 11.2. A *stratified foliation* of a 2-strata space (X, Σ) is a pair $(\mathcal{X}, \mathcal{S})$ such that
(1) $\mathcal{X}$ is a smooth foliation of the top stratum $X - \Sigma$,
(2) $\mathcal{S}$ is a smooth foliation of the singular stratum Σ, and
(3) every point in Σ has an open neighborhood U with a local trivialization $\psi : U \times cL \xrightarrow{\cong} p^{-1}(U)$ as in Definition 2.1 (4), such that the leaves of the product foliation $\mathcal{S}_U \times (c\mathcal{L} - \{c\})$ correspond under ψ to the leaves of $\mathcal{X}_{p^{-1}(U)-\Sigma}$ for some smooth foliation $\mathcal{L}$ on L.

(Note that the leaves of $\mathcal{S}_U \times \{c\}$ are taken to the leaves of $\mathcal{S}_U$ automatically, as ψ is the identity on $U \times \{c\}$.)

Definition 11.3. A *stratified foliation* of a depth-1 space $(X, \Sigma_1, \dots, \Sigma_r)$ is a tuple $(\mathcal{X}, \mathcal{S}_1, \dots, \mathcal{S}_r)$ such that, with $X_i = X - \bigcup_{j \neq i} \Sigma_j$, $(\mathcal{X}_{X_i}, \mathcal{S}_i)$ is a stratified foliation of the 2-strata space (X_i, Σ_i) for every i.

Example 11.4. The following type of foliated 2-strata space plays a role in the work of Farrell and Jones on the topological rigidity of negatively curved manifolds, [**22**]. Let (Y, Σ) be a 2-strata space and let M be a connected manifold whose fundamental group G acts on Y preserving the two strata such that Σ has a G-invariant tube T with equivariant retraction $p : T \to \Sigma$. Let $\widetilde{M}$ be the universal cover of M. The quotient $X = \widetilde{M} \times_G Y$ of $\widetilde{M} \times Y$ under the diagonal action of G is a 2-strata space with top stratum $\widetilde{M} \times_G (Y - \Sigma)$ and bottom stratum $\widetilde{M} \times_G \Sigma$. A stratified foliation $(\mathcal{X}, \mathcal{S})$ of X is given by taking

$$\mathcal{X} = \{p(\widetilde{M} \times \{y\}) \mid y \in Y - \Sigma\} \text{ and } \mathcal{S} = \{p(\widetilde{M} \times \{y\}) \mid y \in \Sigma\},$$

where p is the covering projection $p : \widetilde{M} \times Y \to X$. To see this, trivialize locally the flat Y-bundle $X \to M$ induced by $\widetilde{M} \times Y \to \widetilde{M}$, trivialize locally $p : T \to \Sigma$ and equip the link L with the 0-dimensional foliation $\mathcal{L}$.

Proposition 11.5. *For a stratified foliation* $(\mathcal{X}, \mathcal{S})$ *of a* 2*-strata space* (X, Σ) *with control data* (T, p, ρ)*, the following statements hold:*
(i) *If* v *is a vector at a point in* $T - \Sigma$ *which is tangent to a leaf of* $\mathcal{X}$*, then* $p_*(v)$ *is tangent to a leaf of* $\mathcal{S}$*.*
(ii) *The radial function* ρ *is constant along the leaves of* $\mathcal{X}_{T-\Sigma}$*. In particular,* $\rho_*(v) = 0$ *for* v *tangent to* $\mathcal{X}_{T-\Sigma}$*.*

Proof. (i) Let $U \subset \Sigma$ be a chart such that v is based at a point of $p^{-1}(U) - \Sigma$ and consider the commutative diagram

$$\begin{tikzcd} TU \times T(L \times (0,2)) \arrow[rr, "\psi_*", "\cong"'] \arrow[dr, "\mathrm{proj}_1"'] & & T(p^{-1}(U) - \Sigma) \arrow[dl, "p_*"] \\ & TU. & \end{tikzcd}$$

Let $F \in \mathcal{X}_{p^{-1}(U)-\Sigma}$ be the leaf that v is tangent to. Then by Definition 11.2 (3), there exists a leaf $S \times K \times \{t\}$, $S \in \mathcal{S}_U$, $K \in \mathcal{L}$, $t \in (0, 2)$, such that $\psi(S \times K \times \{t\}) = F$. Hence there is a vector $(u, w) \in TS \oplus TK$ with $\psi_*(u, w, 0) = v$. Then

$$p_*(v) = p_*(\psi_*(u, w, 0)) = \mathrm{proj}_1(u, w, 0) = u$$

with u tangent to S, which is an open subset of a leaf of $\mathcal{S}$.

(ii) It suffices to prove that ρ is locally constant along the leaves of $\mathcal{X}_T$, since leaves are connected. Let F be a leaf in $\mathcal{X}_{p^{-1}(U)-\Sigma}$ and let $S \in \mathcal{S}_U$, $K \in \mathcal{L}$, t be such that $\psi(S \times K \times \{t\}) = F$, as in (i). Using the commutative diagram (1) in Definition 2.1, we have

$$\rho(F) = \rho\psi(S \times K \times \{t\}) = \tau \circ \mathrm{proj}_2(S \times K \times \{t\}) = \tau(K \times \{t\}) = \{t\}.$$

Hence ρ is constant on F. q.e.d.

It follows from this proposition that our definition of a stratified foliation is compatible with the definition of Farrell and Jones as given in [**21**, Def. 1.4]. The latter requires essentially that

(a) for vectors v tangent to $\mathcal{X}_{T-\Sigma}$, the ratio of the length of $p_*(v)^\perp$ to the length of v, where $p_*(v)^\perp$ is the component of $p_*(v)$ perpendicular to the leaves of $\mathcal{S}$, becomes as small as we like by taking the base point of v sufficiently close to Σ as measured by ρ, and

(b) the same statement for the ratio of the size of $\rho_*(v)$ to the length of v.

Note that this definition requires endowing the strata with a system of Riemannian metrics. Suppose that a 2-strata space has a stratified foliation in the sense of our Definition 11.2. As $p_*(v)^\perp = 0$ by Proposition 11.5(i), condition (a) is satisfied. As $\rho_*(v) = 0$ by Proposition 11.5(ii), condition (b) is satisfied as well.

Furthermore, our stratified foliations are compatible with the "conical foliations" of [**35**], which the authors define only for spherical links, that is, for X a manifold. They do allow, however, singular foliations on the links, which we do not. On the other hand, we allow the 0-dimensional foliation on the link, which they disable.

Let $(M, \mathcal{F})$ be a foliated manifold and $N \subset M$ an immersed submanifold. One says that $\mathcal{F}$ is *tangent to* N if for each leaf F in $\mathcal{F}$, either $F \cap N = \varnothing$ or $F \subset N$.

Lemma 11.6. *If $\mathcal{F}$ is tangent to N, then $\mathcal{G} = \{F \in \mathcal{F} \mid F \cap N \neq \varnothing\}$ is a smooth foliation of N.*

Theorem 11.7. *Let (X, Σ) be a 2-strata space endowed with a stratified foliation which is 0-dimensional on the links. Then the restrictions of the link bundle to the leaves of the singular stratum are flat bundles.*

Proof. The total space $E = \rho^{-1}(1)$ of the link bundle $p| : E \to \Sigma$ is a submanifold of $X - \Sigma$ and $\mathcal{X}$ is tangent to E. Indeed, if F is a leaf of $\mathcal{X}$ such that $F \cap E \neq \varnothing$, then there is a point $x \in F$ such that $\rho(x) = 1$. By Proposition 11.5(ii), ρ is constant along F. Thus $\rho|_F \equiv 1$ and so $F \subset E$. By Lemma 11.6, $\mathcal{E} = \{F \in \mathcal{X} \mid F \cap E \neq \varnothing\}$ is a foliation of E. Let S be a leaf in Σ and set $E_S = p^{-1}(S) \cap E$. Then E_S is an immersed submanifold of E. We claim that

$$\mathcal{E} \text{ is tangent to } E_S. \qquad (*)$$

In order to see this, let $F \in \mathcal{E}$ be a leaf that touches E_S, $F \cap E_S \neq \varnothing$. We have to show that $F \subset E_S$. Since $F \cap E_S \neq \varnothing$, there is a point $x_0 \in F$ with $p(x_0) \in S$. We must show that $p(x) \in S$ for all $x \in F$. Since F is connected, we may join x_0 and x by a path $\gamma : [0,1] \to F$, $\gamma(0) = x_0$, $\gamma(1) = x$. The compact space $p\gamma[0,1] \subset \Sigma$ can be covered by finitely many open sets $U_0, \ldots, U_k \subset \Sigma$, each of which comes with a diffeomorphism $\psi_i : U_i \times L \times \{1\} \to p^{-1}(U_i) \cap E$ such that $p\psi_i = \mathrm{proj}_1$. By the Lebesgue number lemma, there is an N such that each $p\gamma(I_j)$, $I_j = [j/N, (j+1)/N]$, lies in some U_i. Then the claim $(*)$ is implied by the following statement:

$$\begin{array}{l}\text{For all } 0 \leq j < N\text{: If } p\gamma(j/N) \in S, \text{ then} \\ p\gamma(t) \in S \text{ for all } t \in I_j. \end{array} \qquad (**)$$

To prove $(**)$, assume that $p\gamma(j/N) \in S$ and let i be such that $p\gamma(I_j) \subset U_i$. Let F_0 be the unique connected component of $F \cap p^{-1}(U_i)$ that contains $\gamma(j/N)$. Then, as $\gamma(I_j)$ is connected and contained in $F \cap p^{-1}(U_i)$, we have $\gamma(t) \in F_0$ for all $t \in I_j$. By the definition of a stratified foliation, there is a leaf S' in $\mathcal{S}$ and a leaf $K \in \mathcal{L}$ such that $\psi_i(S'_0 \times K \times \{1\}) = F_0$, where S'_0 is a connected component of $S' \cap U_i$. Since

$p\gamma(j/N) \in S$ and

$$p\gamma(j/N) = \mathrm{proj}_1 \circ \psi_i^{-1} \circ \gamma(j/N) \in \mathrm{proj}_1 \circ \psi_i^{-1}(F_0) \\ = \mathrm{proj}_1(S'_0 \times K \times \{1\}) = S'_0 \subset S',$$

the leaves S and S' have a point in common, which implies that $S' = S$. In particular, $S'_0 \subset S$. Consequently, as $\gamma(t) \in F_0$ for all $t \in I_j$,

$$p\gamma(t) = \mathrm{proj}_1 \circ \psi_i^{-1} \circ \gamma(t) \in \mathrm{proj}_1 \circ \psi_i^{-1}(F_0) = S'_0 \subset S$$

for all $t \in I_j$, which establishes statement $(**)$, and thus also the claim $(*)$. By Lemma 11.6,

$$\mathcal{E}_S = \{F \in \mathcal{E} \mid F \cap E_S \neq \varnothing\} = \{F \in \mathcal{X} \mid F \cap E_S \neq \varnothing\}$$

is a smooth foliation of E_S. So far, we have not used the assumption that the foliations $\mathcal{L}$ on the links are zero-dimensional. We shall now use that assumption to prove that $(p| : E_S \to S, \mathcal{E}_S)$ is a transversely foliated bundle. Let $s = \dim \mathcal{S}$. For every point $x \in S$, we must find an open neighborhood $V \subset S$, $V \cong \mathbb{R}^s$, and a diffeomorphism $\varphi : V \times L \to p^{-1}(V) \cap E$ such that $p\varphi = \mathrm{proj}_1$ and φ carries the product foliation $\{V \times \{l\}\}_{l \in L}$ to the foliation $(\mathcal{E}_S)_{p^{-1}(V) \cap E}$. This implies that $\mathcal{E}_S$ is transverse to the fibers of the link bundle and that the restriction of p to each leaf of $\mathcal{E}_S$ is a covering map. Let $U \subset \Sigma$ be an open neighborhood of x such that there is a diffeomorphism $\psi : U \times L \times \{1\} \to p^{-1}(U) \cap E$ with $p\psi = \mathrm{proj}_1$. We may moreover take such a U to be the domain of a foliation chart $\phi : U \xrightarrow{\cong} \mathbb{R}^s \times \mathbb{R}^{\dim \Sigma - s}$. Let V be the unique plaque of S in U that contains x. Under ϕ, V is mapped to $\mathbb{R}^s \times \mathrm{pt}$. Let $\varphi : V \times L \to p^{-1}(V) \cap E$ be the restriction of ψ to $V \times L$. A leaf F_0 in $(\mathcal{E}_S)_{p^{-1}(V) \cap E}$ is a connected component of $F \cap p^{-1}(V)$, where F is a leaf of $\mathcal{X}$ which maps to S under p and to 1 under ρ. Let F_1 be the connected component of $F \cap p^{-1}(U)$ which contains F_0. By definition of a stratified foliation, there is a leaf $\{l\}$ in $\mathcal{L}$, $l \in L$, and a plaque V' of S in U such that $\psi(V' \times \{l\} \times \{1\}) = F_1$. We have $p(F_0) \subset V$, as $F_0 \subset F \cap p^{-1}(V)$. Also, $p(F_0) \subset p(F_1) \subset V'$ so that $p(F_0) \subset V \cap V'$. But $V \cap V' = \varnothing$ unless $V = V'$. Since $p(F_0)$ is not empty, we have $V = V'$ and thus $\psi(V \times \{l\} \times \{1\}) = F_1$. In particular, $p(F_1) = p\psi(V \times \{l\} \times \{1\}) = \mathrm{proj}_1(V \times \{l\} \times \{1\}) = V$. Hence $F_1 \subset F \cap p^{-1}(V)$. Since F_1 is connected, $F_0 \subset F_1$, and F_0 is a connected component of $F \cap p^{-1}(V)$, we conclude that $F_1 = F_0$. Thus any leaf F_0 in $(\mathcal{E}_S)_{p^{-1}(V) \cap E}$ corresponds under φ to a leaf of the form $V \times \{l\}$ for some $l \in L$. We have shown that $\mathcal{E}_S$ is a transverse foliation of the link bundle over S. This transverse foliation defines a flat connection on $p| : E_S \to S$, see also [**15**, Theorem 2.1.9].

q.e.d.

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