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NUMBER OF NODAL DOMAINS AND SINGULAR POINTS OF EIGENFUNCTIONS OF NEGATIVELY CURVED SURFACES WITH AN ISOMETRIC INVOLUTION

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Abstract

We prove two types of nodal results for density 1 subsequences of an orthonormal basis $\{\phi_j\}$ of eigenfunctions of the Laplacian on a negatively curved compact surface (M, g). The first result pertains to Riemann surfaces (M, J, σ) with an anti-holomorphic involution σ such that $M - \mathrm{Fix}(\sigma)$ has more than one component. In any genus $\mathfrak{g}$, there is a $(3\mathfrak{g}-3)$-dimensional moduli space of such real Riemann surfaces. Our main result is that, for any negatively curved σ-invariant metric g on M, the number of nodal domains of the even or odd Δ_g-eigenfunctions tends to infinity along a density 1 subsequence. For a generic σ-invariant negatively curved metric g, the multiplicity of all eigenvalues equals 1, and all eigenfunctions are either even or odd, and therefore the result holds for almost any eigenfunction.

The analytical part of the proof shows that the number of zeros of even eigenfunctions restricted to $\mathrm{Fix}(\sigma)$, and the number of singular points of odd eigenfunctions on $\mathrm{Fix}(\sigma)$, tend to infinity. This is a quantum ergodic restriction phenomenon. Our second result generalizes this statement to any negatively curved surface (M, g) and to a generic curve $C \subset M$: the number of zeros of eigenfunctions $\phi_j|_C$ tends to infinity.

The additional step to obtain a growing number of nodal domains in the (M, J, σ) setting is topological. It generalizes an argument of Ghosh, Reznikov, and Sarnak on the modular domain to higher genus.

1. Introduction

Let (M, g) be a compact two-dimensional C^∞ Riemannian surface of genus $\mathfrak{g} \geq 2$, and let ϕ_λ be an L^2-normalized eigenfunction of the Laplacian,

$$-\Delta_g \phi_\lambda = \lambda \phi_\lambda.$$

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We denote its nodal line by

$$Z_{\phi_\lambda} = \{x : \phi_\lambda(x) = 0\},$$

and denote by $N(\phi_\lambda)$ the number of nodal domains of ϕ_λ, i.e., the number of connected components Ω_j of

$$M \backslash Z_{\phi_\lambda} = \bigcup_{j=1}^{N(\phi_\lambda)} \Omega_j.$$

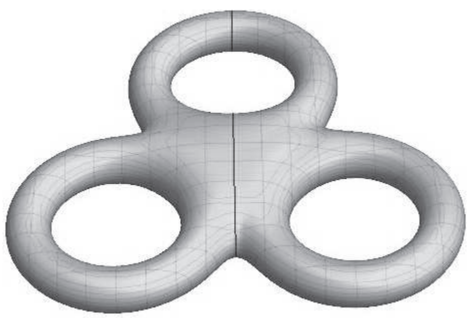
We also let $\{\phi_j\}_{j=1}^{\infty}$ denote an ordered orthonormal basis of eigenfunctions, with respect to the inner product $\langle u, v\rangle = \int_M u\bar{v} dA_g$, ordered by the corresponding sequence of eigenvalues $\lambda_0 = 0 < \lambda_1 \leq \lambda_2 \uparrow \infty$. Here, dA_g is the area form of (M, g). We write $N(\phi_j)$ for $N(\phi_{\lambda_j})$.

It is known that $N(\phi_\lambda)$ need not tend to infinity with λ. Indeed, H. Lewy constructed sequences of spherical harmonics on the standard S^2 with eigenvalues tending to infinity for which the number of nodal domains is ≤ 3 [**25**]. Earlier examples on the flat torus appeared in [**33**, **11**]. Similarly, there exist surfaces possessing sequences of eigenfunctions for which the number of critical points is uniformly bounded independent of the eigenvalue [**21**]. But it seems reasonable to conjecture that for any (M, g), there exists *some* orthonormal sequence $\{\phi_{j_k}\}$ of eigenfunctions for which $N(\phi_{j_k}) \to \infty$ as $k \to \infty$. However, to the authors' knowledge there are almost no known results proving this on any surface except obvious cases such as surfaces of revolution where one may separate variables. The one exception is a new result of Ghosh, Reznikov, and Sarnak [**13**], which will be discussed in detail below. The main result of this article is that there exist Riemann surfaces (M, J) possessing an infinite-dimensional class of negatively curved metrics g on M for which $N(\phi_{j_k}) \to \infty$ along an orthonormal sequence of Δ_g eigenfunctions of density 1.

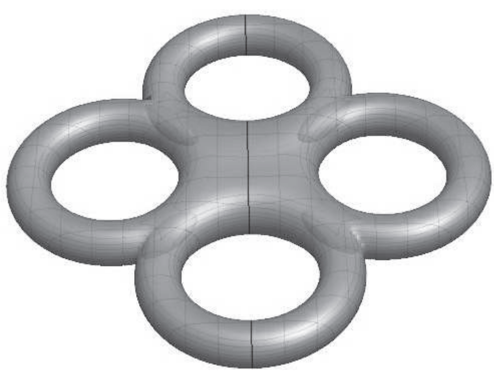
The relevant Riemann surfaces (M, J) are complexifications of real algebraic curves $M(\mathbb{R})$ that *divide* (equivalently, *separate*) M in the sense that $M \backslash M(\mathbb{R})$ has more than one component (in which case it has two components). Such surfaces possess an anti-holomorphic involution σ whose fixed-point set $\mathrm{Fix}(\sigma)$ is the real curve $M(\mathbb{R})$. It is a classical result of F. Schottky [**31**], F. Klein [**24**], and G. Weichold [**37**] that Riemann surfaces (M, J, σ) with anti-holomorphic involution and with dividing fixed-point set $\mathrm{Fix}(\sigma)$ exist in any genus, and that the number of connected components equals 2. Klein refers to a Riemann surface with anti-holomorphic involution as a *real Riemann surface*, and we employ his terminology in what follows. Real Riemann surfaces are sometimes called symmetric Riemann surfaces, with the anti-holomorphic involution called a symmetry. The condition that $M(\mathbb{R}) = \mathrm{Fix}(\sigma)$ be dividing is equivalent to M/σ being orientable. The quotient is sometimes called a Klein surface, and the dividing case is sometimes referred to as *orientable Klein surfaces* (see [**1**, **27**]). Klein and Weichold define the topological type of a real Riemann surface as the triple $(\mathfrak{g}, n, a)$, where $\mathfrak{g}$ is the genus, n is the number of connected components of $M(\mathbb{R})$ and $a = \pm 1$ depending on whether M/σ is orientable or not. Since we often need to refer to this condition, we use the following:

Definition 1.1. We say that a real Riemann surface (M, J, σ) of genus $\mathfrak{g}$ (with anti-holomorphic involution σ) is of type I if $\mathrm{Fix}(\sigma)$ is dividing, i.e., $M - \mathrm{Fix}(\sigma)$ has 2 components. We then call (M, J, σ) a real Riemann surface of type I.

The moduli space of dividing real algebraic curves (i.e., real Riemann surfaces of type I) is non-empty for every $\mathfrak{g} \geq 1$ and is known to be diffeomorphic to $\mathbb{R}^{3\mathfrak{g}-3}$. We refer to §2 for background. Some real Riemann surfaces of type I may be constructed as Schottky doubles of planar multiply connected domains, i.e., by gluing together two copies of the domain along the boundary components [**30**]. Some images drawn with Mathematica are given below.



Surface of genus 3 with dividing $\mathrm{Fix}(\sigma)$.



Surface of genus 4 with dividing $\mathrm{Fix}(\sigma)$.

We define $\mathcal{M}_{M,J,\sigma}$ to be the space of C^∞ σ-invariant *negatively curved* Riemannian metrics on a real Riemann surface (M, J, σ) of type I. Any negatively curved metric g_1 induces a σ-invariant one by averaging, $g_1 \to g = \frac{1}{2}(g_1 + \sigma^* g_1)$. Hence $\mathcal{M}_{M,J,\sigma}$ is an open set in the space of σ-invariant metrics and, in particular, is infinite dimensional. For each $g \in \mathcal{M}_{M,J,\sigma}$, it follows from Harnack's theorem (see [**10**]) that the fixed-point set $\mathrm{Fix}(\sigma)$ is a disjoint union

$$\mathrm{Fix}(\sigma) = \gamma_1 \cup \cdots \cup \gamma_n \tag{1.1}$$

of $0 \leq n \leq \mathfrak{g} + 1$ simple closed geodesics. We use the assumption of negative curvature in two ways: (i) to ensure that the geodesic flow is ergodic (a classic result of Hedlund, Hopf, and others), and (ii) to have good sup norm estimates of eigenfunctions (see §3.2).

The isometry σ acts by translation on $L^2(M, dA_g)$, and we define $L^2_{even}(M)$, resp. $L^2_{odd}(M)$, to denote the subspace of even functions $f(\sigma x) = f(x)$, resp. odd elements $f(\sigma x) = -f(x)$. We define the even, resp. odd, projection by $\Pi_{even} f(x) = \frac{1}{2}(f(x) + f(\sigma x))$, resp. $\Pi_{odd} f(x) = \frac{1}{2}(f(x) - f(\sigma x))$. Translation by any isometry σ commutes with the Laplacian Δ_g, and so these projections commute with Δ_g and the even-odd subspaces are Δ_g-invariant. Hence the even and odd parts of eigenfunctions are eigenfunctions, and all eigenfunctions are linear combinations of even or odd eigenfunctions. We denote by $\{\phi_j\}$ an orthonormal basis of $L^2_{even}(M)$ of even eigenfunctions, resp. $\{\psi_j\}$ an orthonormal basis of $L^2_{odd}(M)$ of odd eigenfunctions.

Remark 1.2. In Proposition 7.1, we prove the (essentially known) result that for generic metrics in $\mathcal{M}_{M,J,\sigma}$, the eigenvalues are simple (multiplicity 1) and therefore all eigenfunctions are either even or odd.

Our main result is the following:

Theorem 1.3. *Let (M, J, σ) be a compact real Riemann surface of genus $\mathfrak{g} \geq 2$ of type I with anti-holomorphic involution σ. Let $\mathcal{M}_{M,J,\sigma}$ be the space of σ-invariant negatively curved C^∞ Riemannian metrics on M. Then for any $g \in \mathcal{M}_{(M,J,\sigma)}$ and any orthonormal Δ_g-eigenbasis $\{\phi_j\}$ of $L^2_{even}(M)$, resp. $\{\psi_j\}$ of $L^2_{odd}(M)$, one can find a density 1 subset A of $\mathbb{N}$ such that*

$$\lim_{\substack{j\to\infty \\ j\in A}} N(\phi_j) = \infty,$$

resp.

$$\lim_{\substack{j\to\infty \\ j\in A}} N(\psi_j) = \infty.$$

For the odd eigenfunctions ψ_j, the conclusion holds under the weaker assumption that $Fix(\sigma) \neq \emptyset$, i.e., for the complexification of any real algebraic curve.

Combining this with Remark 1.2, we conclude

Corollary 1.4. *For generic metrics in $\mathcal{M}_{M,J,\sigma}$, the number of nodal domains tends to infinity along a density 1 subsequence of any orthonormal basis of eigenfunctions.*

There are two main steps in the proof of Theorem 1.3, an analytic step (which is the deepest part) and a topological step. The overall strategy is inspired by the article of Ghosh, Reznikov, and Sarnak [**13**], discussed in §1.3. The analytic part is to prove that for $g \in \mathcal{M}_{M,J,\sigma}$, the number of transversal intersection points of the nodal line of the even eigenfunctions ϕ_j with $\mathrm{Fix}(\sigma)$, resp. the number of singular points of odd eigenfunctions ψ_j on $\mathrm{Fix}(\sigma)$, tends to infinity along a density 1 subsequence of eigenfunctions. These results are of independent interest, and we discuss them in more detail in §1.1. We then discuss a significant generalization to generic smooth curves in §1.2. The principal ingredient in these results is the series of quantum ergodic restriction theorems proved recently in [**8**, **35**].

The second step (Lemma 6.4) is a topological argument. Using the Euler inequality for embedded graphs, we show that the growing number of nodal intersections with $\mathrm{Fix}(\sigma)$ in Theorem 1.6 implies a growing number of nodal domains. This topological argument uses implicitly that $\mathrm{Fix}(\sigma)$ is the common boundary of the two components of $M\backslash\mathrm{Fix}(\sigma)$.

1.1. Number of nodal points or singular points on Fix(σ).

Remark 1.5. Notational Conventions: Throughout the article, γ always denotes a sub-arc of $\mathrm{Fix}(\sigma)$, i.e., a sub-arc of one of the component simple closed geodesics γ_j of (1.1). By a sub-arc we mean the image of a sub-interval under an arc-length parametrization. Also, for $f \in C^1(M)$, $\partial_\nu f$ denotes the derivative $\nabla f \cdot \nu|_\gamma$ by the unit normal $\nu = J\gamma'$ to γ. Here, J is the complex structure.

The analytical part of Theorem 1.3 for even eigenfunctions is the following:

Theorem 1.6. *Let (M, J, σ) be (as above) a real Riemann surface of genus $\mathfrak{g} \geq 2$ of type I. Let $g \in \mathcal{M}_{M,J,\sigma}$ as in Theorem 1.3. Then for any orthonormal eigenbasis $\{\phi_j\}$ of $L^2_{even}(M)$, one can find a density 1 subset A of $\mathbb{N}$ such that*

$$\lim_{\substack{j\to\infty \\ j\in A}} \#\left(Z_{\phi_j} \cap \gamma\right) = \infty$$

for any subarc $\gamma \subset \mathrm{Fix}(\sigma)$. Furthermore, there are an infinite number of zeros where $\phi_j|_\gamma$ changes sign.

Remark 1.7. $Z_{\phi_j} \cap \gamma$ must be a finite set of points. For, if $Z_{\phi_j} \cap \gamma$ contains a curve, then the tangential derivative of ϕ_j along the curve vanishes. Hence, together with $\partial_\nu \phi_j = 0$, we have $d\phi_j(x) = 0$ along the curve, contradicting the upper bound in [**12**] on the number of singular points.

For odd eigenfunctions ψ_j, we prove that the number of singular points on $\text{Fix}(\sigma)$ tends to infinity. By singular points of an eigenfunction ψ_λ of eigenvalue λ we mean the set

$$\Sigma_{\psi_\lambda} = \{x \in Z_{\psi_\lambda} : d\phi_\lambda(x) = 0\}$$

of critical points of ψ_λ that lie on the nodal set Z_{ψ_λ}. It is proved in [**12**] that the number of singular points of ψ_λ is bounded by $C_g\lambda$ on any surface. For generic metrics, the singular set is empty [**36**]. However, for negatively curved surfaces with an isometric involution, odd eigenfunctions ψ always have singular points. Indeed, odd eigenfunctions vanish on γ and have singular points at $x \in \gamma$ where the normal derivative vanishes, $\partial_\nu \psi_\lambda = 0$.

Theorem 1.8. *Let (M, J, σ) be (as above) a compact real Riemann surface of genus $\mathfrak{g} \geq 2$ and of type I. Let $g \in \mathcal{M}_{M,J,\sigma}$. Then for any orthonormal eigenbasis $\{\psi_j\}$ of $L^2_{odd}(M)$, one can find a density 1 subset A of $\mathbb{N}$ such that*

$$\lim_{\substack{j\to\infty \\ j\in A}} \# \left(\Sigma_{\psi_j} \cap \gamma\right) = \infty$$

for any subarc $\gamma \subset \text{Fix}(\sigma)$. Furthermore, there are an infinite number of singular points where $\partial_\nu \psi_j|_\gamma$ changes sign.

One of the principal ingredients in the proofs of Theorems 1.6 and 1.8 is the QER (quantum ergodic restriction) theorem for Cauchy data, proved by H. Christianson, J. Toth, and the second author in [**8**]. Roughly speaking, it says that if $\{u_j\}$ is a quantum ergodic sequence of eigenfunctions of a Riemannian manifold (M, g), and if $H \subset M$ is a smooth hypersurface, then the Cauchy data

$$\{(u_j|_H,\ \partial_\nu u_j|_H)\} \tag{1.2}$$

on H is quantum ergodic along the hypersurface. The first component is known as the Dirichlet data, and the second component is the Neumann data. We review the definition of quantum ergodicity and the statement of the QER theorem for Cauchy data in §4 (see, in particular, Theorem 4.1).

The QER theorem for Cauchy data is applied in Theorems 1.6 and 1.8 to the case where M is a surface and $H = \text{Fix}(\sigma)$. The negative curvature of g guarantees that the geodesic flow is ergodic, and hence that the even orthonormal basis $\{\phi_j\}$ of $L^2_{even}(M)$ and the odd orthonormal basis $\{\psi_j\}$ of $L^2_{odd}(M)$ are quantum ergodic. By the QER theorem, the Cauchy data of the even and odd sequences are quantum ergodic along $\text{Fix}(\sigma)$. The curve $\text{Fix}(\sigma)$ is special because the odd eigenfunctions automatically vanish on it and the even eigenfunctions have vanishing normal derivatives. The QER theorem thus implies that the Dirichlet data of $\{\phi_j\}$ and the Neumann data of $\{\psi_j\}$ are quantum ergodic on $\text{Fix}(\sigma)$.

The relevance of the QER theorem is that it forces the restrictions of even eigenfunctions to $\mathrm{Fix}(\sigma)$ to oscillate more and more rapidly and to have a growing number of sign-changing zeros as $\lambda_j \to \infty$. Similarly, the normal derivatives of the odd eigenfunctions have a growing number of zeros. The proof first uses the Kuzencov trace formula with remainder (Theorem 3.1 and Theorem 3.2; see also [**39**]) to show that $\int_\gamma \phi_j ds$ is "small" (tends to zero at a certain rate) as $j \to \infty$ for any curve γ and for almost all eigenfunctions. On the other hand, the QER theorem shows that $\int_\gamma |\phi_j|^2 ds$ is large (i.e., does not tend to zero). A standard sup norm bound on eigenfunctions of non-positively curved surfaces is then used to compare the integrals $\int_\gamma |\phi_j|^2 ds$ and $\int_\gamma |\phi_j| ds$. The combination of the Kuznecov bound, the QER asymptotics, and the sup norm bound just manages to show that $\int_\gamma |\phi_j| ds > |\int_\gamma \phi_j ds|$ for any geodesic arc γ. Hence there must exist sign-changing zeros. The proof that odd eigenfunctions have a growing number of singular points is similar.

1.2. Quantum ergodic restriction and intersections of nodal lines and generic curves. Theorem 1.6 on intersections of nodal lines and $\mathrm{Fix}(\sigma)$ admits a generalization, Theorem 1.9, in which $\mathrm{Fix}(\sigma)$ is replaced by a C^∞ curve C satisfying a certain generic condition. Although this theorem does not (as yet) have applications to counting nodal domains, we include it here because the proof is very similar to that of Theorem 1.6 and requires almost no additional work.

The generic condition on the curve C is that it is asymmetric with respect to the geodesic flow. The asymmetry condition is stated precisely in Definition 8.1, but we state it somewhat informally as follows: C is asymmetric with respect to the geodesic flow if pairs of geodesics emanating from points $q \in C$ with mirror symmetric initial velocities almost never return to the same point of C at the same time. More precisely, consider any $q \in C$, and any pair of geodesics γ_+, γ_- with initial data $\gamma_\pm(0) = q$, and with $\gamma_\pm'(0)$ having the same tangential component but opposite normal components with respect to T_qC. Then C is said to be asymmetric if the pair $\gamma_\pm(t)$ of geodesics almost never returns to C at the same time to the same place. "Almost never" refers to the natural symplectic surface measure on the space $(q, \xi) \in S^*_C M$ of unit vectors to M with footpoint on C. The only known examples of curves that fail to be asymmetric are the fixed-point sets of orientation-reversing isometric involutions. The asymmetry condition has a natural generalization to any dimension, and it is proved in [**35**] that generic hypersurfaces are asymmetric. For more details, see §8.

Theorem 1.9. *Let (M, g) be a C^∞ compact negatively curved surface, and let C be a closed curve that is asymmetric with respect to the geodesic flow. Then for any orthonormal eigenbasis $\{\phi_j\}$ of Δ_g-eigenfunctions of (M, g), there exists a density* 1 *subset A of $\mathbb{N}$ such* *that*

$$\begin{cases} \lim_{\substack{j\to\infty \\ j\in A}} \#\left(Z_{\phi_j} \cap C\right) = \infty, \\ \\ \lim_{\substack{j\to\infty \\ j\in A}} \#\left\{x \in C : \partial_\nu \phi_j(x) = 0\right\} = \infty. \end{cases}$$

Furthermore, there are an infinite number of zeros where $\phi_j|_C$ (resp. $\partial_\nu \phi_j|_C$) changes sign.

We emphasize that the asymmetry assumption of Theorem 1.9 is opposite to the hypothesis of Theorem 1.6, and that the two intersection theorems are independent of each other. Theorem 1.9 is based on the QER theorem of [**35**] for Dirichlet data, and not the QER theorem for Cauchy data of [**8**] that is used in Theorem 1.6. In dimension 2, the result of [**35**] asserts that if (M, g) is a surface with ergodic geodesic flow and $C \subset M$ is a curve satisfying the asymmetry condition, then the restriction of a density 1 subsequence of eigenfunctions to C is quantum ergodic. Theorem 1.9 is not used in the proof of Theorem 1.3. At present, we do not know if it leads to lower bounds on numbers of nodal domains because we do not know an analogue of the topological argument used in the case $C = \mathrm{Fix}(\sigma)$. However, results on numbers of nodal points along curves seem to us of independent interest.

1.3. Background on counting nodal domains. Having completed the statement of results of this article, we round out the picture by providing background on the history of nodal domain counting problems.

We recall that J. Brüning in [**5**], (and Yau, unpublished) showed that $\mathcal{H}^1(Z_{\phi_\lambda}) \geq c_g\sqrt{\lambda}$, i.e., the length is bounded below by $c_g\sqrt{\lambda}$ for some constant $c_g > 0$.

Let $\{\phi_j\}_{j\geq 0}$ be an orthonormal eigenbasis of $L^2(M)$ with the eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$. According to the Weyl law, we have the following asymptotic:

$$j \sim \frac{Vol(M)}{4\pi}\lambda_j.$$

Therefore, by Courant's general nodal domain theorem [**11**], we obtain an upper bound for $N(\phi_j)$:

$$N(\phi_j) \leq j = \frac{Vol(M)}{4\pi}\lambda_j(1 + o(1)).$$

The bound is not sharp (Pleijel improved the leading coefficient). As was pointed out by T. Hoffmann-Ostenhof [**18**], it is not even known whether for any (M, g) one can find a sequence of eigenfunctions with growing number of nodal domains. As mentioned above, the number of nodal domains does not have to grow with the eigenvalue; i.e., when $M = S^2$ or T^2, there exist eigenfunctions with arbitrarily large eigenvalues with $N(\phi) \leq 3$ ([**33, 25**]). We conjecture that for any Riemannian manifold, there exists a sequence of eigenfunctions ϕ_{j_k} with $N(\phi_{j_k}) \to \infty$. At the present time, this is not even known to hold for generic metrics. However, when M is the unit sphere S^2 and ϕ is a random spherical harmonics, then

$$N(\phi) \sim c\lambda_\phi$$

holds almost surely for some constant $c > 0$ [**29**].

The main article inspiring the present work is that of Ghosh, Reznikov, and Sarnak [**13**], which proves a conditional lower bound on the number of nodal domains of the even Hecke–Maass L^2 eigenfunctions of the Laplacian on the finite-area hyperbolic surface $\mathbb{X} = \Gamma \backslash \mathbb{H}$ for $\Gamma = SL(2, \mathbb{Z})$. The hyperbolic surface $\mathbb{X}$ admits the orientation-reversing, isometric involution $\sigma : (x, y) \to (-x, y)$ with fixed-point set the infinite geodesic $\gamma := \{iy \mid y > 0\}$. Let ϕ be an even Maass–Hecke L^2 eigenfunction on $\mathbb{X} = SL(2, \mathbb{Z}) \backslash \mathbb{H}$. In [**13**], the number of nodal domains which intersect a compact geodesic segment $\beta \subset \gamma$ is denoted by $N^\beta(\phi)$.

Theorem 1.10 ([**13**]). *Assume β is sufficiently long, and assume the Lindelöf hypothesis for the Maass–Hecke L-functions. Then for any $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ depending only on ϵ such that*

$$N^\beta(\phi) > c_\epsilon \lambda_\phi^{\frac{1}{24} - \epsilon}.$$

The proof uses methods of L-functions of arithmetic automorphic forms to obtain lower bounds on the number of sign changes of the even eigenfunctions. A topological argument is then used to relate the number of nodal domains that intersect $\beta = \mathrm{Fix}(\sigma)$ with the number of sign changes on β [**13**, (Theorem 2.2] and Lemma 6.4). The conclusion applies to the entire sequence of even L^2 Maass–Hecke eigenfunctions, but assumes the Lindelöf hypothesis.

Our Theorem 1.3 follows their lead in obtaining lower bounds on numbers of nodal domains by first obtaining lower bounds on intersections of nodal lines with the fixed-point set of an orientation-reversing isometric involution and then by applying a topological argument to relate the number of such intersection points to the number of nodal domains. However, the details differ considerably. First, we use the QER theorems (as well as known results on periods) to prove that there are many nodal points (resp. singular points) along the fixed-point set of the involution. The techniques come from microlocal analysis rather than arithmetic analysis and hold for any negatively curved metric. Second, we use a somewhat different topological argument that works when the genus $\mathfrak{g} \geq 2$ to relate numbers of nodal intersections to numbers of nodal domains. We do adopt their terminology of inert and split nodal domains.

1.4. Further results and open problems. The following unconditional result was proved by one of the authors.

Theorem 1.11 ([**22**]). *Let $\beta \subset \delta$ be any fixed compact geodesic segment. Let ϕ_j be the sequence of even Maass–Hecke cusp forms. For any* *fixed $\epsilon > 0$, one can find a density* 1 *subset A of* $\mathbb{N}$ *such that*

$$N^{\beta}(\phi_j) > \lambda_{\phi_j}^{\frac{1}{16}-\epsilon}$$

for all $j \in A$.

The proof is an application of Quantitative Quantum Ergodicity and the Lindelöf hypothesis on average. It applies to all but a thin subsequence of the even Maass–Hecke cusp forms.

In [**23**] the results and arguments of this article are generalized to non-positively curved Riemannian surfaces with concave boundary in which $\mathrm{Fix}(\sigma)$ is replace by the boundary ∂M of M. The surfaces of [**23**] are not assumed to possess any symmetries. A key point is that the topological argument of this article can be adapted to relate nodal intersections with ∂M to nodal domains intersecting the boundary.

A natural question for future research is to prove quantitative lower bounds on the number of intersections of the nodal line with a curve when the geodesic flow is ergodic. In [**34**] it is shown that the number of nodal intersections in the real analytic case is bounded above by $\sqrt{\lambda}$. By Crofton's formula the bound is achieved for a fixed eigenfunction by a random curve. It seems reasonable to conjecture that for some (perhaps generic) curves, there exists a density 1 subsequence of the eigenfunctions whose nodal lines have $C\sqrt{\lambda}$ intersections with the curve. Some sharp results on flat tori are given by Bourgain and Rudnick in [**4**].

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2. Riemann surfaces with an anti-holomorphic involution and dividing fixed point set

In this section we review the theory of real Riemann surfaces (M, J, σ) of type I in the sense of Definition 1.1. The main point is to cite references that prove that such Riemann surfaces exist.

As mentioned in the introduction, a real Riemann surface (M, J, σ) is a Riemann surface with anti-holomorphic involution with $\mathrm{Fix}(\sigma) \neq \emptyset$. Then $M = M(\mathbb{C})$ is the complexification of the real algebraic curve $M(\mathbb{R}) = \mathrm{Fix}(\sigma)$. The complement $M(\mathbb{C}) - M(\mathbb{R})$ of the real locus in the complex locus has either one or two connected components. $M(\mathbb{R})$ is said to divide (or separate) if $M(\mathbb{C}) - M(\mathbb{R})$ has two connected components. The topological invariants of (M, J, σ) are:

- The genus $\mathfrak{g}$ of M.
- The number $n(M)$ of connected components of $M(\mathbb{R})$ (see (1.1)).
- The number $a = a(M)$ defined by: $a(M) = 0$ if $M(\mathbb{R})$ divides the complex locus and $a(M) = 1$ if $M(\mathbb{C}) - M(\mathbb{R})$ is connected. As in Definition 1.1 we say that (M, J, σ) is of type I if $M - M(\mathbb{R})$ has two components and we denote them by $M_+ \cup M_-$, where $M_\pm$ are connected, where $M_+^0 \cap M_-^0 = \emptyset$ (the interiors are disjoint), where $\sigma(M_+) = M_-$, and where $\partial M_+ = \partial M_- = \mathrm{Fix}(\sigma)$.

The triple $(\mathfrak{g}, n, a)$ is called the topological type of (M, J, σ) and is a complete set of topological invariants of a real algebraic curve. The quotient of $M(\mathbb{C})$ by σ is a connected 2-manifold X with $n(M)$ boundary components. X has Euler characteristic $1 - \mathfrak{g}$ and is orientable if and only if $a(M) = 0$. One has the following constraints:

1) $0 \leq n(X) \leq \mathfrak{g} + 1$.
2) If $n(X) = 0$, then $a(X) = 1$. If $n(X) = \mathfrak{g} + 1$ then $a(X) = 0$.
3) If $a(X) = 0$, then $n(X) = \mathfrak{g} + 1 (\mathrm{mod}\ 2)$.

Klein [**24**] (see also [**15**]) proved that any pair $(n(M), a(M))$ which satisfies these constraints is realized by some real curve of genus $\mathfrak{g}$. We refer to [**20**] for a modern proof and to [**6**] for explicit examples.

Furthermore, it is known that the moduli space of real algebraic curves of a given topological type $(\mathfrak{g}, n, a)$ is diffeomorphic to the quotient of $\mathbb{R}^{3\mathfrak{g}-3}$ by a discrete group action. Theorem 3.3 of [**27**] (see also [**27**, Theorem 6.1]) and Corollary 2.1 of [**28**] express the moduli space of real Riemann surfaces of type $(\mathfrak{g}, n, a)$ in terms of a Teichmüller space modulo a discrete group action. The original result appears to be due to Seppäla (see, e.g., [**32**]).

In [**9**] a geometric interpretation of the moduli parameters is given: it is shown in [**9**] that a Riemann surface with anti-holomorphic involution σ with $\mathrm{Fix}(\sigma) \neq \emptyset$ has a geodesic pants decomposition invariant by the involution; the geodesics are orthogonal to $\mathrm{Fix}(\sigma)$. This occurs when the "twist parameters" are equal to 0 or 1/2. One can then parameterize the moduli space of real algebraic curves of a given topological type by the $3\mathfrak{g} - 3$ lengths of the geodesics in the pants decomposition with respect to the hyperbolic metric.

3. Kuznecov sum formula and sup norm estimates on surfaces

In this section, we review a prior result [**39**] on the asymptotics of the "periods" $\int_C f u_j ds$ of eigenfunctions u_j over C^∞ curves C, where f is a fixed smooth function. The degree of smoothness could be relaxed, but that is irrelevant for our purposes. In the following, (M, g) may be any Riemannian surface and $C \subset M$ any closed curve. We denote by $\{u_j\}$ an orthonormal basis of Δ_g-eigenfunctions. In our applications to

Theorems 1.6, resp. 1.8, $\{u_j\}$ will be an orthonormal basis consisting of the even functions $\{\phi_j\}$ and odd eigenfunctions $\{\psi_j\}$ of a surface with orientation-reversing isometric involution.

Theorem 3.1 ([**39**], Corollary 3.3)**.** *Let (M, g) be a Riemannian surface, let $\{u_j\}$ be an orthonormal basis of Δ_g eigenfunctions, and let $C \subset M$ be a closed curve of a surface M. Let $f \in C^\infty(C)$. Then*

$$\sum_{\lambda_j < \lambda} \left| \int_C f u_j ds \right|^2 = \frac{1}{\pi} \left| \int_C f ds \right|^2 \sqrt{\lambda} + O_f(1).$$

We use only the principal term and not the remainder estimate here. Also, the condition that C be closed is not important, since the support of f could be any arc of any component.

A small modification of the proof of Theorem 3.1 is the following: Let ∂_ν denote the normal derivative along C.

Theorem 3.2. *Let $C \subset M$ be a closed curve of a surface M, and let $f \in C^\infty(C)$. Then*

$$\sum_{\lambda_j < \lambda} \left| \lambda_j^{-1/2} \int_C f \partial_\nu u_j ds \right|^2 = \frac{1}{\pi} \left| \int_C f ds \right|^2 \sqrt{\lambda} + O_f(1).$$

The proofs of Theorems 3.1 and 3.2 are very similar. We sketch the proofs for the sake of completeness, following [**39**, **16**]. In fact, the theorems are valid for hypersurfaces of Riemannian manifolds of any dimension, and we sketch the proofs in the general case.

Let (M, g) be an n-dimensional Riemannian manifold, and let $\{u_j\}$ be an orthonormal basis of Δ_g-eigenfunctions. Given a smooth hypersurface $H \subset M$, define the semi-classical Cauchy data of the eigenfunctions on H by

$$CD(u_j) := \{(u_j|_H,\ h_j D_\nu u_j|_H)\}. \tag{3.1}$$

Here, $h_j = \lambda_j^{-\frac{1}{2}}$ and $D_\nu = \frac{1}{i}\partial_\nu$, where ∂_ν is a fixed choice of a unit normal derivative. As mentioned above (see (1.2)), the first component of the Cauchy data is called the Dirichlet data and the second is called the normalized Neumann data. The term "semi-classical Cauchy data" refers to this normalization.

To prove Theorems 3.1 and 3.2, we employ a standard Fourier cosine Tauberian approach (see [**19**, **39**] for background). First we smooth out the averages over the spectrum by convolving with a suitable test function ρ. We determine the asymptotics of the smoothed spectral average by relating it to the singularity at $t = 0$ of a Fourier-dual sum involving the wave group. Then we use a Tauberian theorem to obtain the asymptotics of the original spectral average plus a remainder of one lower order from the asymptotics of the smoothed spectral average. We only use the leading-order term of the asymptotics in this article and therefore do not give details on the remainder term.

Let $E(t,x,y)$ be the Schwartz kernel of the even part $\cos t\sqrt{-\Delta}$ of the wave group. We define the restricted Schwartz kernels to H by

$$\begin{cases} E^D(t,q,q') = E^D(t,\cdot,\cdot)|_{H\times H}, \\ \\ E^N(t,q,q') = D_{\nu_q} D_{\nu_{q'}} E(t,\cdot,\cdot)|_{H\times H}. \end{cases}$$

For $f \in C^\infty(M)$ and for $B = D, N$ we define the distributions

$$S_f^B(t) : \; = \; \int_H \int_H E^B(t,q,q') f(q) f(q') ds(q) ds(q'),$$

where ds is the surface measure. Thus,

$$\begin{aligned} S_f^D(t) &= \sum_j \cos t\sqrt{\lambda_j}\, |\langle f(q), u_j(q)\rangle_H|^2, \\ \\ S_f^N(t) &= \sum_j \lambda_j^{-1} \cos t\sqrt{\lambda_j}\, \left|\langle f(q), D_{\nu_q} u_j(q)\rangle_H\right|^2. \end{aligned}$$

Here, we use the notation

$$\langle f, g\rangle_H := \int_H f\bar{g} ds$$

employed in Theorem 3.4 of [**16**], which we quote below to obtain the leading coefficients.

We further introduce a smooth cutoff $\rho \in \mathcal{S}(\mathbb{R})$ (Schwartz space) with $\mathrm{supp}\hat{\rho} \subset (-\epsilon, \epsilon)$, where $\hat{\rho}$ is the Fourier transform of ρ, and consider

$$S_f^B(\mu, \rho) := \int_{\mathbb{R}} \hat{\rho}(t)\; S_f^B(t) e^{it\mu} dt.$$

Proposition 3.3. *If* $\mathrm{supp}\hat{\rho}$ *is contained in a sufficiently small interval around* 0, *with* $\hat{\rho} \equiv 1$ *in a smaller interval,* $S_f^B(\lambda, \rho)$ *is a semi-classical Lagrangian distribution with asymptotic expansion as* $\lambda \to \infty$ *given by*

$$S_f^D(\mu, \rho) = \sum_j \rho(\mu - \sqrt{\lambda_j}) |\langle u_j, f\rangle_H|^2 = \tfrac{1}{\pi} ||f||^2_{L^2(H)} + O(\mu^{-1}),$$

$$S_f^N(\mu, \rho) = \sum_j \rho(\mu - \sqrt{\lambda_j}) \lambda_j^{-1} |\langle (D_\nu u_j, f\rangle_H|^2 = \tfrac{1}{\pi} ||f||^2_{L^2(H)} + O(\mu^{-1}).$$

In fact, there exists a complete asymptotic expansion of $S_f^B(\lambda, \rho)$ in powers of μ^{-1}. Proposition 3.3 combines with the Fourier Tauberian Theorem 17.5.6 of [**19**] to prove Theorems 3.1 and 3.2.

We sketch the proof of Proposition 3.3. The first observation is that there exists $\epsilon_0 > 0$ so that

$$\text{sing supp} S_f^B(t) \cap (-\epsilon_0, \epsilon_0) = \{0\}. \tag{3.2}$$

Here, sing supp denotes the singular support; (3.2) states that $S_f^B(t)$ has an isolated singularity at $t = 0$. This follows from propagation of singularities for the wave kernel and its restriction to a hypersurface (curve) in both variables. As discussed in [**19**], the wave front set $WF'(E(t))$ is the graph C_t of the geodesic flow on $T^*M\backslash\{0\}$ (the cotangent bundle minus the zero section). Integration over $H \times H$ is the pushforward of the pullback of $E(t, x, y)$ to $\mathbb{R} \times H \times H$ under the submersion

$$\pi : \mathbb{R} \times H \times H \to \mathbb{R}, \quad \pi(t, q, q') = t.$$

Using the calculus of wave front sets under pullback and pushforward (see [**19**] for background) as in (1.6) of [**39**], the wave front set of $S_f^B(t)$ is given by

$$WF(S_f^B(t)) = \{(t, \tau) \in T^*\mathbb{R} : \exists (x, \xi, y, \eta) \in C_t' \cap N^*H \times N^*H\}$$

in the support of the symbol. Here, N^*H is the co-normal bundle of H. Hence there exists $\epsilon_0 > 0$ so that no trajectory starting orthogonally from H can hit H again at any point. Thus, the only singularity in this time interval is $t = 0$.

The leading coefficient in the asymptotics of Proposition 3.3 is the principal symbol of $S_f(t)$ at $t = 0$. As in [**39**] it may be calculated using the symbol calculus of Fourier integral distributions under pushforward and pullback. But it may also be obtained from the following result of [**16**].

Theorem 3.4 (Completeness of Cauchy data on interior hypersurfaces)**.** *Let $\rho \in \mathcal{S}(\mathbb{R})$ be such that $\hat{\rho}$ is identically 1 near 0 and has sufficiently small support. Then for any $f \in C^\infty(H)$, we have*

$$f(q) = \lim_{\mu \to \infty} \pi \sum_j \rho(\mu - \sqrt{\lambda_j}) \langle u_j, f \rangle_H u_j(q), \quad q \in H$$

and

$$f(q) = \lim_{\mu \to \infty} \pi \sum_j \rho(\mu - \sqrt{\lambda_j})\ \lambda_j^{-1} \langle \partial_\nu u_j, f \rangle_H\ (\partial_\nu u_j)(q),$$

where $\langle \cdot, \cdot \rangle_H$ denotes the inner product in $L^2(H)$.

3.1. Rate of decay of generic terms.

Proposition 3.5. *There exists a subsequence of eigenfunctions ϕ_j of natural density 1 so that, for all $f \in C^\infty(\gamma)$,*

$$\left\{ \begin{array}{l} \left| \int_\gamma f \phi_j ds \right| \\ \\ \lambda_j^{-\frac{1}{2}} \left| \int_\gamma f \partial_\nu \phi_j ds \right| \end{array} \right. = O_f(\lambda_j^{-1/4} (\log \lambda_j)^{1/4}). \tag{3.3}$$

Proof. Denote by $N(\lambda)$ the number of eigenfunctions in $\{j \mid \lambda < \lambda_j < 2\lambda\}$. For each f, we have by Theorem 3.1 and Chebyshev's inequality,

$$\frac{1}{N(\lambda)}|\{j \mid \lambda < \lambda_j < 2\lambda, \left|\int_{\gamma_i} f\phi_j ds\right|^2 \geq \lambda_j^{-1/2}(\log\lambda_j)^{1/2}\}| = O_f(\frac{1}{(\log\lambda_j)^{1/2}}).$$

It follows that the upper density of exceptions to (3.3) tends to zero. We then choose a countable dense set $\{f_n\}$ and apply the diagonalization argument of [**38**, Lemma 3] or [**40**, Theorem 15.5 step (2)] to conclude that there exists a density one subsequence for which (3.3) holds for all $f \in C^\infty(\gamma)$. The same holds for the normal derivative. q.e.d.

3.2. Sup norm estimates. We are assuming our surfaces have negative curvature. This ensures that (M,g) has no conjugate points and that the Bérard–Selberg estimates on sup-norms of eigenfunctions in [**2**] apply.

Theorem 3.6. *Let (M,g) be a non-positively curved compact Riemannian manifold and let $\{\phi_j\}$ be an orthonormal basis of Δ_g eigenfunctions, $-\Delta_g\phi_j = \lambda_j\phi_j$. Then there exists a constant $C_g > 0$ depending only on g such that:*

$$||\phi_j||_{L^\infty(M)} \leq C_g\lambda_j^{\frac{n-1}{4}}/\sqrt{\log\lambda_j}.$$

This estimate is an improvement by $\frac{1}{\sqrt{\log\lambda_j}}$ on the universal sup norm estimates.

4. Review of the QER theorem for Cauchy data

The proof of Theorem 1.3 is based on the following QER theorem for Cauchy data of [**8**]. This is a general result on restrictions of eigenfunctions and their normal derivatives to a hypersurface $H \subset M$ of a Riemannian manifold of any dimension.

Theorem 4.1. *Let (M,g,σ) be a real Riemann surface of type I satisfying the assumptions of Theorem* 1.3. *Let γ be a component of* $\mathrm{Fix}(\sigma)$, *and let $\{\phi_j\}$ be an orthonormal basis of $L^2_{even}(M)$ by even eigenfunctions, and let $\{\psi_j\}$ denote an orthonormal basis of $L^2_{odd}(M)$ by odd eigenfunctions. Then there exists a subsequence ϕ_{j_k} (resp ψ_{j_k}) of $\{\phi_j\}$ (resp ψ_j) of density 1, such that for any $f \in C(\gamma)$,*

$$\begin{cases} \int_\gamma f\,|\phi_{j_k}|_\gamma|^2 \to_{k\to\infty} \frac{2C}{Area(M)}\int_\gamma f(s)ds, \\ \\ \lambda_{j_k}^{-1}\int_\gamma f\,|\psi_{j_k}|_\gamma|^2 \to_{k\to\infty} \frac{2C}{Area(M)}\int_\gamma f(s)ds. \end{cases}$$

Here, $C = \int_0^1(1-\sigma^2)^{-\frac{1}{2}}d\sigma = \frac{\pi}{2}$.

Since f may be assumed to be supported on one component γ_j, there is no essential difference in stating the result for functions on $\mathrm{Fix}(\sigma)$ or for ones supported on a single component of $\mathrm{Fix}(\sigma)$. We state it in the latter form because we plan to use the result on small sub-arcs. We have dropped the subscript γ_j for notational simplicity and also because the result is valid for any sub-arc of a γ_j. We refer to [**35**, **8**] for background and undefined notation for pseudo-differential operators.

4.1. Sketch of proof of Theorem 4.1. Theorem 4.1 is an almost immediate consequence of Theorem 1 of [**8**]. We briefly review that result and how it implies Theorem 4.1. To facilitate comparison to [**8**], we use the semi-classical notation $h_j = \lambda_j^{-\frac{1}{2}}$ of that article. We also drop the j in the subscript.

The QER result pertains to matrix elements of semi-classical pseudo-differential operators along γ with respect to the restricted eigenfunctions. Only multiplication operators are used in this article, but the result is stated for all pseudo-differential operators. Pseudo-differential operators on γ are denoted by $a^w(y, hD_y)$ or $Op_\gamma(a)$.

We define the microlocal lifts of the Neumann data as the linear functionals on semi-classical symbols $a \in S^0_{sc}(\gamma)$ given by the matrix elements

$$\mu_h^N(a) := \int_{B^*\gamma} a \, d\Phi_h^N := \langle Op_\gamma(a) hD_\nu \phi_h|_\gamma, hD_\nu \phi_h|_\gamma \rangle_{L^2(\gamma)}.$$

Here, $Op_\gamma(a)$ is a semi-classical pseudo-differential operator on γ. The matrix element on the right side is the inner product of the restricted eigenfunction with $Op_\gamma(a)$ applied to the restricted eigenfunction. As above, $B^*\gamma$ is the unit co-ball bundle for the metric g. We also define the *renormalized microlocal lifts* of the Dirichlet data by

$$\mu_h^D(a) := \int_{B^*\gamma} a \, d\Phi_h^{RD} := \langle Op_\gamma(a)(1 + h^2\Delta_\gamma)\phi_h|_\gamma, \phi_h|_\gamma \rangle_{L^2(\gamma)}.$$

Here, $h^2\Delta_\gamma$ denotes the negative tangential Laplacian $-h^2 \frac{d^2}{ds^2}$ for the induced metric on γ, so that the symbol $1 - |\sigma|^2$ of the operator $(1 + h^2\Delta_\gamma)$ vanishes on the tangent directions $S^*\gamma$ of γ. Finally, we define the microlocal lift $d\Phi_h^{CD}$ of the Cauchy data to be the sum

$$d\Phi_h^{CD} := d\Phi_h^N + d\Phi_h^{RD}. \tag{4.1}$$

Theorem 1 of [**8**] states that the Cauchy data of a sequence of quantum ergodic eigenfunctions along γ is QER for semi-classical pseudo-differential operators with symbols vanishing on the glancing set $S^*\gamma$, i.e., that

$$d\Phi_h^{CD} \to \omega,$$

where

$$\omega(a) = \frac{2}{\pi \mathrm{Area}(M)} \int_{B^*\gamma} a_0(s, \sigma)(1 - |\sigma|^2)^{1/2} ds d\sigma.$$

Here, a_0 is the principal symbol of $Op_\gamma(a)$, a function on $T^*\gamma$; as above, s is arclength from a fixed basepoint and σ is the dual symplectic coordinate. Also, $B^*\gamma$ denotes the unit co-ball bundle of γ, i.e., tangent vectors to γ of length ≤ 1. Since γ is a curve, $B^*\gamma$ is the interval $\sigma \in (-1,1)$ at each point $\gamma(s)$ in an arc-length parameterization, where σ is the symplectically dual variable.

Theorem 4.2. *Assume that $\{\phi_h\}$ is a quantum ergodic sequence of eigenfunctions on M. Then the sequence $\{d\Phi_h^{CD}\}$ (4.1) of microlocal lifts of the Cauchy data of ϕ_h is quantum ergodic on γ in the sense that for any $a \in S^0_{sc}(\gamma)$,*

$$\langle Op_H(a) hD_\nu \phi_h|_\gamma, hD_\nu\phi_h|_\gamma\rangle_{L^2(\gamma)} + \left\langle Op_\gamma(a)(1+h^2\Delta_\gamma)\phi_h|_\gamma, \phi_h|_\gamma\right\rangle_{L^2(\gamma)}$$

$$\to_{h\to 0^+} \tfrac{4}{\mu(S^*M)} \textstyle\int_{B^*\gamma} a_0(s,\sigma)(1-|\sigma|^2)^{1/2} ds d\sigma,$$

where a_0 is the principal symbol of $Op_\gamma(a)$.

Under our assumption in Theorem 1.6 that g has negative curvature, the geodesic flow of (M,g) is ergodic, and therefore a density 1 subsequence of eigenfunctions is quantum ergodic in the ambient manifold. We refer to [**38**, **40**] for background on this standard result. In this case, Theorem 4.2 implies that the Cauchy data is quantum ergodic along γ.

In Theorem 1.3, we apply the QER theorem to the hypersurface (curve) $H = \mathrm{Fix}(\sigma) \subset M$, i.e., the fixed-point set (1.1) of the anti-holomorphic involution σ of (M, J, σ) of type I and to metrics $g \in \mathcal{M}_{(M,J,\sigma)}$.

We first apply the QER result to odd eigenfunctions.

Corollary 4.3. *Let (M, g) be a negatively curved surface with an orientation-reversing isometric involution σ for which* $\mathrm{Fix}(\sigma)$ *is dividing. Let γ be one of components of* $\mathrm{Fix}(\sigma)$. *Then there exists a density 1 sequence $\{\psi_{j_k}\}$ of the odd eigenfunctions of (M, g) so that, for any $f \in C(\gamma)$,*

$$\lambda_{j_k}^{-1} \int_\gamma f|(D_\nu \psi_{j_k})|_\gamma|^2 ds \to_{h\to 0^+} \frac{4}{\mu(S^*M)} \int_{B^*\gamma} f(s)(1-|\sigma|^2)^{1/2} ds d\sigma.$$

This follows from Theorem 4.2 since the Dirichlet data term

$$\left\langle Op_\gamma(a)(1+h^2\Delta_\gamma)\phi_h|_\gamma, \phi_h|_\gamma\right\rangle_{L^2(\gamma)}$$

vanishes if ψ_j is odd.

If we apply Theorem 4.2 to even eigenfunctions, the Neumann data drops out and we get the following:

Corollary 4.4. *Let (M, g) have an orientation-reversing isometric involution with separating fixed-point set* $\mathrm{Fix}(\sigma)$, *and let γ be one of its* *components. Then for any sequence of even quantum ergodic eigenfunctions of* (M, g)*,*

$$\left\langle Op_\gamma(a)(1+h^2\Delta_\gamma)\phi_h|_\gamma, \phi_h|_\gamma \right\rangle_{L^2(\gamma)}$$

$$\to_{h\to 0^+} \tfrac{4}{\mu(S^*M)} \textstyle\int_{B^*\gamma} a_0(s,\sigma)(1-|\sigma|^2)^{1/2} ds d\sigma.$$

This is not quite the result we wish to apply, since the Dirichlet data term involves $(1+h^2\Delta_\gamma)\phi_h|_\gamma$. However, Theorem 2 of [**8**] transfers this operator from the Dirichlet data to the Neumann data and provides the result we need. To be precise, in [**8**] we further define the microlocal lift $d\Phi_h^D \in \mathcal{D}'(B^*\gamma)$ of the Dirichlet data of ϕ_h,

$$\int_{B^*\gamma} a\, d\Phi_h^D := \langle Op_\gamma(a)\phi_h|_\gamma, \phi_h|_\gamma \rangle_{L^2(\gamma)},$$

and the renormalized microlocal lift of the Neumann data,

$$\int_{B^*\gamma} a\, d\Phi_h^{RN} := \langle (1+h^2\Delta_\gamma + i0)^{-1} Op_\gamma(a) hD_\nu \phi_h|_\gamma, hD_\nu\phi_h|_\gamma \rangle_{L^2(\gamma)}.$$

Theorem 2 of [**8**] then states:

Theorem 4.5. *Assume that* $\{\phi_h\}$ *is a quantum ergodic sequence on* M*. Then there exists a sub-sequence* ϕ_h *of density 1 as* $h \to 0^+$ *such that for all* $a \in S^0_{sc}(\gamma)$*,*

$$\begin{aligned}&\left\langle (1+h^2\Delta_\gamma + i0)^{-1} Op_\gamma(a) hD_\nu\phi_h|_H, hD_\nu\phi_h|_\gamma \right\rangle_{L^2(\gamma)}\\ &\quad + \langle Op_\gamma(a)\phi_h|_\gamma, \phi_h|_\gamma \rangle_{L^2(\gamma)}\\ &\to_{h\to 0^+} \frac{4}{2\pi \mathrm{Area}(M)} \int_{B^*\gamma} a_0(s,\sigma)(1-|\sigma|^2)^{-1/2} ds d\sigma.\end{aligned}$$

In our application, (M, g) has negative curvature and ergodic geodesic flow. Hence (as mentioned above) by the standard quantum ergodicity theorem [**38**, **40**], there exists a quantum ergodic subsequence of the even eigenfunctions of density one. We note that the volume form $(1-|\sigma|^2)^{-1/2} ds d\sigma$ on the right side is singular at $\sigma = 1$ but the singularity is integrable. The Neumann data vanishes for even eigenfunctions, and by letting $a_0(s,\sigma) = f(s)$ for some continuous function f we get the following:

Corollary 4.6. *Let* (M, g) *be a negatively curved surface with an orientation-reversing isometric involution whose fixed-point set is dividing, and let* γ *be one of its components. Then there exists a density 1 subsequence* ϕ_{j_k} *of even eigenfunctions of* (M, g) *such that, for any* $f \in C(\gamma)$*,*

$$\textstyle\int_\gamma f|\phi_h|_\gamma|^2 \to_{h\to 0^+} \frac{2C}{Area(M)} \int_\gamma f(s) ds.$$

Here, $C = \int_0^1 (1-\sigma^2)^{-\frac{1}{2}} d\sigma = \frac{\pi}{2}$*.*

Theorem 4.1 follows from Corollary 4.3 together with Corollary 4.6.

5. Proof of Theorems 1.6 and 1.8

We now have all of the ingredients for the proof of Theorem 1.6 and Theorem 1.8. We first consider the even eigenfunctions.

Proof. We may assume that the sequence $\{\phi_j\}$ has the quantum ergodic restriction property of Theorem 4.1 such that for any $f \in C_0^\infty(\mathrm{Fix}(\sigma))$,

$$\lim_{j\to\infty} \int_{\mathrm{Fix}(\sigma)} f|\phi_j|^2 ds = B \int_{\mathrm{Fix}(\sigma)} f ds \tag{5.1}$$

for some constant $B > 0$ (namely, $B = \frac{2C}{Area(M)}$ in the notation of Theorem 4.1). From Proposition 3.5, we may further assume that for any $f \in C_0^\infty(\mathrm{Fix}(\sigma))$,

$$\left| \int_{\mathrm{Fix}(\sigma)} f\phi_j ds \right| = O_f(\lambda_j^{-1/4}(\log \lambda_j)^{1/4}). \tag{5.2}$$

Fix $R \in \mathbb{N}$. Let $\beta_1, \cdots, \beta_R$ be any partition of a subarc γ of $\mathrm{Fix}(\sigma)$ into subarcs. Let $f_1, \cdots, f_R \in C_0^\infty(\gamma)$ be chosen such that

$$\mathrm{supp}\{f_i\} = \beta_i$$
$$f_i \geq 0 \text{ on } \gamma.$$

Note that

$$||\phi_j||_{L^\infty(M)} \int_{\beta_i} f_i|\phi_j| ds \geq \int_{\beta_i} f_i|\phi_j|^2 ds,$$

and hence by Theorem 3.6 and (5.1), there exists a constant $c > 0$ (depending on R) such that the following estimate holds for any $i = 1, \cdots, R$ for all sufficiently large j:

$$\int_{\beta_i} f_i|\phi_j| > c\lambda_j^{-1/4}\sqrt{\log \lambda_j}.$$

Therefore by (5.2), for all sufficiently large j, ϕ_j has at least one sign change on each subarc β_i. It follows that ϕ_j has at least R sign changes on γ for all sufficiently large j. Since R can be chosen arbitrarily, we conclude that

$$\lim_{j\to\infty} \# \left(Z_{\phi_j} \cap \gamma \right) = \infty.$$

q.e.d.

The proof of Theorem 1.8 is almost the same. We use the second statement of Theorem 4.1 i.e., Corollary 4.3, together with the Kuznecov-type formula for normal derivatives of Theorem 3.2. From these, we may assume that for any $f \in C_0^\infty(\mathrm{Fix}(\sigma))$,

$$\lim_{j\to\infty} \lambda_j^{-1} \int_{\mathrm{Fix}(\sigma)} f|\partial_\nu \psi_j|^2 ds = B \int_{\mathrm{Fix}(\sigma)} f ds$$

for some constant $B > 0$, and that

$$\lambda_j^{-\frac{1}{2}} \left| \int_{\mathrm{Fix}(\sigma)} f \partial_\nu \psi_j ds \right| = O_f(\lambda_j^{-1/4} (\log \lambda_j)^{1/4}).$$

A slight modification of the Selberg–Bérard sup norm estimate, with no essential change in the proof, gives $||\lambda_j^{-\frac{1}{2}} \nabla \psi_j||_{L^\infty(M)} = O(\lambda_j^{1/4}/ (\log \lambda_j)^{1/2})$ on a surface of negative curvature. Hence there exists a constant $c > 0$ (depending on R) such that

$$\lambda_j^{-\frac{1}{2}} \int_{\beta_i} f_i |\partial_\nu \psi_j| ds > c\lambda_j^{-1/4} (\log \lambda_j)^{1/2}$$

holds for any $i = 1, \cdots, R$ for all sufficiently large j.

Therefore as in the even case, $\partial_\nu \psi_j$ has at least one sign change on each subarc β_i for all sufficiently large j; hence $\# \left(\Sigma_{\psi_j} \cap \gamma\right) \geq R$ is satisfied for all sufficiently large j. Again, since R can be chosen arbitrarily, we have

$$\lim_{j\to\infty} \# \left(\Sigma_{\psi_j} \cap \gamma\right) = \infty.$$

6. Local structure of nodal sets in dimension 2

The next step is to use a topological argument to conclude that ϕ_j (resp. ψ_j) has many nodal domains from the fact that ϕ_j has many zeros (resp. ψ_j has many singular points) on $\mathrm{Fix}(\sigma)$.

We first review some local results on the structure of nodal sets. We then use the local results to convert Z_{ϕ_λ} into a graph embedded into the surface M. The nodal domains become the faces of the surface-embedded graph. In §6.3 we review the general definitions of surface-embedded graphs and Euler's identity for surface-embedded graphs whose faces are simply connected. For general surface-embedded graphs, the Euler identity is not valid but there exists Euler's inequality [**14**]. This inequality is sufficient for our lower bound on the number of nodal domains. We prove that the number of faces is bounded from below by the number of vertices having degree at least 4, and we use this to bound $N(\phi_j)$ (resp. ψ_j) in terms of the number of zeros of ϕ_j (resp. the number of singular points of ψ_j) on $Fix(\sigma)$.

6.1. Local structure of nodal sets in dimension 2. We will need a classical result on the local structure of nodal sets in dimension 2.

Proposition 6.1. [**3**, **17**, **7**] *Assume that ϕ_λ vanishes to order k at x_0. Let $\phi_\lambda(x) = \phi_k^{x_0}(x) + \phi_{k+1}^{x_0} + \cdots$ denote the C^∞ Taylor expansion* *of ϕ_λ into homogeneous terms $\phi_n^{x_0}$ of degree n in normal coordinates x centered at x_0. Then $\phi_k^{x_0}(x)$ is a Euclidean harmonic homogeneous polynomial of degree k.*

To prove this, one substitutes the homogeneous expansion into the equation $-\Delta\phi_\lambda = \lambda^2\phi_\lambda$ and Taylor expands the coefficients of Δ around x_0. The zeroth-order approximation Δ_{x_0} to Δ (obtained by freezing the coefficients at x_0) is the Euclidean Laplacian known as the osculating Laplacian. It lowers the order of $\phi_k^{x_0}$ by two units. Since there is no corresponding term on the right side, $\Delta_{x_0}\phi_k^{x_0} = 0$.

In dimension 2, a homogeneous harmonic polynomial of degree N is the real or imaginary part of the unique holomorphic homogeneous polynomial z^N of this degree—i.e., $p_N(r,\theta) = r^N \sin N\theta$. As observed in [**7**], there exists a C^1 local diffeormorphism χ in a disc around a zero x_0 so that $\chi(x_0) = 0$ and so that $\phi_N^{x_0} \circ \chi = p_N$. It follows that the restriction of ϕ_λ to a curve H is C^1 equivalent around a zero to p_N restricted to $\chi(H)$. The nodal set of p_N around zero consists of N rays, $\{r(\cos\theta, \sin\theta) : r > 0, p_N|_{S^1}(v) = 0\}$. It follows that the local structure of the nodal set in a small disc around a singular point p is C^1 equivalent to N equi-angular rays emanating from p. We refer to [**7**] for further details.

6.2. Sign changing zeros and singular points. We next draw some conclusions about local structure of nodal and singular sets of even/odd eigenfunctions of real Riemann surfaces (M, J, σ) of type I with σ-invariant metric g.

As mentioned in the introduction, $\mathrm{Fix}(\sigma)$ consists of a union of closed geodesics of g (1.1). Let $\gamma \subset \mathrm{Fix}(\sigma)$ be any component geodesic.

We recall that a singular point $x_0 \in M$ for an eigenfunction ϕ_λ is a point where $\phi_\lambda(x_0) = d\phi_j(x_0) = 0$. A non-singular zero is called a regular zero.

Lemma 6.2. *Let ϕ_λ be an even eigenfunction, and let $x_0 = \gamma(s_0)$ be a zero of $\phi_\lambda|_\gamma$. Then if x_0 is a regular zero, then $\phi_\lambda|_\gamma$ changes sign. That is, if the even eigenfunction does not change sign at the zero x_0 along γ, x_0 must be a singular point.*

Indeed, since ϕ is even, its normal derivative vanishes everywhere on γ. If ϕ does not change sign at x_0, then γ is tangent to Z_{ϕ_j} at x_0, i.e., $\frac{d}{ds}\phi_j(\gamma(s)) = 0$, so that x_0 is a singular point.

Next we consider odd eigenfunctions and let ψ_λ be an odd eigenfunction. As above, let γ be a component of $\mathrm{Fix}(\sigma)$. Then $\psi_\lambda \equiv 0$ on γ and the zeros of $\partial_\nu\psi_\lambda$ on γ are singular points of ψ_λ.

Lemma 6.3. *Let ψ_λ be an odd eigenfunction. Then the zeros of the normal derivative $\partial_\nu\psi_\lambda$ on γ are intersection points of the nodal set of ψ_λ in $M\backslash\gamma$ with γ, i.e., points where at least two nodal branches cross.*

Proof. If x_0 is a singular point, then $\phi_j(x_0) = d\phi_j(x_j) = 0$, so the zero set of ϕ_λ is similar to that of a spherical harmonic of degree $k \geq 2$, which consists of $k \geq 2$ arcs meeting at equal angles at 0. It follows that at least two transverse branches of the nodal set of an odd eigenfunction meet at each singular point on γ. q.e.d.

6.3. Surface-embedded graph and Euler's inequality. We recall that an embedded graph G in a surface M is a finite set $V(G)$ of vertices and a finite set $E(G)$ of edges that are simple (non-self-intersecting) curves in M such that any two distinct edges have at most one endpoint and no interior points in common. The *faces* f of G are the connected components of $M \backslash V(G) \cup \bigcup_{e \in E(G)} e$. The set of faces is denoted $F(G)$. An edge $e \in E(G)$ is *incident* to f if the boundary of f contains an interior point of e. Every edge is incident to at least one and to at most two faces; if e is incident to f, then $e \subset \partial f$. We refer the reader to [**26**] for a detailed discussion on surface-embedded graphs.

The faces are not assumed to be cells and the sets $V(G), E(G), F(G)$ are not assumed to form a CW complex. If every face is homeomorphic to an open disc, then we have the Euler's identity

$$|V(G)| - |E(G)| + |F(G)| - 1 = 1 - 2\mathfrak{g},$$

where $\mathfrak{g}$ is the genus of the surface M.

For a general G imbedded into M, denoting by $m(G)$ the number of connected components of G, we have

$$|V(G)| - |E(G)| + |F(G)| - m(G) \geq 1 - 2\mathfrak{g}. \tag{6.1}$$

This can be proven by considering a supergraph $G' \supset G$ whose faces are homeomorphic to an open disc [**14**, **26**]. Let $G' = G_0 \supset G_1 \cdots \supset G_k = G$ be a sequence of surface-imbedded graphs such that G_{i+1} is obtained after deleting either an edge or an isolated vertex. For G', we have the equality

$$|V(G')| - |E(G')| + |F(G')| - m(G') = 1 - 2\mathfrak{g},$$

and one can check that

$$|V(G_i)| - |E(G_i)| + |F(G_i)| - m(G_i)$$

is non-decreasing in i, and therefore (6.1) holds.

6.4. Graph structure of the nodal set and completion of proof of Theorem 1.3. We now prove the inequality for even (resp. odd) eigenfunctions in the higher-genus case of a Riemann surface with an orientation-reversing isometric involution with non-empty fixed-point set.

From Proposition 6.1, we can define a *nodal graph* $V(\phi), E(\phi)$ from Z_{ϕ_λ} as follows.

1) For each connected component of Z_{ϕ_λ} that is homeomorphic to a circle and that does not intersect γ, we add a vertex.

2) Each singular point is a vertex.
3) If $\gamma \not\subset Z_{\phi_\lambda}$, then each intersection point in $\gamma \cap Z_{\phi_\lambda}$ is a vertex.
4) Edges are the arcs of Z_{ϕ_λ} ($Z_{\phi_\lambda} \cup \gamma$, when ϕ_λ is even) that join the vertices listed above.

Let $V(\phi)$ and $E(\phi)$ be the finite set of vertices and the finite set of edges given above, respectively. This way, we obtain a nodal graph $V(\phi), E(\phi)$ of ϕ embedded into the surface M. Note that every vertex of a nodal graph has degree at least 2 by Proposition 6.1.

Indeed, the faces $F(\phi)$ of the nodal graph of odd eigenfunctions are nodal domains, which do not have to be simply connected. In the even case, the faces that do not intersect γ are nodal domains, and the ones that do are inert nodal domains which are cut in two by γ.

We apply Euler's inequality (6.1) to give a lower bound for the number of nodal domains for even and odd eigenfunctions.

Lemma 6.4. *For an odd eigenfunction ψ_j,*

$$N(\psi_j) \geq \#\left(\Sigma_{\psi_j} \cap \gamma\right) + 2 - 2\mathfrak{g},$$

and for an even eigenfunction ϕ_j,

$$N(\phi_j) \geq \frac{1}{2}\#\left(Z_{\phi_j} \cap \gamma\right) + 1 - \mathfrak{g}.$$

Proof. **Odd case.** For an odd eigenfunction ψ_j, $\gamma \subset Z_{\psi_j}$. Therefore, $|F(\psi_j)| = N(\psi_j)$. Let $n(\psi_j) = \#\left(\Sigma_{\psi_j} \cap \gamma\right)$ be the number of singular points on γ. These points correspond to vertices $\in V(\psi_j)$ having degree at least 4 and every vertex of a nodal graph has degree at least 2. Therefore, we have that

$$\begin{aligned} 0 &= \sum_{v \in V(\psi_j)} \deg(v) - 2|E(\psi_j)| \\ &\geq 2\left(|V(\psi_j)| - n(\psi_j)\right) + 4n(\psi_j) - 2|E(\psi_j)| \end{aligned}$$

and

$$|E(\psi_j)| - |V(\psi_j)| \geq n(\psi_j).$$

Now we apply (6.1) with $m(\psi_j) \geq 1$ to obtain

$$N(\psi_j) \geq n(\psi_j) + 2 - 2\mathfrak{g}.$$

Even case.

Following the terminology of [**13**], a nodal domain of an even eigenfunction is called *inert* if it is σ-invariant, in which case it intersects γ in a segment. Otherwise, it is called *split*.

For an even eigenfunction ϕ_j, let $N_{in}(\phi_j)$ be the number of nodal domain U that satisfies $\sigma U = U$ (inert nodal domains). Let $N_{sp}(\phi_j)$ be the number of the rest (split nodal domains). From the assumption that $Fix(\sigma)$ is separating, inert nodal domains intersect $\mathrm{Fix}(\sigma)$ on simple segments, and $\mathrm{Fix}(\sigma)$ divides each nodal domain into two connected components. This implies that, because $\gamma \subset \mathrm{Fix}(\sigma)$ is added when defining the nodal graph, the inert nodal domain may correspond to two faces on the graph, depending on whether the nodal domain intersects γ or not. Therefore, $|F(\phi_j)| \leq 2N_{in}(\phi_j) + N_{sp}(\phi_j)$.

Observe that each point in $Z_{\phi_j} \cap \gamma$ corresponds to a vertex $\in V(\phi_j)$ having degree at least 4 on the nodal graph. Hence by the same reasoning as the odd case, we have

$$N(\phi_j) \geq N_{in} + \frac{1}{2}N_{sp}(\phi_j) \geq \frac{|F(\phi_j)|}{2} \geq \frac{n(\phi_j)}{2} + 1 - \mathfrak{g},$$

where $n(\phi_j) = \#\left(Z_{\phi_j} \cap \gamma\right)$. q.e.d.

Now Theorem 1.3 follows from Theorem 1.6, Theorem 1.8, and Lemma 6.4.

6.5. Comparison to Ghosh, Reznikov, and Sarnak. For the sake of completeness, let us briefly summarize the argument in [**13**] for genus 0 surfaces. We emphasize that their argument does not work in our setting where $\mathfrak{g} \geq 2$.

The number of inert nodal domains of ϕ is denoted R_ϕ. The number of sign changes of ϕ on γ is denoted n_ϕ. The main result of Section 2 of [**13**] in genus 0 is that $R_\phi \geq \frac{1}{2}n_\phi + 1$.

The proof starts with the case where the nodal set is regular. In that case, the nodal line emanating from a regular sign-change zero on γ must intersect γ again at another sign-change zero. The nodal lines intersect γ orthogonally in the regular case. Applying σ to the curve produces an inert nodal domain, and the inequality follows. The remainder of the proof is to show that when singular points occur, $R_\phi - \left(\frac{1}{2}n_\phi + 1\right)$ never increases when arcs between singular points are removed. Hence $R_\phi - \left(\frac{1}{2}n_\phi + 1\right)$ is greater than or equal to the regular case, which is ≥ 0. We note that the local characterization of nodal sets rules out the cusped nodal crossing of Figure 7 of [**13**].

7. Generic simplicity of eigenvalues

In this section, we prove the genericity result stated in Remark 1.2:

Proposition 7.1. *Let* (M, J, σ) *be a real Riemann surface of type I—i.e., with an anti-holomorphic involution with dividing fixed-point set* $\mathrm{Fix}(\sigma)$*. Then for generic negatively* σ*-invariant curved metrics, the Laplace eigenfunctions are either even or odd.*

Here, as usual, *generic* means that the set includes a residual set in the Banach space of C^k negatively curved metrics for some sufficiently large k (or in C^∞, if one is accustomed to residual sets in Frechet spaces).

Proof. Any eigenfunction may be decomposed as a sum of its even part and its odd part and both parts are, of course, also eigenfunctions.

To prove the Proposition it suffices to show that for a residual set of non-positively σ-invariant curved metrics, the multiplicity of each eigenvalue is equal to one. The eigenfunction is then unique up to scalar multiple and must be either even or odd. In a standard way [**36**], it suffices to show that for each j there exists an open dense set of such metrics for which the jth eigenvalue is simple.

Openness is simple since a sufficiently small perturbation of a metric for which the jth eigenvalue is simple also has a simple jth eigenvalue. Regarding density, assume that one cannot split the eigenvalue at some negatively curved metric g_0. The small perturbation of g is then also negatively curved. If we cannot separate the eigenvalue, then for any infinitesimal area-preserving σ-invariant perturbation we have $\int_M \dot{\rho}|\phi_j^1|^2 = \int_M \dot{\rho}|\phi_j^2|^2$, where ϕ_j^1 and ϕ_j^2 are two distinct σ-invariant eigenfunctions corresponding to the same eigenvalue. But this says that $|\phi_j^1|^2 - |\phi_j^2|^2$ is orthogonal to all σ-invariant functions $\dot{\rho}$ so that $\int_M \dot{\rho} dV_g = 0$. Since $|\phi_j^1|^2, |\phi_j^2|^2$ are also σ-invariant, we take the quotient by the $\mathbb{Z}_2$ action defined by σ and find that $\int_{M/\mathbb{Z}_2} \dot{\rho}(|\phi_j^1|^2 - \dot{\rho}|\phi_j^2|^2)dV = 0$ for all smooth $\dot{\rho}$ on $M/\mathbb{Z}_2$ such that $\int \dot{\rho} = 0$. That is, $|\phi_j^1|^2 - |\phi_j^2|^2 = C$ for some constant C on $M/\mathbb{Z}_2$. Integrating over M shows that $C = 0$ and therefore $\phi_j^1 = \epsilon\phi_j^2$, where $\epsilon = \pm 1$. The sign must be constant by regularity, and we then get a contradiction. q.e.d.

8. Proof of Theorem 1.9

At this point, we have completed the proof of the main result Theorem 1.3 on nodal domains. The purpose of this section is to prove much more general versions of Theorems 1.6 and 1.8, in which no symmetry assumptions are assumed. At this time of writing, we do not know how to obtain lower bounds on numbers of nodal domains from this result. However, the proof is almost the same as for Theorem 1.6 and applies to generic curves. It seems to us of independent interest and in the future it may have applications to counting nodal domains.

In place of QER Theorem 4.1 for Cauchy data, we use the QER theorem of [**35**] for Dirichlet data. It asserts that if the geodesic flow of (M, g) is ergodic, then restrictions $\phi_j|_H$ of eigenfunctions (or their normal derivatives) to a hypersurface $H \subset M$ is quantum ergodic on H when H satisfies a certain generic asymmetry condition.

As mentioned above, this QER theorem is quite distinct in terms of its hypotheses from the one for Cauchy data used in Theorem 4.1.

8.1. Quantum ergodic restriction theorems for Dirichlet data. Roughly speaking, the QER theorem for Dirichlet data says that restrictions of eigenfunctions to hypersurfaces $H \subset M$ for (M, g) with ergodic geodesic flow are quantum ergodic along H as long as H is asymmetric for the geodesic flow. By this is meant that a tangent vector ξ to H of length ≤ 1 is the projection to TH of two unit tangent vectors $\xi_\pm$ to M. Then $\xi_\pm = \xi + r\nu$ where ν is the unit normal to H and $|\xi|^2 + r^2 = 1$. There are two possible signs of r corresponding to the two choices of “inward” resp. “outward” normal. Asymmetry of H with respect to the geodesic flow G^t means that the two orbits $G^t(\xi_\pm)$ almost never return at the same time to the same place on H. A generic hypersurface is asymmetric. The fixed-point set of an isometry σ of course fails to be asymmetric and is the model for a “symmetric” hypersurface. We refer to [**35**, Definition 1] for the precise definition of “positive measure of microlocal reflection symmetry” of H. By asymmetry we mean that this measure is zero.

We now state the special cases relevant to Theorem 1.9. To keep to the notation of [**35**], we write $h_j = \lambda_j^{-\frac{1}{2}}$ and employ the calculus of semiclassical pseudo-differential operators [**40**] where the pseudo-differential operators on H are denoted by $a^w(y, hD_y)$ or $Op_{h_j}(a)$. The unit co-ball bundle of H is denoted by B^*H.

The QER theorem for Dirichlet data involves a hypothesis on the (orientable) hypersurface H (or curve, in the case where M is a surface). We denote by ν_+ a choice of unit normal field to H.

We denote by $T^*_H M$, resp. T^*H, the covectors to M with footpoint on H, resp. the unit covectors to H. If $(s, \sigma) \in B^*H$ (the co-ball bundle of H), there exist two unit covectors $\xi_\pm(s, \sigma) \in S^*_s M$ such that $|\xi_\pm(s, \sigma)| = 1$ and $\xi|_{T_sH} = \sigma$. In the orthogonal decomposition of $T_sM = T_sH \oplus N_sH$ into the tangent resp. normal space to H, they are given by

$$\xi_\pm(s, \sigma) = \sigma \pm \sqrt{1 - |\sigma|^2}\nu_+(s).$$

We define the reflection involution through H of covectors to M based at points of H by

$$r_H : T^*_H M \to T^*_H M, \quad r_H(s, \mu\, \xi_\pm(s, \sigma)) = (s, \mu\, \xi_\mp(s, \sigma)), \;\; \mu \in \mathbb{R}_+.$$

Its fixed-point set is T^*H.

We denote by G^t the homogeneous geodesic flow of (M, g), i.e., the Hamiltonian flow on $T^*M - 0$ generated by $|\xi|_g$. We then put $\exp_x t\xi = \pi \circ G^t(x, \xi)$.

Definition 8.1. We say that H is asymmetric with respect to the geodesic flow if

$$\mu_{L,H}\left(\bigcup_{j\neq 0}^{\infty}\{(s,\xi) \in S^*_H M : r_H G^{T^{(j)}(s,\xi)}(s,\xi) = G^{T^{(j)}(s,\xi)} r_H(s,\xi)\}\right) = 0.$$

In other words, if we launch a pair of geodesics with initial conditions $\xi_+(s, \sigma)$, resp. $\xi_-(s, \sigma)$, then almost surely in (s, σ) they do not return at the same time to the same place. Almost surely is with respect to the natural surface measure on $S^*_H M$.

Theorem 8.2. *Let (M, g) be a compact surface with ergodic geodesic flow, and let $C \subset M$ be a closed curve that is asymmetric with respect to the geodesic flow. Then there exists a density* 1 *subset A of $\mathbb{N}$ such that for $a \in S^{0,0}(T^*C \times [0, h_0))$,*

$$\lim_{\substack{j\to\infty \\ j\in A}} \langle Op_{h_j}(a)\phi_{h_j}|_C, \phi_{h_j}|_C\rangle_{L^2(C)} = \omega(a),$$

where

$$\omega(a) = \frac{4}{vol(S^*M)} \int_{B^*C} a_0(s,\sigma)\,(1-|\sigma|^2)^{-\frac{1}{2}}\, ds d\sigma.$$

In particular, this holds for multiplication operators f.

There is a similar result for normalized Neumann data. The normalized Neumann data of an eigenfunction along C is denoted by

$$\lambda_j^{-\frac{1}{2}} D_\nu \phi_j|_C.$$

Here, $D_\nu = \frac{1}{i}\partial_\nu$ is a fixed choice of unit normal derivative.

We define the microlocal lifts of the Neumann data as the linear functionals on semi-classical symbols $a \in S^0_{sc}(C)$ given by

$$\mu_h^N(a) := \int_{B^*C} a\, d\Phi_h^N := \langle Op_C(a) h D_\nu \phi_h|_C, h D_\nu \phi_h|_C\rangle_{L^2(C)}.$$

Theorem 8.3. *Let (M, g) be a compact surface with ergodic geodesic flow, and let $C \subset M$ be a closed curve that is asymmetric with respect to the geodesic flow. Then there exists a density* 1 *subset A of $\mathbb{N}$ such that for $a \in S^{0,0}(T^*C \times [0, h_0))$,*

$$\lim_{\substack{h_j\to 0^+ \\ j\in A}} \mu_h^N(a) \to \omega(a),$$

where

$$\omega(a) = \frac{4}{vol(S^*M)} \int_{B^*C} a_0(s,\sigma)\,(1-|\sigma|^2)^{\frac{1}{2}}\, ds d\sigma.$$

In particular, this holds for multiplication operators f.

8.2. Conclusion of the proof of Theorem 1.9. The proof of Theorem 1.9 is now the same as the proof of Theorem 1.6, using Theorem 8.2 in place of Theorem 4.1.

References

[1] Norman L. Alling and Newcomb Greenleaf. *Foundations of the theory of Klein surfaces.* Lecture Notes in Mathematics, Vol. 219. Springer-Verlag, Berlin-New York, 1971, MR0333163, Zbl 0225.30001.

[2] Pierre H. Bérard. On the wave equation on a compact Riemannian manifold without conjugate points. *Math. Z.*, 155(3):249–276, 1977, MR0455055, Zbl 0341.35052.

[3] Lipman Bers. Local behavior of solutions of general linear elliptic equations. *Comm. Pure Appl. Math.*, 8:473–496, 1955, MR0075416, Zbl 0066.08101.

[4] Jean Bourgain and Zeév Rudnick. Nodal intersections and L^p restriction theorems on the torus. *arXiv:1308.4247*, 2013.

[5] Jochen Brüning. Über Knoten von Eigenfunktionen des Laplace-Beltrami-Operators. *Math. Z.*, 158(1):15–21, 1978, MR0478247, Zbl 0349.58012.

[6] Emilio Bujalance, Francisco Javier Cirre, José Manuel Gamboa, and Grzegorz Gromadzki. *Symmetries of compact Riemann surfaces*, volume 2007 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010, MR2683160, Zbl 1208.30002.

[7] Shiu Yuen Cheng. Eigenfunctions and nodal sets. *Comment. Math. Helv.*, 51(1):43–55, 1976, MR0397805, Zbl 0334.35022.

[8] Hans Christianson, John A. Toth, and Steve Zelditch. Quantum ergodic restriction for Cauchy data: interior que and restricted que. *Math. Res. Lett.*, 20(3):465–475, 2013, MR3162840, Zbl 1288.58017.

[9] Antonio F. Costa and Hugo Parlier. A geometric characterization of orientation-reversing involutions. *J. Lond. Math. Soc. (2)*, 77(2):287–298, 2008, MR2400392, Zbl 1154.30030.

[10] Antonio F. Costa and Hugo Parlier. On Harnack's theorem and extensions: a geometric proof and applications. *Conform. Geom. Dyn.*, 12:174–186, 2008, MR2448264, Zbl 1193.30054.

[11] Richard Courant and David Hilbert. *Methods of mathematical physics. Vol. I.* Interscience Publishers, Inc., New York, N.Y., 1953, MR0065391, Zbl 0053.02805.

[12] Rui-Tao Dong. Nodal sets of eigenfunctions on Riemann surfaces. *J. Differential Geom.*, 36(2):493–506, 1992, MR1180391, Zbl 0776.53024.

[13] Amit Ghosh, Andre Reznikov, and Peter Sarnak. Nodal domains of Maass forms I. *Geom. Funct. Anal.*, 23(5):1515–1568, 2013, MR3102912, Zbl 06228492.

[14] Martin Grohe. Fixed-point definability and polynomial time on graphs with excluded minors. *J. ACM*, 59(5):Art. 27, 64, 2012, MR2995826, Zbl 1281.68129.

[15] Benedict H. Gross and Joe Harris. Real algebraic curves. *Ann. Sci. École Norm. Sup. (4)*, 14(2):157–182, 1981, MR631748, Zbl 0533.14011.

[16] Xiaolong Han, Andrew Hassell, Hamid Hezari, and Steve Zelditch. Completeness of boundary traces of eigenfunctions. *arXiv:1311.0935, to appear Proc. L. M. S.*, 2013.

[17] Philip Hartman and Aurel Wintner. On the local behavior of solutions of non-parabolic partial differential equations. III. Approximations by spherical harmonics. *Amer. J. Math.*, 77:453–474, 1955, MR0076156, Zbl 0066.08001.

[18] Thomas Hoffmann-Ostenhof. Problems for the number of nodal domains in Geometric Aspects of Spectral Theory xiv. *Oberwolfach Reports*, 33:2068–2070, 2012.

[19] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003, MR1996773, Zbl 1028.35001. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[20] Harris Jaffee. Real algebraic curves. *Topology*, 19(1):81–87, 1980, MR559478, Zbl 0426.14013.

[21] Dmitry Jakobson and Nikolai Nadirashvili. Eigenfunctions with few critical points. *J. Differential Geom.*, 53(1):177–182, 1999, MR1776094, Zbl 1038.58036.

[22] Junehyuk Jung. Quantitative Quantum Ergodicity and the nodal lines of Maass-Hecke cusp forms. *arXiv:1301.6211*, 2013.

[23] Junehyuk Jung and Steve Zelditch. Number of nodal domains of eigenfunctions on non-positively curved surfaces with concave boundary. *arXiv:1401.4520*, 2014.

[24] Felix Klein. Ueber Realitätsverhältnisse bei der einem beliebigen Geschlechte zugehörigen Normalcurve der φ. *Math. Ann.*, 42(1):1–29, 1893, MR1510765, Zbl 25.0689.03.

[25] Hans Lewy. On the minimum number of domains in which the nodal lines of spherical harmonics divide the sphere. *Comm. Partial Differential Equations*, 2(12):1233–1244, 1977, MR0477199, Zbl 0377.31008.

[26] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces.* Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001, MR1844449, Zbl 0979.05002.

[27] Sergey M. Natanzon. Klein surfaces. *Uspekhi Mat. Nauk*, 45(6(276)):47–90, 189, 1990, MR1101332, Zbl 0734.30037.

[28] Sergey M. Natanzon. *Moduli of Riemann surfaces, real algebraic curves, and their superanalogs*, volume 225 of *Translations of Mathematical Monographs.* American Mathematical Society, Providence, RI, 2004, MR2075914, Zbl 1056.14033. Translated from the 2003 Russian edition by Sergei Lando.

[29] Fedor Nazarov and Mikhail Sodin. On the number of nodal domains of random spherical harmonics. *Amer. J. Math.*, 131(5):1337–1357, 2009, MR2555843, Zbl 1186.60022.

[30] Menahem Schiffer and Donald C. Spencer. *Functionals of finite Riemann surfaces.* Princeton University Press, Princeton, N. J., 1954, MR0065652, Zbl 0059.06901.

[31] Friedrich Schottky. Ueber eine specielle Function, welche bei einer bestimmten linearen Transformation ihres Argumentes unverändert bleibt. *J. Reine Angew. Math.*, 101:227–272, 1887, Zbl 19.0424.02.

[32] Mika Seppälä and Robert Silhol. Moduli spaces for real algebraic curves and real abelian varieties. *Math. Z.*, 201(2):151–165, 1989, MR997218, Zbl 0645.14012.

[33] Antonie Stern. Bemerkungen über asymptotisches Verhalten von Eigenwerten und Eigenfunktionen. Math.- naturwiss. Diss. Göttingen, 30 S (1925)., 1925, Zbl 51.0356.01.

[34] John A. Toth and Steve Zelditch. Counting nodal lines which touch the boundary of an analytic domain. *J. Differential Geom.*, 81(3):649–686, 2009, MR2487604, Zbl 1180.35395.

[35] John A. Toth and Steve Zelditch. Quantum ergodic restriction theorems: manifolds without boundary. *Geom. Funct. Anal.*, 23(2):715–775, 2013, MR3053760, Zbl 1277.53088.

[36] Karen Uhlenbeck. Generic properties of eigenfunctions. *Amer. J. Math.*, 98(4):1059–1078, 1976, MR0464332, Zbl 0355.58017.

[37] Guido Weichold. Über symmetrische Riemannsche Flächen und die Periodizitätsmodulen der zugerhörigen Abelschen Normalintegrale ersters Gattung. *Leipziger dissertation*, 1883.

[38] Steve Zelditch. Uniform distribution of eigenfunctions on compact hyperbolic surfaces. *Duke Math. J.*, 55(4):919–941, 1987, MR916129, Zbl 0643.58029.

[39] Steve Zelditch. Kuznecov sum formulae and Szegő limit formulae on manifolds. *Comm. Partial Differential Equations*, 17(1-2):221–260, 1992, MR1151262, Zbl 0749.58062.

[40] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012, MR2952218, Zbl 1252.58001.

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