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THE CALABI-YAU EQUATION ON THE KODAIRA-THURSTON MANIFOLD, VIEWED AS AN S^1-BUNDLE OVER A 3-TORUS

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Abstract

We prove that the Calabi-Yau equation on the Kodaira-Thurston manifold has a unique solution for every S^1-invariant initial datum.

1. Introduction and statement of the result

The celebrated Calabi–Yau theorem affirms that given a compact Kähler manifold (M^n, Ω, J) with first Chern class $c_1(M^n)$, every $(1,1)$-form $\tilde{\rho} \in 2\pi c_1(M^n)$ is the *Ricci form* of a unique Kähler metric whose Kähler form belongs to the cohomology class $[\Omega]$. This theorem was conjectured by Calabi in [**4**] and subsequently proved by Yau in [**15**]. The Calabi–Yau theorem can be alternatively reformulated in terms of symplectic geometry by saying that, given a compact Kähler manifold (M^n, Ω, J) and a volume form σ satisfying the normalizing condition

$$\int_{M^n} \sigma = \int_{M^n} \Omega^n,$$

then there exists a unique Kähler form $\tilde{\Omega}$ on (M^n, J) solving

$$\tilde{\Omega}^n = \sigma, \qquad [\tilde{\Omega}] = [\Omega]. \tag{1}$$

Equation (1) still makes sense in the *almost-Kähler* case, when J is merely an almost-complex structure. In this more general context (1) is usually called the *Calabi–Yau equation.*

In [**5**] Donaldson described a project about compact symplectic 4-manifolds involving the Calabi–Yau equation and showed the uniqueness of the solutions. Donaldson's project is principally based on a conjecture stated in [**5**] whose confirmation would lead to new fundamental results in symplectic geometry. Donaldson's project was partially confirmed by Taubes in [**9**] and strongly motivates the study of the Calabi–Yau equation on non-Kähler 4-manifolds.

In [**16**] Weinkove proved that the Calabi–Yau equation can be solved if the torsion of J is sufficiently small, and in [**13**] Tosatti, Weinkove,

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and Yau proved the Donaldson conjecture assuming an extra condition on the curvature and the torsion of the almost-Kähler metric. Furthermore, Tosatti and Weinkove solved in [**12**] the Calabi–Yau equation on the Kodaira–Thurston manifold assuming the initial datum σ invariant under the action of a 2-dimensional torus T^2. The Kodaira–Thurston is historically the first example of symplectic manifold without Kähler structures (see [**11**, **1**]) and it is defined as the direct product of a compact quotient of the 3-dimensional Heisenberg group by a lattice with the circle S^1. In [**6**] it is proved that when σ is T^2-invariant, the Calabi–Yau equation on the Kodaira–Thurston manifold can be reduced to a Monge–Ampère equation on a torus which has always a solution. Moreover, in [**6**, **3**] the same equation is studied in every T^2-fibration over a 2-torus.

The *Kodaira–Thurston manifold* is defined as the compact 4-manifold

$$M = \mathrm{Nil}^3/\Gamma \times S^1,$$

where Nil^3 is the 3-dimensional real Heisenberg group

$$\mathrm{Nil}^3 = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \; : \; x, y, z \in \mathbb{R} \right\}$$

and Γ is the lattice in Nil^3 of matrices having integers entries.

Therefore, M is parallelizable and has the global left-invariant coframe

$$e^1 = dy, \quad e^2 = dx, \quad e^3 = dt, \quad e^4 = dz - xdy \tag{2}$$

satisfying the structure equations

$$de^1 = de^2 = de^3 = 0, \quad de^4 = e^{12}, \tag{3}$$

with

$$e^{ij} = e^i \wedge e^j.$$

Since $\mathrm{Nil}^3/\Gamma \times S^1 = (\mathrm{Nil}^3 \times \mathbb{R})/(\Gamma \times \mathbb{Z})$, the Kodaira–Thurston manifold M is a 2-step nilmanifold and every left-invariant almost-Kähler structure on $\mathrm{Nil}^3 \times \mathbb{R}$ projects to an almost-Kähler structure on M. Moreover, the compact 3-dimensional manifold $N = \mathrm{Nil}^3/\Gamma$ is the total space of an S^1-bundle over a 2-dimensional torus T^2 with projection $\pi_{xy} \colon N \to T^2_{xy}$ and M inherits a structure of principal S^1-bundle over the 3-dimensional torus $T^3 = T^2_{xy} \times S^1_t$, i.e.,

$$\begin{array}{ccc} S^1 & \hookrightarrow & N \times S^1 = M \\ & & \downarrow \\ & & T^2 \times S^1 = T^3. \end{array}$$

Then it makes sense to consider differential forms invariant by the action of the fiber S^1_z. A k-form ϕ on M is invariant by the action of the fiber

S^1_z if its coefficients with respect to the global basis $e^{j_1}\wedge\cdots\wedge e^{j_k}$ do not depend on the variable z.

These observations allow us to extend the analysis in [**12**, **6**] from T^2-invariant to S^1-invariant data σ.

Consider on M the canonical metric

$$g=\sum_{k=1}^{4} e^k\otimes e^k \tag{4}$$

and the compatible symplectic form

$$\Omega = e^{13}+e^{42}.$$

The pair (Ω, g) specifies an almost-complex structure J making (Ω, J) an almost-Kähler structure. Observe that

$$Je^1=e^3 \quad \text{and} \quad Je^4=e^2.$$

Then we can consider the Calabi–Yau equation

$$(\Omega+d\alpha)^2=\mathrm{e}^F\,\Omega^2, \tag{5}$$

where the unknown α is a smooth 1-form on M such that

$$J(d\alpha)=d\alpha \tag{6}$$

and the datum F is a smooth function on M satisfying

$$\int_M \mathrm{e}^F\,\Omega^2=\int_M \Omega^2. \tag{7}$$

We have the following theorem.

Theorem 1. *The Calabi–Yau equation* (5) *has a unique solution* $\tilde\omega=\Omega+d\alpha$ *for every* S^1*-invariant volume form* $\sigma=\mathrm{e}^F\,\Omega^2$ *such that*

$$\int_{T^3}\mathrm{e}^F\,dV=1, \tag{8}$$

where dV *is the volume form* $dx\wedge dy\wedge dt$ *on* T^3*.*

Since uniqueness follows from a general result in [**5**], we need only to prove existence. This will be done in two steps. First, in Section 2 we reduce equation (5) to a fully nonlinear PDE on the 3-dimensional base torus T^3. Then, in Section 4 we show that such an equation is solvable. Section 3 concerns the *a priori* estimates needed in Section 4.

With some minor changes in the proof, it is possible to generalize Theorem 1 to the larger class of invariant almost-Kähler structures on the Kodaira–Thurston manifold. All positively oriented invariant almost-Kähler structures compatible with the canonical metric (4) can be obtained by rotating the symplectic form $\Omega=e^{13}+e^{42}$. Indeed, since the three forms

$$\Omega=e^{13}+e^{42}\,,\quad \Omega'=e^{14}+e^{23}\,,\quad \Omega''=e^{12}+e^{34}$$

are a basis of invariant self-dual 2-forms, every positively oriented invariant 2-form ω compatible with g can be written as

$$\omega = A\Omega + B\Omega' + C\Omega''$$

for some constants A, B, C satisfying $A^2 + B^2 + C^2 = 1$. The condition $d\omega = 0$ is equivalent to $C = 0$, and therefore every positively oriented symplectic 2-form compatible with g can be written as

$$\omega_\theta = (\cos\theta\, e^1 + \sin\theta\, e^2) \wedge e^3 - (-\sin\theta\, e^1 + \cos\theta\, e^2) \wedge e^4,$$

for some $\theta \in [0, 2\pi)$.

Theorem 2. *Assume either* $\cos\theta = 0$ *or* $\tan\theta \in \mathbb{Q}$. *Then the Calabi–Yau equation*

$$(\omega_\theta + d\alpha)^2 = \mathrm{e}^F \omega_\theta^2, \quad J_\theta(d\alpha) = 0$$

has a unique solution $\tilde{\omega} = \omega_\theta + d\alpha$ *for every* S^1*-invariant volume form* $\sigma = \mathrm{e}^F\, \omega_\theta^2$ *satisfying* (8).

In Section 5 we give some details on how to modify the proof of Theorem 1 in order to prove Theorem 2.

Observe that for $\theta = 0$, ω_0 is the form $\Omega = e^{13} + e^{42}$ considered in Theorem 1, while $\omega_{\pi/2} = e^{14} + e^{23}$ is the symplectic form Ω'.

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2. Reduction to a single elliptic equation

The dual frame of (2) is

$$e_1 = \partial_y + x\partial_z, \quad e_2 = \partial_x, \quad e_3 = \partial_t, \quad e_4 = \partial_z.$$

If u is S^1-invariant, it does not depend on z, and we have

$$e_1 u = \partial_y u = u_y, \quad e_2 u = \partial_x u = u_x, \quad e_3 u = \partial_t u = u_t, \quad e_4 u = 0.$$

It is convenient to set

$$\partial_1 = \partial_y, \quad \partial_2 = \partial_x, \quad \partial_3 = \partial_t, \tag{9}$$

so the differential can be written as

$$du = \sum_{i=1}^{3} \partial_i u\, e^i.$$

Theorem 3. *Given a smooth function* $u : T^3 \to \mathbb{R}$ *such that*

$$\int_{T^3} u\, dV = 0, \tag{10}$$

set

$$\alpha = d^c u - u e^1. \tag{11}$$

Then the 1*-form* (11) *satisfies equation* (6)*. Moreover,* α *solves equation* (5) *if and only if* u *is a solution to the fully non-linear PDE*

$$(u_{xx} + 1)(u_{yy} + u_{tt} + u_t + 1) - u_{xy}^2 - u_{xt}^2 = \mathrm{e}^F. \tag{12}$$

Proof. Thanks to (3) we have

$$\begin{aligned} dd^c u &= \sum_{i=1}^3 \sum_{j=1}^3 \partial_i \partial_j u\, e^i \wedge J e^j - \partial_2 u\, e^{12} \\ &= \sum_{i=1}^3 \sum_{j=1}^3 \partial_i \partial_j u\, e^i \wedge J e^j + d(u e^1) + \partial_3 u\, e^{13}. \end{aligned}$$

Therefore, $d\alpha$ is of type $(1,1)$ and

$$\begin{aligned} d\alpha &= \sum_{i=1}^3 \sum_{j=1}^3 \partial_i \partial_j u\, e^i \wedge J e^j + \partial_3 u\, e^{13} \\ &= (u_{yy} + u_{tt} + u_t) e^{13} - u_{xx} e^{24} + u_{xy}(e^{23} - e^{14}) + u_{xt}(e^{12} - e^{34}). \end{aligned}$$

Then a simple computation shows that α satisfies (5) if and only if u satisfies (12). q.e.d.

We end this section by proving ellipticity of equation (12).

First we fix some notation. Functions on the 3-torus can be identified with functions $u : \mathbb{R}^3 \to \mathbb{R}$ that are 1-periodic in each variable.

For any non-negative integer n, we denote by $C^n(T^3)$ the Banach space of C^n functions $u\colon T^3 \to \mathbb{R}$ equipped with norm

$$\|u\|_{C^n} = \max_{m \le n} |u|_{C^m},$$

where

$$|u|_{C^m} = \max_{|\kappa| = m} \sup_{q \in \mathbb{R}^3} \left|\partial^\kappa u(q)\right|.$$

Given $0 < \rho < 1$ and $u \in C^0(T^3)$, we set

$$\left[\!\left[u(q) \right]\!\right]_\rho = \sup_{0 < |h| \le 1} \left| u(q+h) - u(q) \right| \, |h|^{-\rho}.$$

Here we employ the multi-index notation $\partial^\kappa = \partial_1^{\kappa_1} \partial_2^{\kappa_2} \partial_3^{\kappa_3}$ and $|\kappa| = \kappa_1 + \kappa_2 + \kappa_3$.

For every non-negative integer n and real number $0 < \rho < 1$, define the space $C^{n+\rho}(T^3)$ of functions $u \in C^n(T^3)$ such that

$$|u|_{C^{n+\rho}} = \max_{|\kappa|=n} \sup_{q\in\mathbb{R}^3} \left[\!\left[\partial^\kappa u(q)\right]\!\right]_\rho < \infty.$$

$C^{n+\rho}(T^3)$ is a Banach space with respect to the norm

$$\|u\|_{C^{n+\rho}} = \max\Big\{ \|u\|_{C^n}\,, |u|_{C^{n+\rho}} \Big\}.$$

In conclusion, we have defined $C^\sigma(T^3)$ for every non-negative real number σ.

Finally, we denote by $\tilde{C}^\sigma(T^3)$ the closed subspace of all $u \in C^\sigma(T^3)$ satisfying

$$\int_{T^3} u\, dV = 0.$$

Proposition 1. *Let $u \in \tilde{C}^2(T^3)$ be a solution to* (12)*. Then we have*

$$u_{xx} > -1 \tag{13}$$

and

$$u_{yy} + u_{tt} + u_t > -1. \tag{14}$$

Proof. Indeed, from equation (12) we have

$$(u_{yy} + u_{tt} + u_t + 1)(u_{xx} + 1) \geq \mathrm{e}^F > 0.$$

This implies that $u_{yy} + u_{tt} + u_t + 1$ and $u_{xx} + 1$ have always the same sign. But at a point where u attains its minimum, we must have

$$u_{xx} + 1 \geq 1.$$

q.e.d.

Let

$$\Delta u = u_{xx} + u_{yy} + u_{tt}$$

be the standard Laplacian in $\mathbb{R}^3$.

Now we prove ellipticity of equation (12).

Proposition 2. *Let $u \in \tilde{C}^2(T^3)$ be a solution to equation* (12)*. Then we have*

$$0 < 2\mathrm{e}^{F/2} \leq \Delta u + u_t + 2 \tag{15}$$

and

$$\begin{aligned}(u_{xx} + 1)(\eta^2 + \tau^2) + (u_{yy} + u_{tt} + u_t + 1)\xi^2 - 2u_{xy}\xi\eta - 2u_{xt}\xi\tau \geq \\ \geq \Lambda(u)\big(\xi^2 + \eta^2 + \tau^2\big), \qquad \text{for all } (\xi,\eta,\tau) \in \mathbb{R}^3,\end{aligned} \tag{16}$$

where

$$\Lambda(u) = \frac{1}{2}\Big(\Delta u + u_t + 2 - \sqrt{(\Delta u + u_t + 2)^2 - 4\mathrm{e}^F}\Big). \tag{17}$$

REMARK. The left-hand side of (16) is the principal symbol of the linearization of (12) at the solution u. Since a non-linear equation is elliptic on a set S if its linearization at any $u \in S$ is elliptic, we have that equation (12) is elliptic on the set of all of its solutions $u \in \tilde{C}^2(T^3)$.

Proof. Inequality (15) follows from (13), (14), and (12).

A simple computation shows that the characteristic polynomial of the matrix

$$P(u) = \begin{bmatrix} u_{yy} + u_{tt} + u_t + 1 & u_{xy} & u_{xt} \\ u_{xy} & u_{xx} + 1 & 0 \\ u_{xt} & 0 & u_{xx} + 1 \end{bmatrix}$$

associated to the quadratic form on the left-hand side of (16) is

$$\big(\lambda - (u_{xx} + 1)\big)\big(\lambda^2 - (\Delta u + u_t + 2)\lambda + \mathrm{e}^F\big).$$

Then the eigenvalues of $P(u)$ are

$$\lambda_\pm = \frac{1}{2}\Big(\Delta u + u_t + 2 \pm \sqrt{(\Delta u + u_t + 2)^2 - 4\mathrm{e}^F}\Big)$$

and $u_{xx} + 1$. Since

$$\begin{aligned}(\Delta u + u_t + 2)^2 - 4\mathrm{e}^F &= \big((u_{yy} + u_{tt} + u_t + 1) - (u_{xx} + 1)\big)^2 + u_{xy}^2 + u_{xt}^2 \\ &\geq \big((\Delta u + u_t + 2) - 2(u_{xx} + 1)\big)^2,\end{aligned}$$

we have

$$\lambda_- \leq u_{xx} + 1 \leq \lambda_+,$$

and the proof is complete. q.e.d.

3. *A priori* estimates

3.1. C^0-estimate.

Proposition 3. *We have*

$$|u_x| \leq 1, \tag{18}$$

for all solution u to (12).

Proof. Fix $(x, y, t) \in \mathbb{R}^3$, and consider the periodic function

$$v(s) = u(x + s, y, t).$$

We have

$$v''(s) = u_{xx}(x + s, y, t) \geq -1.$$

Let $s_0 \in [0, 1]$ be a critical point of v. Then we have

$$v'(s) = \int_{s_0}^{s} v''(r)\, r \begin{cases} \geq -(s - s_0) \geq -1, & s_0 \leq s \leq s_0 + 1, \\ \leq -(s - s_0) \leq 1, & s_0 - 1 \leq s \leq s_0. \end{cases}$$

By periodicity we get that these estimates hold everywhere; in particular, we obtain

$$|u_x(x, y, t)| = |v'(0)| \leq 1.$$

q.e.d.

Denote by

$$\nabla u = \begin{bmatrix} u_x \\ u_y \\ u_t \end{bmatrix}$$

the standard gradient of u. We have

$$|\nabla u|^2 = u_x^2 + u_y^2 + u_t^2$$

thus, if we set

$$|\nabla u|_{C^0} = \big||\nabla u|\big|_{C^0},$$

we have

$$|u|_{C^1} \leq |\nabla u|_{C^0} \leq \sqrt{3}\; |u|_{C^1}\,.$$

In this paper all L^p norms are taken on the torus T^3. In particular, we set

$$\|\nabla u\|_{L^2}^2 = \int_{T^3} |\nabla u|^2 \; dV = \int_{T^3} (u_x^2 + u_y^2 + u_t^2)\, dV.$$

Theorem 4. *Given a real number $p \geq 2$, we have*

$$\big\|\nabla\, |u|^{p/2}\big\|_{L^2}^2 \leq \frac{p^2}{16}\; \|u\|_{L^p}^p + \frac{5p^3}{16}\; \big|1 + \mathrm{e}^F\big|_{C^0}\, \|u\|_{L^p}^{p-1}\,, \tag{19}$$

for all $u \in \tilde{C}^2(T^3)$ satisfying equation (12).

Proof. From Theorem 3 we have that

$$\alpha = d^c u - u e^1 \tag{20}$$

solves equation (5), which can be rewritten as

$$(\mathrm{e}^F - 1)\, \Omega^2 = d\alpha \wedge (\Omega + \tilde{\Omega}),$$

where

$$\tilde{\Omega} = \Omega + d\alpha.$$

Since

$$\begin{aligned} d\big(u\, |u|^{p-2}\big) &= |u|^{p-2}\; du + u\, (p-2)\, |u|^{p-3}\, \frac{u}{|u|}\, du \\ &= (p-1)\, |u|^{p-2}\; du, \qquad \text{for } u \neq 0, \end{aligned}$$

we have

$$\begin{aligned} &\int_{T^3} d\Big(\big(u\, |u|^{p-2}\, \alpha\big) \wedge (\Omega + \tilde{\Omega})\Big) = \\ &= (p-1) \int_{T^3} |u|^{p-2}\; du \wedge \alpha \wedge (\Omega + \tilde{\Omega}) + \int_{T^3} |u|^{p-2}\, u (\mathrm{e}^F - 1)\, \Omega^2, \end{aligned} \tag{21}$$

and Stokes' theorem implies

$$\int_{T^3} |u|^{p-2}\; du \wedge \alpha \wedge (\Omega + \tilde{\Omega}) = \frac{1}{p-1} \int_{T^3} (1 - \mathrm{e}^F)\, |u|^{p-2}\, u\, \Omega^2. \tag{22}$$

Taking into account that

$$\tilde\Omega = (u_{yy} + u_{tt} + u_t + 1)e^{13} - (u_{xx} + 1)e^{24}, \\ + u_{xy}(e^{23} - e^{14}) + u_{xt}(e^{12} - e^{34}), \tag{23}$$

we have

$$du \wedge \alpha \wedge \Omega = \frac{1}{2}\left(u_x^2 + u_y^2 + u_t(u_t + u)\right)\Omega^2 \tag{24}$$

and

$$\begin{aligned} du \wedge \alpha \wedge \tilde\Omega =& \frac{1}{2}\left(u_y^2 + \left(u_t + \frac{1}{2}\,u\right)^2\right)(u_{xx} + 1)\Omega^2 \\ &+ \frac{1}{2}\,u_x^2(u_{yy} + u_{tt} + u_t + 1)\Omega^2 \\ &- \left(u_x u_y u_{xy} + u_x\left(u_t + \frac{1}{2}\,u\right)u_{xt}\right)\Omega^2 \\ &- \frac{1}{8}\,u^2(u_{xx} + 1)\Omega^2. \end{aligned} \tag{25}$$

Thanks to (16), we obtain from (25) that

$$du \wedge \alpha \wedge \tilde\Omega \geq -\frac{1}{8}\,u^2(u_{xx} + 1)\Omega^2.$$

Then from (22) and (24) we get

$$\begin{aligned} \int_{T^3} |u|^{p-2}\left(u_x^2 + u_y^2 + u_t(u_t + u)\right) dV \leq& \\ \leq \frac{1}{4}\int_{T^3} |u|^p\,(u_{xx} + 1)\,dV &+ \frac{2}{p-1}\int_{T^3} (1 - \mathrm{e}^F)\,|u|^{p-2}\,u\,dV. \end{aligned} \tag{26}$$

An integration by parts gives

$$\int_{T^3} |u|^{p-2}\,uu_t\,dV = (1-p)\int_{T^3} |u|^{p-2}\,uu_t\,dV,$$

and therefore we have

$$\int_{T^3} |u|^{p-2}\,uu_t\,dV = 0.$$

Since, moreover,

$$\int_{T^3} |u|^p\,u_{xx}\,dV = -p\int_{T^3} |u|^{p-2}\,uu_x^2\,dV,$$

estimates (18) and (26) imply

$$\begin{aligned} \int_{T^3} |u|^{p-2}\,|\nabla u|^2\;dV \leq& \frac{1}{4}\int_{T^3} |u|^p\;dV + \\ &+ \left(\frac{p}{4} + \frac{2}{p-1}\;\left|1 - \mathrm{e}^F\right|_{C^0}\right)\int_{T^3} |u|^{p-1}\;dV. \end{aligned} \tag{27}$$

But the left-hand side can be rewritten as

$$\int_{T^3} |u|^{p-2} |\nabla u|^2 \, dV = \frac{4}{p^2} \int_{T^3} \left|\nabla |u|^{p/2}\right|^2 dV.$$

Moreover,

$$\frac{p}{4} + \frac{2}{p-1} \left|1 - \mathrm{e}^F\right|_{C^0} \leq \frac{5p}{4} \left|1 + \mathrm{e}^F\right|_{C^0}, \qquad \text{for } p \geq 2;$$

then (27) becomes

(28)
$$\int_{T^3} \left|\nabla |u|^{p/2}\right|^2 dV \leq \frac{p^2}{16} \int_{T^3} |u|^p \, dV + \frac{5p^3}{16} \left|1 + \mathrm{e}^F\right|_{C^0} \int_{T^3} |u|^{p-1} \, dV.$$

Since T^3 has measure 1, we have

(29) $$\|u\|_{L^{p-1}} \leq \|u\|_{L^p}.$$

Estimate (19) follows from (28) and (29). q.e.d.

It is rather natural to compare estimate (19) with the classical *a priori* Yau's estimate

$$\left\|\nabla |\varphi|^{p/2}\right\|_{L^2}^2 \leq \frac{mp^2}{4p-1} \left(\left|1 - \mathrm{e}^F\right|_{C^0}\right) \|\varphi\|_{L^p}^{p-1}$$

involving the solutions φ to the complex Monge–Ampère equation $(\omega + dd^c\varphi)^m = \mathrm{e}^F \, \omega^m$ in $2m$-dimensional Kähler manifolds (see, for instance, [**8**, Proposition 5.4.1]). The right-hand side of (19) contains the extra term $\frac{p^2}{16} \|u\|_{L^p}^p$ due to the presence of $-ue^1$ in (11). This is a problem in the first step of the C^0-estimate, i.e., with $p = 2$. We take care of this in the next proposition.

From the Strong Maximum Principle Δu constant implies u constant, and then $-\Delta$ is an operator from $\tilde{C}^2(T^3)$ into $\tilde{C}^0(T^3)$. As such, its first eigenvalue is $4\pi^2$. This implies the inequality

(30) $$4\pi^2 \|u\|_{L^2}^2 \leq \int_{T^3} -\Delta u \, u \, dV = \|\nabla u\|_{L^2}^2, \qquad \text{for all } u \in \tilde{C}^2(T^3).$$

Proposition 4. *We have*

(31) $$\|u\|_{L^2} \leq \left|1 + \mathrm{e}^F\right|_{C^0},$$

for all $u \in \tilde{C}^2(T^3)$ satisfying equation (12).

Proof. Since

$$\left\|\nabla |u|\right\|_{L^2}^2 = \|\nabla u\|_{L^2}^2,$$

from (19) with $p = 2$ and (30) we obtain

$$4\pi^2 \|u\|_{L^2}^2 \leq \frac{1}{4} \|u\|_{L^2}^2 + \frac{5}{2} \left|1 + \mathrm{e}^F\right|_{C^0} \|u\|_{L^2},$$

which implies (31). q.e.d.

Now we are ready to prove an a priori C^0 estimate for the solutions to (12):

Theorem 5. *Given $F \in C^2(T^3)$ satisfying condition* (8)*, there exists a positive constant C_0, depending only on $|F|_{C^0}$, such that*

$$|u|_{C^0} \le C_0, \tag{32}$$

for all $u \in \tilde{C}^2(T^3)$ satisfying equation (12)*.*

Proof. From the Sobolev Imbedding Theorem (see, for instance, [**2**, Theorem 5.4]), there exists a positive constant K such that

$$\|w\|^2_{L^6} \le K\Big(\|w\|^2_{L^2} + \|\nabla w\|^2_{L^2}\Big), \tag{33}$$

for all w in the Sobolev space $W^{1,2}(T^3)$.

Then from (19) and (33) we have

$$\begin{aligned} \|u\|^p_{L^{3p}} &\le K\Big(1 + \frac{p^2}{16}\Big)\|u\|^p_{L^p} + K\frac{5p^3}{16}\left|1 + \mathrm{e}^F\right|_{C^0}\|u\|^{p-1}_{L^p} \\ &\le K\, p^3\, \|u\|^p_{L^p}\Big(1 + \left|1 + \mathrm{e}^F\right|_{C^0}\|u\|^{-1}_{L^2}\Big), \qquad \text{for all } p \ge 2. \end{aligned} \tag{34}$$

It follows that

$$\frac{\|u\|_{L^{3p_k}}}{\|u\|_{L^{p_k}}} \le (Mp_k^3)^{1/p_k}, \qquad \text{for all } k \in \mathbb{Z}_+,$$

with

$$M = K\left(1 + \left|1 + \mathrm{e}^F\right|_{C^0}\|u\|^{-1}_{L^2}\right) \tag{35}$$

and

$$p_k = 2 \cdot 3^k.$$

Then

$$\frac{\|u\|_{L^{3p_n}}}{\|u\|_{L^2}} \le \prod_{k=0}^{n}(Mp_k^3)^{1/p_k}, \qquad \text{for all } n \in \mathbb{Z}_+.$$

But

$$\prod_{k=0}^{\infty}(Mp_k^3)^{1/p_k} = \exp\left(\sum_{k=0}^{\infty}\frac{1}{2\cdot 3^k}\Big(\log(8M) + 3k\log 3\Big)\right) = (8M)^{3/4}3^{3\mu/2},$$

with

$$\mu = \sum_{k=1}^{\infty}\frac{k}{3^k} < \infty.$$

Then

$$|u|_{C^0} = \sup_{n\in\mathbb{N}}\|u\|_{L^{p_n}} \le (8M)^{3/4}3^{3\mu/2}\,\|u\|_{L^2}\,. \tag{36}$$

Now from (35) and (31) we have

$$M^{3/4}\left\|u\right\|_{L^2} = K^{3/4}\left(\left\|u\right\|_{L^2} + \left|1+\mathrm{e}^F\right|_{C^0}\right)^{3/4}\left\|u\right\|_{L^2}^{1/4} \leq (2K)^{3/4}\left|1+\mathrm{e}^F\right|_{C^0},$$

and (32) follows from (36). q.e.d.

3.2. Estimate of gradient and Laplacian.

We make use of the tensor product notation. In particular, $(\nabla\otimes\nabla)u$ is the Hessian matrix of u, and $\mathrm{tr}(\nabla\otimes\nabla)=\Delta$ is the Laplacian.

Observe that

$$(\nabla\otimes\nabla)(uv) = v\,(\nabla\otimes\nabla)u + u\,(\nabla\otimes\nabla)v + (\nabla u\otimes\nabla v) + (\nabla v\otimes\nabla u).$$

Theorem 6. *Given $F\in C^2(T^3)$ satisfying condition (8), there exists a positive constant C_1, depending only on $\|F\|_{C^2}$, such that*

$$|\Delta u|_{C^0} \leq C_1\big(1+|u|_{C^1}\big), \tag{37}$$

for all $u\in\tilde{C}^4(T^3)$ satisfying equation (12).

Proof. From equation (12) we obtain

$$\begin{aligned}(38)\quad &\big(\Delta F + |\nabla F|^2 + F_t\big)\mathrm{e}^F = \\ &= (u_{yy}+u_{tt}+u_t+1)(\Delta u_{xx}+u_{xxt}) \\ &\quad + (u_{xx}+1)(\Delta u_{yy}+u_{yyt}+\Delta u_{tt}+u_{ttt}) \\ &\quad + (u_{xx}+1)(\Delta u_t+u_{tt}) + 2\nabla u_{xx}\cdot\nabla(u_{yy}+u_{tt}+u_t) \\ &\quad - 2u_{xy}(\Delta u_{xy}+u_{xyt}) - 2|\nabla u_{xy}|^2 - 2u_{xt}(\Delta u_{xt}+u_{xtt}) - 2|\nabla u_{xt}|^2.\end{aligned}$$

Consider

$$\Phi = (\Delta u + u_t + 2)\mathrm{e}^{-\mu u}, \tag{39}$$

where

$$\mu = \frac{\epsilon}{\max(\Delta u + u_t + 2)} \tag{40}$$

and $0<\epsilon<1$ is a constant to be chosen later. Differentiating (39) yields

$$\nabla\Phi = \mathrm{e}^{-\mu u}\Big(\nabla(\Delta u + u_t) - \mu(\Delta u + u_t + 2)\nabla u\Big)$$

and

$$\begin{aligned}(\nabla\otimes\nabla)\Phi = &-\mu\mathrm{e}^{-\mu u}\Big(\nabla u\otimes\nabla(\Delta u+u_t) + \nabla(\Delta u+u_t)\otimes\nabla u\Big) \\ &+ \mu^2\mathrm{e}^{-\mu u}\Big((\Delta u+u_t+2)\nabla u\otimes\nabla u\Big) + \\ &+ \mathrm{e}^{-\mu u}\Big((\nabla\otimes\nabla)(\Delta u+u_t) - \mu(\Delta u+u_t+2)(\nabla\otimes\nabla)u\Big).\end{aligned}$$

Consider now a point (x_0,y_0,t_0), where Φ attains its maximum value.

We have $\nabla\Phi = 0$ and $(\nabla \otimes \nabla)\Phi \leq 0$, so that

$$\nabla(\Delta u + u_t) = \mu(\Delta u + u_t + 2)\nabla u, \tag{41}$$

and

$$(\nabla \otimes \nabla)(\Delta u + u_t) \leq \mu(\Delta u + u_t + 2)\Big((\nabla \otimes \nabla)u + \mu\nabla u \otimes \nabla u\Big). \tag{42}$$

In particular, we obtain

$$\begin{aligned}\Big(\mu(\Delta u + u_t + 2)(u_{xy} + \mu u_x u_y) - (\Delta u_{xy} + u_{xyt})\Big)^2 \leq \\ \leq \Big(\mu(\Delta u + u_t + 2)(u_{xx} + \mu u_x^2) - (\Delta u_{xx} + u_{xxt})\Big)\cdot \\ \cdot \Big(\mu(\Delta u + u_t + 2)(u_{yy} + \mu u_y^2) - (\Delta u_{yy} + u_{yyt})\Big)\end{aligned} \tag{43}$$

and

$$\begin{aligned}\Big(\mu(\Delta u + u_t + 2)(u_{xt} + \mu u_x u_t) - (\Delta u_{xt} + u_{xtt})\Big)^2 \leq \\ \leq \Big(\mu(\Delta u + u_t + 2)(u_{xx} + \mu u_x^2) - (\Delta u_{xx} + u_{xxt})\Big)\cdot \\ \cdot \Big(\mu(\Delta u + u_t + 2)(u_{tt} + \mu u_t^2) - (\Delta u_{tt} + u_{ttt})\Big).\end{aligned} \tag{44}$$

From (42) we have, in particular, that

$$\mu(\Delta + u_t + 2)(\partial_i\partial_j u + \mu\partial_i u\partial_j u) - (\Delta\partial_i\partial_j u + \partial_t\partial_i\partial_j u) \geq 0,$$

for all $1 \leq i, j \leq 3$. Then, form (43), (44), and (16) with

$$\begin{cases} \xi = \Big(\mu(\Delta u + u_t + 2)(u_{xx} + \mu u_x^2) - (\Delta u_{xx} + u_{xxt})\Big)^{1/2}, \\ \eta = \Big(\mu(\Delta u + u_t + 2)(u_{yy} + \mu u_y^2) - (\Delta u_{yy} + u_{yyt})\Big)^{1/2}, \\ \tau = \Big(\mu(\Delta u + u_t + 2)(u_{tt} + \mu u_t^2) - (\Delta u_{tt} + u_{ttt})\Big)^{1/2}, \end{cases}$$

we obtain

$$\begin{aligned}&(u_{yy} + u_{tt} + u_t + 1)(\Delta u_{xx} + u_{xxt}) \\ &+ (u_{xx} + 1)(\Delta u_{yy} + u_{yyt} + \Delta u_{tt} + u_{ttt}) \\ &- 2u_{xy}(\Delta u_{xy} + u_{xyt}) - 2u_{xt}(\Delta u_{xt} + u_{xtt}) \leq \\ &\leq \mu(\Delta u + u_t + 2)(u_{yy} + u_{tt} + u_t + 1)(u_{xx} + \mu u_x^2) \\ &\quad + \mu(\Delta u + u_t + 2)(u_{xx} + 1)(u_{yy} + \mu u_y^2 + u_{tt} + \mu u_t^2) \\ &\quad - 2\,\mu(\Delta u + u_t + 2)\Big(u_{xy}(u_{xy} + \mu u_x u_y) + u_{xt}(u_{xt} + \mu u_x u_t)\Big).\end{aligned} \tag{45}$$

Substituting (41) and (45) into (38), and using (15), we get

$$
\begin{aligned}
(46)\quad & \big(\Delta F + |\nabla F|^2 + F_t\big)\mathrm{e}^F \leq \\
& \leq \mu(\Delta u + u_t + 2)(u_{yy} + u_{tt} + u_t + 1)(u_{xx} + \mu u_x^2) \\
& \quad + \mu(\Delta u + u_t + 2)(u_{xx} + 1)\big(u_{yy} + u_{tt} + \mu(u_y^2 + u_t^2)\big) \\
& \quad + \mu(\Delta u + u_t + 2)(u_{xx} + 1)u_t + 2\nabla u_{xx} \cdot \nabla(u_{yy} + u_{tt} + u_t) \\
& \quad - 2\mu(\Delta u + u_t + 2)\Big(u_{xy}(u_{xy} + \mu u_x u_y) + u_{xt}(u_{xt} + \mu u_x u_t)\Big).
\end{aligned}
$$

On the other side, from (41) we have

$$
\begin{aligned}
(47)\quad & \mu^2(\Delta u + u_t + 2)^2 |\nabla u|^2 = |\nabla(\Delta u + u_t)|^2 = \\
& = |\nabla u_{xx}|^2 + |\nabla(u_{yy} + u_{tt} + u_t)|^2 + 2\nabla u_{xx} \cdot \nabla(u_{yy} + u_{tt} + u_t) \\
& \geq 2\nabla u_{xx} \cdot \nabla(u_{yy} + u_{tt} + u_t).
\end{aligned}
$$

Eventually, from (46) and (47) we obtain

$$
\begin{aligned}
(48)\quad & \big(\Delta F + |\nabla F|^2 + F_t\big)\mathrm{e}^F \leq \\
\leq & \mu(\Delta u + u_t + 2)\Big((u_{yy} + u_{tt} + u_t + 1)u_{xx} + (u_{xx} + 1)(u_{yy} + u_{tt} + u_t)\Big) \\
& - 2\,\mu(\Delta u + u_t + 2)(u_{xy}^2 + u_{xt}^2) \\
& + 2\,\mu^2(\Delta u + u_t + 2)\Big((u_{yy} + u_{tt} + u_t + 1)u_x^2 + (u_{xx} + 1)(u_y^2 + u_t^2)\Big) \\
& + \mu^2(\Delta u + u_t + 2)^2 |\nabla u|^2 \\
\leq & 2\mu(\Delta u + u_t + 2)\mathrm{e}^F - \mu(\Delta u + u_t + 2)^2 + \mu^2(\Delta u + u_t + 2)^2 |\nabla u|^2 .
\end{aligned}
$$

Set

$$M = \Delta u(x_0, y_0, t_0) + u_t(x_0, y_0, t_0) + 2$$

and

$$u_0 = u(x_0, y_0, t_0),$$

so that

$$\max \Phi = M\mathrm{e}^{-\mu u_0}.$$

From (48) we get

$$(49)\qquad \mu M^2 \leq \left|(\Delta F + F_t)\mathrm{e}^F\right|_{C^0} + 2\mu M \left|\mathrm{e}^F\right|_{C^0} + \mu^2 M^2 |\nabla u|^2_{C^0} .$$

Denote by $\tilde{u}$ the value of u at a point where $\Delta u + u_t + 2$ attains its maximum value. Then, thanks to Theorem 5, we have

$$(50)\qquad M \leq \max(\Delta u + u_t + 2) \leq M\mathrm{e}^{\mu(\tilde{u} - u_0)} \leq M\mathrm{e}^{2\mu C_0}.$$

Moreover, (40) and (15) imply

$$2\mu = \frac{2\epsilon}{\max(\Delta u + u_t + 2)} \leq \epsilon\,\mathrm{e}^{-\min F/2} \leq \mathrm{e}^{-\min F/2},$$

and then (50) yields

$$\epsilon \exp\Big(- \mathrm{e}^{-\min F/2}\, C_0\Big) \leq \mu M \leq \epsilon \tag{51}$$

and

$$\exp\Big(- \mathrm{e}^{-\min F/2}\, C_0\Big) \max(\Delta u + u_t + 2) \leq M. \tag{52}$$

Eventually, from (49), (51), and (52) we obtain

$$\begin{aligned}\epsilon \exp\Big(- 2\mathrm{e}^{-\min F/2}\, C_0\Big) &\max(\Delta u + u_t + 2) \leq \\ &\leq \left|(\Delta F + F_t)\mathrm{e}^F\right|_{C^0} + 2\epsilon \left|\mathrm{e}^F\right|_{C^0} + \epsilon^2 \left|\nabla u\right|^2_{C^0},\end{aligned}$$

i.e.,

$$\begin{aligned}&\max(\Delta u + u_t + 2) \leq \\ &\leq \exp\Big(2\mathrm{e}^{-\min F/2}\, |u|_{C^0}\Big)\Big(\frac{1}{\epsilon}\left|(\Delta F + F_t)\mathrm{e}^F\right|_{C^0} + 2\left|\mathrm{e}^F\right|_{C^0} + 3\epsilon \left|\nabla u\right|^2_{C^0}\Big).\end{aligned} \tag{53}$$

Since

$$|\Delta u|_{C^0} \leq \max(\Delta u + u_t + 2) + |\nabla u|_{C^0} + 2,$$

estimate (37) follows from (53), with

$$\epsilon = \frac{1}{1 + |\nabla u|_{C^0}}.$$

q.e.d.

To prove the next theorem, we need the following estimate.

Proposition 5. *Given $0 < \mu < 1$, there exists a positive K_0, depending only on μ, such that*

$$|u|_{C^{1+\mu}} \leq K_0\Big(\|u\|_{C^0} + |\Delta u|_{C^0}\Big), \qquad \text{for all } u \in C^2(T^3). \tag{54}$$

Proof. Let $p = \frac{3}{1-\mu}$. Since $p > 3$, the Morrey inequality gives

$$|u|_{C^{1+\mu}} \leq C \, \|u\|_{W^{2,p}},$$

where the constant C depends only on μ. On the other hand, elliptic L^p estimates for the Laplacian give

$$\|u\|_{W^{2,p}} \leq C'\big(\|u\|_{L^p} + \|\Delta u\|_{L^p}\big),$$

where again C' depends only on μ.

Finally, if $u \in C^2(T^3)$, we have

$$\|u\|_{L^p} + \|\Delta u\|_{L^p} \leq |u|_{C^0} + |\Delta u|_{C^0}.$$

q.e.d.

Theorem 7. *Consider $F \in C^2(T^3)$ satisfying condition (8). Then there exists a positive constant C_2, depending only on $\|F\|_{C^2}$, such that*

$$|u|_{C^1} \leq C_2, \tag{55}$$

for all $u \in \tilde{C}^4(T^3)$ satisfying equation (12).

Proof. Let $0 < \mu < 1$. Thanks to standard interpolation theory (see [**7**, section 6.8]), for all $\epsilon > 0$ there exists a positive constant M_ϵ such that

$$|u|_{C^1} \leq M_\epsilon \, |u|_{C^0} + \epsilon \, |u|_{C^{1+\mu}} \,, \qquad \text{for all } u \in C^{1+\mu}(T^3).$$

Then, thanks to Theorem 5 and Proposition 5, we have

$$\begin{aligned} |u|_{C^1} &\leq M_\epsilon C_0 + \epsilon K_0 \Big(C_0 + |u|_{C^1} + |\Delta u|_{C^0} \Big) \\ &\leq M_\epsilon C_0 + \epsilon K_0 \Big(C_0 + |u|_{C^1} + C_1(1 + |u|_{C^1}) \Big) \\ &= M_\epsilon C_0 + \epsilon K_0 (C_0 + C_1) + \epsilon K_0 (1 + C_1) \, |u|_{C^1} \,, \end{aligned}$$

which implies (55), if we choose

$$\epsilon < \frac{1}{K_0(1 + C_1)}.$$

q.e.d.

Corollary 1. *Under the hypotheses of Theorem* 7*, we have that equation* (12) *is uniformly elliptic on the set* $\mathcal{S}$ *of all solutions* $u \in \tilde{C}^4(T^3)$*, in the sense that*

$$\inf_{u \in \mathcal{S}} \Lambda(u) > 0,$$

where Λ *is defined in* (17)*.*

Proof. It follows from Proposition 2 and Theorems 5 and 7. q.e.d.

3.3. $C^{2+\rho}$-estimate.

We begin by recalling a theorem of [**14**], which greatly simplifies the estimate of derivatives up to second order. In [**14**] the theorem has been stated locally, but on compact manifolds it holds globally.

Theorem 8 ([**14**, Theorem 5.1])**.** *Let* $\tilde{\Omega}$ *be be the solution of the Calabi–Yau equation*

$$\tilde{\Omega}^n = \mathrm{e}^F \Omega^n, \qquad [\tilde{\Omega}] = [\Omega],$$

on a compact almost-Kähler manifold (M^{2n}, Ω, J)*.*

Assume there are two constants $\tilde{C}_0 > 0$ *and* $0 < \rho_0 < 1$ *such that* $F \in C^{\rho_0}(M^{2n})$ *and*

$$\operatorname{tr} \tilde{g} \leq \tilde{C}_0,$$

where $\tilde{g}$ *is the Riemannian metric associated to* $\tilde{\Omega}$*.*

Then there exist two constants $\tilde{C} > 0$ *and* $0 < \rho < 1$*, depending only on* M^{2n}*,* Ω*,* J*,* C_0 *and* $\|F\|_{C^{\rho_0}}$*, such that*

$$\|\tilde{g}\|_{C^\rho} \leq \tilde{C}.$$

Using this Theorem we easily prove the following estimate.

Theorem 9. *Given $F \in C^2(T^3)$ satisfying condition (8), there exist constants $C_3 > 0$ and $\rho > 0$, both depending only on $\|F\|_{C^2}$, such that*

$$\|u\|_{C^{2+\rho}} \leq C_3, \tag{56}$$

for all $u \in \tilde{C}^4(T^3)$ satisfying equation (12).

Proof. From (23) we obtain that the Riemannian metric $\tilde{g}$ is represented by the matrix

$$\tilde{g} = \begin{bmatrix} u_{yy} + u_{tt} + u_t + 1 & u_{xy} & 0 & u_{xt} \\ u_{xy} & u_{xx} + 1 & u_{xt} & 0 \\ 0 & u_{xt} & u_{yy} + u_{tt} + u_t + 1 & -u_{xy} \\ u_{xt} & 0 & -u_{xy} & u_{xx} + 1 \end{bmatrix}.$$

Then

$$\operatorname{tr} \tilde{g} = 2(\Delta u + u_t + 2).$$

Thanks to Theorems 5 and 7 we can apply Theorem 8 and get that

$$\max\left\{\|1 + u_{xx}\|_{C^\rho}, \|1 + u_{yy} + u_{tt} + u_t\|_{C^\rho}, \|u_{xy}\|_{C^\rho}, \|u_{xt}\|_{C^\rho}\right\} \leq \tilde{C}, \tag{57}$$

where $\tilde{C}$ depends only on $\|F\|_{C^2}$.

Now the estimates of second-order derivatives can be obtained as follows. Given a solution u of equation (12), we have that u can be viewed as a solution to the linear PDE

$$Pu_{xx} + Q(u_{yy} + u_{tt}) - 2Ru_{xy} - 2Su_{xt} + Qu_t = f \tag{58}$$

with

$$P = u_{yy} + u_{tt} + u_t + 1, \quad Q = u_{xx} + 1, \quad R = u_{xy}, \quad S = u_{xt},$$

and

$$f = 2\mathrm{e}^F - (\Delta u + u_t + 2).$$

Thanks to Proposition 2, Corollary 1 and estimate (57), standard Schauder theory gives the estimate (56). q.e.d.

4. Proof of Theorem 1

Proposition 6. *Assume $u \in \tilde{C}^{2+\rho}(T^3)$ is a solution to equation (12) with $\rho > 0$. If $F \in C^\infty(T^3)$, then $u \in \tilde{C}^\infty(T^3)$.*

Proof. From Proposition 2 we have that equation (12) is elliptic. Then from [**10**, Theorem 4.8, Chapter 14], it follows that u belongs to the Sobolev space $W^{n,2}(T^3)$, for all $n \in \mathbb{Z}_+$. But this implies that $u \in C^\infty(T^3)$. q.e.d.

Thanks to Theorem 3, Theorem 1 is an immediate consequence of the following theorem.

Theorem 10. *Let $F \in C^\infty(T^3)$ satisfy (8). Then equation (12) has a solution $u \in \tilde{C}^\infty(T^3)$.*

Proof. We apply the continuity method (see [**7**, Section 17.2]). For $0 \le \tau \le 1$, let

(59)
$$\mathfrak{S}_\tau = \Big\{ u \in \tilde{C}^\infty(T^3) \ : \ (u_{yy} + u_{tt} + u_t + 1)(u_{xx} + 1) - u_{xy}^2 - u_{xt}^2 = \mathrm{e}^{F_\tau} \Big\},$$

where

$$F_\tau = \log(1 - \tau + \tau\, \mathrm{e}^F).$$

Note that $0 \in \mathfrak{S}_0$ and that $\mathfrak{S}_1$ consists of the solutions to (12) lying in $\tilde{C}^\infty(T^3)$. Since

$$\max_{0 \le \tau \le 1} \|F_\tau\|_{C^2} < \infty,$$

and

$$\int_{T^3} \mathrm{e}^{F_\tau}\, dV = \int_{T^3} \big(1 - \tau + \tau\, \mathrm{e}^F\big)\, dV = 1,$$

by Theorem 9 there exists a real number $\rho > 0$ such that

$$\sup_{u \in \mathfrak{S}} \|u\|_{C^{2+\rho}} < \infty, \tag{60}$$

with

$$\mathfrak{S} = \bigcup_{0 \le \tau \le 1} \mathfrak{S}_\tau \neq \emptyset.$$

Since $0 \in \mathfrak{S}_0$, the set $\big\{\tau \in [0,1] \ : \ \mathfrak{S}_\tau \neq \emptyset\big\}$ is not empty and we can define

$$\mu = \sup\big\{\tau \in [0,1] \ : \ \mathfrak{S}_\tau \neq \emptyset\big\}.$$

In order to compete the proof we have to show that $\mathfrak{S}_\mu \neq \emptyset$ and $\mu = 1$.

- $\mathfrak{S}_\mu \neq \emptyset$. By the definition of μ there exist two sequences $(\tau_k) \subset [0,1]$ and $(u_k) \subset \tilde{C}^\infty(T^3)$ such that (μ_k) is increasing and $u_k \in \mathfrak{S}_{\tau_k}$ for all k. Thanks to (60), the sequence (u_k) is bounded in $\tilde{C}^\rho(T^3)$; then by the Ascoli–Arzelà Theorem there exists a subsequence (u_{k_j}) convergent in $\tilde{C}^{2+\rho/2}(T^3)$. Let $v = \lim u_{k_j}$. Then v belongs to $\tilde{C}^{2+\rho/2}(T^3)$ and satisfies the equation

  $$(v_{yy} + v_{tt} + v_b + 1)(v_{xx} + 1) - v_{xy}^2 - v_{xt}^2 = \mathrm{e}^{F_\mu}.$$

  By Proposition 6 v belongs to $\tilde{C}^\infty(T^3)$. In particular, v belongs to $\mathfrak{S}_\mu$, which turns out to be not empty.
- $\mu = 1$. Assume by contradiction $\mu < 1$, and define the non-linear C^∞ operator

  $$\begin{cases} T : \tilde{C}^\rho(T^3) \times [0,1] \to \tilde{C}^{\rho-2}(T^3), \\ T(u,\tau) = (u_{yy} + u_{tt} + u_t + 1)(u_{xx} + 1) - u_{xy}^2 - u_{xt}^2 - \mathrm{e}^{F_\tau}. \end{cases}$$

  Observe that the condition $\int_{T^3} T(u,\tau)\, dV = 0$ follows from the identities $\int_{T^3} (u_{xy}^2 + u_{xt}^2)\, dV = \int_{T^3} (u_{yy} + u_{tt}) u_{xx}\, dV$ and $\int_{T^3} \mathrm{e}^{F_\tau}\, dV = 1$.

Since $\mathfrak{S}_\mu$ is not empty, there exists $v \in \mathfrak{S}_\mu$ such that $T(v, \mu) = 0$. Compute

$$\partial_1 T(v, \mu) w = Lw,$$

where

$$Lw = P w_{xx} + Q(w_{yy} + w_{tt}) - 2R w_{xy} - 2S w_{xt} + Q w_t = f$$

with

$$P = v_{yy} + v_{tt} + v_t + 1, \quad Q = v_{xx} + 1, \quad R = v_{xy}, \quad S = v_{xt},$$

Since $v \in \mathfrak{S}_\mu$, we know that $L : \tilde{C}^{2+\rho}(T^3) \to \tilde{C}^{\rho}(T^3)$ is elliptic. Then by the Strong Maximum Principle $L = 0$ implies that u is constant. This shows that L is is one-to-one on $\tilde{C}^{2+\rho}$. Moreover, by ellipticity, L has closed range, and thus Schauder Theory and the Continuity Method (see [**7**, Theorem 5.2]) show that L is onto. Therefore, by the Implicit Function Theorem there exists an $\epsilon > 0$ such that

$$T(u, \tau) = 0$$

is solvable with respect to u for every $\tau \in (\mu - \epsilon, \mu + \epsilon)$. Thanks to Proposition 6, these solutions belong to $\tilde{C}^\infty(T^3)$. Then $\mathfrak{S}_\tau \neq \emptyset$ for all $\mu < \tau < \mu + \epsilon$, in contradiction with the definition of μ.

q.e.d.

5. Outline of the proof of Theorem 2

Let θ as in the statement of Theorem 2. Then we can write

$$\omega_\theta = f^{13} - f^{24},$$

with

$$f^1 = \cos\theta\, e^1 + \sin\theta\, e^2, \quad f^2 = -\sin\theta\, e^1 + \cos\theta\, e^2, \quad f^3 = e^3, \quad f^4 = e^4.$$

Since

$$df^4 = de^4 = e^{12} = f^{12},$$

one easily obtains that

$$\alpha = d^{\mathrm{c}} u - u f^1$$

satisfies (6) and (5) if and only if $u \in \tilde{C}^2(T^3)$ is a solution to the fully non-linear PDE

(61)

$$\Big((\cos\theta\,\partial_x - \sin\theta\,\partial_y)^2 u + 1\Big)\Big((\sin\theta\,\partial_x + \cos\theta\,\partial_y)^2 u + \partial_t^2 u + \partial_t u + 1\Big) - \\ - \Big((\cos\theta\,\partial_x - \sin\theta\,\partial_y)(\sin\theta\,\partial_x + \cos\theta\,\partial_y) u\Big)^2 - \\ - \Big((\cos\theta\,\partial_x - \sin\theta\,\partial_y)\partial_t u\Big)^2 = \mathrm{e}^F.$$

Let

$$v(p, q, t) = u(x, y, t),$$

with

$$\begin{cases} x = \cos\theta\, p + \sin\theta\, q, \\ y = -\sin\theta\, p + \cos\theta\, q. \end{cases}$$

Then

$$\begin{cases} \partial_p v = \cos\theta\, \partial_x u - \sin\theta\, \partial_y u, \\ \partial_q v = \sin\theta\, \partial_x u + \cos\theta\, \partial_y u. \end{cases}$$

This implies that (61) can be rewritten as

$$(v_{pp} + 1)(v_{qq} + v_{tt} + v_t + 1) - v_{pq}^2 - v_{pt}^2 = \mathrm{e}^G, \tag{62}$$

where

$$G(p, q, t) = F(x, y, t).$$

Equation (62) is formally the same as equation (12). There is, however, a big difference in periodicity conditions, which become

$$v(p + \cos\theta\, m + \sin\theta n, q + \sin\theta\, m + \cos\theta\, n, t + k) = v(p, q, t),$$

for all $m, n, k \in \mathbb{Z}$.

In particular, this implies that the proof of Proposition 3 fails, unless v is periodic with respect to the first variable p. An elementary argument shows that this happens if and only if either $\cos\theta = 0$ or $\tan\theta \in \mathbb{Q}$, i.e., if and only if there exist two integers m and n such that

$$m^2 + n^2 > 0$$

and

$$\cos\theta = \frac{m}{\sqrt{m^2 + n^2}}, \qquad \sin\theta = \frac{n}{\sqrt{m^2 + n^2}}.$$

Then

$$v(p + \sqrt{m^2 + n^2}, q, t) = v(p, q, t),$$

and from $v_{pp} > -1$ we get the estimate

$$|v_p| \leq \sqrt{m^2 + n^2}.$$

The rest of the proof of Theorem 2 can be obtained by a slight modification of the argument used to prove Theorem 1 and is left to the reader.

References

[1] E. Abbena, An example of an almost Kähler manifold which is not Kählerian, *Boll. Un. Mat. Ital. A (6)* **3** (1984), no. 3, 383–392, MR 769169, Zbl 0559.53023.

[2] R.A. Adams, Sobolev Spaces, *Pure and Applied Mathematics,* vol. **65**, Academic Press, Orlando, 1975, MR 2424078, Zbl 1098.46001.

[3] E. Buzano, A. Fino & L. Vezzoni, The Calabi–Yau equation for T^2-bundles over the non-Lagrangian case, *Rend. Semin. Mat. Univ. Politec. Torino* **69** (2011), no. 3, 281–298, MR 2945685, Zbl 1247.53085.

[4] E. Calabi, On Kähler manifolds with vanishing canonical class, in *Algebraic geometry and topology, symposium in honor of S. Lefschetz*, pp. 78–89, Princeton University Press, Princeton, N.J., 1957, MR 0085583, Zbl 0080.15002.

[5] S.K. Donaldson, Two-forms on four-manifolds and elliptic equations, Inspired by S.S. Chern, 153–172, *Nankai Tracts Math.* **11**, World Scientific, Hackensack N.J., 2006, MR 2313334, Zbl 1140.58018.

[6] A. Fino, Y.Y. Li, S. Salamon & L. Vezzoni, The Calabi–Yau equation on 4-manifolds over 2-tori, *Trans. Amer. Math. Soc.* **365** (2013), no. 3, 1551–1575, MR 3003274, Zbl 1269.53047.

[7] D. Gilbarg & N.S. Trudinger, Elliptic partial differential equations of second order, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. **224**, Springer-Verlag, Berlin, 1983, MR 0737190, Zbl 0361.35003.

[8] D.D. Joyce, Compact manifolds with special holonomy, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000. xii+436 pp, MR 1787733, Zbl 1027.53052.

[9] C.H. Taubes, Tamed to compatible: symplectic forms via moduli space integration, *J. Symplectic Geom.* **9** (2011), no. 2, 161–250, MR 2811651, Zbl 1232.57021.

[10] M.E. Taylor, Partial Differential Equations III, Nonlinear Equations, *Applied Mathematical Sciences*, vol. V **117**, Springer, New York, NY, 1996, MR 1477408, Zbl 1206.35004.

[11] W.P. Thurston, Some simple examples of symplectic manifolds, *Proc. Amer. Math. Soc.* **55** (1976), 467–468, MR 0402764, Zbl 0324.53031.

[12] V. Tosatti & B. Weinkove, The Calabi–Yau equation on the Kodaira–Thurston manifold, *J. Inst. Math. Jussieu* **10** (2011), no. 2, 437–447, MR 2787695, Zbl 1219.53072.

[13] V. Tosatti, B. Weinkove & S.T. Yau, Taming symplectic forms and the Calabi–Yau equation, *Proc. London Math. Soc.* **97** (2008), no. 2, 401–424, MR 2439667, Zbl 1153.53054.

[14] V. Tosatti, Y. Wang, B. Weinkove & X. Yang, $C^{2,\alpha}$ estimates for nonlinear elliptic equations in complex and almost-complex geometry, `arXiv:1402.0554`.

[15] S.T. Yau, On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, *Comm. Pure Appl. Math.* **31** (1978), no. 3, 339–411, MR 480350, Zbl 0369.53059.

[16] B. Weinkove, The Calabi–Yau equation on almost-Kähler four-manifolds, *J. Differential Geom.* **76** (2007), no. 2, 317–349, MR 2330417, Zbl 1123.32015.

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