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BOUNDS ON GENUS AND GEOMETRIC INTERSECTIONS FROM CYLINDRICAL END MODULI SPACES

SAŠO STRLE

Abstract

In this paper we present a way of computing a lower bound for the genus of any smooth representative of a homology class of positive self-intersection in a smooth four-manifold X with second positive Betti number $b_2^+(X) = 1$. We study the solutions of the Seiberg-Witten equations on the cylindrical end manifold which is the complement of the surface representing the class. The result can be formulated as a form of generalized adjunction inequality. The bounds obtained depend only on the rational homology type of the manifold, and include the Thom conjecture as a special case. We generalize this approach to derive lower bounds on the number of intersection points of n algebraically disjoint surfaces of positive self-intersection in manifolds with $b_2^+(X) = n$.

Introduction

Seiberg-Witten theory has proved very useful in the study of the minimal genus problem. After Kronheimer-Mrowka's proof of the Thom conjecture [6], regarding the minimal genus problem in the complex projective plane, the following result (the so-called generalized Thom conjecture) was obtained by Morgan-Szabo-Taubes [13] (for classes of nonnegative self-intersection) and Ozsvath-Szabo [18] (the general case): any smooth symplectic curve in a closed symplectic four-manifold minimizes the genus in its homology class. Their proofs depend on results of Taubes [22] about Seiberg-Witten theory of symplectic manifolds, specifically the basic classes of such manifolds. In this paper we present a way of deriving genus bounds that does not depend on any special

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structure on the manifold. Rather than working over a closed manifold, we study Seiberg-Witten equations on an associated cylindrical end manifold; this approach is related to the work of Frøyshov [2]. The bounds obtained in this way depend only on the rational homology type of the manifold. Since the results are so general, the information about a possible symplectic structure on X is lost; in particular, the bounds are independent of (the sign of) the canonical class of X, whereas the bounds coming from the generalized Thom conjecture detect differences in canonical classes. An important advantage of our approach is that it can be used in manifolds with vanishing Seiberg-Witten invariants, in particular to study geometric intersections of surfaces.

Consider a divisible homology class $d\xi \in H_2(X;\mathbf{Z})$ of positive self-intersection in a smooth four-manifold X with $b_1(X)=0$ and $b_2^+(X)=1$. The divisibility $d>1$ of the homology class is crucial (for technical reasons) while studying Seiberg-Witten equations on the cylindrical end manifold $Z=X-\Sigma$, where Σ is a smooth embedded surface representing $d\xi$. The end of Z is modeled on a nontrivial circle bundle Y over Σ, and we work with Seiberg-Witten solutions on Z that exponentially decay to solutions on Y. This depends on description of the perturbed Seiberg-Witten moduli spaces on Y obtained by Mrowka-Ozsvath-Yu [15].

Even though the method requires us to consider a divisible class, the main result concerning genus bounds holds for primitive classes as well. The bound can be stated in the form of a generalized adjunction inequality.

Theorem A. *Let X be a smooth closed oriented four-manifold with $b_1(X)=0$ and $b_2^+(X)=1$. If $\Sigma \subset X$ is a smooth embedded surface of positive self-intersection, then*

$$\chi(\Sigma)+[\Sigma]^2 \leq |\langle c,[\Sigma]\rangle| \tag{1}$$

for any characteristic vector $c \in H^2(X)$ that satisfies $c^2>\sigma(X)$.

Based on this inequality it is straightforward to derive minimal genus formulae in (rational homology) $\mathbf{CP}^2$, $S^2\times S^2$ and $\mathbf{CP}^2\#\overline{\mathbf{CP}}^2$. In manifolds with rational homology of rational surfaces $\mathbf{CP}^2\#n\overline{\mathbf{CP}}^2$ with $2\leq n\leq 9$ the results are easiest to state for reduced classes, defined by Li-Li [8]. In particular, we prove that for any $g>0$ there are only finitely many reduced classes of minimal genus g (see Proposition 14.2; also see [7]). We note that genuine rational surfaces mentioned above are 'genus-minimal' in the sense that minimal genus representatives in these manifolds have the smallest possible genus among all manifolds with the same rational homology type.

Above considerations generalize to manifolds X with $b_2^+(X) = n$ in a way that allows us to study a collection of n surfaces in X. The counterpart of the adjunction inequality is the following result.

Theorem B. *Let X be a smooth closed connected four-manifold with $b_1(X) = 0$ and $b_2^+(X) = n > 1$, and let $\Sigma_1, \ldots, \Sigma_n$ be disjoint embedded surfaces in X with positive self-intersections. If $c \in H^2(X)$ is a characteristic vector satisfying*

$$c^2 > \sigma(X) \quad \textit{and} \quad \langle c, [\Sigma_i] \rangle \geq 0 \quad \textit{for all } i,$$

then

$$\chi(\Sigma_i) + [\Sigma_i]^2 \leq \langle c, [\Sigma_i] \rangle \tag{2}$$

holds for at least one i.

We use this to derive a lower bound on the number of intersection points of surfaces of low genus. For example, suppose that classes $(p, q, 0, 0)$ and $(0, 0, r, s)$ in $H_2(S^2 \times S^2 \# S^2 \times S^2)$ are represented by spheres in the connected sum $S^2 \times S^2 \# S^2 \times S^2$. If $p, q, r, s \geq 2$ and $p + q \geq r + s$, then the number of intersection points of the two spheres is at least

$$pq + (r-1)(s-1).$$

In general, the lower bound on the number of intersection points obtained in this way is roughly by a factor of 2 better than the bounds obtained via the G-signature Theorem (see [4]). We also give an example where the bound on the number of intersection points is optimal.

This paper is divided in two parts. Part I is concerned with technical aspects of Seiberg-Witten moduli spaces over cylindrical end manifolds. The main results of this part are the dimension formula for the moduli space (Corollary 8.2) and compactness and regularity results of Section 9. In Part II we use results of Part I to derive genus bounds and bounds on the number of intersection points of surfaces. We first present a derivation of a genus bound in $\mathbf{CP}^2$ (which is equivalent to the Thom conjecture) and then proceed to the general case. This is described in Theorem 11.1, which can be rephrased as a generalized adjunction inequality stated above (see Section 12). After that we consider several examples in which one can derive explicit formulae for genus bounds and address the question of representability. In the last section of the paper we turn to geometric intersections of surfaces.

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Part I: Cylindrical end moduli spaces

1. The setup

Throughout X will denote a smooth closed connected oriented four-manifold. If a smooth oriented surface Σ is embedded in X so that the image of its fundamental homology class $[\Sigma]$ in $H_2(X)$ is not a torsion class, we say that Σ *represents* this homology class. Denote by $N \subset X$ a compact tubular neighborhood of Σ. It will be convenient to distinguish between ∂N, oriented as the boundary of N, and $Y = -\partial N$, oriented as the 'boundary' of Z. More precisely, let Z_0 be the closure of the complement of N in X; then $Y = \partial Z_0$ and we think of $Z = X - \Sigma$ as Z_0 with a half-infinite cylinder attached, $Z = Z_0 \cup_Y [0, \infty) \times Y$. We refer to $[0, \infty) \times Y$ as the *cylindrical end* of Z and say that the end of Z is *modeled on* Y. The following proposition summarizes the relevant cohomological information about these spaces. Unless specified otherwise all the (co)homology groups have integer coefficients.

Proposition 1.1. *Let X be a closed oriented four-manifold and let $\Sigma \subset X$ be an embedded surface representing the class $d\xi$, where $\xi \in H_2(X)$ is a primitive class of nonzero self-intersection and $d \geq 1$ is an integer. Denote by N a compact tubular neighborhood of Σ and by $n = (d\xi)^2$ the degree of the circle bundle $\partial N \to \Sigma$. Then*

$$H^1(\partial N) \cong H^1(\Sigma), \quad H^2(\partial N) \cong H^2(\Sigma)/n[\Sigma]^* \oplus H^1(\Sigma),$$

$$H^1(Z) \cong H^1(X), \quad H^2(Z) \cong H^2(X)/d\alpha \oplus F,$$

where $[\Sigma]^*$ *denotes the fundamental cohomology class of* Σ, α *is the Poincaré dual of* ξ *and* F *is a subgroup of* $H^1(\Sigma)$. *The restriction homomorphism* $H^2(X) \to H^2(N)$ *sends* α *to* $d\xi^2[\Sigma]^*$ *and its image is a subgroup of index* d. *Moreover, the restriction homomorphism* $H^2(Z) \to H^2(\partial N)$ *is injective on* F.

Proof. The cohomology groups of ∂N follow easily from the Gysin exact sequence of the circle bundle $S^1 \hookrightarrow \partial N \xrightarrow{p} \Sigma$ with the Chern class $c_1 = n[\Sigma]^*$.

To determine the cohomology of Z, use the Poincaré duality and excision isomorphisms: $H^2(Z) \cong H_2(Z, \partial N) \cong H_2(X, N)$. The last group can be computed using the exact sequence of the pair (X, N):

$$H_3(X) \xrightarrow{\cong} H_3(X,N) \to H_2(N) \to H_2(X) \to H_2(X,N) \to H_1(N).$$

Since $H_1(N)$ is a free abelian group, we have $H_2(X,N) \cong H_2(X)/d\xi \oplus F'$ for some subgroup F' of $H_1(N)$. Similarly we obtain $H^1(Z) \cong H_3(X,N) \cong H^1(X)$.

From $\langle \alpha, [\Sigma] \rangle = \xi \cdot d\xi$ it follows that $\alpha \in H^2(X)$ restricts to $d\xi^2[\Sigma]^* \in H^2(N)$. The remaining claims follow easily. q.e.d.

2. Seiberg-Witten solutions on a circle bundle

We describe the structure of the moduli spaces of solutions of certain *perturbed* Seiberg-Witten equations on a circle bundle $p\colon Y \to \Sigma$ of degree $n \neq 0$ over an oriented smooth surface Σ, studied by Mrowka-Ozsvath-Yu [15]. The purpose of the perturbation is to make the equations behave as if the bundle Y were a product. This is achieved by choosing a 'product' connection as the background connection in T^*Y in place of the Levi-Civita connection.

To define the background connection, choose a constant curvature metric g_Σ of volume 1 on Σ; denote by vol_Σ the corresponding volume form and let $\omega = p^*(\mathrm{vol}_\Sigma)$ be its pull-back to Y. The circle bundle Y admits a connection 1-form $i\varphi\colon TY \to i\mathbf{R}$ of constant curvature; observe that $d\varphi = -2\pi n\omega$, since Y has degree n. This connection determines a splitting $T^*Y = \mathbf{R}\varphi \oplus H$, where $H \cong p^*T^*\Sigma$ is the horizontal distribution. A metric on Y, compatible with this splitting, is given by $g_Y = \varphi^2 + p^*g_\Sigma$; the corresponding volume form is $\mathrm{vol}_Y = \varphi \wedge \omega$. Note that the radius of a fiber circle with respect to this metric is 1. The *product connection* on Y is defined by $\nabla^Y = d \oplus p^*\nabla^\Sigma$, where ∇^Σ is the

Levi-Civita connection of (Σ, g_Σ). Connection ∇^Y is compatible with the splitting and the metric. However, it is not torsion-free as $Y \to \Sigma$ is a nontrivial bundle.

The *trivial* Spin^c structure on Y is the one with the trivial bundle of spinors W^Y. By viewing W^Y as the pull-back of a spinor bundle over Σ, one can endow it with a ∇^Y-compatible spin connection that we denote by ∇^Y as well. The Clifford multiplication in W^Y is as follows: vectors in H act via the pull-back action, while φ acts by $\pm i$ on $(W^Y)^\pm$, where the splitting $W^Y = (W^Y)^+ \oplus (W^Y)^-$ is induced by the splitting of the spinor bundle over Σ.

Given a hermitian line bundle $E \to Y$ we say that the Spin^c structure on Y with the bundle of spinors $W_E^Y = W^Y \otimes E$ is *determined* by E. A unitary connection A in E induces a spin connection $\nabla^Y \otimes A$ in $W^Y \otimes E$; the Dirac operator of this connection is denoted by D_A. We need to understand the moduli space of solutions of the perturbed 3-dimensional Seiberg-Witten equations on Y (i.e., the equations defined using the above Dirac operator) in a given Spin^c structure. The space of reducible solutions in the Spin^c structure determined by E is nonempty only for torsion bundles E (i.e., the ones with torsion Chern class). If A_0 is a smooth flat connection in E, then $A_0 + \mathbf{i}\alpha$ is a reducible solution if and only if α is a closed one-form on Y. This gives an identification between the space of reducible solutions and the space of closed one-forms on Y. The moduli space of reducible solutions, obtained by dividing the space of solutions by the gauge group action, is therefore identified with $H^1(Y;\mathbf{R})/H^1(Y) \cong H^1(\Sigma; S^1)$ via the choice of A_0.

Recall that the three-dimensional Seiberg-Witten equations are the equations for the critical points of the Chern-Simons-Dirac functional on $\mathbf{R}\times Y$ (see [13]). This means that their linearization is self-adjoint, so for positive dimensional moduli spaces the linearization of the equations is not surjective. The appropriate notion of non-degeneracy of the moduli space is the following.

Definition 2.1. A component $\mathcal{N}$ of the moduli space of Seiberg-Witten solutions on Y is (Morse-Bott) *nondegenerate*, if the kernel of the linearization of the Seiberg-Witten equations at (A, Φ) is isomorphic to the tangent space to $\mathcal{N}$ at $[A, \Phi]$ for any point $[A, \Phi] \in \mathcal{N}$.

The following theorem, proved in [15], describes the moduli spaces of solutions to the perturbed Seiberg-Witten equations on Y for various Spin^c structures.

Theorem 2.2. *Let Y be a circle bundle of degree $n \neq 0$ over a surface Σ. The space of solutions in the* Spin^c *structure determined by $E \to Y$ is nonempty only if E is the pull-back of a line bundle $F \to \Sigma$. Fix such an E and let $c = c_1(F)$.*

(a) *The moduli space $\mathcal{R}(E)$ of reducible solutions is homeomorphic to the dual torus $H^1(\Sigma; S^1)$. Moreover, if $c \not\equiv 0 \bmod n$ or if $\Sigma \cong S^2$, this space is nondegenerate. In particular, the above identification is a diffeomorphism.*

(b) *The irreducible components of the moduli space are parameterized by line bundles $E_0 \to \Sigma$ satisfying*

$$c_1(E_0) \equiv c \bmod n \quad \textit{and} \quad 0 < |c_1(E_0)| < g.$$

All of these components are compact and nondegenerate; they arise as the pull-backs of solutions to the vortex equations on Σ.

(c) *Since the Chern class of the* Spin^c *structure is torsion, the Chern-Simons-Dirac functional descends to a real-valued function on the moduli space. If we normalize it so that it equals 0 on $\mathcal{R}(E)$, then its value on the component corresponding to a line bundle E_0 is $8\pi^2(c_1(E_0))^2/n$.*

3. Seiberg-Witten solutions on a cylinder

The first step in understanding the structure of the space of Seiberg-Witten solutions on a manifold with a cylindrical end is to study the solutions on the cylindrical part. A standard approach which guarantees good limiting behavior of solutions at infinity is to consider only finite energy solutions (see [13] and [12]).

Let (A, Ψ) be a configuration on $[0, \infty) \times Y$ in a temporal gauge; denote by (A_t, Ψ_t) the path of configurations on Y obtained by restricting (A, Ψ) to the slices $t \times Y$. Recall that the Seiberg-Witten equations on the cylinder take the form

$$\frac{\partial}{\partial t}(A_t, \Psi_t) = (*_Y(q(\Psi_t) - F_{A_t}), D_{A_t}\Psi_t).$$

The *energy* of a configuration (A, Ψ) on $[0, \infty) \times Y$ in a temporal gauge is given by the square of the L^2-norm of the right-hand side in the above equation.

Any solution on Y, being a critical point of the above equation, gives rise to a static solution on the cylinder; such a solution clearly has finite energy. Moreover, any finite energy solution on the cylinder converges to a static solution exponentially fast. This result is the Seiberg-Witten analogue of the exponential decay results established by Morgan-Mrowka-Ruberman [12] in Donaldson's theory. For our purposes, however, we do not need to know that all solutions are exponentially decaying to solutions on Y. That is, without referring to exponential decay results, we will consider only those configurations on the cylinder $[0, \infty) \times Y$ that decay exponentially to solutions on Y, for some appropriately chosen decay constant.

4. Seiberg-Witten equations on a cylindrical end manifold

Let Z_0 denote a compact oriented 4-manifold with boundary, Y, a circle bundle of degree $n \neq 0$ over a surface Σ. Choose a collar $[-1, 0] \times Y \subset Z_0$ and equip $Z = Z_0 \cup [0, \infty) \times Y$ with a cylindrical end metric that agrees with $dt^2 + g_Y$ on $[-1, \infty) \times Y$, where g_Y is the metric on Y described in Section 2. As the background connection ∇^Z for the Dirac operator we use a metric compatible connection that agrees with the Levi-Civita connection on the complement of $(-1, \infty) \times Y$ and agrees with the pull-back of the connection ∇^Y on the cylinder $[0, \infty) \times Y$.

Given a Spin^c structure on Z we denote the corresponding bundles of spinors by $W = W^+ \oplus W^-$ and the determinant line by $L = \det(W^+)$. As the configuration space for the Seiberg-Witten equations on Z we choose the subset of uniformly *exponentially decaying configurations.* The restriction of an exponentially decaying configuration to the end $[0, \infty) \times Y$ differs from some static solution on the cylinder by a term that converges to zero exponentially fast along the cylinder (see Definition 5.1 for details). We will specify the rate of convergence in Proposition 6.2.

Suppose now that we are in the situation from Section 1: Z is the complement of a representative Σ of a multiple class $d\xi$ in a closed manifold X. In this case the following proposition shows that any Spin^c structure on Z which admits exponentially decaying solutions arises as the restriction of a Spin^c structure on X. We will use this to express the dimension of the moduli space of Seiberg-Witten solutions on Z in terms of the invariants of X.

Proposition 4.1. *Let $\Sigma \subset X$ be a smooth representative of the class $d\xi$ and let $n = (d\xi)^2$; denote by N a compact tubular neighborhood of Σ and by $Z = X - \Sigma$ the cylindrical end manifold with the end modeled on Y. Let $\mathcal{S}$ be a* Spin^c *structure on Z for which the space of exponentially decaying Seiberg-Witten solutions is nonempty.*

(a) *The restriction of $\mathcal{S}$ to Y is determined by a torsion line bundle and it extends to a* Spin^c *structure on the disk bundle N. We fix the extension W^N of the trivial* Spin^c *structure W^Y on Y to a* Spin^c *structure on N, which is uniquely determined by requiring that the Chern class of its determinant line is equal to $n[\Sigma]^*$. Through this choice we obtain a canonical extension of the given* Spin^c *structure $\mathcal{S}$ on Z to a* Spin^c *structure on X.*

(b) *Any two* Spin^c *structures on X that induce the given* Spin^c *structure $\mathcal{S}$ on Z differ by a power of the line bundle on X with the Chern class $d\alpha$, where α denotes the Poincaré dual of ξ.*

Proof. The induced Spin^c structure on Y is determined by a line bundle $E \to Y$ satisfying $L|_Y = E^2$. Note that the existence of exponentially decaying Seiberg-Witten solutions on Z implies the existence of solutions on Y. Combining this with Theorem 2.2 we conclude that the line bundle E is torsion and hence the pull-back of a line bundle on Σ.

An extension of the trivial Spin^c structure W^Y on Y to a Spin^c structure on N is given as follows. Denote by $\widehat{N} \to \Sigma$ the normal bundle of Σ in X, considered as a complex line bundle of degree n. As a complex manifold, the line bundle $\widehat{N}$ carries a canonical Spin^c structure with the bundles of spinors $W^+ = \Lambda^0 \oplus \Lambda^{0,2}$ and $W^- = \Lambda^{0,1}$. These bundles are determined up to isomorphism by their restrictions to the zero-section $\Sigma \subset \widehat{N}$. Note that $W^-|_\Sigma \cong \widehat{N} \oplus K_\Sigma^{-1}$, where K_Σ is the canonical bundle of Σ. The determinant line is therefore isomorphic to $\widehat{N} \otimes K_\Sigma^{-1}$. Since the pull-back of $\widehat{N}$ is trivial over Y, we need to change the Spin^c structure by a square root of the pull-back of the canonical bundle of Σ.

A different extension of a Spin^c structure on Z to a Spin^c structure on X can be obtained by changing the Spin^c structure on N by a power of the pull-back of $\widehat{N}$ (since this operation preserves the Spin^c structure on Y). From $c_1(\widehat{N}) = (d\xi)^2[\Sigma]^*$ and Proposition 1.1 it follows that the Chern class of the auxiliary bundle on X changes under this operation by a multiple of $d\alpha$. To see that these are the only possibilities, consider two Spin^c structures on X that differ by a line bundle E. If they restrict to give the same Spin^c structure on Z, then $E|_Z$ is trivial, hence $c_1(E)$ lies in the kernel of the restriction homomorphism $H^2(X) \to H^2(Z)$. Recall that this kernel is generated by $d\alpha$. q.e.d.

As in the case of Spin^c structures on Y we will say that a Spin^c structure on N is *determined* by a line bundle $E \to N$ (or by a line bundle $E_0 \to \Sigma$) if the bundle of spinors is of the form $W^N \otimes E$ (or $W^N \otimes p^* E_0$) .

Given a Spin^c structure on X, denote its determinant line by Det. With the above notation, we write $\langle c_1(\mathrm{Det}), \xi\rangle = k + (2s+1)d\xi^2$ for some $k \in \{0, \ldots, 2d\xi^2 - 1\}$ and $s \in \mathbf{Z}$. Note that k is the representative of the residue class

$$\langle c_1(\mathrm{Det}), \xi\rangle + d\xi^2 \bmod 2d\xi^2$$

in $\{0, \ldots, 2d\xi^2 - 1\}$; the purpose of the shift by $d\xi^2$ is to make k directly related to the induced Spin^c structure on Y. Indeed, the induced bundle of spinors over N is determined by a line bundle $E_0 \to \Sigma$ with $c_1(E_0) = e$ satisfying $2e = d(k + 2sd\xi^2)$.

Possible values of k are constrained by the fact that $c_1(\mathrm{Det})$ is a characteristic class; this motivates the following definition.

Definition 4.2. We call $k \in \{0, \ldots, 2d\xi^2 - 1\}$ a *characteristic number* for (X, ξ, d), if there exists a Spin^c structure on X whose determinant line Det satisfies $\langle c_1(\mathrm{Det}), \xi\rangle = k + (2s+1)d\xi^2$ for some $s \in \mathbf{Z}$.

Clearly the parity of a characteristic number is uniquely determined. The following characterization of characteristic numbers is easily verified.

Lemma 4.3. *Let X be a closed four-manifold, $\xi \in H_2(X)$ a primitive class of positive self-intersection, and $d \geq 1$ an integer. If ξ^2 is even, then the set of characteristic numbers for (X, ξ, d) consists of all even numbers in $\{0, \ldots, 2d\xi^2 - 1\}$; if ξ^2 is odd, then the set of characteristic numbers consists of all numbers in $\{0, \ldots, 2d\xi^2 - 1\}$ with the parity opposite to that of d. In particular, kd is even in either case.*

5. Configuration space over a cylindrical end manifold

In order to be able to consider the Seiberg-Witten equations on Z as an elliptic system of equations we need to choose appropriate function spaces in which to study the equations. Since we chose to work with configurations on Z that uniformly exponentially decay to configurations on Y, we can work with the weighted Sobolev spaces $L^2_{r,\delta}$ for some small $\delta > 0$. We cannot choose $\delta = 0$ because with this choice the operator on Y, associated to the linearization of the Seiberg-Witten equations on Z, has a nontrivial kernel and so the equations are not Fredholm (cf. [11]). Recall that the L^2_δ norm of a function f can be defined by $||f||^2_\delta = \int_Z |f|^2 e^{\delta\tau}$, where $\tau\colon Z \to \mathbf{R}$ is a smooth function that is equal to -1 on the complement of $(-1,\infty) \times Y$ and agrees with the t coordinate on the cylinder $[0,\infty) \times Y$; we further assume that τ depends only on t and is non-decreasing. Sobolev norms for $r > 0$ are defined analogously.

We are interested in the space of configurations on Z decaying to reducible solutions on Y. Fix for now a Spin^c structure on Z with nonempty space of exponentially decaying Seiberg-Witten solutions with reducible limits. For (A, Ψ) a smooth solution on Z denote its asymptotic value by $(B_\infty, 0)$. The space of all asymptotic values is an affine space whose underlying vector space is the space of imaginary-valued closed one-forms on Y. We let B be a smooth unitary connection in L (the determinant line of the Spin^c structure on Z) which agrees with the pull-back of B_∞ on the cylinder $[0,\infty) \times Y$. We use the connection B (resp. B_∞) to identify the space of unitary connections in L (resp. $L|_Y$) with the imaginary-valued 1-forms on Z (resp. Y).

Rather than working with the full configuration space, we restrict the possible asymptotic values of configurations (by fixing the gauge at infinity). Specifically, we replace the space of imaginary-valued closed one-forms on Y by the subspace $\mathcal{H}$ of imaginary-valued harmonic one-forms. Note that the subgroup of the gauge group $\mathcal{G}^Y$ that acts on $\mathcal{H}$ consists of harmonic gauge transformations and is therefore isomorphic to $S^1 \times H^1(Y)$, where S^1 corresponds to the constant gauge transformations. However, apart from the constant gauge transformations, the only harmonic gauge transformations on Y that extend to gauge transformations on Z are those that correspond to the classes in the image of $H^1(Z) \to H^1(Y)$. In our application of the cylindrical end moduli space to the genus problem we will assume that $b_1(Z) = 0$. Assuming the latter, the gauge group over Z for the restricted (i.e., gauge-fixed at infinity) configuration space consists of the gauge transformations that converge to the constant gauge transformations on Y. We will see later that the space of solutions on Z contains reducible configurations, so the subgroup of constant gauge transformations does not act freely on it. For this reason we consider the *based moduli space*, obtained by dividing the space of solutions by the action of the gauge group based at infinity. The group of constant gauge transformations still acts on the based moduli space and we will use this action to obtain our results.

Definition 5.1. The exponentially decaying *configuration space* on Z, corresponding to the reducible solutions on Y, is defined to be

$$\mathcal{H} \times L^2_{2,\delta}(\mathbf{i}\Lambda^1(Z) \oplus W^+).$$

More precisely, the *configuration* associated to an element (h, α, Φ) is

$$(A, \Psi) = (B + \alpha + \tilde{r}h, \Phi).$$

A gauge transformation $\sigma \in L^2_{3,loc}(Z, S^1)$ belongs to the *gauge group based at infinity*, $\mathcal{G}_\infty$, if there exist a $T > 0$ and an $f \in L^2_{3,\delta}([T, \infty) \times Y, \mathbf{iR})$ so that the restriction of σ to the cylinder $[T, \infty) \times Y$ is given by $\sigma = \exp(f)$.

6. Deformation complex

Recall that Spin^c structures on Z with nonempty exponentially decaying configuration space are induced from X. Moreover, we will consider only Spin^c structures on X for which the induced Spin^c structure on Y is nontrivial, unless Σ is a sphere; this way the moduli space of reducible solutions on Y is always nondegenerate (Theorem 2.2). Suppose $(B_\infty, 0)$ is the asymptotic value of a smooth solution (A, Ψ) on Z. We use the configuration space on Z as described in Definition 5.1. Since the configurations on Z converge to reducible solutions on Y, Seiberg-Witten equations on Z give rise to a map

$$\text{(3)} \qquad \mathrm{SW}\colon \mathcal{H} \oplus L^2_{2,\delta}(\mathbf{i}\Lambda^1(Z) \oplus W^+) \to L^2_{1,\delta}(\mathbf{i}\Lambda^{2,+}(Z) \oplus W^-).$$

This is well-defined because all the terms in the Seiberg-Witten map exponentially decay to 0 (recall that the multiplication $L^2_{2,\delta} \otimes L^2_{2,\delta} \to L^2_{1,\delta}$ is continuous). The *deformation complex* $\mathcal{D}_{(A,\Psi)}$ of the solution (A, Ψ), taking into account the action of $\mathcal{G}_\infty$, is

$$\text{(4)} \quad 0 \to L^2_{3,\delta}(Z, \mathbf{iR}) \xrightarrow{K_{(A,\Psi)}} \mathcal{H} \oplus L^2_{2,\delta}(\mathbf{i}\Lambda^1(Z) \oplus W^+) \to$$
$$\xrightarrow{T_{(A,\Psi)}\mathrm{SW}} L^2_{1,\delta}(\mathbf{i}\Lambda^{2,+}(Z) \oplus W^-) \to 0,$$

where $K_{(A,\Psi)}(f) = (0, 2df, -f\Psi)$ is the infinitesimal gauge group action and

$$T_{(A,\Psi)}\mathrm{SW}(\widehat{\alpha}, \psi) = (d^+\widehat{\alpha} - 2Q(\Psi, \psi), D_A\psi + \widehat{\alpha} \cdot \Psi)$$

is the linearization of the Seiberg-Witten map at (A, Ψ). Here Q is the bilinear map associated to the quadratic map q in the Seiberg-Witten equations and $\widehat{\alpha} = \alpha + \dot{\tau} h$ is the one-form on Z corresponding to $(h, \alpha) \in \mathcal{H} \oplus L^2_{2,\delta}(\mathbf{i}\Lambda^1(Z))$. The cohomology groups of this complex provide some local information about the based moduli space as described below. We first make the following observation.

Lemma 6.1. *The zeroth cohomology group of the deformation complex is trivial.*

Proof. If $f \in L^2_{3,\delta}(Z, \mathbf{iR})$ is in the kernel of $K_{(A,\Psi)}$, then $df = 0$. Thus f is constant and since it converges to 0 at infinity, it must be identically equal to zero. q.e.d.

The first cohomology group of the deformation complex (4) is called the *Zariski tangent space* of the moduli space and the second cohomology group is called the *obstruction space*. If (A, Ψ) is a regular point for the Seiberg-Witten map, then the obstruction space vanishes and the first cohomology of the complex is isomorphic to the (geometric) tangent space of the moduli space at $[A, \Psi]$.

We will compute the index of the deformation complex $\mathcal{D}_{(A,\Psi)}$ via the index of the *fiber complex* $\mathcal{F}_{(A,\Psi)}$ associated to it; the latter is defined by the following exact sequence of complexes

$$0 \to \mathcal{F}_{(A,\Psi)} \to \mathcal{D}_{(A,\Psi)} \to \mathcal{H} \to 0,$$

where $\mathcal{H}$ denotes the deformation complex of the asymptotic value $(B_\infty, 0)$ of (A, Ψ) and the morphism to $\mathcal{H}$ corresponds to taking limits at infinity. Here we identified the complex $\mathcal{H}$ with its only nonzero group (in dimension 1), namely the group of harmonic one-forms on Y. The fiber complex differs from the full deformation complex by a finite dimensional space $\mathcal{H}$ (of dimension $2g$, where g is the genus of Σ), hence it suffices to compute the index of the fiber complex. The Fredholm properties of the fiber complex are determined by the asymptotic behavior of its 'wrapped-up' form [11], given by (5) below.

Proposition 6.2. *There exists $\delta_0 > 0$ so that for any $\delta \in (0, \delta_0]$, the Seiberg-Witten map* (3), *considered as a map of the spaces of L^2_δ sections, is Fredholm and its index is constant on the configuration space. Moreover, the asymptotic map of the linearization has trivial kernel.*

Proof. Let (A,Ψ) be a configuration on Z with a reducible asymptotic value $(B_\infty,0)$ on Y. The L^2_δ adjoint K^*_δ of the infinitesimal gauge group action $K_{(A,\Psi)}$ is defined with respect to the following inner products: for imaginary-valued forms α and β let

$$\langle\alpha,\beta\rangle_\delta = \int_Z \alpha\wedge *\beta\, e^{\delta\tau},$$

where $*$ is the complex anti-linear extension of the Hodge star-operator; for spinors ψ and ϕ let

$$\langle\psi,\phi\rangle_\delta = 2\mathrm{Re}\int_Z \langle\psi(z),\phi(z)\rangle e^{\delta\tau}\mathrm{vol}_Z.$$

Then $K^*_\delta(\alpha,\psi) = 2e^{-\delta\tau}d^*e^{\delta\tau}\alpha + 2\mathbf{i}\mathrm{Im}\langle\Psi,\psi\rangle$; after dividing by 2 we obtain the 'wrapped-up' fiber complex

$$(5)\qquad \begin{gathered} F_\delta\colon L^2_{2,\delta}(\mathbf{i}\Lambda^1(Z)\oplus W^+)\to L^2_{1,\delta}(\mathbf{i}\Lambda^0(Z)\oplus\mathbf{i}\Lambda^{2,+}(Z)\oplus W^-),\\ F_\delta(\alpha,\psi) = \big(e^{-\delta\tau}d^*e^{\delta\tau}\alpha+\mathbf{i}\,\mathrm{Im}\,\langle\Psi,\psi\rangle, d^+\alpha-2Q(\Psi,\psi), D_A\psi+\alpha\cdot\Psi\big).\end{gathered}$$

To analyze this map we conjugate it by the isometry $T_\delta = e^{-\varepsilon\tau}\colon L^2\to L^2_\delta$, where $\varepsilon=\delta/2$. This gives a map F between the spaces of L^2 sections that sends (α,ψ) to

$$\begin{pmatrix} d^*\alpha-\varepsilon\langle\alpha,d\tau\rangle+\mathbf{i}\,\mathrm{Im}\,\langle\Psi,\psi\rangle\\ d^+\alpha-\varepsilon(d\tau\wedge\alpha)^+-2Q(\Psi,\psi)\\ D_A\psi-\varepsilon d\tau\cdot\psi+\alpha\cdot\Psi\end{pmatrix}.$$

For the purpose of computing the asymptotic operator of F we only need to understand its form on the cylinder $C=[0,\infty)\times Y$. Recall that $W^+\cong p_2^*W^Y_E$ over C, where W^Y_E is the bundle of spinors on Y and $p_2\colon[0,\infty)\times Y\to Y$ is the projection. Moreover, the Clifford multiplication by dt induces an isomorphism between W^+ and W^-. The bundles of forms on the cylinder are given by $\Lambda^1(C)=p_2^*(\Lambda^0(Y)\oplus\Lambda^1(Y))$ and $\Lambda^{2,+}(C)\cong p_2^*\Lambda^1(Y)$, where the last isomorphism follows from the fact that any self-dual two-form on the cylinder is of the form $dt\wedge\gamma_t+*_3\gamma_t$ for some path γ_t of one-forms on Y. Writing $\alpha=\mathbf{i}f dt+\mathbf{i}\beta$, $F(\mathbf{i}f,\mathbf{i}\beta,\psi)$ is given by

$$\left(-\mathbf{i}\Big(\frac{\partial f}{\partial t}-d_3^*\beta+\varepsilon f\Big),\mathbf{i}\Big(\frac{\partial\beta}{\partial t}-d_3f+*_3d_3\beta-\varepsilon\beta\Big),\frac{\partial\psi}{\partial t}-D_{B_\infty}\psi-\varepsilon\psi\right)+o(1).$$

Up to an obvious isomorphism, F is of the form $\frac{\partial}{\partial t} - G + o(1)$, where the *asymptotic operator* G acts on the space of sections of $\Lambda^0(Y) \oplus \Lambda^1(Y) \oplus W_E^Y$ via the matrix operator

$$\begin{bmatrix} -\varepsilon & d_3^* & 0 \\ d_3 & -*_3 d_3 + \varepsilon & 0 \\ 0 & 0 & D_{B_\infty} + \varepsilon \end{bmatrix}.$$

Notice that G splits as the sum of (the perturbations of) the asymptotic operators corresponding to the anti-self-duality (ASD) operator and the Dirac operator on Y. By results of Lockhart and McOwen [11], the operator F (and hence F_δ) is Fredholm if the kernel of G is trivial. For the ASD part this follows from the computation of the spectrum of this operator (see the proof of Proposition 7.2): for $\varepsilon = 0$ the ASD asymptotic operator has nontrivial kernel, whereas for positive ε the kernel is trivial. For the Dirac part recall that by our choice of the Spin^c structure on X, the space of reducible solutions on Y in the induced Spin^c structure is nondegenerate, hence the kernel of D_{B_∞} is trivial. Moreover, the spectrum of D_{B_∞} depends only on the gauge equivalence class of B_∞; from compactness of the fundamental domain for the action of the gauge group on the space of flat connections on Y, invertibility of the operators, and the fact that D_{B_∞} has discrete spectrum, it follows that G has trivial kernel for all small enough positive ε.

Above we showed that the operator F (and hence F_δ), associated to an arbitrary configuration (A, Ψ), is Fredholm. To finish the proof we only need to show that the operator F_δ depends continuously on the configuration (A, Ψ). Suppose (A', Ψ') is another configuration; denote the difference $(A', \Psi') - (A, \Psi)$ by (a, ϕ). Then the difference of the two linearizations is a bilinear map in $((a, \phi), (\alpha, \psi))$. The required continuity now follows from the continuity of the Sobolev multiplication $L^2_{2,\delta} \times L^2_{2,\delta} \to L^2_{1,\delta}$. q.e.d.

7. Index of the deformation complex

We use the Atiyah-Patodi-Singer index formula (see [1]) to compute the index of the fiber complex. This requires the operator to be independent of the t variable along the cylinder, which is not the case at a solution to the Seiberg-Witten equations. However, according to Proposition 6.2, the index can be computed using any configuration with a reducible limit, in particular one which agrees with the pull-back of a reducible configuration on Y along the cylinder. Recall that the Atiyah-Patodi-Singer index can be expressed in terms of the (extended) L^2 solutions; since by Proposition 6.2 the asymptotic operator on L^2 has no kernel, the Atiyah-Patodi-Singer index of the fiber complex agrees with its Fredholm index.

We first recall the relevant results from [1]. Let E_0, $E_1 \to Z_1$ be hermitian vector bundles of the same fiber dimension over $Z_1 = \{z \in Z \mid \tau(z) \leq 1\}$ and let $F\colon \Gamma(E_0) \to \Gamma(E_1)$ be an elliptic operator. Assume that on the cylinder $[0,1] \times Y \subset Z_1$ we have $F = \sigma(\frac{\partial}{\partial t} - G)$, for some self-adjoint operator $G\colon \Gamma(E) \to \Gamma(E)$, where $E = E_0|_Y$ and σ is an isomorphism between E_0 and E_1 over the cylinder. The new ingredient in the index formula (compared to the formula over a closed manifold) is a boundary correction term, which is a spectral function of G. More precisely, let $\eta(s) = \sum_{\lambda \neq 0} \operatorname{sign}(\lambda)|\lambda|^{-s}$, where λ runs over the spectrum of G, be the *eta function* of G. This series defines a holomorphic function in a half-plane $Re(z) > z_0$ and extends to a meromorphic function on the entire plane; this extension has a finite value at 0. The correction term is defined in terms of $\eta(0)$ and the dimension h of the kernel of G.

The Atiyah-Patodi-Singer domain of F, denoted by $\Gamma(E_0, P)$, consists of all the sections of E_0 whose restriction to the boundary $1 \times Y$ of Z_1 lies in the kernel of P. Here $P = P_{\geq 0}\colon \Gamma(E) \to \Gamma(E)$ denotes the spectral projection of G, corresponding to the nonnegative eigenvalues, i.e., the orthogonal projection onto the subspace spanned by the eigenvectors of G with nonnegative eigenvalues. Notice that with this choice of domain any solution of $F = 0$ extends to an L^2 solution on Z.

Theorem 7.1. *With the above notation, $F\colon \Gamma(E_0, P) \to \Gamma(E_1)$ has a finite index given by*

$$\operatorname{ind}_{APS} F = \int_{Z_1} k - \frac{h + \eta(0)}{2},$$

where k is a differential form on Z_1 determined by F, called the index density *of F. Moreover, $\operatorname{ind}_{APS} F = h(E_0) - h(E_1) - h_\infty(E_1)$, where $h(E_0)$ is the dimension of the kernel of F on the space of L^2 sections on Z, $h(E_1)$ is the corresponding dimension for the adjoint F^* of F, and $h_\infty(E_1)$ is the dimension of the space of asymptotic values of extended L^2 solutions of F^*.*

To compute the index of the fiber complex we need the signature eta invariant of Y, which was computed by Komuro [5], and the eta invariant of the perturbed Dirac operator on Y (see Section 2), which was computed by Nicolaescu [17]; the latter paper also contains a derivation of the index formula.

Proposition 7.2. *Let X be a closed four-manifold and $\Sigma \subset X$ a smooth surface representing the homology class $d\xi$, where $d > 1$ and $\xi \in H_2(X)$ is a primitive class of positive self-intersection. Given a* Spin^c *structure on X with the determinant line* Det, *we write*

$$\langle c_1(\mathrm{Det}), \xi \rangle = k + (2s+1)d\xi^2$$

for some characteristic number k and some $s \in \mathbf{Z}$. The index of the fiber complex (5), *associated to the space of Seiberg-Witten solutions on the cylindrical end manifold $Z = X - \Sigma$, that along the end converge to a fixed reducible solution $(B_\infty, 0)$, is*

$$(6) \quad \frac{1}{4}\int_Z c_1(B)^2 - \frac{\sigma(X)}{4} + \frac{(k - d\xi^2)^2}{4\xi^2} + 1 + b_1(X) - b_2^+(X) - 2g,$$

where B is a unitary connection in $\mathrm{Det}|_Z$, *which agrees with the pull-back of B_∞ on the end $[0, \infty) \times Y$.*

Remark. By rewriting the above dimension formula, one can obtain the Frøyshov invariant (see [2]) of the circle bundle Y for a range of Spin^c structures.

Proof. We will compute the (real) index of the linearization of the Seiberg-Witten map at a configuration $(B, 0)$ with asymptotic value $(B_\infty, 0)$. In this case the associated operator F (defined in the proof of Proposition 6.2) on the spaces of L^2 sections takes the form

$$F(\alpha, \psi) \mapsto \big(d^*\alpha, d^+\alpha, D_B\psi\big) - \varepsilon\big(\langle \alpha, d\tau\rangle, (d\tau \wedge \alpha)^+, d\tau \cdot \psi\big).$$

Clearly F splits as the sum $F_0 \oplus F_1$, where F_0 is a zeroth-order perturbation of the anti-self-duality operator $\mathcal{A} = d^* \oplus d^+$, and F_1 is a zeroth-order perturbation of the Dirac operator D_B. Hence we can split the index computation accordingly.

Index of the anti-self-dual part: For the purpose of invoking the Atiyah-Patodi-Singer index theorem we complexify the spaces of forms. The index density in the statement of Theorem 7.1 depends only on the principal symbol of the operator; for F_0 it is therefore determined by $\mathcal{A}$. The difference in the indices of F_0 and $\mathcal{A}$ comes from the correction term in the index formula and can be described as the spectral flow of a family of associated asymptotic operators as made precise below.

Notice that $\mathcal{A}$ is (isomorphic to) the adjoint of the operator A_+ from [1] and we will use the following fact from there. The Atiyah-Patodi-Singer index computation gives

$$\frac{1}{2}(\sigma(Z)+\chi(Z)) = \int_Z k - \frac{1}{2}\eta_{\mathrm{sign}}(0),$$

where $\sigma(Z)$ and $\chi(Z)$ are the signature and the Euler characteristic of Z respectively, k is the index density of $\mathcal{A}$ and $\eta_{\mathrm{sign}}(0)$ is its eta invariant. Therefore we obtain

$$\mathrm{ind}_{APS}(\mathcal{A}) = -\int_Z k - \frac{1}{2}(h - \eta_{\mathrm{sign}}(0)) = -\frac{1}{2}(\sigma(Z)+\chi(Z)+h),$$

where h is the dimension of the kernel of the asymptotic operator

$$G_0 = \begin{bmatrix} 0 & d_3^* \\ d_3 & -*_3\, d_3 \end{bmatrix} : \Omega^0(Y)\oplus\Omega^1(Y) \to \Omega^0(Y)\oplus\Omega^1(Y).$$

A pair (f,β) in the kernel of G_0 satisfies $d_3^*\beta = 0$ and $d_3 f = *_3 d_3\beta = 0$. Hence the kernel of G_0 consists of harmonic forms and we have $h = 1+2g$. Using this, along with $\sigma(X) = \sigma(Z)+1$, $\chi(X) = \chi(Z)+2-2g$, and $\sigma(X)+\chi(X) = 2 - 2b_1(X) + 2b_2^+(X)$, we obtain $\sigma(Z)+\chi(Z) = -1+2b_1(X)+2b_2^+(X)+2g$; thus $\mathrm{ind}_{APS}(\mathcal{A}) = b_1(X) - b_2^+(X) - 2g$.

Notice that the asymptotic operator of F_0 is of the form $G_0 + \varepsilon E$, where $E = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. It will be convenient to consider the family of operators $G_u = G_0 + u\varepsilon E$ for $u \in [0,1]$, connecting the limiting operators G_0 and G_1 of $\mathcal{A}$ and F_0 respectively. The difference in the correction terms for G_0 and G_1 is equal to the spectral flow of the family G_u. More precisely, let $\widehat{\eta}_u = \frac{1}{2}(h_{G_u} + \eta_{G_u}(0))$ denote the *reduced eta invariant* of G_u. From the definition it is clear that $\widehat{\eta}_u$ has a jump at $u = a$ only if some eigenvalue $\lambda(u)$ of G_u vanishes at a (but is not zero at least on one side of $u = a$). The case of interest to us is when any eigenvalue $\lambda(u)$ behaves in one of the following ways: $\lambda(u) = 0$ for all u, $\lambda(u) \neq 0$ for all u, or $\lambda(u) = 0$ iff $u = 0$ and this zero is transverse. Assume first that $\lambda(u)$ is the only eigenvalue crossing 0 at $u = 0$. If $\lambda'(0) > 0$, then $\widehat{\eta}_1 = \widehat{\eta}_0$, since at $u = 0$ the eigenvalue $\lambda(0)$ contributes $+1$ to h and nothing to η, whereas for $u > 0$ it contributes $+1$ to η and nothing to h. Similar considerations in the case $\lambda'(0) < 0$ imply that $\widehat{\eta}_1 = \widehat{\eta}_0 - 1$, since in this case $\lambda(u)$ contributes to h and η with the opposite signs. This clearly generalizes to a finite number of eigenvalues crossing 0 at $u = 0$.

To determine the difference of indices of $\mathcal{A}$ and F_0 we therefore need to understand the behavior of the eigenvalues of the family G_u. A pair $(f, \beta) \in \Omega^0(Y) \oplus \Omega^1(Y)$ in the kernel of G_u satisfies

$$d_3^*\beta = \varepsilon u f, \quad d_3 f - *_3 d_3 \beta = -\varepsilon u \beta.$$

This implies $d_3^* d_3 f = -u^2\varepsilon^2 f$, which has no solutions for $u \neq 0$ since the Laplace operator $d_3^* d_3$ is positive definite. For $u = 0$ we computed above that the dimension of the kernel is $1 + 2g$. The eigenvalue $\lambda_1(u) = -\varepsilon u$, corresponding to the space of constant functions, has multiplicity 1 (at 0), whereas the multiplicity of $\lambda_2(u) = \varepsilon u$, corresponding to the space of harmonic one-forms, is $2g$. We conclude from the previous paragraph that $\widehat{\eta}_1 = \widehat{\eta}_0 - 1$, and hence the index of F_0 is $1 + b_1(X) - b_2^+(X) - 2g$.

Index of the Dirac part: Using similar considerations as above we see that the index of F_1 is equal to the index of D_B; the reason for this is that all the asymptotic operators $D_{B_\infty} + u\varepsilon$ are invertible (for $u \in [0,1]$) by Theorem 2.2 and our choice of ε. The index density of D_B is $\frac{1}{8}(c_1(B)^2 - L(\nabla^Z))$, where $L(\nabla^Z) = \frac{1}{3}p_1(\nabla^Z)$ is the Hirzebruch L-class associated to the background connection ∇^Z in Z; recall that ∇^Z agrees with the Levi-Civita connection ∇^{LC} on the complement of $(-1, \infty) \times Y$ and agrees with the pull-back of ∇^Y on $[0, \infty) \times Y$. From Theorem 7.1 we get $\mathrm{ind}_{APS}(D_B) = \frac{1}{8}\int_Z (c_1(B)^2 - L(\nabla^Z)) - \frac{1}{2}\eta_{D_{B_\infty}}(0)$. The eta invariant of the Dirac operator on Y, coupled to the flat connection B_∞, was computed in [17] and is equal to

$$\eta_{D_{B_\infty}}(0) = -\frac{ad^2}{6} - \frac{k^2}{4a} + \frac{kd}{2},$$

where we used the fact that the radius of the fiber circles in Y is 1. The only other term we need to interpret is the integral of the L-class. If we were using the Levi-Civita connection as the background connection on Z, then we could use the fact that $L(\nabla^{LC})$ is the index density of the signature operator on Z. In particular, we have $\sigma(Z) = \int_Z L(\nabla^{LC}) - \eta_{\mathrm{sign}}(0)$. The signature eta invariant $\eta_{\mathrm{sign}}(0)$ for Y was computed in [5] and is given by

$$\eta_{\mathrm{sign}}(0) = 1 - \frac{ad^2}{3} + \frac{2ad^2}{3}(a^2d^4 + 2g - 2).$$

In our case, however, there is another term coming from the difference in the L-classes of the two connections: we have

$$\int_{Z_0} L(\nabla^Z) = \int_{Z_{-1}} L(\nabla^{LC}) + \int_{[-1,0]\times Y} L(\nabla^Z).$$

The last term in this expression can be computed explicitly; write $\nabla^Z = \nabla^\infty + f(t)\alpha$, where ∇^∞ denotes the pull-back connection, $\alpha = \nabla^\infty - \nabla^{LC}$ and f is a smooth non-decreasing function that maps $[-1, 0]$ onto itself. The computation can be done with respect to a local orthonormal coframe $(\varphi_1, \varphi_2, \varphi)$ on Y, where $\mathbf{i}\varphi$ is the connection of the circle bundle $Y \to \Sigma$ and (φ_1, φ_2) is the pull-back of a local coframe on Σ. This yields

$$\int_{[-1,0]\times Y} L(\nabla^Z) = -\frac{2ad^2}{3}(a^2d^4 + 2g - 2)$$

(see [17] for more details). Note that the index formula gives the complex index of the operator F_1. q.e.d.

8. Dimension of the cylindrical end moduli space

To express the formal dimension of the moduli space of Seiberg-Witten solutions on the cylindrical end manifold $Z = X - \Sigma$ in terms of the data on the closed manifold X we need the following result.

Lemma 8.1. *Let X be a closed four-manifold and let $\Sigma \subset X$ be an embedded surface with self-intersection $n \neq 0$. Denote by Z the complement $X - \Sigma$, thought of as a manifold with a cylindrical end $[0, \infty) \times Y$. Given a* Spin^c *structure on X, let $p = \langle c_1(\text{Det}), [\Sigma]\rangle \in \mathbf{Z}$, where* Det $\to X$ *denotes the determinant line of the* Spin^c *structure. For any unitary connection B on $L =$* Det $|_Z$*, whose restriction to the cylinder $[0, \infty) \times Y$ agrees with the pull-back of a flat connection in* Det $|_Y$*, we have*

$$\int_Z c_1(B)^2 = c_1(\text{Det})^2 - \frac{p^2}{n}. \tag{7}$$

Proof. Let $Z_1 = \{z \in Z \,|\, \tau(z) \leq 1\}$ and let Y be the oriented boundary of Z_1. We think of X as the union of Z_1 and a compact tubular neighborhood N of Σ in X. Denote by $[0, 1] \times Y$ the oriented collar to the boundary Y of N. Suppose A is a connection in Det $|_N$ that in a neighborhood of the boundary $0 \times Y$ agrees with the pull-back of B_∞, the latter being the limit of B. Then A and B together define a (smooth) connection in Det and we have $c_1(\text{Det})^2 = \int_Z c_1(B)^2 + \int_N c_1(A)^2$. We will evaluate the second integral. Combining Proposition 6.2 and the index formula (6), we see that, for the purpose of the computation, we can choose A to be any connection in Det $|_N$ which in a neighborhood of

Y agrees with the pull-back of some flat connection in $\mathrm{Det}|_Y$. In what follows we use the notation from Section 2.

Let A_1 be the pull-back of a constant curvature connection in $\mathrm{Det}\,|_\Sigma$ to $\mathrm{Det}\,|_N$; then $F_{A_1} = -2\pi \mathbf{i} p\omega$. We let $A = A_1 + \mathbf{i}\frac{p}{n} f(t)\varphi$, where $i\varphi$ is a constant curvature connection of the circle bundle $Y \to \Sigma$ and $f\colon [0,1] \to [0,1]$ is a smooth non-increasing function which is identically 1 in a neighborhood of 0 and identically 0 in a neighborhood of 1. Since the degree of Y is $-n$, we have $d\varphi = 2\pi n\omega$, so A is flat close to the boundary of N. A simple computation shows that $F_A = 2\pi \mathbf{i} p(f(t) - 1)\omega + \mathbf{i}\frac{p}{n} f'(t)dt \wedge \varphi$ and hence

$$F_A \wedge F_A = -4\pi \frac{p^2}{n}\big(f(t) - 1\big) f'(t)\, dt \wedge \mathrm{vol}_Y.$$

Using this along with $\mathrm{vol}\,(Y) = 2\pi$ and $c_1(A) = \frac{\mathbf{i}}{2\pi} F_A$ gives

$$\int_Z c_1(A)^2 = 2\frac{p^2}{n}\int_0^1 (f(t) - 1) f'(t) dt = \frac{p^2}{n}.$$

q.e.d.

Now we can obtain a convenient formula for the (formal) dimension of the moduli space of Seiberg-Witten solutions on Z with reducible limits.

Corollary 8.2. *Let X be a closed four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Suppose $\Sigma \subset X$ is a smooth surface representing the homology class $d\xi$, where $d > 1$ and $\xi \in H_2(X)$ is a primitive class of positive self-intersection. Let $Z = X - \Sigma$, thought of as a cylindrical end manifold. Given a* Spin^c *structure on X, let $p = \langle c_1(\mathrm{Det}), \xi\rangle$, where* Det $\to X$ *denotes the determinant line of the* Spin^c *structure; we write $p = k + (2s + 1)d\xi^2$ for some characteristic number k and some $s \in$* **Z**. *The formal dimension of the based moduli space of Seiberg-Witten solutions on Z with reducible limits is given by*

$$\frac{c_1(\mathrm{Det})^2 - \sigma(X)}{4} + \frac{(k - d\xi^2)^2 - p^2}{4\xi^2}. \tag{8}$$

Moreover, this dimension depends only on the induced Spin^c *structure on Z, i.e., it is independent of s.*

Proof. Starting with the index of the fiber complex, given by (6), recall from the discussion preceding Proposition 6.2 that we need to add $2g$ to it (where g is the genus of Σ) to obtain the index of the full deformation complex. The formula now follows from the lemma above with $n = (d\xi)^2$ and p replaced by pd.

Considering $c_1(\mathrm{Det})$ as a class in $H^2(X;\mathbf{R})$, we write

$$c_1(\mathrm{Det}) = \frac{p}{a}\alpha + c$$

for some $c \in H^2(X;\mathbf{R})$, where α is the Poincaré dual of ξ and $a = \xi^2$. It follows that $c \cup \alpha = \langle c, \xi \rangle = 0$, hence $c_1(\mathrm{Det})^2 = p^2/a + c^2$, which implies the following expression for the formal dimension:

$$\frac{c^2 - \sigma(X)}{4} + \frac{(k - d\xi^2)^2}{4\xi^2}.$$

Note that since α and c are orthogonal and $b_2^+(X) = 1$, c^2 cannot be positive.

Finally, consider another Spin^c structure on X which induces the same Spin^c structure on Z. Then its determinant line, denoted by Det$'$, satisfies $c_1(\mathrm{Det}') = c_1(\mathrm{Det}) + 2sd\alpha$ for some $s \in \mathbf{Z}$ (Proposition 4.1). This shows that c', defined analogously as c above, equals c; the last assertion of the corollary then follows from the above expression for the formal dimension. q.e.d.

9. Compactness and regularity of the cylindrical end moduli space

Let Σ be an embedded surface of positive self-intersection in a closed four-manifold X with $b_1(X) = 0$. Denote by Z the complement $X - \Sigma$, thought of as a manifold with a cylindrical end $[0,\infty)\times Y$. We topologize the moduli space of exponentially decaying Seiberg-Witten solutions on Z by the weakest topology, containing the topology of uniform C^k convergence on compact subsets for some $k \geq 3$, with respect to which the Chern-Simons-Dirac functional along the cylinder is continuous at infinity.

Proposition 9.1. *Fix a* Spin^c *structure on X for which there are no irreducible Seiberg-Witten solutions on Y in the induced* Spin^c *structure. Then the moduli space of exponentially decaying Seiberg-Witten solutions on Z (in the induced* Spin^c *structure) with reducible limits is compact.*

Proof. Given a sequence (A_n, Ψ_n) of Seiberg-Witten solutions on Z, observe that this sequence has a convergent subsequence on any submanifold $Z_t \subset Z$ (after appropriate changes of gauge). This is essentially a consequence of [6, Lemma 4]. The uniform boundedness of $|\Psi_n|$ on Z follows from the fact that the configurations converge to zero at infinity, and from a standard maximum principle argument [6, Lemma 2]. The only difference is that we are not using the Levi-Civita connection as the background connection. However, the two Dirac operators differ by Clifford multiplication by a one-form $f(t)dt$, where f is a smooth bounded function [15, Lemma 5.2.1], and it easily follows from this that the result holds for the perturbed Dirac operator as well.

By the diagonal argument we can find a subsequence of (A_n, Ψ_n) which, after appropriate changes of gauge, converges on all compact subsets of Z to some solution (A, Ψ) of the Seiberg-Witten equations on Z; we still denote this subsequence by (A_n, Ψ_n). Potential noncompactness therefore arises from the behavior of solutions on the end $[0, \infty) \times Y$. In particular, the convergence of the sequence (A_n, Ψ_n) depends on the convergence of the sequence of its asymptotic values. Recall that the moduli space of asymptotic values is identified with the space $\mathcal{H}$ of imaginary valued harmonic one-forms on Y.

We first prove that the sequence of the asymptotic values of (A_n, Ψ_n) is bounded; in fact, the moduli space is contained in the fiber of the projection to the space of asymptotic values. Let (A, Ψ) and (B, Φ) be two solutions with reducible asymptotic values. Then $\alpha = A - B$ exponentially decays to a form $\alpha_\infty \in \mathcal{H}$. To prove that $\alpha_\infty = 0$ it suffices to show that $\int_K \alpha_\infty = 0$ for any embedded circle $K \subset Y$ representing a homology class in the kernel of the morphism $H_1(Y) \to H_1(Z)$. Since such a K is the boundary of a surface $S \subset Z$, this is equivalent to $\int_S (F_A - F_B) = 0$. As both F_A and F_B represent the same relative cohomology class on Z, the claim follows.

Suppose now that $[(A_n, \Psi_n)]$ does not converge to $[(A, \Psi)]$ in the topology of the moduli space. This means that the Chern-Simons-Dirac functional has different values on the asymptotic configurations of (A_n, Ψ_n) and (A, Ψ). Then the asymptotic value of (A, Ψ) is irreducible (see [13, Proposition 8.5] for more details), which contradicts the assumption that the space of solutions on Y consists entirely of reducibles. q.e.d.

Now we turn to the question of regularity of the moduli space. We are particularly interested in the behavior of the Dirac operator at a reducible solution; a more general result regarding such Dirac operators on a closed manifold is proved in [21]. For technical reasons we choose to work with $L^2_{k,\delta}$ ($k \geq 4$) configuration space.

Proposition 9.2. *Let X be a closed oriented four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Suppose that $\Sigma \subset X$ is a smooth surface of positive self-intersection, and let $Z = X - \Sigma$, thought of as a cylindrical end manifold. Then for any small enough $\delta > 0$ there exists a second category subset Ω of imaginary-valued $L^2_{k-1,\delta}$ self-dual two-forms on Z, such that for any $\omega \in \Omega$ the following holds. For any exponentially decaying connection A in the determinant line $L \to Z$ satisfying $F_A^+ = \omega$, the Dirac operator D_A is either injective or surjective. Moreover, the irreducible part of the ω-perturbed moduli space of exponentially decaying Seiberg-Witten solutions on Z with reducible asymptotic values is a smooth orientable finite dimensional manifold.*

Proof. Fix a smooth unitary connection A_0 in the determinant line $L \to Z$ and let $\mathcal{N}$ be the manifold of all exponentially decaying configurations (A, Ψ) (with reducible limits) satisfying

$$T(A, \Psi) = (d^*(A - A_0), D_A\Psi) = 0, \quad \Psi \neq 0.$$

To prove that $\mathcal{N}$ is a smooth manifold we need to verify that the differential

$$D_{(A,\Psi)}T(\alpha, \psi) = (d^*\alpha, D_A\psi + \alpha \cdot \Psi)$$

of T at (A, Ψ) is onto. The adjoint of this operator is

$$(f, \chi) \mapsto (df + \mathbf{i}\langle \_ \cdot \Psi, \chi\rangle - \delta f\, d\tau, D_A\chi + \delta d\tau \cdot \chi),$$

where we used the usual L^2_δ inner product on the space of imaginary valued one-forms and the real part of the hermitian L^2_δ inner product on spinors. Expression $\mathbf{i}\langle \_ \cdot \Psi, \chi\rangle$ denotes the imaginary valued one-form characterized by

$$\langle \alpha, \mathbf{i}\langle \_ \cdot \Psi, \chi\rangle\rangle = \mathrm{Re}\,\langle \alpha \cdot \Psi, \chi\rangle,$$

for all imaginary valued one-forms α.

Rather than proving directly that the kernel of the adjoint is trivial, we will replace the adjoint by the operator of the same kind with $\delta = 0$. The injectivity of thus obtained operator D implies the injectivity of the adjoint for $\delta > 0$ small enough.

Suppose that $D(f, \chi) = 0$. Computing with respect to a local orthonormal coframe $\{\varphi_1, \dots, \varphi_4\}$ on Z we obtain

$$\mathbf{i}d^* df = \operatorname{Re}\langle \mathbf{i}D_A\Psi, \chi\rangle + \langle \mathbf{i}\Psi, D_A^*\chi\rangle \\ - \operatorname{Re}\left\langle \mathbf{i}\sum_j (\nabla_{\varphi_j}\varphi_j + (*d*\varphi_j)\varphi_j)\cdot\Psi, \chi\right\rangle.$$

The first two terms in this expression vanish as Ψ and χ are harmonic. Moreover, the one-form inside the last term vanishes for Levi-Civita connection. The background connection we are using differs from the Levi-Civita connection on the cylinder $[0, \infty) \times Y$ by a multiple of

$$\begin{pmatrix} 0 & \varphi_3 & \varphi_2 & 0 \\ -\varphi_3 & 0 & -\varphi_1 & 0 \\ -\varphi_2 & \varphi_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(see [15, Lemma 5.2.1]), where we took (φ_1, φ_2) to be the pull-back of an orthonormal coframe on Σ and $\mathbf{i}\varphi_3$ to be the connection one-form of the circle bundle Y. So the last term in the above expression vanishes as well. This implies that f is constant, and since f exponentially decays to zero, $f = 0$. Finally, since $\langle \alpha\cdot\Psi, \chi\rangle = 0$ for any α, where Ψ and χ are harmonic, it follows from unique continuation property for harmonic spinors that $\chi = 0$ (see [14, Lemma 6.2.1]). This proves that $\mathcal{N}$ is smooth. In particular, at any point $(A, \Psi) \in \mathcal{N}$,

$$\operatorname{im} D_A + \{\alpha\cdot\Psi \mid d^*\alpha = 0\} = L^2_{3,\delta}(W^-).$$

Let Ω_0 be the set of regular values of the map $\mathcal{N} \to L^2_{3,\delta}(\mathbf{i}\Lambda^{2,+}(Z))$, $(A, \Psi) \mapsto F_A^+$. Given $\omega \in \Omega_0$, let $(A, \Psi) \in \mathcal{N}$ be a point satisfying $F_A^+ = \omega$. Now the differential d^+ from the tangent space to $\mathcal{N}$ at (A, Ψ) to the space of imaginary self-dual two-forms is onto by the choice of ω. Further, from Hodge decomposition of $L^2_{k,\delta}$ forms on Z (see [10]; note that we chose δ so that the Laplace operator is Fredholm), and the assumption $H^1(Z) = 0$, it follows that the space of co-closed one-forms maps isomorphically onto the space of self-dual two-forms. Thus for any co-closed one-form α there exists a spinor ψ so that $D_A\psi + \alpha\cdot\Psi = 0$. Combining this with the observation at the end of the previous paragraph proves that D_A is onto.

The proof of regularity of the irreducible part proceeds as for a closed manifold (see [14]); we get a set Ω_1 of regular perturbation parameters and let $\Omega = \Omega_0 \cap \Omega_1$. q.e.d.

Part II: Genus bounds

10. An example: genus bounds in $\mathbf{CP}^2$

Before stating and proving the main theorem (in the next section), we will demonstrate the argument in the simplest possible case, for $X = \mathbf{CP}^2$. Let $\xi = [\mathbf{CP}^1]$ be the standard generator of $H_2(X)$; then $\xi^2 = 1$. We fix $d > 1$ and consider a smooth genus g representative Σ of the class $d\xi$. A Spin^c structure on X is uniquely determined by $p = \langle c_1(\mathrm{Det}), \xi\rangle$. As before we write $p = k+d$ for some characteristic number k. We will see in the proof of Theorem 11.1 that it suffices to consider $k \in \{0, \ldots, d\}$. Recall from the discussion preceding Definition 4.2 that the line bundle, determining the induced Spin^c structure on Y, is the pull-back of a line bundle $E_0 \to \Sigma$ with $c_1(E_0) = kd/2$. According to part (b) of Theorem 2.2, irreducible solutions in the given Spin^c structure on Y exist only if

$$g > \frac{kd}{2}. \tag{9}$$

The general formula (8) for the formal dimension of the based moduli space of Seiberg-Witten solutions with reducible limits on Z in the case we are considering becomes

$$\frac{(k-d)^2 - 1}{4}. \tag{10}$$

Notice that this number is even and by 1 greater than the expected dimension of the moduli space.

Using inequality (9) and the dimension formula (10) we obtain a lower bound on the genus g of Σ based on the following observation (explained in the proof of Theorem 11.1): if the moduli space of Seiberg-Witten solutions with reducible limits on Z is compact and positive dimensional, this leads to a contradiction. In other words, using Proposition 9.1, if the moduli space is positive dimensional, then (9) must hold.

The dimension (10) of the based moduli space is positive for $k < d-1$ in the range of k's considered. Since d and k have different parity, we conclude (for $d \geq 3$) that

$$g > \frac{(d-3)d}{2}.$$

Since the classes $d[\mathbf{CP}^1]$ for $d = 1, 2$ are represented by spheres, this inequality is equivalent to the *Thom conjecture*, which was first established by Kronheimer and Mrowka [6].

Theorem 10.1. *Let $\Sigma \subset \mathbf{CP}^2$ represent the class $d[\mathbf{CP}^1]$ for some $d \geq 1$. Then the genus g of Σ satisfies*

$$g \geq \frac{(d-1)(d-2)}{2}. \tag{11}$$

Moreover, this lower bound is attained by a smooth holomorphic curve representing this homology class.

Remark.

(a) Note that analogous genus bound holds for classes $d[\mathbf{CP}^1]$ with $d \leq -1$, where d gets replaced by $|d|$ in (11). This observation is true in general, since we can always replace ξ by $-\xi$.

(b) From above computations we see that genus bound (11) holds in any X which is a rational homology $\mathbf{CP}^2$: by possibly changing the orientation we can assume that X is positive definite and let ξ be a generator of $H_2(X)$. However, it is not true in general that this bound is the best possible for any rational homology $\mathbf{CP}^2$. As an example, consider Mumford surface [16], which is an algebraic surface of general type with the canonical class 3. According to the generalized symplectic Thom conjecture (see [13]) the minimal genus in the class of multiplicity $d > 0$ equals

$$\frac{(d+1)(d+2)}{2}.$$

11. The main theorem

In the previous section we saw how the moduli space of Seiberg-Witten solutions on the complement of an embedded surface, representing a given homology class in $\mathbf{CP}^2$, can be used to derive a lower bound for the genus of any smooth representative of this class. Below we generalize this result to a bigger class of four-manifolds with $b_2^+ = 1$. The bound is not explicit, but it can be effectively computed in many specific cases. The idea of the proof is analogous to Kronheimer's proof of Donaldson's Theorem on definite intersection forms of closed manifolds (see [21]).

Theorem 11.1. *Let X be a smooth closed oriented four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. Suppose $\Sigma \subset X$ is a smooth surface representing homology class $d\xi$, where $d > 1$ and $\xi \in H_2(X)$ is a primitive class of positive self-intersection. Let K be the set of all characteristic numbers $k \in \{0, \ldots, d\xi^2\}$ for (X, ξ, d) which satisfy the following condition: there exists a* Spin^c *structure on X such that*

$$c_1(\mathrm{Det})^2 > \sigma(X) + 4kd, \tag{12}$$

and $\langle c_1(\mathrm{Det}), \xi\rangle = k + d\xi^2$, where Det *is the determinant line of the* Spin^c *structure. Suppose that K is not empty and let k_0 be the maximum of K. Then the genus g of Σ satisfies*

$$g > \frac{k_0 d}{2}. \tag{13}$$

Proof. We will use our standard notation $Z = X - \Sigma$ for the cylindrical end manifold and Y for the boundary of a tubular neighborhood of Σ (oriented as the 'boundary' of Z). Choose a regular perturbation $\omega \in \Omega$ (see Proposition 9.2); then the irreducible part $\mathcal{M}^*$ of the perturbed moduli space $\mathcal{M}$ is smooth. Notice that (12) is equivalent to the dimension of the (perturbed) based moduli space $\widetilde{\mathcal{M}}$ being positive (by Corollary 8.2); since this dimension is even, this also implies that the dimension of $\mathcal{M}$ is positive. With this remark, the statement of the theorem is equivalent to the following claim, which we prove below: if the moduli space of Seiberg-Witten solutions on Z is positive dimensional for a given Spin^c structure, then it is not compact.

Suppose contrary to the statement of the theorem that for some $k \in K$, $kd/2 \geq g$. Fix a Spin^c structure on X for which (12) holds for k. According to part (b) of Theorem 2.2 there are no irreducible solutions on Y in the induced Spin^c structure and therefore the moduli space $\mathcal{M}$ is compact (see Proposition 9.1).

Next we show that $\mathcal{M}$ contains a unique reducible point $[A, 0]$. Since $\omega \in \Omega$, d^+ is a surjection from the space of extended one-forms to the space of self-dual two-forms. Hence the equation $F_A^+ = \omega$ has a solution. Suppose now that $(A', 0)$ is another solution and write $A' = A + \mathbf{i}\alpha$ for some one-form α on Z; clearly $d^+\alpha = 0$ and therefore $d\alpha = 0$. By the choice of gauge the asymptotic value h of α is harmonic. Since the class of α is trivial in $H^1(Z; \mathbf{R})$, h represents the trivial class in $H^1(Y; \mathbf{R})$, hence $h = 0$. Thus there exists a function f on Z, exponentially decaying to 0 at infinity, such that $\alpha = 2df$, i.e., $(A, 0)$ and $(A', 0)$ are gauge equivalent.

To finish the proof of the claim, we need to understand the structure of $\mathcal{M}$ at the reducible point $[A, 0]$. Recall that the index of the ASD part of the linearization at $(A, 0)$ is zero. Since its kernel also vanishes (by an argument similar to the one in the previous paragraph), the (Zariski) tangent space and the obstruction space at $(A, 0)$ correspond to the kernel and cokernel of D_A respectively. As we assumed that the index is positive, Proposition 9.2 implies that the cokernel of the Dirac operator vanishes, so the based moduli space $\widetilde{\mathcal{M}}$ is smooth. The action of the group S^1 of constant gauge transformations on the kernel of D_A is by complex multiplication, hence a closed neighborhood $\mathcal{V}$ of $[A, 0]$ in $\mathcal{M}$ is a cone on some projective space $\mathbf{CP}^n$. Let $\mathcal{N}$ be the smooth compact submanifold of $\mathcal{M}$ with the boundary $\mathbf{CP}^n$, obtained as the closure of the complement of $\mathcal{V}$. Denote by c the Chern form of the S^1 bundle $\widetilde{\mathcal{N}} \to \mathcal{N}$, where $\widetilde{\mathcal{N}} \subset \widetilde{\mathcal{M}}$ is the preimage of $\mathcal{N}$. Note that the induced S^1 bundle over the boundary $\mathbf{CP}^n$ is the tautological bundle. So $\int_{\mathbf{CP}^n} c^n = \int_{\mathcal{N}} d(c^n) = 0$ is a contradiction.

We remark that for $k = 0$, the condition $g > 0$ that we obtain from the argument above is consistent with the assumption that Σ is a sphere (needed for non-degeneracy of solutions on Y).

Finally, we check that it suffices to consider characteristic numbers $\leq ad$, i.e., that for $ad < k < 2ad$ we do not get any new restrictions on the genus (here $a = \xi^2$). Suppose that for some k in $(ad, 2ad)$ condition (12) holds for some Spin^c structure on X. Then we claim that there exists a Spin^c structure on X with the characteristic number $k' = 2ad - k$, for which (12) holds as well, hence $k' \in K$. To see this first change the given Spin^c structure by the line bundle E on X with $c_1(E) = -d\alpha$, where α is the Poincaré dual of ξ. The characteristic number of the inverse of thus obtained Spin^c structure is k'. The expression for the dimension of the moduli space is unaffected by these changes of the Spin^c structure. For the first change this follows from Corollary 8.2. For the second note that $c_1(-\mathrm{Det}) = -c_1(\mathrm{Det})$, hence the class c, defined by $\langle c_1(\mathrm{Det}), \xi\rangle = \frac{p}{a}\alpha + c$, changes sign. The claim now follows from the alternative form of the dimension formula in the proof of Corollary 8.2. Since $k' \in K$, we have seen above that $g > k'd/2 = |k - 2ad|d/2$; however, by part (b) of Theorem 2.2, this implies that there exist irreducible Seiberg-Witten solutions on Y in the Spin^c structure induced by the one given on X and hence the moduli space need not be compact. q.e.d.

Remark. We note that the leading term in the genus bound for a divisible class $d\xi$, obtained from the above theorem, equals $(d\xi)^2/2$, which is by a factor of 2 better than the bounds obtained via the G-signature Theorem (cf. Rohlin [19]).

A special case of interest occurs when the class $\xi \in H_2(X)$ is *characteristic*, that is, its Poincaré dual is a characteristic class.

Corollary 11.2. *With notation as in Theorem* 11.1, *assume that* $H_1(X) = 0$, *the signature of* X *is negative, and that* ξ *is characteristic. Then the genus* g *of any smooth surface* Σ *representing* $d\xi$ *satisfies*

$$g > \binom{d}{2}\xi^2.$$

Proof. By Furuta's 10/8 Theorem [3] (in fact by a Theorem of Donaldson), X is odd. Consider the Spin^c structure on X characterized by $c_1(\mathrm{Det}) = (2d-1)\alpha$, where α is the Poincaré dual of ξ. From $\langle c_1(\mathrm{Det}), \xi\rangle = (2d-1)a = k + ad$ we get $k = a(d-1)$, where $a = \xi^2$. Since the class c (defined in the proof of Corollary 8.2) is equal to 0 in this case, (12) is equivalent to $(k-ad)^2 > a\sigma(X)$, which is clearly true for $k = ad - a$ and $\sigma(X) < 0$. This implies $g > a(d-1)d/2$. q.e.d.

12. Geometric constructions

Let X be a smooth four-manifold. For a class $\xi \in H_2(X)$ denote by $g_\xi(d)$ the minimal genus of a smooth representative of $d\xi$; we write g_ξ for $g_\xi(1)$. In this section we will show, using some simple geometric constructions, that asymptotically $g_\xi(d)$ does not grow faster than $(d\xi)^2/2$. Combining this with genus bounds from Theorem 11.1 we conclude that $(d\xi)^2/2$ describes the dominant term in $g_\xi(d)$ (as $d \to \infty$) in a manifold X with $b_2^+(X) = 1$.

Proposition 12.1. *Let* X *be a smooth four-manifold and let* $\xi \in H_2(X)$ *be a class of positive self-intersection. Then*

$$g_\xi(d) \le \frac{(d\xi)^2}{2} - \left(\frac{\xi^2}{2} + 1 - g_\xi\right)d + 1 \tag{14}$$

for any $d > 1$. *Moreover, there exists a smooth representative of* $d\xi$ *with the genus given by the right-hand side of the above inequality.*

Proof. Let $\Sigma \subset X$ be a smooth embedded surface of genus g_ξ representing ξ and let Σ' denote Σ with the interiors of $a := \xi^2$ disjoint disks removed. Think of the normal bundle ν_Σ of Σ in X as being obtained from the product bundle over Σ' by adding a degree 1 bundle over a 2-disk for each puncture. To construct d copies Σ_i of Σ in general position, we choose d distinct parallel copies Σ_i' of Σ'. Over each 2-disk we cap-off Σ_i' by adding a disk in such a way that any two disks intersect transversely in a single point and any intersection point is common to two disks only. It is clear from the construction that the total number of intersection points between the surfaces Σ_i thus obtained equals $a\binom{d}{2}$. A neighborhood of each intersection point looks like a pair of transversely intersecting disks. Removing these and replacing them by annuli gives $\Sigma(d)$. Since the surfaces, obtained from Σ_i, $i = 1, \dots, d$ by removing small (disjoint) disks around the intersection points are disjoint, we need $d-1$ annuli to make a connected surface; each of the remaining annuli increases the genus by 1. q.e.d.

As an immediate consequence of the above inequality we obtain the following bound on the genus of a representative of a primitive class.

Corollary 12.2. *Let X be a smooth four-manifold and $\xi \in H_2(X)$ a class of positive self-intersection. If $g_\xi(d) > (d\xi^2 - \Delta_d)d/2$ for some $d > 1$, then $g_\xi > (\xi^2 - \Delta_d)/2$.*

Assume now that X is a smooth closed oriented four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. If $\xi \in H_2(X)$ is a primitive class of positive self-intersection a, then for any integer $d > 1$ we know from Theorem 11.1 that $g_\xi(d) > (ad - \Delta_d)d/2$ for some Δ_d. For example, if the signature of X is negative and ξ is characteristic, we can take $\Delta_d = a$ (see Corollary 11.2). The previous corollary then implies that a characteristic class in X is not represented by an embedded sphere.

In general we obtain the following consequence of Theorem 11.1. This result is equivalent to the *generalized adjunction inequality* of Theorem A.

Corollary 12.3. *Let X be a smooth closed oriented four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 1$. For any (primitive) class $\xi \in H_2(X)$ of positive self-intersection there exists $\Delta \geq 0$ so that*

$$g_\xi(d) > \frac{(d\xi^2 - \Delta)d}{2}$$

for all $d \geq 1$.

Proof. We first check that the set K in the statement of Theorem 11.1 is nonempty for d large enough. Choose a Spin^c structure on X with $c_1(\mathrm{Det}) = 2d\alpha - \gamma$, where γ is a characteristic vector satisfying

$\gamma^2 > \sigma(X)$ and $\gamma \cup \alpha \geq 0$. Such a characteristic vector clearly exists – starting with any characteristic vector we can get one satisfying these conditions by adding to it a large enough multiple of α. Then $k = ad - \langle \gamma, \xi \rangle$ belongs to $[0, ad]$ for large enough d. We need to check that (12) also holds:

$$c_1(\mathrm{Det})^2 - \sigma(X) - 4kd = \gamma^2 - \sigma(X) > 0.$$

Fix some d for which K is nonempty and a characteristic number $k \in K$ for (X, ξ, d); let $\Delta := ad - k$. Denote by $c_1 = c_1(\mathrm{Det})$ the Chern class of the Spin^c structure that satisfies (12) with k and d, and let $c_1(\mathrm{Det}') = c_1 + 2n\alpha$ for some integer n. Then $\Delta' = a(d+n) - k' = \Delta$ and it follows from the alternative form of the dimension formula in the proof of Corollary 8.2 that the Spin^c structure with determinant Det' satisfies (12) with k' and $d+n$ in place of k and d. This implies that the genus bound in the statement of the corollary holds for all multiplicities $d+n$ for which $k' \geq 0$; it clearly holds for the rest. q.e.d.

13. Manifolds with signature zero

In this section X denotes a smooth closed oriented four-manifold with $b_1(X) = 0$, $b_2^+(X) = 1$ and signature $\sigma(X) = 0$. Up to isomorphism, there are only two possible intersection forms such a manifold can have, distinguished by the parity. The even intersection form is given by $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and is realized for example by $S^2 \times S^2$; the odd intersection form is given by $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and is realized for example by $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$. Genus bounds that follow from Theorem A depend only on the intersection pairing of the manifold, so we need only consider two cases.

Proposition 13.1. *Suppose the intersection pairing of X is isomorphic to H; let $\{\xi_1, \xi_2\}$ be a basis of $H_2(X)$ modulo the torsion subgroup with respect to which the intersection pairing is given by H. Then any class $\xi \in H_2(X)$, whose image in $H_2(X;\mathbf{R})$ is given by $p\xi_1 + q\xi_2$ with $pq \neq 0$, satisfies*

$$g_\xi \geq (|p| - 1)(|q| - 1).$$

Proof. After possibly changing the orientation of X we may assume that the self-intersection $2pq$ of ξ is positive; then we can further assume

$p, q > 0$. Denote by $\xi_i^* \in H^2(X; \mathbf{R})$ the Hom-dual of ξ_i and consider a characteristic vector c satisfying $c = 2\xi_1^* + 2\xi_2^*$ in $H^2(X; \mathbf{R})$. Since $c^2 > 0$, the claimed bound follows from adjunction inequality (1). q.e.d.

The following result for $S^2 \times S^2$ and the corresponding result regarding classes in $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$ (see below) was proved independently by Ruberman [20] and by Li-Li [8].

Corollary 13.2. *With notation as in the previous proposition, assume further that ξ_1 and ξ_2 are represented by spheres which intersect transversely in one point. Then*

$$g_\xi = \begin{cases} (|p|-1)(|q|-1); & pq \neq 0 \\ 0; & pq = 0. \end{cases}$$

Proof. Let Σ_i be a representative of ξ_i as in the statement. Since the self-intersection of ξ_i is zero, any class with $pq = 0$ is represented by a sphere. Suppose now that $pq \neq 0$; we may assume $p, q > 0$. To construct a representative of ξ with genus $(p-1)(q-1)$, take p disjoint copies of Σ_1 and q disjoint copies of Σ_2, so that any copy of Σ_1 intersects any copy of Σ_2 in exactly one point. Resolving the intersection points gives the required representative. q.e.d.

Proposition 13.3. *Suppose the intersection pairing of X is isomorphic to E; let $\{\xi_1, \xi_2\}$ be a basis of $H_2(X)$ modulo the torsion subgroup with respect to which the intersection pairing is given by E. Then any class $\xi \in H_2(X)$ of positive self-intersection satisfies*

$$g_\xi > \frac{p^2 - q^2 - 3|p| + |q|}{2},$$

where $p\xi_1 + q\xi_2$ is the image of ξ in $H_2(X; \mathbf{R})$.

Proof. We may assume (by possibly changing the sign of ξ_i) that $p > q \geq 0$. Let c be a characteristic class whose real image is $3\xi_1^* - \xi_2^*$, where $\xi_i^* \in H^2(X; \mathbf{R})$ denotes the Hom-dual of ξ_i. As $c^2 > 0$, the adjunction inequality implies the claimed genus bound, except for $q = 0$ and $p \leq 2$. In the latter cases the claimed bound states $g_\xi \geq 0$, which is the best possible bound since in $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$ these classes are represented by spheres. q.e.d.

Corollary 13.4. *With notation as in the previous proposition, assume further that ξ_i are represented by disjoint spheres. Then*

$$g_\xi = \begin{cases} \dfrac{p^2 - q^2 - 3|p| + |q|}{2} + 1; & |p| > |q| \\ \dfrac{q^2 - p^2 - 3|q| + |p|}{2} + 1; & |q| > |p| \\ 0; & |p| = |q|. \end{cases}$$

Proof. Let Σ_i be a representative of ξ_i as in the statement. Note that any class of the form $(\pm 1, \pm 1)$ is represented by a sphere of self-intersection 0. Hence any class (p, q) with $|p| = |q|$ is represented by a sphere.

Suppose that $|p| > |q|$ (the remaining case is analogous). We may assume $p > q > 0$ for the purpose of construction; if $q = 0$, the situation is as in $\mathbf{CP}^2$. To construct a representative of ξ with the stated genus, decompose ξ as $(p, q) = q(1, 1) + (p - q)(1, 0)$. Represent $q(1, 1)$ by q disjoint spheres, and $(p-q)(1, 0)$ by a surface Σ of genus $(p-q-1)(p-q-2)/2$ which intersects each of the spheres in $p - q$ points. Finally resolve the intersection points. q.e.d.

14. Manifolds with negative signature

Let X be a smooth closed oriented four-manifold with $b_1(X) = 0$, $b_2^+(X) = 1$ and signature $\sigma(X) = 1 - n$ with $n \geq 2$. We will assume that the intersection form of X is odd. This is always the case for $n \leq 8$, since any such form is odd. Without the restriction on n, the assumption holds for manifolds without 2-torsion according to Furuta's 10/8 Theorem [3].

Fix a primitive class $\xi \in H_2(X)$ of positive self-intersection and choose a basis $\{\xi_0, \ldots, \xi_n\}$ of $H_2(X)$ (modulo the torsion) with respect to which the intersection form is given by $\langle 1 \rangle \oplus n\langle -1 \rangle$, and $\xi = (p, q_1, \ldots, q_n)$ with $p > 0$ and $q_i \geq q_{i+1} \geq 0$; then $\xi^2 = p^2 - \sum q_i^2$. Denote by $m = m_\xi$ the number of nonzero q_i's.

It turns out that the genus bounds obtained from adjunction inequality (1) with c the 'canonical class' $(3, -1, \ldots, -1)$ are only optimal for reduced classes. The notion of a reduced class, used by Li-Li [8] to study genus bounds in rational surfaces $\mathbf{CP}^2 \# n\overline{\mathbf{CP}}^2$ for $n \leq 9$, extends naturally to manifolds we are considering.

Definition 14.1. A class $\xi \in H_2(X)$ as above is called *reduced* with respect to the basis $\{\xi_0, \ldots, \xi_n\}$ provided $m_\xi \leq 9$ and $p \geq q_1 + q_2 + q_3$, where $q_3 = 0$ if $n = 2$.

It is proved in [8, Lemma 4.1] that in rational surfaces with $n \leq 9$ any class of positive self-intersection can be mapped to a reduced class (with respect to the standard basis) by an orientation preserving diffeomorphism. The argument there also proves that any class $\xi \in H_2(X)$ with $m_\xi \leq 9$ is reduced with respect to some basis as above.

Proposition 14.2. *With above notation, suppose that* $2 \leq m \leq 9$ *and* ξ *is reduced. Then*

$$g_\xi(d) > \frac{(d\xi^2 - 3p + \sum q_i)d}{2}.$$

Moreover, $g_\xi(d) > 0$ *unless* $d = 1$, $m = 2$ *and* $\xi = (p, p-1, 1)$ *for some* $p > 1$. *Excluding the latter classes, given any* $g > 0$ *there is only a finite number of reduced classes with minimal genus no greater than* g.

Remark. The last statement gives an affirmative answer to a conjecture of Li and Li [8]. In fact, B.H. Li proved [7] that the above lower bound is sharp in rational surfaces $X = \mathbf{CP}^2 \# n\overline{\mathbf{CP}}^2$ for $n \leq 9$.

Proof. Let c be a characteristic vector whose real image with respect to the Hom-dual basis is $(3, -1, \ldots, -1)$; clearly $c^2 > \sigma(X)$. Let $\Delta = \langle c, \xi \rangle = 3p - \sum q_i$, where the sum, as all other sums over i, runs from 1 to m. Since ξ is reduced, $\Delta \geq 0$.

Suppose first that $m \geq 3$. Since ξ is reduced,

$$p - \frac{3}{2} \geq \left(q_1 - \frac{1}{2}\right) + \left(q_2 - \frac{1}{2}\right) + \left(q_3 - \frac{1}{2}\right).$$

Using this along with $q_i \geq q_{i+1}$, we obtain

$$\xi^2 - \Delta \geq (9 - m)(q_3^2 - q_3) \geq 0,$$

thus $g_\xi(d) \geq 1$ for any $d \geq 1$. Note in general that to establish the last claim of this proposition, it suffices to show that an upper bound on minimal genus implies an upper bound on q_1; this is enough since increasing the value of p (while keeping q_i's fixed) increases $\xi^2 - \Delta$. The argument is simple if $m \leq 8$, as then $\xi^2 - \Delta > (q_1 - 1/2)(q_2 - 1/2)$. The last inequality implies that an upper bound on minimal genus yields an upper bound on q_1. For $m = 9$ the first inequality of this paragraph gives

$$\xi^2 - \Delta \geq \sum_{i=4}^{7} (q_1 - q_i) \left(q_i - \frac{1}{2} \right) + \sum_{i=8}^{9} (q_2 - q_i) \left(q_i - \frac{1}{2} \right);$$

we need to consider several cases. If $q_1 > q_7$, a bound on minimal genus implies a bound on q_1. Same holds if $q_1 = q_7$, but $q_1 > q_9$. Finally, if $q_1 = q_9 = q$,

$$\xi^2 - \Delta = (p - 3q)(p + 3q - 3),$$

but positive square condition implies $p > 3q$; again it follows that there is only a finite number of such vectors whose minimal genus is at most g.

If $m = 2$, then $p \geq q_1 + q_2$ implies

$$\xi^2 - \Delta + 2 \geq 2(q_1 - 1)(q_2 - 1),$$

which is strictly positive unless $q_2 = 1$. Note also that for $q_2 > 1$ there are only finitely many classes (p, q_1, q_2) with minimal genus at most g. For $q_2 = 1$ we get

$$\xi^2 - \Delta + 2 = (p - q_1 - 1)(p + q_1 - 2);$$

this equals 0 only if $p = q_1 + 1$, and for $p \geq q_1 + 2$ there are only finitely many classes $(p, q_1, 1)$ with minimal genus no greater than g. q.e.d.

One can verify that the bound in the above proposition is the best possible bound obtainable from Theorem A. However, this also follows from work of Li-Li [8] and [9], where they prove that the minimal genus bound for a reduced class $\xi = (p, q_1, \ldots, q_m)$ of positive self-intersection in a rational surface $\mathbf{CP}^2 \# n \overline{\mathbf{CP}}^2$ with $n \leq 9$ is given by

$$g_\xi(d) = \frac{(d\xi^2 - 3p + \sum q_i)d}{2} + 1.$$

In the next proposition we give a construction of a minimal genus representative in a special case.

Proposition 14.3. *Let X, ξ_i, $\xi = (p, q_0, \ldots, q_m)$ and $m \leq 9$ be as above, and let $\overline{m}$ be the largest value of i for which $q_i > 2$. Assume further that $\sum_{i=1}^{\overline{m}} q_i \leq p$ and that for $i = 0, \ldots, m$ the classes ξ_i are represented by disjoint spheres. Then ξ has a representative of genus*

$$g_\xi = \left(\xi^2 - 3p + \sum q_i \right) / 2 + 1.$$

Proof. Assume first that $\overline{m} = m$ and let $q := \sum_{i=1}^{\overline{m}} q_i$. Then ξ can be decomposed as $(p-q)\xi_0 + \sum q_i(\xi_0+\xi_i)$. By assumption ξ_0 and $\xi_0+\xi_i$ (for $i = 1, \ldots, m$) can be represented by spheres Σ_i ($i = 0, \ldots, m$) any two of which intersect transversely in one point. Moreover, the spheres Σ_i for $i \geq 1$ have self-intersection zero. To construct a representative for ξ, take $p - q$ copies of Σ_0 and q_i disjoint copies of Σ_i, so that the whole collection of spheres is in general position and any two spheres that intersect have exactly one point in common. Note that the total number of intersection points of these p spheres is

$$\binom{p-q}{2} + q(p-q) + \sum_{i<j} q_i q_j,$$

so after resolving the intersection points we obtain a minimal genus representative.

If $m > \overline{m}$, for any $q_i = 2$ take two spheres representing ξ_i that intersect transversely in one point. Then connect one of the two spheres representing ξ_i to a sphere Σ_j for some $j \leq \overline{m}$, obtaining a surface Σ. Now cancel the -1 intersection point with one of the $+1$ intersection points of Σ_j. To this end choose a curve $\gamma \subset \Sigma$ connecting the two intersection points and replace the complements of small disks around the intersection points (cut out from the other surfaces, not Σ) by a tube which is the restriction to γ of the normal circle bundle of Σ. Resolving the remaining intersection points again gives a minimal genus representative. Finally, if q_i is 1, connect the corresponding sphere to the surface constructed before. q.e.d.

15. Geometric intersections of surfaces

Let X be a smooth closed connected four-manifold. We say that a collection of classes $\xi_1, \ldots, \xi_n \in H_2(X)$ is *algebraically disjoint* if $\xi_i \cdot \xi_j = 0$ for all pairs $i \neq j$. Classes in an algebraically disjoint collection can clearly be represented by disjoint surfaces – starting with any choice of representatives in general position, we can eliminate a pair of ± 1 intersection points between two surfaces by adding a one-handle to one of them. An important question is whether the classes can be represented by disjoint surfaces of low genus. It turns out that the minimal genus representatives of the classes intersect in general and we will derive a lower bound for the number of pairs of ± 1 intersection points.

Proof of Theorem B. We assume for the purpose of the proof that Σ_i is representing a divisible class $d_i\xi_i$ for some $d_i > 1$. The result for $d_i = 1$ then follows as in Corollary 12.2. Consider $Z = X - \cup_{i=1}^n \Sigma_i$ as a cylindrical end manifold with cylindrical ends $[0,\infty) \times Y_i$, where Y_i is the boundary of a tubular neighborhood of Σ_i (with the opposite orientation). Note that Z is negative semi-definite, so we can adapt the argument that we used to derive genus bounds to this context.

We work with exponentially decaying Seiberg-Witten configuration space on Z with reducible asymptotic values; the background connection on Z agrees on the end $[0,\infty)\times Y_i$ with the pull-back of the product connection on Y_i (see the proof of Proposition 7.2). By gauge fixing at infinity we can assume that the asymptotic values differ by imaginary-valued harmonic one-forms on each end. We use gauge group based at infinity on the first end $[0,\infty) \times Y_1$. To compute the index of the deformation complex, we follow the proof of Proposition 7.2. The quotient of the deformation complex by the fiber complex is

$$0 \to \oplus_{i=2}^n \mathbf{R} \to \oplus_{i=1}^n \mathcal{H}(Y_i) \to 0 \to 0,$$

where $\mathcal{H}(Y_i)$ denotes the space of imaginary-valued one-forms on Y_i. Hence the index of the deformation complex is the index of the fiber complex plus $2\sum g_i - (n-1)$, where g_i is the genus of Σ_i. The computation of the fiber index splits in the anti self-dual part and the Dirac part. Note for the first that each end contributes 1 to the spectral flow; it follows that the fiber index of the ASD part equals $-2\sum g_i + n - 1$. The index of the deformation complex is thus given by

$$\frac{1}{4}(c_1(\mathrm{Det}) - \sigma(X)) - \sum_{i=1}^n k_i d_i$$

(see also Lemma 8.1), where Det is the determinant line of a Spin^c structure on X satisfying

$$k_i = \langle c_1(\mathrm{Det}), \xi_i \rangle - d_i\xi_i^2 \in [0, d_i\xi_i^2]$$

for all i.

Next we argue as in the proof of Theorem 11.1. The based moduli space of Seiberg-Witten solutions on Z is smooth (after choosing an appropriate perturbation) and contains a unique reducible point. From the structure of the moduli space close to the reducible point we conclude that if the moduli space is positive dimensional, then it is not compact. Therefore $g_i > k_i d_i/2$ for at least one i. Finally, setting $c_1(\mathrm{Det}) = \sum 2d_i\alpha_i - c$, where α_i is the Poincaré dual of ξ_i, we see that the formal dimension of the (based) moduli space is positive if and only if $c^2 > \sigma(X)$ (see the proof of Corollary 12.3). The condition at the end of the previous paragraph then becomes $\langle c, [\Sigma_i]\rangle \in [0, [\Sigma_i]^2]$, while the genus bound can be expressed as $\chi(\Sigma_i) + [\Sigma_i]^2 \leq \langle c, [\Sigma_i]\rangle$. q.e.d.

In what follows we will restrict our attention to manifolds with $b_2^+(X) = 2$ in order to keep the discussion simple.

Theorem 15.1. *Let X be a smooth closed connected four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 2$, and let Σ_1, Σ_2 be embedded surfaces in general position, representing algebraically disjoint classes of positive self-intersection. Suppose that a characteristic vector $c \in H^2(X)$ satisfies*

$$c^2 > \sigma(X), \qquad \langle c, [\Sigma_i]\rangle \geq 0 \ \textit{for } i = 1, 2,$$

$$\textit{and} \quad \chi(\Sigma_i) + [\Sigma_i]^2 > \langle c, [\Sigma_i]\rangle \ \textit{for } i = 1, 2.$$

Then

$$g(\Sigma_1) + g(\Sigma_2) + N \geq \frac{[\Sigma_1]^2 + [\Sigma_2]^2 - \langle c, [\Sigma_1] + [\Sigma_2]\rangle}{2} + 1,$$

where N denotes the number of pairs of ± 1 intersection points between Σ_1 and Σ_2.

Proof. If condition (2) fails for both surfaces, then by Theorem B the surfaces are not disjoint. To construct disjoint representatives, we trade pairs of ± 1 intersection points for one-handles – this way elimination of a pair of intersection points increases the genus of one of the surfaces by 1. We add the maximal possible number of handles to Σ_1, so that the resulting surface still does not satisfy (2), and add the rest of the handles to Σ_2. Since the sum of the genera of thus constructed disjoint surfaces equals $g(\Sigma_1) + g(\Sigma_2) + N$, the claimed inequality follows from Theorem B. q.e.d.

We compare this to bounds obtained using G-signature Theorem. As is the case for genus bounds, the result we obtained is roughly by a factor of 2 better. Specifically, we state the following consequence of a Theorem of Gilmer [4].

Proposition 15.2 (Gilmer [4]). *Let X be a smooth closed connected four-manifold with $b_1(X) = 0$ and $b_2^+(X) = 2$, and let Σ_1, Σ_2 be embedded surfaces in general position, representing algebraically disjoint* *classes of positive self-intersection. If Σ_1 and Σ_2 are not disjoint and $[\Sigma_1]+[\Sigma_2]$ is divisible by 2, then*

$$g(\Sigma_1)+g(\Sigma_2)+N \geq \frac{[\Sigma_1]^2+[\Sigma_2]^2}{4}-1,$$

where N denotes the number of pairs of ± 1 intersection points between Σ_1 and Σ_2.

15.1 Examples

(1) Let $X=(S^2\times S^2)\#(S^2\times S^2)$ and let $\xi_1=(p,q,0,0)$ and $\xi_2=(0,0,r,s)$ be classes of positive self-intersection, expressed with respect to the standard basis for $H_2(X)$. We may assume that $p,q,r,s>0$. If ξ_i is primitive, it is represented by an embedded sphere in X, according to a Theorem of Wall [23]; however, for $p,q,r,s\geq 2$ it is not represented by a sphere in its summand. Let Σ_1 and Σ_2 be smooth representatives of ξ_1 and ξ_2 in general position. Denote by g_i the genus of Σ_i and by N the number of pairs of ± 1 intersection points between Σ_1 and Σ_2. Using characteristic vectors $c=(2,2,0,0)$ and $c=(0,0,2,2)$ in Theorem 15.1 gives the following lower bounds for g_1+g_2+N:

$$\begin{cases} (p-1)(q-1)+rs; & p,q\geq 2,\ g_1<(p-1)(q-1) \text{ and } g_2\leq rs \\ pq+(r-1)(s-1); & r,s\geq 2,\ g_1\leq pq \text{ and } g_2<(r-1)(s-1). \end{cases}$$

In particular, if ξ_1 and ξ_2 with $p,q,r,s\geq 2$ are represented by spheres Σ_1 and Σ_2 in X, then

$$N\geq \max\{(p-1)(q-1)+rs, pq+(r-1)(s-1)\}.$$

(2) Consider now $X=\mathbf{CP}^2\#\mathbf{CP}^2$ and let $\xi_1=(p,q)$ and $\xi_2=(q,-p)$ for some $p,q>0$, expressed with respect to the standard basis for $H_2(X)$. Note that ξ_i has a smooth representative Σ_i of genus

$$g=\frac{p^2+q^2-3(p+q)}{2}+2,$$

obtained as the connected sum of minimal genus representatives for classes of divisibility p and q in $\mathbf{CP}^2$. Let Σ_1 and Σ_2 be genus g representatives of ξ_1 and ξ_2 in general position, and let N be the number of pairs of ± 1 intersection points. Using characteristic vectors $c = (3, -1)$, $c = (3, 1)$ and $c = (1, -3)$ in Theorem 15.1, we obtain

$$N \geq \begin{cases} p + 2q - 3; & p \geq 2 \\ 2p + q - 3; & q \geq 2 \text{ and } p \leq 3q \\ p + 4q - 3; & q \geq 2 \text{ and } p \geq 3q. \end{cases}$$

In particular, for $q = 1$ we have $N \geq p - 1$ and this bound is sharp, which can be seen as follows. Decompose $(p, 1) = (p - 1)(1, 0) + (1, 1)$ and $(1, -p) = (1, -1) - (p - 1)(0, 1)$. Since the classes $(1, 1)$ and $(1, -1)$ can be represented by disjoint embedded spheres, $N = p - 1$.

(3) As the last example consider $X = 2\mathbf{CP}^2 \# 2\overline{\mathbf{CP}}^2$, and let $\xi_1 = (p, 0, q, 0)$ for some $p > q > 0$ and $\xi_2 = (0, r, 0, s)$ for some $r > s > 0$. These classes are represented by spheres in X (according to the Theorem of Wall mentioned above), but not in their copy of $\mathbf{CP}^2 \# \overline{\mathbf{CP}}^2$, unless $p = q + 1$ and $r = s + 1$. By choosing representatives Σ_i for ξ_i of small genus and in general position, we get the following bound based on Theorem 15.1 (using $c = (3, 1, -1, -1)$ and $c = (1, 3, -1, -1)$):

$$g_1 + g_2 + N \geq \begin{cases} \dfrac{p^2 - q^2 - 3p + q}{2} + \dfrac{r^2 - s^2 - r + s}{2} + 1; & p \geq q + 2 \\ \dfrac{p^2 - q^2 - p + q}{2} + \dfrac{r^2 - s^2 - 3r + s}{2} + 1; & r \geq s + 2. \end{cases}$$

Small genus here means that for a formula to hold, g_1 has to be no greater than the first summand and g_2 has to be no greater than the second summand.

Remark. It is an interesting question whether the bounds obtained in the above examples are optimal.

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McMaster University
1280 Main Street West
Hamilton, Ontario L8S 4K1
Canada