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DEFORMATIONS OF FUCHSIAN ADS REPRESENTATIONS ARE QUASI-FUCHSIAN

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Abstract

Let Γ be a finitely generated group, and let $\mathrm{Rep}(\Gamma, \mathrm{SO}(2,n))$ be the moduli space of representations of Γ into $\mathrm{SO}(2,n)$ ($n \geq 2$). An element $\rho : \Gamma \to \mathrm{SO}(2,n)$ of $\mathrm{Rep}(\Gamma, \mathrm{SO}(2,n))$ is *quasi-Fuchsian* if it is faithful, discrete, preserves an acausal $(n-1)$-sphere in the conformal boundary Ein_n of the anti-de Sitter space, and if the associated globally hyperbolic anti-de Sitter space is spatially compact—a particular case is the case of *Fuchsian representations*, *i.e.,* composition of a faithful, discrete, and cocompact representation $\rho_f : \Gamma \to \mathrm{SO}(1,n)$ and the inclusion $\mathrm{SO}(1,n) \subset \mathrm{SO}(2,n)$.

In [**10**] we proved that quasi-Fuchsian representations are precisely representations that are Anosov as defined in [**29**]. In the present paper, we prove that the space of quasi-Fuchsian representations is open and closed, *i.e.,* that it is a union of connected components of $\mathrm{Rep}(\Gamma, \mathrm{SO}(2,n))$.

The proof involves the following fundamental result: Let Γ be the fundamental group of a globally hyperbolic spatially compact spacetime locally modeled on AdS_n, and let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be the holonomy representation. Then, if Γ is Gromov hyperbolic, the $\rho(\Gamma)$-invariant achronal limit set in Ein_n is acausal.

Finally, we also provide the following characterization of representations with zero-bounded Euler class: they are precisely the representations preserving a closed achronal subset of Ein_n.

1. Introduction

Let $\mathrm{SO}_0(1,n)$, $\mathrm{SO}_0(2,n)$ denote the identity components of, respectively, $\mathrm{SO}(1,n)$, $\mathrm{SO}(2,n)$ ($n \geq 2$). Let Γ be a cocompact torsion-free lattice in $\mathrm{SO}_0(1,n)$. For any Lie group G we consider the moduli space of representations of Γ into G modulo conjugacy, equipped with the usual topology as an algebraic variety (see, for example, [**25**]):

$$\mathrm{Rep}(\Gamma, G) := \mathrm{Hom}(\Gamma, G)/G$$

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1.1. Discrete representations. In the case $G = \mathrm{SO}_0(2, n)$ we distinguish the *Fuchsian representations:* they are the representations obtained by composition of the natural embedding $\mathrm{SO}_0(1, n) \subset \mathrm{SO}_0(2, n)$ and any faithful and discrete representation of Γ into $\mathrm{SO}_0(1, n)$. The space of faithful and discrete representations of Γ into $\mathrm{SO}_0(1, n)$ is the union of two connected components of $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(1, n))$: for $n \geq 3$, it follows from the Mostow rigidity theorem, and for $n = 2$, it follows from the connectedness of the Teichmüller space—observe that there are indeed two connected components, one corresponding to representations such that $\rho^*\xi = \xi$, and the other to representations for which $\rho^*\xi = -\xi$, where ξ is a generator of $\mathrm{H}^n(\mathrm{SO}_0(1, n), \mathbb{Z})$.

It follows that the space of Fuchsian representations is the union of two connected subsets of $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2, n))$. Therefore, one can consider the union $\mathrm{Rep}_0(\Gamma, G)$ of connected components of $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2, n))$ containing all the Fuchsian representations. The main result of the present paper is the following theorem, which provides a positive answer to Question 8.1 in [**10**].

Theorem 1.1. *Every deformation of a Fuchsian representation, i.e., every element of* $\mathrm{Rep}_0(\Gamma, \mathrm{SO}_0(2, n))$*, is faithful and discrete.*

This Theorem is actually a particular case of a more general result, Theorem 1.2, that we will state in the next section.

If one compares this result with the *a priori* similar theory of deformations of Fuchsian representations into $\mathrm{SO}_0(1, n+1)$, one observes that the situation is at first glance completely different: it is well known that large deformations of Fuchsian representations are **not** faithful and discrete; Fuchsian representations actually can be deformed to the trivial representation!

On the other hand, Theorem 1.1 is very similar to the principal theorem in [**29**] in the case $G = \mathrm{SL}(n, \mathbb{R})$, and where Γ is a cocompact lattice in $\mathrm{SO}_0(1, 2)$, *i.e.,* a closed surface group. In this situation, Fuchsian representations are induced by the inclusion $\Gamma \subset \mathrm{SO}_0(1, 2)$ and the morphism $\mathrm{SO}_0(1, 2) \to \mathrm{SL}(n, \mathbb{R})$ corresponding to the unique n-dimensional irreducible representation of $\mathrm{SO}_0(1, 2)$. The connected component of $\mathrm{Rep}(\Gamma, \mathrm{SL}(n, \mathbb{R}))$ containing the Fuchsian representations is the *Hitchin component,* and its elements are called *quasi-Fuchsian representations.* In [**29**] F. Labourie proves that quasi-Fuchsian representations are *hyperconvex, i.e.,* that they are faithful, have discrete image, and preserve some curve in the projective space $\mathbb{P}(\mathbb{R}^n)$ with some very strong convexity properties (in particular, this curve is strictly convex). Later, O. Guichard proved in [**26**] that conversely hyperconvex representations are quasi-Fuchsian.

1.2. Anosov representations. At the very heart of the theory is the notion of (G, P)*-Anosov representation* (or simply Anosov representation when there is no ambiguity about the pair (G, P)), where G is a

Lie group acting on any topological space P. The group Γ in general is a Gromov hyperbolic finitely generated group ([**28**]; see also Sect. 8 in [**10**])—typically, a closed surface group, or, more generally, a cocompact lattice in $\mathrm{SO}_0(1,k)$ for some k.

Unfortunately, the terminology is not uniform in the literature. For example, what is called a $(\mathrm{SO}_0(1,n+1),\partial\mathbb{H}^{n+1})$-Anosov representation in [**27**] would be called $(G,\mathcal{Y})$-Anosov in the terminology of [**7**] or [**10**], where $\mathcal{Y}$ is the space of geodesics of $\mathbb{H}^{n+1}$, and also P-Anosov in the terminology of [**28**], where P is the stabilizer of a point in $\partial\mathbb{H}^{n+1}$. Here we adopt the definition used in [**28**], and the terminology of [**27**].

Simple, general arguments ensure that Anosov representations are faithful, with discrete image formed by loxodromic elements, and that they form an open domain in $\mathrm{Rep}(\Gamma,G)$. As a matter of fact, quasi-Fuchsian representations into $\mathrm{SL}(n,\mathbb{R})$ are $(\mathrm{SL}(n,\mathbb{R}),\mathcal{F})$-Anosov, where $\mathcal{F}$ is the frame variety. However, the converse is not necessarily true: see [**7**] for the study of a family of non-hyperconvex $(\mathrm{SL}(3,\mathbb{R}),\mathcal{F})$-Anosov representations.

The *quasi-Fuchsian* terminology is inherited from hyperbolic geometry: a representation $\rho:\Gamma\to\mathrm{SO}_0(1,n+1)$ is quasi-Fuchsian if it is faithful, discrete, and preserves a topological $(n-1)$-sphere in $\partial\mathbb{H}^{n+1}$. It is well known to the experts that quasi-Fuchsian representations into $\mathrm{SO}_0(1,n+1)$ are precisely the $(\mathrm{SO}_0(1,n+1),\partial\mathbb{H}^{n+1})$-Anosov representations, and a proof can be obtained by adapting the arguments used in [**10**]. It is also a direct consequence of Theorem 1.8 in [**28**].

The anti-de Sitter space AdS_{n+1} is the Lorentzian analogue of the hyperbolic space $\mathbb{H}^{n+1}$ (see Sect. 2.1 for a brief review on basic facts about AdS_{n+1}). It is a Lorentzian manifold, of constant sectional curvature -1. Whereas in hyperbolic space pairs of points are classified up to isometry by their distance, in anti-de Sitter space we have to distinguish three types of pairs of points, according to the nature of the geodesic joining the two points: this geodesic may be spacelike, lightlike, or timelike—in the last two cases, the points are said to be *causally related.* Moreover, AdS_{n+1} is oriented and admits also a *time orientation, i.e.,* an orientation of every non-spacelike geodesic. The group $\mathrm{SO}_0(2,n)$ is precisely the group of orientation and time orientation preserving isometries of AdS_{n+1}.

The anti-de Sitter space AdS_{n+1} admits a conformal boundary called the *Einstein universe* and denoted by Ein_n, which plays a role similar to that of the conformal boundary $\partial\mathbb{H}^{n+1}$ for the hyperbolic space. The Einstein universe is a conformal Lorentzian spacetime, and is also subject to a causality notion: in particular, a subset Λ of the Einstein space Ein_n is called *acausal* if any pair of distinct points in Λ are the extremities of a spacelike geodesic in AdS_{n+1}.

Once these fundamental notions are introduced, we can state the main content of [**10**]: Let Γ be a Gromov hyperbolic group. For any representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ the following notions coincide:

– $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov,

– $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is faithful, discrete, and preserves an **acausal** closed subset Λ in the conformal boundary Ein_n of AdS_{n+1}.

If, furthermore, Γ is isomorphic to the fundamental group of a closed manifold of dimension n, then Λ is a topological $(n-1)$-sphere.

In particular, when Γ is a uniform lattice in $\mathrm{SO}_0(1,n)$, a representation of Γ into $\mathrm{SO}_0(2,n)$ is called *quasi-Fuchsian* if it is faithful, discrete, and preserves an acausal topological $(n-1)$-sphere in Ein_n. In other words, Theorem 1.1 can be restated as follows: *Deformations (large or small) of Fuchsian representations into* $\mathrm{SO}_0(2,n)$ *are all quasi-Fuchsian.* It will be a corollary of the following more general statement, which will be proved in Sect. 6:

Theorem 1.2. *Let $n \geq 2$, and let Γ be a Gromov hyperbolic group of cohomological dimension $\geq n$. Then the moduli space*

$$\mathrm{Rep}_{an}(\Gamma, \mathrm{SO}_0(2,n))$$

of $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$*-Anosov representations is open and closed in the moduli space* $\mathrm{Rep}(\Gamma, \mathrm{SO}_0(2,n))$.

Remark 1.3. The reason for the hypothesis on the cohomological dimension is to ensure that the invariant closed acausal subset is a topological $(n-1)$-sphere. It will follow from the proof that actually, under this hypothesis, if $\mathrm{Rep}_{an}(\Gamma, \mathrm{SO}_0(2,n))$ is non-empty, then Γ is the fundamental group of a closed manifold, and its cohomological dimension is precisely n.

1.3. GHC-regular representations. In order to present the ideas involved in the proof of Theorem 1.2, we need to recall a bit further a few classical definitions in Lorentzian geometry. By *spacetime* we mean here an oriented Lorentzian manifold with a time orientation given by a smooth timelike vector field. This allows us to define the notion of future- and past-directed causal curves. A subset Λ in (M,g) is *achronal* (respectively *acausal*) if there every timelike curve (respectively causal curve) joining two points in Λ is necessarily trivial, *i.e.,* reduced to one point. A *time function* is a function $t : M \to \mathbb{R}$ that is strictly increasing along any causal curve. A spacetime (M,g) is *globally hyperbolic spatially compact* (abbreviated to GHC) if it admits a time function whose level sets are all compact.

Spatially compact global hyperbolicity is notoriously equivalent to the existence of a *compact Cauchy hypersurface*, *i.e.,* is a compact achronal set S that intersects every inextendible timelike curve at exactly one point. This set is then automatically a locally Lipschitz hypersurface (see [**31**, Sect. 14, Lemma 29]).

Observe that all these notions are not really associated to the Lorentzian metric g, but to its conformal class $[g]$. Hence they are relevant to the Einstein universe, which is naturally equipped with an $\mathrm{SO}_0(2,n)$-invariant conformal class of Lorentzian metric, but without any $\mathrm{SO}_0(2,n)$-invariant representative.

The key fact used in [**10**] is that $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov representations are holonomy representations of GHC spacetimes locally modeled on AdS_{n+1}. Thanks to the work of G. Mess and his followers [**30**, **2**] the classification of GHC locally AdS spacetimes has been almost completed: they are in one-to-one correspondence with *GHC-regular representations.*

More precisely: Let Γ be a torsion-free finitely generated group of cohomological dimension n. A morphism $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is a GHC-regular representation if it is faithful, discrete, and preserves an **achronal** closed $(n-1)$-topological sphere Λ in Ein_n. Define the invisible domain $E(\Lambda)$ as the domain in AdS_{n+1} consisting of the points that are not causally related to any element of Λ (cf. Sect. 3.1). The action of $\rho(\Gamma)$ on $E(\Lambda)$ is then free and properly discontinuous; the quotient space, denoted by $M_\rho(\Lambda)$, is GHC. Moreover, every maximal GHC spacetime locally modeled on AdS has this form. Also observe that Λ only depends on ρ; there is at most one such invariant achronal sphere. Finally, if the limit set Λ is acausal, then the group Γ is Gromov hyperbolic (actually, in this case, Γ acts properly and cocompactly on a $\mathrm{CAT}(-1)$ metric space; see Proposition 8.3 in [**10**]).

Therefore, the only reason a GHC-regular representation may fail to be $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov is that the achronal sphere Λ might be non-acausal. A crucial step of the present paper, which is proved in Sect. 5.3, and from which Theorem 1.2 follows quite directly, is the following:

Theorem 1.4. *Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a GHC-regular representation, where Γ is a Gromov hyperbolic group. Then the achronal limit set Λ is acausal, i.e., ρ is $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov.*

Even if not logically relevant to the proofs in the present paper, we point out that there are examples of GHC-regular representations with non-acausal limit set Λ. Let us describe briefly in this introduction the family detailed in Sect. 4.6: Let (p,q) be a pair of positive integers such that $p+q=n$, and let Γ be a cocompact lattice of $\mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q)$. The natural inclusion of $\mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q)$ into $\mathrm{SO}_0(2,n)$ arising from the orthogonal splitting $\mathbb{R}^{2,n} = \mathbb{R}^{1,p} \oplus \mathbb{R}^{1,p}$ induces a representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ that is GHC-regular but where the invariant achronal limit set Λ is not acausal. The quotient space $M_\rho(\Lambda) := \rho(\Gamma)\backslash E(\Lambda)$ is a GHC spacetime, called a *split* AdS *spacetime,* and the representation is a *split regular representation* (Definition 4.28).

1.4. Bounded cohomology. Finally, in the last section, we give another characterization of GHC-regular representations. There is a fundamental bounded cohomology class ξ in $\mathrm{H}^2_b(\mathrm{SO}_0(2,n),\mathbb{Z})$, the *bounded Euler class.* It can be alternatively defined as the bounded cohomology class induced by the natural Kähler form ω of the symmetric $2n$-dimensional space

$$\mathcal{T}_{2n} := \mathrm{SO}_0(2,n)/(\mathrm{SO}_0(2)\times \mathrm{SO}_0(n)),$$

or as the one associated to the central exact sequence

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SO}}_0(2,n) \to \mathrm{SO}_0(2,n) \to 1.$$

If $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is GHC-regular, the pull-back $\rho^*(\xi)$ (the *Euler class* $\mathrm{eu}_b(\rho)$) is necessarily trivial.

Theorem 1.5. *Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a faithful and discrete representation, where Γ is the fundamental group of a negatively curved closed manifold M of dimension n. The following assertions are equivalent:*

1) *ρ is $(\mathrm{SO}_0(2,n),\mathrm{Ein}_n)$-Anosov,*
2) *ρ is GHC-regular,*
3) *the bounded Euler class $\mathrm{eu}_b(\rho)$ vanishes.*

Observe that the equivalence between items (1) and (2) follows from the main result in [**10**], Theorem 1.4, and the fact that fundamental groups of negatively curved closed manifolds are Gromov hyperbolic.

As a last comment, we recall part of the conjecture already proposed in [**10**, Conjecture 8.11]: We expect that GHC-regular representations of hyperbolic groups are all quasi-Fuchsian; in other words, that if a hyperbolic group Γ admits a GHC-regular representation into $\mathrm{SO}_0(2,n)$, then it must be isomorphic to a uniform lattice in $\mathrm{SO}_0(1,n)$.

We expect actually a bit more. According to Theorems 1.2 and 1.4, the space of GHC-regular representations is open and closed, hence a union of connected components of $\mathrm{Rep}(\Gamma,\mathrm{SO}_0(2,n))$. It would be interesting to prove eventually that it coincides with $\mathrm{Rep}_0(\Gamma,\mathrm{SO}_0(2,n))$, *i.e.,* that quasi-Fuchsian representations are all deformations of Fuchsian representations.

1.5. Overview of the paper. Section 2 introduces the preliminary material on anti-de Sitter space, Einstein space, and their Klein models and conformal models. In Sect. 3 we define the convex hull $\mathrm{Conv}(\Lambda)$ and the invisible domain $E(\Lambda)$ associated to a closed achronal subset Λ of Ein_n. We describe how one fits inside the other one, and show that they are dual one to each other. In Sect. 4 we study the specific case where Λ is a topological sphere: the invisible domain $E(\Lambda)$ is then globally hyperbolic, and admits a *regular cosmological time,* whose gradient lines form a remarkable family of timelike geodesics—the *cosmological lines.* They form an interesting embedded surface in the space of timelike geodesics of AdS_{n+1}, which is the symmetric space $\mathcal{T}_{2n}$ associated to $\mathrm{SO}_0(2,n)$. We conclude this section with the description of split AdS spacetimes, and the description of crowns and their realms, which are in one-to-one correspondence with maximal flats in $\mathcal{T}_{2n}$.

In Sect. 5 we really start the proofs of the main theorems. In Sect. 5.1 we show that if Λ is preserved by a GHC-regular representation of a Gromov hyperbolic group, then it contains no crown. Then, in the following section, we use this result to show that under these hypotheses, the only common part between the closure of the invisible domain and the convex hull is Λ. These are the main results we need for the proof of Theorem 1.4, which we present in Sect. 5.3.

In Sect. 6 we prove Theorem 1.2. The point is that it is quite easy to show that a limit of GHC-regular representations is GHC-regular, and Theorem 1.4 ensures that this limit is Anosov.

The last section, 7, is devoted to the proof of Theorem 1.5. We end the paper by showing in Sect. 7.3 that Theorem 1.5 gives a quick proof of the fact that two representations of the same group Γ (Gromov hyperbolic or not) in $\mathrm{PSL}(2,\mathbb{R})$ are semiconjugate if and only if they have the same bounded Euler class.

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2. Preliminaries on the anti-de Sitter space and the Einstein universe

We assume the reader sufficiently acquainted with basic causality notions in Lorentzian manifolds like *causal* or *timelike* curves, *inextendible* causal curves, *Lorentzian length* of causal curves, *time orientation, future* and *past* of subsets, *time function, achronal* subsets, etc., so that the brief description provided in the introduction above is sufficient. We refer to [**11**] or [**31**, section 14] for further details.

Definition 2.1. A *spacetime* is a connected, oriented, and time-oriented Lorentzian manifold.

2.1. Anti-de Sitter space. Let $\mathbb{R}^{2,n}$ be the vector space of dimension $n+2$, with coordinates $(u, v, x_1, \ldots, x_n)$, endowed with the quadratic form

$$\mathrm{q}_{2,n}(u, v, x_1, \ldots, x_n) := -u^2 - v^2 + x_1^2 + \ldots + x_n^2.$$

We denote by $\langle \mathsf{x} \mid \mathsf{y} \rangle$ the associated scalar product. For any subset A of $\mathbb{R}^{2,n}$ we denote $A^\perp$ the orthogonal of A, *i.e.,* the set of elements y in $\mathbb{R}^{2,n}$ such that $\langle \mathsf{y} \mid \mathsf{x} \rangle = 0$ for every x in A. We also denote by $\mathcal{C}_n$ the isotropic cone $\{\mathsf{x} \in \mathbb{R}^{2,n} / \mathrm{q}_{2,n}(\mathsf{x}) = 0\}$.

Definition 2.2. The anti-de Sitter space AdS_{n+1} is the hypersurface $\{\mathsf{x} \in \mathbb{R}^{2,n} / \mathrm{q}_{2,n}(\mathsf{x}) = -1\}$ endowed with the Lorentzian metric obtained by restriction of $\mathrm{q}_{2,n}$.

For every element x of AdS_{n+1}, there is a canonical identification between the tangent space $T_{\mathsf{x}} \mathrm{AdS}_{n+1}$ and the $\mathrm{q}_{2,n}$-orthogonal $\mathsf{x}^\perp$.

We will also consider the coordinates $(r, \theta, x_1, \ldots, x_n)$ with

$$u = r\cos(\theta), v = r\sin(\theta).$$

We equip AdS_{n+1} with the time orientation defined by this vector field, *i.e.,* the time orientation such that the timelike vector field $\dfrac{\partial}{\partial\theta}$ is everywhere future oriented.

Observe the analogy with the definition of hyperbolic space $\mathbb{H}^n$. Moreover, for every real number θ_0, the subset $H_{\theta_0} := \{(r, \theta, x_1, \ldots, x_n)/\theta = \theta_0\} \subset \mathbb{R}^{2,n}$ is a totally geodesic copy of $\mathbb{H}^n$ embedded in AdS_{n+1}. More generally, the totally geodesic subspaces of dimension k in AdS_{n+1} are connected components of the intersections of AdS_{n+1} with the linear subspaces of dimension $(k+1)$ in $\mathbb{R}^{2,n}$.

Remark 2.3. In particular, geodesics are intersections with 2-planes. Timelike geodesics can all be described in the following way: Let x, y two elements of AdS_{n+1} such that $\langle \mathsf{x} \mid \mathsf{y} \rangle = 0$. Then, when θ describes $\mathbb{R}/2\pi\mathbb{Z}$, the points $c(\theta) := \cos(\theta)\mathsf{x} + \sin(\theta)\mathsf{y}$ describe a future oriented timelike geodesic containing x (for $\theta = 0$) and y (for $\theta = \pi/2$), parametrized by unit length: the Lorentzian length of the restriction of c to $]0, \theta[$ is θ.

2.2. Conformal model.

Proposition 2.4. *The anti-de Sitter space* AdS_{n+1} *is conformally equivalent to* $(\mathbb{S}^1 \times \mathbb{D}^n, -d\theta^2 + ds^2)$*, where* $d\theta^2$ *is the standard Riemannian metric on* $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$*, where* ds^2 *is the standard metric (of curvature* $+1$*) on the sphere* $\mathbb{S}^n$ *and* $\mathbb{D}^n$ *is the open upper hemisphere of* $\mathbb{S}^n$*.*

Proof. In the $(r, \theta, x_1, \ldots, x_n)$-coordinates the AdS metric is

$$-r^2\,\mathrm{d}\theta^2 + \mathrm{ds}^2_{hyp}$$

where ds^2_{hyp} is the hyperbolic metric, *i.e.*, the induced metric on $H_0 = \{(r, \theta, x_1, \dots, x_n)/\theta = 0\} \approx \mathbb{H}^n$. More precisely, H_0 is a sheet of the hyperboloid $\{(r, x_1, \dots, x_n) \in \mathbb{R}^{1,n}/ - r^2 + x_1^2 + \cdots + x_n^2 = -1\}$. The map $(r, x_1, \dots, x_n) \to (1/r, x_1/r, \dots, x_n/r)$ sends this hyperboloid to $\mathbb{D}^n$, and an easy computation shows that the pull-back by this map of the standard metric on the hemisphere is $r^{-2}\,\mathrm{ds}^2_{hyp}$. The proposition follows. q.e.d.

Proposition 2.4 shows in particular that AdS_{n+1} contains many closed causal curves (including all timelike geodesics; cf. Remark 2.3). But the universal covering $\widetilde{\mathrm{AdS}}_{n+1}$, conformally equivalent to $(\mathbb{R} \times \mathbb{D}^n, -d\theta^2 + ds^2)$, contains no periodic causal curve. However, it is not globally hyperbolic (see Definition 4.5).

2.3. Einstein universe. The Einstein universe Ein_{n+1} is the product $\mathbb{S}^1 \times \mathbb{S}^n$ endowed with the metric $-d\theta^2 + ds^2$, where ds^2 is as above the standard spherical metric. The universal Einstein universe $\widetilde{\mathrm{Ein}}_{n+1}$ is the cyclic covering $\mathbb{R} \times \mathbb{S}^n$ equipped with the lifted metric still denoted $-d\theta^2 + ds^2$, but where θ now takes value in $\mathbb{R}$. Observe that for $n \geq 2$, $\widetilde{\mathrm{Ein}}_{n+1}$ is the universal covering, but it is not true for $n = 1$. According to this definition, Ein_{n+1} and $\widetilde{\mathrm{Ein}}_{n+1}$ are Lorentzian manifolds, but it is more adequate to consider them as conformal Lorentzian manifolds. We fix a time orientation: the one for which the coordinate θ is a time function on $\widetilde{\mathrm{Ein}}_{n+1}$.

In the sequel, we denote by $\mathrm{p} : \widetilde{\mathrm{Ein}}_{n+1} \to \mathrm{Ein}_{n+1}$ the cyclic covering map. Let $\delta : \widetilde{\mathrm{Ein}}_{n+1} \to \widetilde{\mathrm{Ein}}_{n+1}$ be a generator of the Galois group of this cyclic covering. More precisely, we select δ so that for any $\tilde{x}$ in $\widetilde{\mathrm{Ein}}_{n+1}$ the image $\delta(\tilde{x})$ is in the future of $\tilde{x}$.

Even if the Einstein universe is seen merely as a conformal Lorentzian spacetime, one can define the notion of *photons, i.e.,* (non-parameterized) lightlike geodesics. We can also consider the causality relation in Ein_{n+1} and $\widetilde{\mathrm{Ein}}_{n+1}$. In particular, we define for every x in Ein_{n+1} the *lightcone* $C(x)$: it is the union of all photons containing x. If we write x as a pair $(\theta, \bar{x})$ in $\mathbb{S}^1 \times \mathbb{S}^n$, the lightcone $C(x)$ is the set of pairs $(\theta', \bar{y})$ such that $|\theta' - \theta| = d(\bar{x}, \bar{y})$, where d is the distance function for the spherical metric ds^2.

There is only one point in $\mathbb{S}^n$ at distance π of $\bar{x}$: the antipodal point $-\bar{x}$. Above this point, there is only one point in Ein_{n+1} contained in $C(x)$: the antipodal point $-x = (\theta + \pi, -\bar{x})$. The lightcone $C(x)$ with the points x, $-x$ removed is the union of two components:

– The *future cone:* It is the set $C^+(x) := \{(\theta', \bar{y})/\theta < \theta' < \theta + \pi,\ d(\bar{x}, \bar{y}) = \theta' - \theta\}$.

– The *past cone:* It is the set $C^-(x) := \{(\theta', \bar{y})/\theta - \pi < \theta' < \theta,\ d(\bar{x}, \bar{y}) = \theta - \theta'\}$.

Observe that the future cone of x is the past cone of $-x$, and that the past cone of x is the future cone of $-x$.

According to Proposition 2.4 AdS_{n+1} (respectively $\widetilde{\mathrm{AdS}}_{n+1}$) conformally embeds in Ein_{n+1} (respectively $\widetilde{\mathrm{Ein}}_{n+1}$). Hence we will sometimes write by the letter x elements of AdS_{n+1} instead of the letter x, which is the notation for elements of $\mathbb{R}^{2,n}$. Observe that this embedding preserves the time orientation. Since the boundary $\partial\mathbb{D}^n$ is an equatorial sphere, the boundary $\partial\widetilde{\mathrm{AdS}}_{n+1}$ is a copy of the Einstein universe $\widetilde{\mathrm{Ein}}_n$. In other words, one can attach a "Penrose boundary" $\partial\widetilde{\mathrm{AdS}}_{n+1}$ to $\widetilde{\mathrm{AdS}}_{n+1}$ such that $\widetilde{\mathrm{AdS}}_{n+1} \cup \partial\widetilde{\mathrm{AdS}}_{n+1}$ is conformally equivalent to $(\mathbb{S}^1 \times \overline{\mathbb{D}}^n, -d\theta^2 + ds^2)$, where $\overline{\mathbb{D}}^n$ is the closed upper hemisphere of $\mathbb{S}^n$.

The restrictions of p and δ to $\widetilde{\mathrm{AdS}}_{n+1} \subset \widetilde{\mathrm{Ein}}_{n+1}$ are, respectively, a covering map over AdS_{n+1} and a generator of the Galois group of the covering; we will still denote them by p and δ.

2.4. Achronal subsets. Recall that a subset of a conformal Lorentzian manifold is *achronal* (respectively *acausal*) if there is no timelike (respectively causal) curve joining two distinct points of the subset. In $\widetilde{\mathrm{Ein}}_n \approx (\mathbb{R} \times \mathbb{S}^{n-1}, -d\theta^2 + ds^2)$, it is quite easy to show that every achronal subset is precisely the graph of a 1-Lipschitz function $f : \Lambda_0 \to \mathbb{R}$ where Λ_0 is a subset of $\mathbb{S}^{n-1}$ endowed with its canonical metric d. In particular, the achronal embedded topological hypersurfaces in $\partial\widetilde{\mathrm{AdS}}_{n+1}$ are exactly the graphs of the 1-Lipschitz functions $f : \mathbb{S}^{n-1} \to \mathbb{R}$: they are topological $(n-1)$-spheres.

Similarly, achronal subsets of $\widetilde{\mathrm{AdS}}_{n+1}$ are graphs of 1-Lipschitz functions $f : \Lambda_0 \to \mathbb{R}$ where Λ_0 is a subset of $\mathbb{D}^n$ and achronal topological hypersurfaces are graphs of 1-Lipschitz maps $f : \mathbb{D}^n \to \mathbb{R}$.

Stricto sensu, there is no achronal subset in Ein_{n+1} since closed timelike curves through a given point cover the entire Ein_{n+1}. Nevertheless, we can keep track of this notion in Ein_{n+1} by defining "achronal" subsets of Ein_{n+1} as projections of genuine achronal subsets of $\widetilde{\mathrm{Ein}}_{n+1}$. This definition is justified by the following results:

Lemma 2.5 (Lemma 2.4 in [**10**]). *The restriction of* p *to any achronal subset of* $\widetilde{\mathrm{Ein}}_{n+1}$ *is injective.*

Corollary 2.6 (Corollary 2.5 in [**10**]). *Let $\widetilde{\Lambda}_1$, $\widetilde{\Lambda}_2$ be two achronal subsets of* $\widetilde{\mathrm{Ein}}_{n+1}$ *admitting the same projection in* Ein_{n+1}. *Then there is an integer k such that*

$$\widetilde{\Lambda}_1 = \delta^k \widetilde{\Lambda}_2,$$

where δ is the generator of the Galois group introduced above.

2.5. The Klein model $\mathbb{ADS}_{n+1}$ of the anti-de Sitter space. We now consider the quotient $\mathbb{S}(\mathbb{R}^{2,n})$ of $\mathbb{R}^{2,n} \setminus \{0\}$ by positive homotheties. In other words, $\mathbb{S}(\mathbb{R}^{2,n})$ is the double covering of the projective space

$\mathbb{P}(\mathbb{R}^{2,n})$. We denote by $\mathbb{S}$ the projection of $\mathbb{R}^{2,n} \setminus \{0\}$ onto $\mathbb{S}(\mathbb{R}^{2,n})$. For every x, y in $\mathbb{S}(\mathbb{R}^{2,n})$, we denote by $\langle \mathrm{x} \mid \mathrm{y} \rangle$ the **sign** of the real number $\langle \mathsf{x} \mid \mathsf{y} \rangle$, where $\mathsf{x}, \mathsf{y} \in \mathbb{R}^{2,n}$ are representatives of x, y. The *Klein model* $\mathbb{ADS}_{n+1}$ of the anti-de Sitter space is the projection of AdS_{n+1} to $\mathbb{S}(\mathbb{R}^{2,n})$, endowed with the induced Lorentzian metric, *i.e.*,

$$\mathbb{ADS}_{n+1} := \{\mathrm{x} \in \mathbb{S}(\mathbb{R}^{2,n}) \;/\; \langle \mathrm{x} \mid \mathrm{x} \rangle < 0\}.$$

The topological boundary of $\mathbb{ADS}_{n+1}$ in $\mathbb{S}(\mathbb{R}^{2,n})$ is the projection of the isotropic cone $\mathcal{C}_n$; we will denote this boundary by $\partial\mathbb{ADS}_{n+1}$. The projection $\mathbb{S}$ defines a one-to-one isometry between AdS_{n+1} and $\mathbb{ADS}_{n+1}$. The continuous extension of this isometry is a canonical homeomorphism between $\mathrm{AdS}_{n+1} \cup \partial\,\mathrm{AdS}_{n+1}$ and $\mathbb{ADS}_{n+1} \cup \partial\mathbb{ADS}_{n+1}$.

For every linear subspace F of dimension $k+1$ in $\mathbb{R}^{2,n}$, we denote by $\mathbb{S}(F) := \mathbb{S}(F \setminus \{0\})$ the corresponding projective subspace of dimension k in $\mathbb{S}(\mathbb{R}^{2,n})$. The geodesics of $\mathbb{ADS}_{n+1}$ are the connected components of the intersections of $\mathbb{ADS}_{n+1}$ with the projective lines $\mathbb{S}(F)$ of $\mathbb{S}(\mathbb{R}^{2,n})$. More generally, the totally geodesic subspaces of dimension k in $\mathbb{ADS}_{n+1}$ are the connected components of the intersections of $\mathbb{ADS}_{n+1}$ with the projective subspaces $\mathbb{S}(F)$ of dimension k of $\mathbb{S}(\mathbb{R}^{2,n})$.

Definition 2.7. For every $\mathrm{x} = \mathbb{S}(\mathsf{x})$ in $\mathbb{ADS}_{n+1}$, we define the *affine domain* (also denoted by $U(\mathsf{x})$)

$$U(\mathrm{x}) := \{\mathrm{y} \in \mathbb{ADS}_{n+1} \;/\; \langle \mathrm{x} \mid \mathrm{y} \rangle < 0\}.$$

In other words, $U(\mathrm{x})$ is the connected component of $\mathbb{ADS}_{n+1} \setminus \mathbb{S}(\mathsf{x}^{\perp})$ containing x. Let $V(\mathrm{x})$ (also denoted by $V(\mathsf{x})$) be the connected component of $\mathbb{S}(\mathbb{R}^{2,n}) \setminus \mathbb{S}(\mathsf{x}^{\perp})$ containing $U(\mathrm{x})$. The boundary $\partial U(\mathrm{x}) \subset \partial\mathbb{ADS}_{n+1}$ of $U(\mathrm{x})$ in $V(\mathrm{x})$ is called the *affine boundary* of $U(\mathrm{x})$.

Remark 2.8. We can assume that $\mathsf{x} = (1, 0, \dots, 0) \in \mathbb{R}^{2,n}$, so that $\mathbb{S}(\mathsf{x}^{\perp})$ is the projection of the hyperplane $\{u = 0\}$ in $\mathbb{R}^{2,n}$ and $V(\mathsf{x})$ is the projection of the region $\{u > 0\}$ in $\mathbb{R}^{2,n}$. The map

$$(u, v, x_1, x_2, \dots, x_{n+1}) \mapsto (t, \bar{x}_1, \dots, \bar{x}_n) := \left(\frac{v}{u}, \frac{x_1}{u}, \frac{x_2}{u}, \dots, \frac{x_n}{u}\right)$$

induces a diffeomorphism between $V(\mathsf{x})$ and $\mathbb{R}^{n+1}$ mapping the affine domain $U(\mathsf{x})$ to the region $\{(t, \bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^{n+1} |\, \mathrm{q}_{1,n}(t, \bar{x}_1, \dots, \bar{x}_n) < 1\}$, where $\mathrm{q}_{1,n}$ is the Minkowski norm. The affine boundary $\partial U(\mathsf{x})$ corresponds to the hyperboloid $\{(t, \bar{x}_1, \dots, \bar{x}_n |\, \mathrm{q}_{1,n}(t, \bar{x}_1, \dots, \bar{x}_n) = 1\}$. The intersections between $U(\mathsf{x})$ and the totally geodesic subspaces of $\mathbb{ADS}_{n+1}$ correspond to the intersections of the region $\{(t, \bar{x}_1, \dots, \bar{x}_n) \in \mathbb{R}^{n+1} |\, \mathrm{q}_{1,n}(t, \bar{x}_1, \dots, \bar{x}_n) < 1\}$ with the affine subspaces of $\mathbb{R}^{n+1}$.

Lemma 2.9 (Lemma 10.13 in [**3**])**.** *Let U be an affine domain in $\mathbb{ADS}_{n+1}$, and let $\partial U \subset \partial\mathbb{ADS}_{n+1}$ be its affine boundary. Let* x *be be a point in ∂U, and let* y *be a point in $U \cup \partial U$. There exists a causal* *(respectively timelike) curve joining* x *to* y *in* $U \cup \partial U$ *if and only if* $\langle \mathrm{x} \mid \mathrm{y} \rangle \geq 0$ *(respectively* $\langle \mathrm{x} \mid \mathrm{y} \rangle > 0$*).*

Remark 2.10. The boundary of $U(\mathrm{x})$ in $\mathbb{ADS}_{n+1}$ is $\mathbb{S}(x^{\perp}) \cap \mathbb{ADS}_{n+1}$. It has two components: the *past component* $H^{-}(\mathrm{x})$ and the *future component* $H^{+}(\mathrm{x})$. These components are characterized by the following property: timelike geodesics enter $U(\mathrm{x})$ through $H^{-}(\mathrm{x})$ and exit through $H^{+}(\mathrm{x})$.

They can also be defined as follows: Let $\widetilde{U}(\mathrm{x})$ be a lifting in $\widetilde{\mathrm{AdS}}_{n+1}$ of $U(\mathrm{x})$, and let $\widetilde{H}^{\pm}(\mathrm{x})$ be the lifts of $H^{\pm}(\mathrm{x})$. Then $\widetilde{U}(\mathrm{x})$ is the intersection between the future of $H^{-}(\mathrm{x})$ and the past of $H^{+}(\mathrm{x})$.

The boundary components $H^{\pm}(\mathrm{x})$ are totally geodesic embedded copies of $\mathbb{H}^n$. They are also called *hyperplanes dual to* x*,* and we distinguish the hyperplane past-dual $H^{-}(\mathrm{x})$ from the hyperplane future-dual $H^{+}(\mathrm{x})$.

Last but not least, $H^{\pm}(\mathrm{x})$ have also the following characteristic property: every future oriented (respectively past oriented) timelike geodesic starting at x reaches $H^{+}(\mathrm{x})$ (respectively $H^{-}(\mathrm{x})$) at time $\pi/2$ (see Remark 2.3). In other words, $H^{\pm}(\mathrm{x})$ is the set of points at Lorentzian distance $\pm\pi/2$ from x.

2.6. The Klein model of the Einstein universe. Similarly, the Einstein universe has a Klein model: the projection $\mathbb{S}(\mathcal{C}_n)$ in $\mathbb{S}(\mathbb{R}^{2,n})$ of the isotropic cone $\mathcal{C}_n$ in $\mathbb{R}^{2,n}$. The conformal Lorentzian structure can be defined in terms of the quadratic form $\mathrm{q}_{2,n}$ (for more details, see [**21**, **9**]).

An immediate corollary of Lemma 2.9 as follows:

Corollary 2.11. *For* $\Lambda \subseteq \mathrm{Ein}_n$*, the following assertions are equivalent:*

1) Λ *is achronal (respectively acausal).*
2) *When we see* Λ *as a subset of* $\mathbb{S}(\mathcal{C}_n) \approx \mathrm{Ein}_n$ *the scalar product* $\langle \mathrm{x} \mid \mathrm{y} \rangle$ *is non-positive (respectively negative) for every distinct* $\mathrm{x}, \mathrm{y} \in \Lambda$*.*

Remark 2.12. Concerning the notation: In the sequel, we always have in mind the identifications $\mathrm{Ein}_n \approx \mathbb{S}(\mathcal{C}_n)$ and $\mathrm{AdS}_{n+1} \approx \mathbb{ADS}_{n+1}$; and also the conformal identification of AdS_{n+1} with the open domain $\mathbb{D}^n \times \mathbb{S}^1$ of Ein_{n+1}, and we will frequently switch from one model to the other.

We will from now denote by x elements of Ein and AdS, using the notation x when we want to insist on the Klein model, and x for elements of AdS_{n+1} when we see them as elements of $\mathbb{R}^{2,n}$.

2.7. Isometry groups. Every element of $\mathrm{SO}(2,n)$ induces an isometry of AdS_{n+1}, and every isometry of AdS_{n+1} comes from an element of $\mathrm{O}(2,n)$. Similarly, for $n \geq 2$, conformal transformations of Ein_{n+1} are projections of elements of $\mathrm{O}(2,n+1)$ acting on $\mathcal{C}_{n+1}$ (still for $n \geq 2$).

In the sequel we will only consider isometries preserving the orientation and the time orientation, *i.e.,* elements of the neutral component $\mathrm{SO}_0(2,n)$ (or $\mathrm{SO}_0(2,n+1)$).

Let $\widetilde{\mathrm{SO}}_0(2,n)$ be the group of orientation and time orientation preserving isometries of $\widetilde{\mathrm{AdS}}_{n+1}$ (or conformal transformations of $\widetilde{\mathrm{Ein}}_n$). There is a central exact sequence

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SO}}_0(2,n) \to \mathrm{SO}_0(2,n) \to 1$$

where the left term is generated by the transformation δ generating the Galois group of $\mathrm{p} : \widetilde{\mathrm{Ein}}_n \to \mathrm{Ein}_n$ defined previously. Observe that for $n \geq 3$, $\widetilde{\mathrm{SO}}_0(2,n)$ is the universal covering of $\mathrm{SO}_0(2,n)$.

Remark 2.13. Let x_0 be any element of $\mathrm{Ein}_n \approx \mathbb{S}(\mathcal{C}_n)$. Then the open domain defined by

$$\mathrm{Mink}(\mathrm{x}_0) = \{\mathrm{x} \in \mathbb{S}(\mathcal{C}_n) \ / \ \langle \mathrm{x}_0 \mid \mathrm{x} \rangle < 0\}$$

is conformally isometric to the Minkowski space $\mathbb{R}^{1,n-1}$ (see [**21**, **9**]).

In particular, the stabilizer G_0 of x_0 in $\mathrm{SO}_0(2,n)$ is isomorphic to the group of conformal automorphisms of $\mathbb{R}^{1,n-1}$, *i.e.,* of affine transformations whose linear part has the form $x \mapsto \lambda g(x)$, where λ is a positive real number and g an element of $\mathrm{SO}_0(1,n-1)$.

3. Regular AdS manifolds

In all this section, $\widetilde{\Lambda}$ is a closed achronal subset of $\partial\widetilde{\mathrm{AdS}}_{n+1}$, and Λ is the projection of $\widetilde{\Lambda}$ in $\partial\,\mathrm{AdS}_{n+1}$. We describe the *invisibility domain* of $\widetilde{\Lambda}$ (or Λ) and describes their geometric properties. Roughly speaking, they are the region in $\widetilde{\mathrm{AdS}}_{n+1}$ (or AdS_{n+1}) consisting of the points which are not causally related to any point in $\widetilde{\Lambda}$ (or Λ). We also show (section 3.3) that the invisible domain $E(\Lambda)$ can be defined as the convex domain dual to the convex hull of Λ (considered in the Klein model).

3.1. AdS regular domains. We denote by $\widetilde{E}(\widetilde{\Lambda})$ the *invisible domain* of $\widetilde{\Lambda}$ in $\widetilde{\mathrm{AdS}}_{n+1}$, *i.e.,*

$$\widetilde{E}(\widetilde{\Lambda}) =: \widetilde{\mathrm{AdS}}_{n+1} \setminus \left(J^-(\widetilde{\Lambda}) \cup J^+(\widetilde{\Lambda})\right),$$

where $J^-(\widetilde{\Lambda})$ and $J^+(\widetilde{\Lambda})$ are the causal past and the causal future of $\widetilde{\Lambda}$ in $\widetilde{\mathrm{AdS}}_{n+1} \cup \partial\widetilde{\mathrm{AdS}}_{n+1} = (\mathbb{R} \times \overline{\mathbb{D}}^{n-1}, -d\theta^2 + ds^2)$. We denote by $\mathrm{Cl}(\widetilde{E}(\widetilde{\Lambda}))$ the closure of $\widetilde{E}(\widetilde{\Lambda})$ in $\widetilde{\mathrm{AdS}}_{n+1} \cup \partial\widetilde{\mathrm{AdS}}_{n+1}$ and denote by $E(\Lambda)$ the projection of $\widetilde{E}(\widetilde{\Lambda})$ in AdS_{n+1} (according to Corollary 2.6, $E(\Lambda)$ only depends on Λ, not on the choice of the lifting $\widetilde{\Lambda}$).

Definition 3.1. A $(n+1)$-dimensional *AdS regular domain* is a domain of the form $E(\Lambda)$, where Λ is the projection in $\partial\,\mathrm{AdS}_{n+1}$ of an achronal subset $\widetilde{\Lambda} \subset \partial\widetilde{\mathrm{AdS}}_{n+1}$ containing at least two points. If $\widetilde{\Lambda}$ is a topological $(n-1)$-sphere, then $E(\Lambda)$ is *GH-regular* (this definition is motivated by Theorem 4.12 and Proposition 4.14).

Remark 3.2. The invisible domain $\widetilde{E}(\widetilde{\Lambda})$ is causally convex in of $\widetilde{\mathrm{AdS}}_{n+1}$; *i.e.,* every causal curve joining two points in $\widetilde{E}(\widetilde{\Lambda})$ is entirely contained in $\widetilde{E}(\widetilde{\Lambda})$. This is an immediate consequence of the definitions.

Remark 3.3. Recall that $\widetilde{\Lambda}$ is the graph of a 1-Lipschitz function $f : \Lambda_0 \to \mathbb{R}$, where Λ_0 is a closed subset of $\mathbb{S}^{n-1}$ (Section 2.4). Define two functions $f^-, f^+ : \overline{\mathbb{D}}^n \to \mathbb{R}$ as follows:

$$f^-(\bar{x}) := \mathrm{Sup}_{\bar{y}\in\Lambda_0}\{f(\bar{y}) - d(\bar{x},\bar{y})\},$$
$$f^+(\bar{x}) := \mathrm{Inf}_{\bar{y}\in\Lambda_0}\{f(\bar{y}) + d(\bar{x},\bar{y})\},$$

where d is the distance induced by $\mathrm{d}s^2$ on $\overline{\mathbb{D}}^n$. It is easy to check that

$$\widetilde{E}(\widetilde{\Lambda}) = \{(\theta,\bar{x}) \in \mathbb{R}\times\mathbb{D}^n \mid f^-(\bar{x}) < \theta < f^+(\bar{x})\}.$$

Remark 3.4. Keeping the notation of the previous remark, observe that the graph of the restriction of f^+ (or f^-) to $\partial\mathbb{D}^n$ is a closed achronal $(n-1)$-sphere $\widetilde{\Lambda}^+$ (or $\widetilde{\Lambda}^-$) in $\widetilde{\mathrm{AdS}}_{n+1}$ that contains the initial achronal subset $\widetilde{\Lambda}$. They project to achronal $(n-1)$-spheres $\Lambda^\pm$ in $\partial\,\mathrm{AdS}_{n+1}$ that contain Λ.

Furthermore, any element g of $\mathrm{SO}_0(2,n)$ preserving Λ must preserve $E(\Lambda)$, hence the graphs of $f^\pm$, and therefore must preserve Λ^+ and Λ^-.

Definition 3.5. The graph of f^- (respectively f^+) is a closed achronal subset of $\widetilde{\mathrm{AdS}}_{n+1}$, called the *lifted past* (respectively *future*) *horizon* of $\widetilde{E}(\widetilde{\Lambda})$, and denoted $\mathcal{H}^-(\widetilde{\Lambda})$ (respectively $\mathcal{H}^+(\widetilde{\Lambda})$).

The projections in AdS_{n+1} of $\widetilde{\mathcal{H}}^\pm(\widetilde{\Lambda})$ are called *past* and *future horizons* of $E(\Lambda)$, and denoted $\mathcal{H}^\pm(\Lambda)$.

The following lemma is a refinement of Lemma 2.5:

Lemma 3.6 (Corollary 10.6 in [**3**])**.** *For every (non-empty) closed achronal set $\widetilde{\Lambda} \subset \partial\widetilde{\mathrm{AdS}}_{n+1}$, the projection of $\widetilde{E}(\widetilde{\Lambda})$ onto $E(\Lambda)$ is one-to-one.*

Definition 3.7. $\widetilde{\Lambda}$ is *purely lightlike* if the associated subset Λ_0 of $\mathbb{S}^n$ contains two antipodal points $\bar{x}_0$ and $-\bar{x}_0$ such that, for the associated 1-Lipschitz map $f : \Lambda_0 \to \mathbb{R}$ the equality $f(\bar{x}_0) = f(-\bar{x}_0) + \pi$ holds.

If $\widetilde{\Lambda}$ is purely lightlike, for every element $\bar{x}$ of $\overline{\mathbb{D}}^n$ we have $f^-(\bar{x}) = f^+(\bar{x}) = f(-\bar{x}_0) + d(-\bar{x}_0,\bar{x}) = f(\bar{x}_0) - d(\bar{x}_0,\bar{x})$, implying that $\widetilde{E}(\widetilde{\Lambda})$ is empty. Conversely, we have the following:

Lemma 3.8 (Lemma 3.6 in [**10**])**.** *$\widetilde{E}(\widetilde{\Lambda})$ is empty if and only if $\widetilde{\Lambda}$ is purely lightlike. More precisely, if for some point $\bar{x}$ in $\mathbb{D}^n$ the equality $f^+(\bar{x}) = f^-(\bar{x})$ holds, then $\widetilde{\Lambda}$ is purely lightlike.*

Observe that a purely lightlike achronal subset of $\widetilde{\Lambda}$ is contained in the union of lightlike geodesics joining two antipodal points of Ein_n.

3.2. AdS regular domains as subsets of $\mathbb{ADS}_{n+1}$. The canonical homeomorphism between $\mathrm{AdS}_{n+1} \cup \partial\,\mathrm{AdS}_{n+1}$ and $\mathbb{ADS}_{n+1} \cup \partial\mathbb{ADS}_{n+1}$ allows us to see AdS regular domains as subsets of $\mathbb{ADS}_{n+1}$.

Putting together the definition of the invisible domain $E(\Lambda)$ of a set $\Lambda \subset \partial\,\mathrm{AdS}_{n+1}$ and Lemma 2.9, one gets the following:

Proposition 3.9 (Proposition 10.14 in [**3**])**.** *If we see Λ and $E(\Lambda)$ in the Klein model $\mathbb{ADS}_{n+1} \cup \partial\mathbb{ADS}_{n+1}$, then*

$$E(\Lambda) = \{\mathrm{y} \in \mathbb{ADS}_{n+1} \textit{ such that } \langle \mathrm{y} \mid \mathrm{x}\rangle < 0 \textit{ for every } \mathrm{x} \in \Lambda\}.$$

3.3. Convex core of AdS regular domains. In this section, we assume that Λ is not purely lightlike and not reduced to a single point. The following notions are classical and well known:

Definition 3.10. A subset Ω of $\mathbb{S}(\mathbb{R}^{2,n})$ is *convex* if there is a convex cone J of $\mathbb{R}^{2,n}$ such that $\Omega = \mathbb{S}(J)$. The *relative interior* of Ω, denoted by Ω°, is the convex subset $\mathbb{S}(J^\circ)$, where J° is the interior of J in the subspace spanned by J.

It is well known that the closure of a convex subset is still convex, and that it coincides with the closure of the relative interior.

Theorem-Definition 3.11. Let $\Omega = \mathbb{S}(J)$ be a convex subset of $\mathbb{S}(\mathbb{R}^{2,n})$. The following assertions are equivalent:

- J contains no complete affine line.
- There is an affine hyperplane H in $\mathbb{S}(\mathbb{R}^{2,n})$ such that $H \cap J$ is relatively compact in H and such that $\Omega = \mathbb{S}(J \cap H)$.
- The closure of Ω contains no pair of opposite points.

If one of these equivalent properties hold, then Ω is *salient*.

Definition 3.12. Let $\Omega = \mathbb{S}(J)$ a convex subset of $\mathbb{S}(\mathbb{R}^{2,n})$. The *dual* of Ω is the closed convex subset $\mathbb{S}(J^* \setminus \{0\})$ where

$$J^* = \{\mathsf{x} \in \mathbb{R}^{2,n} \ / \ \forall \mathsf{y} \in J,\ \langle \mathsf{x} \mid \mathsf{y}\rangle \leq 0\}.$$

Proposition 3.13. *Let Ω be a convex subset of $\mathbb{S}(\mathbb{R}^{2,n})$. Then the bidual Ω^{**} is the closure $Cl(\Omega)$ of Ω in $\mathbb{S}(\mathbb{R}^{2,n})$. The relative interior Ω° is open in $\mathbb{S}(\mathbb{R}^{2,n})$ if and only if Ω^* is salient.*

Let $\hat{\Lambda}$ be the preimage of $\Lambda \subset \mathrm{Ein}_n = \mathbb{S}(\mathcal{C}_n)$ by $\mathbb{S}$. The convex hull of $\hat{\Lambda}$ is a convex cone $\mathrm{Conv}(\hat{\Lambda})$ in $\mathbb{R}^{2,n}$, whose projection is a compact convex subset of $\mathbb{S}(\mathbb{R}^{2,n})$, denoted by $\mathrm{Conv}(\Lambda)$, and called *the convex hull of* Λ and the *convex core of* $E(\Lambda)$.

Lemma 3.14. *The intersection of* $\mathrm{Conv}(\Lambda)$ *and* Ein_n *is the union of lightlike segments in* Ein_n *joining two elements of Λ. The relative interior* $\mathrm{Conv}(\Lambda)^\circ$ *is contained in* $\mathbb{ADS}_{n+1}$.

Proof. Elements of $\mathrm{Conv}(\hat{\Lambda})$ are linear combinations $\mathsf{x} = \sum_{i=1}^{k} t_i \mathsf{x}_i$, where t_i are non-negative real numbers and x_i elements of $\hat{\Lambda}$.

$$\mathrm{q}_{2,n}(\mathsf{x}) \quad = \quad \sum_{i,j=1}^{k} t_i t_j \langle \mathsf{x}_i \mid \mathsf{x}_j \rangle$$

Since every $\langle \mathsf{x}_i \mid \mathsf{x}_j \rangle$ is non-positive (cf. Lemma 2.9), we have $\mathrm{q}_{2,n}(\mathsf{x}) \leq 0$.

Moreover, if $\mathrm{q}_{2,n}(\mathsf{x}) = 0$, then every $\langle \mathsf{x}_i \mid \mathsf{x}_j \rangle$ must be equal to 0, *i.e.,* the vector space spanned by the x_i's is isotropic, hence either a line, or an isotropic plane in $\mathcal{C}_n$. In the first case, $\mathbb{S}(\mathsf{x})$ is an element of Λ, and in the second case, $\mathbb{S}(\mathsf{x})$ lies on a lightlike geodesic of Ein_n joining two elements of Λ.

Finally, assume that $\mathrm{Conv}(\Lambda)^\circ$ is not contained in $\mathbb{ADS}_{n+1}$. Since $\mathrm{q}_{2,n}(\mathsf{x}) \leq 0$ for every x in $\hat{\Lambda}$, it follows that $\mathrm{Conv}(\hat{\Lambda})$ is contained in $\mathcal{C}_n$, and more precisely, by the argument above, in an isotropic 2-plane. This is a contradiction since Λ by hypothesis is not purely lightlike. q.e.d.

Actually, the case where $\mathrm{Conv}(\Lambda)^\circ$ is not an open subset of AdS_{n+1} is exceptional:

Lemma 3.15 (Lemma 3.13 in [**10**])**.** *If* $\mathrm{Conv}(\Lambda) \cap \mathrm{AdS}_{n+1}$ *has empty interior, then it is contained in a totally geodesic spacelike hypersurface of* AdS_{n+1}.

Proposition 3.9 can be rewritten as follows:

Proposition 3.16 (Proposition 10.17 in [**3**])**.** *The domain* $E(\Lambda)$ *is the intersection* $\mathbb{ADS}_{n+1} \cap (\mathrm{Conv}(\Lambda)^*)^\circ$.

Remark 3.17. A corollary of Proposition 3.16 is that the invisible domain $E(\Lambda)$ is convex, and hence contains $\mathrm{Conv}(\Lambda)^\circ$.

Hence, if x lies in the interior of $\mathrm{Conv}(\Lambda)$, the affine domain $U(\mathrm{x})$ contains the closure of $E(\Lambda)$. Therefore, we have the following:

Proposition 3.18. *Assume that* Λ *is not the boundary of a totally geodesic copy of* $\mathbb{H}^n$ *in* AdS_{n+1}. *Then the restriction of* $\mathrm{p} : \widetilde{\mathrm{AdS}}_{n+1} \to \mathrm{AdS}_{n+1}$ *to the closure of* $\widetilde{E}(\widetilde{\Lambda})$ *is one-to-one.*

In particular, $\mathrm{p} : \widetilde{\mathcal{H}}^{\pm}(\widetilde{\Lambda}) \to \mathcal{H}^{\pm}(\Lambda)$ *is injective.*

The boundary of $E(\Lambda)$ in AdS_{n+1} has two components: the past and future horizons $\mathcal{H}^{\pm}(\Lambda)$ (cf. Definition 3.5). Since $E(\Lambda)$ is convex, every point x in $\mathcal{H}^{-}(\Lambda)$ lies in a support hyperplane for $E(\Lambda)$, *i.e.,* a totally geodesic hyperplane H tangent to $\mathcal{H}^{-}(\Lambda)$ at x. According to Proposition 3.16, H is the hyperplane dual to an element p of $\partial\,\mathrm{Conv}(\Lambda)$, and hence H is either spacelike (if $p \in \mathrm{AdS}_{n+1}$) or degenerate (if $p \in \mathrm{Ein}_n$).

Remark 3.19. For every achronal subset Λ, the intersection

$$\mathrm{Conv}(\Lambda) \cap \mathrm{Ein}_n$$

that is a union of lightlike geodesic segments joining elements of Λ is still achronal (since $\langle \sum s_i\,\mathrm{x}_i \mid \sum t_j\,\mathrm{y}_j\rangle = \sum s_i t_j \langle \mathrm{x}_i \mid \mathrm{y}_j\rangle \leq 0$ for $s_i, t_j \geq 0$, x_i, $\mathrm{y}_j \in \Lambda$). We call it the *filling of* Λ and denote it by $\mathrm{Fill}(\Lambda)$. According to Proposition 3.16,

$$E(\mathrm{Fill}(\Lambda)) = E(\Lambda)$$

Hence we can always assume without loss of generality that Λ is *filled*, *i.e.,* $\Lambda = \mathrm{Fill}(\Lambda)$.

Observe also that any filled purely lightlike achronal subset of Ein_n can be described as the union of the lightlike geodesic joining two given antipodal points (see the end of Sect. 3.1).

4. Globally hyperbolic AdS spacetimes

In all this section, Λ *is a topological achronal* $(n-1)$*-sphere in the boundary* $\partial\,\mathrm{AdS}_{n+1}$ *that is not purely lightlike.* In particular, it implies that Λ is filled (cf. Remark 3.19).

Proposition 4.1 (Corollary 10.7 in [**3**])**.** *For every achronal topological* $(n-1)$*-sphere* $\Lambda \subset \partial\,\mathrm{AdS}_{n+1}$*, the intersection between the closure* $Cl\,(E(\Lambda))$ *of* $E(\Lambda)$ *in* Ein_{n+1} *and* $\mathrm{Ein}_n = \partial\,\mathrm{AdS}_{n+1}$ *is reduced to* Λ*.*

Proposition 4.1 implies that $(\mathrm{Conv}(\Lambda)^*)^\circ \subset \mathrm{AdS}_{n+1}$. Thus, when Λ is a topological sphere,

$$E(\Lambda) = \mathbb{ADS}_{n+1} \cap (\mathrm{Conv}(\Lambda)^*)^\circ = (\mathrm{Conv}(\Lambda)^*)^\circ.$$

Remark 4.2. It follows from Proposition 4.1 that the GH-regular domain $E(\Lambda)$ characterizes Λ, *i.e.,* invisible domains of different achronal $(n-1)$-spheres are different. We call Λ the *limit set of* $E(\Lambda)$.

In this section, we give a description of how the convex hull $\mathrm{Conv}(\Lambda)$ fits inside $E(\Lambda)$ (Proposition 4.3). We introduce the notion of global hyperbolicity and show that $E(\Lambda)$ is globally hyperbolic. Furthermore, it admits a regular cosmological time is the sense of [**1**]. We do a detailed study of the *cosmological time* and show that its restriction to the *past tight region* is a *Cauchy time function* (Proposition 4.18) and $C^{1,1}$ (Lemma 4.19).

We then clarify what is a *GH-regular* or *GHC-regular* spacetime, strictly or not.

We introduce (Section 4.5) the space $\mathcal{T}_{2n}$ of timelike geodesics, which is actually the symmetric space associated to $\mathrm{SO}_0(2,n)$. We describe afterwards some examples of non-strictly GHC-spacetimes, the *split* AdS *spacetimes*, which are closely related to the notion of *crowns,* and their *realms,* defined in Section 4.7. An important feature in the proof of our main result (Theorem 1.4) is that crowns correspond to flats in $\mathcal{T}_{2n}$ (Remark 4.32).

4.1. More on the convex hull of achronal topological $(n-1)$**-spheres.** Recall that there are two maps f^-, f^+ such that $\widetilde{E}(\widetilde{\Lambda}) = \{(\theta, \bar{x})/f^-(\bar{x}) < \theta < f^+(\bar{x})\}$ (cf. Definition 3.3).

Proposition 4.3. *The complement of* Λ *in the boundary* $\partial\,\mathrm{Conv}(\Lambda)$ *has two connected components. Both are closed achronal subsets of* AdS_{n+1}. *More precisely, in the conformal model their liftings to* $\widetilde{\mathrm{AdS}}_{n+1}$ *are graphs of 1-Lipschitz maps* F^+, F^- *from* $\mathbb{D}^n$ *into* $\mathbb{R}$ *such that*

$$f^- \leq F^- \leq F^+ \leq f^+. \tag{1}$$

Proof. See Proposition 3.14 in [**10**]. Observe that in [**10**], Proposition 3.14 is proved in the case where Λ is acausal, and not Fuchsian (the Fuchsian case being the case where Λ is the boundary of a totally geodesic hypersurface in $\widetilde{\mathrm{AdS}}_{n+1}$). Inequalities in Equation (1) are then all strict inequalities, which is false in the general case, as we will see later (section 4.6) in the case of split AdS spacetimes, hence the case where $F^+ = F^-$ everywhere. Nevertheless, the proof of Proposition 3.14 in [**10**] can easily be adapted, providing a proof of Proposition 4.3. q.e.d.

We have already observed that $\partial E(\Lambda) \setminus \Lambda$ is the union of two achronal connected components $\mathcal{H}^\pm(\Lambda)$; in a similar way, $\partial\,\mathrm{Conv}(\Lambda) \setminus \Lambda$ is the union of two achronal n-dimensional topological disks: the *past component* $S^-(\Lambda)$ (the graph of F^-) and the *future component* $S^+(\Lambda)$. Since $E(\Lambda)$ and $\mathrm{Conv}(\Lambda)$ are convex and dual one to another, for every element x in $S^-(\Lambda)$ (respectively $S^+(\Lambda)$) there is an element p of Λ or $\mathcal{H}^+(\Lambda)$ (respectively $\mathcal{H}^-(\Lambda)$) such that $H^-(p)$ (respectively $H^+(p)$) is a support hyperplane for $S^-(\Lambda)$ (respectively $S^+(\Lambda)$) at x: these support hyperplanes are either totally geodesic copies of $\mathbb{H}^n$ (if $p \in \mathrm{AdS}_{n+1}$) or degenerate (if $p \in \Lambda$).

Similarly, at every element x of $\mathcal{H}^-(\Lambda)$ (respectively $\mathcal{H}^+(\Lambda)$) there is a support hyperplane $H^-(p)$ (respectively $H^+(p)$) where p is an element of $S^+(\Lambda) \cup \Lambda$ (respectively $S^-(\Lambda) \cup \Lambda$) (see Figure 1).

Remark 4.4. For every p in $\mathcal{H}^-(\Lambda)$, $H^+(p)$ is a support hyperplane for $\mathrm{Conv}(\Lambda)$, but it could be at a point in Λ. Elements of $\mathcal{H}^-(\Lambda)$ that are dual to support hyperplanes for $\mathrm{Conv}(\Lambda)$ at a point inside AdS_{n+1}, *i.e.,* in $S^+(\Lambda)$, form an interesting subset of $\mathcal{H}^-(\Lambda)$, the *initial singularity set* (cf. [**12**]).

4.2. Global hyperbolicity.

Definition 4.5. A spacetime (M, g) is *globally hyperbolic* (abbreviation GH) if:

- (M, g) is *causal*, *i.e.,* contains no timelike loop,

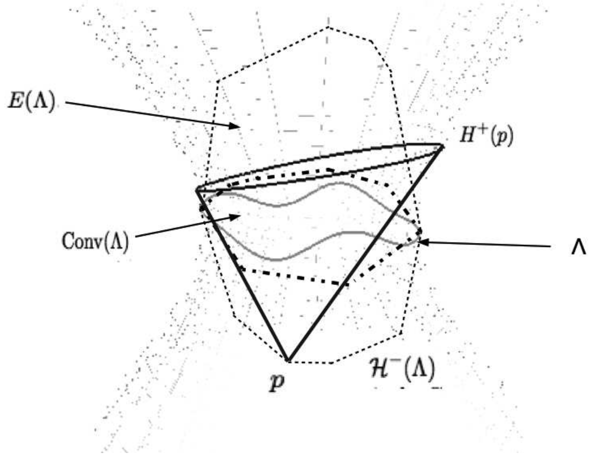

Figure 1. The global situation. The dotted hyperboloid represents the boundary of an affine domain of AdS_{n+1} containing the invisible domain $E(\Lambda)$. The limit set Λ is represented by a topological circle turning around the hyperboloid, and $\mathrm{Conv}(\Lambda)^\circ$ is a convex subset inside the (dual) convex subset $E(\Lambda)$. The future-dual plane $H^+(p)$ for p in the past horizon $\mathcal{H}^-(\Lambda)$ is a support hyperplane of $S^+(\Lambda)$.

- for every x, y in M, the intersection $J^+(x) \cap J^-(y)$ is empty or compact.

Definition 4.6. Let (M, g) be a spacetime. A *Cauchy hypersurface* is a closed acausal subset $S \subset M$ that intersects every inextendible causal curve in (M, g) in one and only one point.

A *Cauchy time function* is a time function $T : M \to \mathbb{R}$ such that every level set $T^{-1}(a)$ is a Cauchy hypersurface in (M, g).

Theorem 4.7 ([**20**], [**14**, **15**, **16**])**.** *Let (M, g) be a spacetime. The following assertions are equivalent:*

1) *(M, g) is globally hyperbolic.*
2) *(M, g) contains a Cauchy hypersurface.*
3) *(M, g) admits a Cauchy time function.*
4) *(M, g) admits a smooth Cauchy time function.*

In a GH spacetime, the Cauchy hypersurfaces are homeomorphic one to the other. In particular, if one of them is compact, all of them are compact.

Definition 4.8. A spacetime (M, g) is *globally hyperbolic spatially compact* (abbrev. GHC) if it contains a compact Cauchy hypersurface.

Proposition 4.9. *A spacetime (M, g) is GHC if and only if it contains a time function $T : M \to \mathbb{R}$ such that every level set $T^{-1}(a)$ is compact.*

4.3. Cosmological time functions. In any spacetime (M, g), one can define the *cosmological time function* as follows (see [**1**]):

Definition 4.10. The *cosmological time function* of a spacetime (M, g) is the function $\tau : M \to [0, +\infty]$ defined by

$$\tau(x) := \mathrm{Sup}\{L(c) \mid c \in \mathcal{R}^-(x)\},$$

where $\mathcal{R}^-(x)$ is the set of past-oriented causal curves starting at x and $L(c)$ is the Lorentzian length of the causal curve c.

Definition 4.11. A spacetime (M, g) with cosmological time function τ is *CT-regular* if

1) M has *finite existence time in the past,* $\tau(x) < \infty$ for every x in M, and
2) for every past-oriented inextendible causal curve $c : [0, +\infty[\to M$, $\lim_{t\to\infty} \tau(c(t)) = 0$.

Theorem 4.12 ([**1**])**.** *If a spacetime (M, g) with cosmological time function τ is CT-regular, then:*

1) *M is globally hyperbolic.*
2) *τ is a time function, i.e., τ is continuous and is strictly increasing along future-oriented causal curves.*
3) *For each x in M, there is at least one realizing geodesic, i.e., a future-oriented timelike geodesic $c :]0, \tau(x)] \to M$ realizing the distance from the "initial singularity," i.e., c has unit speed, is geodesic, and satisfies:*

$$c(\tau(x))) = x \text{ and } \tau(c(t)) = t \text{ for every } t.$$

4) *τ is locally Lipschitz, and admits first and second derivative almost everywhere.*

However, τ is not always a Cauchy time function (see the comment after Corollary 2.6 in [**1**]).

A very nice feature of CT-regularity is that it is preserved by isometries (and thus, by Galois automorphisms):

Proposition 4.13 (Proposition 4.4 in [**10**])**.** *Let $(\widetilde{M}, \tilde{g})$ be a CT-regular spacetime. Let Γ be a torsion-free discrete group of isometries of $(\widetilde{M}, \tilde{g})$ preserving the time orientation. Then the action of Γ on $(\widetilde{M}, \tilde{g})$ is properly discontinuous. Furthermore, the quotient spacetime (M, g) is CT-regular. More precisely, if $\mathrm{p} : \widetilde{M} \to M$ denotes the quotient map, the cosmological times $\tilde{\tau} : \widetilde{M} \to [0, +\infty[$ and $\tau : M \to [0, +\infty[$ satisfy*

$$\tilde{\tau} = \tau \circ \mathrm{p}.$$

Recall that in this section Λ denotes a non-purely lightlike topological achronal $(n-1)$-sphere in $\partial\,\mathrm{AdS}_{n+1}$.

Proposition 4.14 (Proposition 11.1 in [**3**]). *The GH-regular AdS domain $E(\Lambda)$ is CT-regular.*

Hence, according to Theorem 4.12, GH-regular domains are globally hyperbolic. Furthermore:

Definition 4.15. The region $\{\tau < \pi/2\}$ is denoted $E_0^-(\Lambda)$ and called the *past tight* region of $E(\Lambda)$.

Proposition 4.16 (Proposition 11.5 in [**3**]). *Let x be an element of the past tight region $E_0^-(\Lambda)$. Then there is a unique realizing geodesic for x. More precisely, there is one and only one element $r(x)$ in the past horizon $\mathcal{H}^-(\Lambda)$—called the cosmological retract of x—such that the segment $]r(x), x]$ is a timelike geodesic whose Lorentzian length is precisely the cosmological time $\tau(x)$.*

Proposition 4.17 (Proposition 11.6 in [**3**]). *Let $c :]0, T] \to E_0^-(\Lambda)$ be a future oriented timelike geodesic whose initial extremity $p := \lim_{t\to 0} c(t)$ is in the past horizon $\mathcal{H}^-(\Lambda)$. Then the following assertions are equivalent:*

1) *For every $t \in]0, T]$, $c_{|]0,t]}$ is a realizing geodesic for the point $c(t)$.*
2) *There exists $t \in]0, T]$ such that $c_{|]0,t]}$ is a realizing geodesic for the point $c(t)$.*
3) *c is orthogonal to a support hyperplane of $E(\Lambda)$ at $p := \lim_{t\to 0} c(t)$.*

The following proposition was known in the case $n = 2$ ([**30**, **12**] and was implicitly admitted in the few previous papers devoted to the higher-dimensional case (for example, [**3**, **10**]):

Proposition 4.18. *The past tight region $E_0^-(\Lambda)$ is the past in $E(\Lambda)$ of the future boundary component $S^+(\Lambda)$ of the convex core (in particular, it contains $\mathrm{Conv}(\Lambda)^\circ$). The restriction of the cosmological time to $E_0^-(\Lambda)$ is a Cauchy time, taking all values in $]0, \pi/2[$.*

Proof. Let x be an element of $E_0^-(\Lambda)$. In the sequel, we will consider x as an element of $\mathbb{R}^{2,n}$ (but, in order to slightly simplify the redaction, we didn't use the notation x). According to Propositions 4.16 and 4.17 there is a realizing geodesic $]r(x), x]$ orthogonal to a spacelike support hyperplane H tangent to $\mathcal{H}^-(\Lambda)$ at $r(x)$. As described in Section 4.1, this support hyperplane is the hyperplane $H^-(p)$ past-dual to an element p of $S^+(\Lambda)$. The realizing geodesic is contained in the geodesic $\theta \mapsto c(\theta) = \cos(\theta)r(x) + \sin(\theta)p(x)$ (cf. Remark 2.3). For θ in $]0, \pi/2[$ sufficiently close to $\pi/2$, $c(\theta)$ belongs to $\mathrm{Conv}(\Lambda) \subset E(\Lambda)$, and since $E(\Lambda)$ is convex, every $c(\theta)$ $(\theta \in]0, \pi/2[)$ lies in $E(\Lambda)$. Moreover, according to Proposition 4.17, for every θ_0 in $]0, \pi/2[$, the restriction of c to $]0, \theta_0[$ is a realizing geodesic.

Hence

$$\forall\theta \in]0,\pi/2[, \quad \tau(c(\theta)) = \theta.$$

Hence every value in $]0,\pi/2[$ is attained by τ. Moreover, x lies in the past of $p(x)$, hence of $S^+(\Lambda)$. We have

$$E_0^-(\Lambda) \subset I^-(S^+(\Lambda)) \cap E(\Lambda).$$

Conversely, for every x in $I^-(S^+(\Lambda)) \cap E(\Lambda)$, there is a (not necessarily unique) realizing geodesic $c :]0, \tau(x)[\to E(\Lambda)$ such that $c(\tau(x)) = x$ (cf. item (3) in Theorem 4.12). Then the curve c being a timelike geodesic inextendible (in $E(\Lambda)$) towards the past, for $t \to 0$ the points $c(t)$ converge to a limit point $c(0)$ in $\mathcal{H}^-(\Lambda)$. If $\tau(x) \geq \pi/2$, on the one hand, we observe that $c(\pi/2)$ lies in the past of $x = c(\tau(x))$, hence in $I^-(S^+(\Lambda))$. On the other hand

$$\langle c(\pi/2) \mid c(0)\rangle = 0$$

Hence $c(\pi/2)$ is in the hyperplane dual to an element of $\mathcal{H}^-(\Lambda)$ and therefore, by Proposition 3.16, belongs to $S^+(\Lambda)$. But it is a contraction since $S^+(\Lambda)$ is achronal and $c(\pi/2) \in I^-(S^+(\Lambda))$. Hence $\tau(x) < \pi/2$, *i.e.*,

$$I^-(S^+(\Lambda)) \cap E(\Lambda) \subset E_0^-(\Lambda).$$

In order to conclude, we have to prove that τ is a Cauchy time function. Let $c_0 :]a, b[\to E_0^-(\Lambda)$ be an inextendible future-oriented causal curve. The image of $\tau \circ c_0$ is an interval $]\alpha, \beta[$. According to item (2) of Definition 4.11, $\alpha = 0$. We aim to prove $\beta = \pi/2$; hence we assume by contradiction that $\beta < \pi/2$. The curve c is contained in the compact subset $\mathrm{Cl}(E(\Lambda))$ of $\mathrm{AdS}_{n+1} \cup \partial\,\mathrm{AdS}_{n+1} \subset \mathrm{Ein}_{n+1}$, and hence admits a future limit point $c(b)$ in $\mathrm{AdS}_{n+1} \cup \partial\,\mathrm{AdS}_{n+1}$. If $c(b)$ lies in $\mathrm{Ein}_n = \partial\,\mathrm{AdS}_{n+1}$, then it is in Λ (cf. Proposition 4.1). Some element of $E(\Lambda)$ (for example, $c(\dfrac{a+b}{2})$) would be causally related to an element of Λ. This contradiction shows that $c(b)$ lies in AdS_{n+1}—more precisely, in the boundary of $E_0^-(\Lambda)$ in AdS_{n+1}. Since c is future oriented, it follows that $c(b)$ has to be an element of the future boundary $S^+(\Lambda)$.

For every t in $]a, b[$, we denote by $r(t)$ the cosmological retract $r(c(t))$ of $c(t)$, and we consider the unique realizing geodesic segment $\delta_t :=]r(t), c(t)[$. We extract a subsequence t_n converging to b such that $r(t_n)$ converges to an element r_0 of $\mathrm{Cl}\big(\mathcal{H}^-(\Lambda)\big) = \mathcal{H}^-(\Lambda) \cup \Lambda$. Then δ_{t_n} converge to a geodesic segment $\delta_0 = (r_0, c(b))$. Since every δ_{t_n} is timelike, δ_0 is non-spacelike.

For every t in $]a, b[$ we have $c(t) = \cos\tau(c(t))r(t) + \sin\tau(c(t))p(t)$ (where $p(t)$ is the dual of the hyperplane orthogonal to the realizing geodesic at $r(t)$, see above). Hence

$$\langle r(t_n) \mid c(t_n)\rangle = -\cos\tau(c(t_n)).$$

In the limit,

$$\langle r_0 \mid c(b)\rangle = -\cos(\beta) < 0 \quad (\text{since}\, \beta < \pi/2\,)$$

It follows that δ_0 is not lightlike, but timelike. Since timelike geodesics in AdS_{n+1} do not meet $\partial\,\mathrm{AdS}_{n+1}$, it follows that r_0 lies in $\mathcal{H}^-(\Lambda)$.

Finally, every δ_{t_n} is orthogonal to a support hyperplane at $r(t_n)$, and hence at the limit δ_0 is orthogonal to a support hyperplane, which is spacelike since δ_0 is timelike. According to Proposition 4.17, δ_0 is a realizing geodesic. At the beginning of the proof, we have shown that every realizing geodesic can be extended to a timelike geodesic of length $\pi/2$ entirely contained in $E_0^-(\Lambda)$; hence there is an element p_0 in $S^+(\Lambda)\cap H^+(r_0)$ such that the geodesic $]r_0, p_0[$ contains δ_0—in particular, $c(b)$. Hence $[c(b), p_0]$ is a non-trivial timelike geodesic segment joining two elements of the achronal subset $S^+(\Lambda)$, which is a contradiction.

This contradiction proves $\beta = \pi/2$, *i.e.,* that the restriction of τ to every inextendible causal curve is surjective. In other words, τ is a Cauchy time function. The Proposition is proved. q.e.d.

Lemma 4.19. *The restriction of τ to $E_0^-(\Lambda)$ is $C^{1,1}$ (i.e., differentiable with locally Lipschitz derivative), and the realizing geodesics are orthogonal to the level sets of τ.*

Proof. In this proof, we consider, for any subset X of $E(\Lambda)$, the *strict* past set $I^-(X)$ (to be distinguished from the causal past $J^-(X)$), which is the set of final points of past oriented *timelike* curves contained in $E(\Lambda)$ and starting from an element of X. Since $E(\Lambda)$ is globally hyperbolic, $I^-(\Lambda)$ is actually the interior of $J^-(\Lambda)$.

Let x be an element of $E_0^-(\Lambda)$, and let $]r(x), x]$ be the unique realizing geodesic for x. As proven during the proof of Proposition 4.18, there is an element $p(x)$ of $S^+(\Lambda)$ such that $]r(x), p(x)[$ is a timelike geodesic containing $x = \cos(\tau(x))r(x) + \sin(\tau(x))p(x)$ and entirely contained in $E_0^-(\Lambda)$.

Let U be the affine domain $U(p(x))$; the past boundary component H of U is a support hyperplane of $\mathcal{H}^-(\Lambda)$ at $r(x)$ (see Definition 2.7, Remark 2.10). Let $\tau_1 : U \to]0, \pi[$ be the cosmological time function of U: for every y in U, $\tau_1(y)$ is the Lorentzian distance between y and H. Let W be the future of $r(x)$ in U, and let τ_0 be the cosmological time function in W: for every y in W, $\tau_0(y)$ is the Lorentzian length of the timelike geodesic $[r(x), y]$. We have

$$\tau_0(x) = \tau(x) = \tau_1(x).$$

Moreover,

$$\forall y \in W, \quad \tau_0(y) \leq \tau(y) \leq \tau_1(y).$$

A direct computation shows that τ_0 and τ_1 have the same derivative at x: by a standard argument (see, for example, [**17**, Proposition 1.1]) it follows that τ is differentiable at x, with derivative $d_x\tau = d_x\tau_0 = d_x\tau_1$.

Furthermore, the gradient of τ_0 and τ_1 at x is $-\nu(x)$, where $\nu(x)$ is the future-oriented timelike vector tangent at x to the realizing geodesic $[x, r(x)[$ of Lorentzian norm -1, *i.e.*,

$$\forall v \in T_x W, \quad -\langle v \mid \nu(x)\rangle = d_x\tau_0(v)x = d_x\tau(v).$$

Therefore, $-\nu(x)$ is also the Lorentzian gradient of τ. It follows that realizing geodesics are orthogonal to the level sets of τ.

In order to prove that τ is $C^{1,1}$, *i.e.*, that ν is locally Lipschitz, we adapt the argument used in the flat case in [**4**]. We consider first the restriction of ν to the level set $S_{\pi/4} = \tau^{-1}(\pi/4)$ equipped with the induced Riemannian metric. For every x in $S_{\pi/4}$, we have $x = \dfrac{r(x)+p(x)}{\sqrt{2}}$. Observe that $\dfrac{p(x)-r(x)}{\sqrt{2}}$ is then an element of $\mathbb{R}^{2,n}$ of norm -1, orthogonal to x, hence representing an element of $T_x \operatorname{AdS}_{n+1}$. This tangent vector is future oriented and orthogonal to $S_{\pi/4}$: hence $\dfrac{p(x)-r(x)}{\sqrt{2}}$ represents $\nu(x)$.

Let $c :]-1,1[\to S_{\pi/4}$ be a C^1 curve in $S_{\pi/4}$. Since r is the projection onto $\mathcal{H}^-(\Lambda)$, and since $\mathcal{H}^-(\Lambda)$ is locally Lipschitz, the path $r \circ c$ is differentiable almost everywhere in $]-1,1[$. We denote by $\dot{r}$, $\dot{p}$, $\dot{\nu}$ the derivatives of r, p, $\nu = \dfrac{p-r}{\sqrt{2}}$ along c. Almost everywhere, we have

$$\begin{aligned} \mathrm{q}_{2,n}(\dot{\nu}) &= \mathrm{q}_{2,n}\left(\frac{\dot{p}-\dot{r}}{\sqrt{2}}\right) \\ &= \frac{1}{2}(\mathrm{q}_{2,n}(\dot{p}) + \mathrm{q}_{2,n}(\dot{r}) - 2\langle \dot{r} \mid \dot{p}\rangle). \end{aligned}$$

But the derivative of c is

$$\begin{aligned} \mathrm{q}_{2,n}(\dot{c}) &= \mathrm{q}_{2,n}\left(\frac{\dot{r}+\dot{p}}{\sqrt{2}}\right) \\ &= \frac{1}{2}(\mathrm{q}_{2,n}(\dot{r}) + \mathrm{q}_{2,n}(\dot{p}) + 2\langle \dot{r} \mid \dot{p}\rangle). \end{aligned}$$

Now, since $\mathcal{H}^-(\Lambda)$ is locally convex, the quantity $\langle \dot{r} \mid \dot{p}\rangle$, wherever it is defined, is non-negative. Therefore,

$$\mathrm{q}_{2,n}(\dot{\nu}) \leq \mathrm{q}_{2,n}(\dot{c}).$$

It follows that ν is 1-Lipschitz along $S_{\pi/4}$.

On other level sets $S_t = \tau^{-1}(t)$ with $t \in (0, \pi/2)$, every element is of the form $x = \cos(t)r(x) + \sin(t)p(x)$, and $x_{\pi/4} = \dfrac{r(x)+p(x)}{\sqrt{2}}$ is a point in $S_{\pi/4}$. Geometrically, $x_{\pi/4}$ is the unique point in the realizing geodesic for x at cosmological time $\pi/4$. The unit normal vectors $\nu(x)$ and $\nu(x_{\pi/4})$ are parallel one to the other along the realizing geodesic

$]r(x), p(x)[$; hence the variation of $\nu(x)$ along S_t is controlled by the distortion of the map $x \to x_{\pi/4}$ and the variation of ν along $S_{\pi/4}$. The lemma follows. q.e.d.

4.4. GH-regular and quasi-Fuchsian representations. Let Γ be a finitely generated torsion-free group, and let $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ be a faithful, discrete representation, such that $\rho(\Gamma)$ preserves Λ. According to Proposition 4.13, the quotient space $M_\rho(\Lambda) := \rho(\Gamma)\backslash E(\Lambda)$ is globally hyperbolic. Observe that, moreover, Cauchy hypersurfaces of $M_\rho(\Lambda)$ are quotients of Cauchy hypersurfaces in $E(\Lambda)$, which are contractible (since they are graphs of maps from $\mathbb{D}^n$ into $\mathbb{R}$). Hence the cohomological dimension of Γ is $\leq n$, and it is n if and only if the Cauchy hypersurfaces are compact, *i.e.,* $M_\rho(\Lambda)$ is spatially compact.

Conversely, in his celebrated preprint [**30**, **2**], G. Mess proved that any globally hyperbolic spatially compact AdS spacetime embeds isometrically in such a quotient space $\Gamma\backslash E(\Lambda)$. Actually, G. Mess only deals with the case where $n = 2$, but his arguments also apply in higher dimension. For a detailed proof, see [**5**, Corollary 11.2].

Definition 4.20. Let Γ be a torsion-free discrete group. A representation $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ is *GH-regular* if it is faithful, discrete, and preserves a non-empty GH-regular domain $E(\Lambda)$ in AdS_{n+1}. If, moreover, the $(n-1)$-sphere Λ is acausal, then the representation is *strictly GH.*

Definition 4.21. A (strictly) GH-regular representation $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ is (strictly) *GHC-regular* if the quotient space $\rho(\Gamma)\backslash E(\Lambda)$ is spatially compact.

Hence a reformulation of Mess's result is the following:

Proposition 4.22. *A representation $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ is GHC-regular if and only if it is the holonomy of a GHC AdS spacetime.*

There is an interesting special case of strictly GHC-regular representations: the case of *quasi-Fuchsian representations.*

Definition 4.23. A strictly GHC-regular representation $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ is *quasi-Fuchsian* if Γ is isomorphic to a uniform lattice in $\mathrm{SO}_0(1, n)$.

This terminology is motivated by the analogy with the hyperbolic case.

There is a particular case: the case where Λ is a "round sphere" in $\partial\,\mathrm{AdS}_{n+1}$, *i.e.,* the boundary of a totally geodesic spacelike hypersurface $\mathbb{S}(v^\perp) \cap \mathrm{AdS}_{n+1}$:

Definition 4.24. Let Γ be a uniform lattice in $\mathrm{SO}_0(1, n)$. A *Fuchsian* representation $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ is the composition of the natural inclusions $\Gamma \subset \mathrm{SO}_0(1,n)$ and $\mathrm{SO}_0(1,n) \subset \mathrm{SO}_0(2,n)$, where in the latter $\mathrm{SO}_0(1,n)$ is considered as the stabilizer in $\mathrm{SO}_0(2,n)$ of a point in AdS_{n+1}.

In other words, a quasi-Fuchsian representation is Fuchsian if and only if it admits a global fixed point in AdS_{n+1}.

4.5. The space of timelike geodesics. Timelike geodesics in AdS_{n+1} are intersections between $\mathrm{AdS}_{n+1} \subset \mathbb{R}^{2,n}$ and 2-planes P in $\mathbb{R}^{2,n}$ such that the restriction of $\mathrm{q}_{2,n}$ to P is negative definite. The action of $\mathrm{SO}_0(2,n)$ on negative 2-planes is transitive, and the stabilizer of the (u,v)-plane is $\mathrm{SO}(2) \times \mathrm{SO}(n)$. Therefore, the space of timelike geodesics is the symmetric space

$$\mathcal{T}_{2n} := \mathrm{SO}_0(2,n)/\,\mathrm{SO}(2) \times \mathrm{SO}(n).$$

$\mathcal{T}_{2n}$ has dimension $2n$. We equip it with the Riemannian metric $g_{\mathcal{T}}$ induced by the Killing form of $\mathrm{SO}_0(2,n)$. It is well known that $\mathcal{T}_{2n}$ has non-positive curvature and rank 2: the maximal flats (*i.e.,* totally geodesic embedded Euclidean subspaces) have dimension 2. It is also naturally Hermitian. More precisely: let $\mathcal{G} = \mathfrak{so}(2,n)$ be the Lie algebra of $G = \mathrm{SO}_0(2,n)$, and let $\mathcal{K}$ be the Lie algebra of the maximal compact subgroup $K := \mathrm{SO}(2) \times \mathrm{SO}(n)$. We have the Cartan decomposition

$$\mathcal{G} = \mathcal{K} \oplus \mathcal{K}^{\perp}$$

where $\mathcal{K}^{\perp}$ is the orthogonal of $\mathcal{K}$ for the Killing form. Then $\mathcal{K}^{\perp}$ is naturally identified with the tangent space at the origin of G/K. The adjoint action of the $\mathrm{SO}(2)$ term in the stabilizer defines a K-invariant almost-complex structure on $\mathcal{K}^{\perp} \approx T_K(G/K)$ that propagates through left translations to a genuine complex structure J on $\mathcal{T}_{2n} = G/K$. Therefore, $\mathcal{T}_{2n}$ is naturally equipped with a structure of n-dimensional complex manifold, together with a J-invariant Riemannian metric, *i.e.,* a hermitian structure.

Let us consider once more the achronal $(n-1)$-dimensional topological sphere Λ. Then it is easy to prove that every timelike geodesic in AdS_{n+1} intersects $E(\Lambda)$ (cf. [**10**, Lemma 3.5]), and since $E(\Lambda)$ is convex, this intersection is connected, *i.e.,* is a single inextendible timelike geodesic of $E(\Lambda)$. In other words, one can consider $\mathcal{T}_{2n}$ as the space of timelike geodesics of $E(\Lambda)$.

Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a GH-regular representation preserving Λ. The (isometric) action of $\rho(\Gamma)$ on $\mathcal{T}_{2n}$ is free and proper, and the quotient $\mathcal{T}_{2n}(\rho) := \rho(\Gamma)\backslash\mathcal{T}_{2n}$ is naturally identified with the space of inextendible timelike geodesics of $M_\rho(\Lambda) = \rho(\Gamma)\backslash E(\Lambda)$.

Definition 4.25. Let S be a differentiable Cauchy hypersurface in a GH-regular spacetime $M_\rho(\Lambda)$ of dimension $n+1$. The *Gauss map* of S is the map $\nu : S \to \mathcal{T}_{2n}(\rho)$ that maps every element x of $M_\rho(\Lambda)$ to the unique timelike geodesic of $M_\rho(\Lambda)$ orthogonal to S at x.

When S is $C^{1,1}$ (for example, a level set $\tau^{-1}(t)$ of the cosmological time for $t < \pi/2$), then one can define for every C^1 curve c in S the *Gauss length* as the length in $\mathcal{T}_{2n}(\rho)$ of the Lipschitz curve $\nu \circ c$. It defines on S a length metric, called the *Gauss metric* (of course, if S is C^r with $r \geq 2$, then ν is C^{r-1}, and the Gauss metric is a C^{r-1} Riemannian metric).

Since every timelike geodesic intersects S at most once, the Gauss map is always injective. The image of the Gauss map is actually the set of timelike geodesics that are orthogonal to S. Since every timelike geodesic intersects S, it follows easily that the image of the Gauss map is closed and that the Gauss map is actually an embedding.

Remark 4.26. For every $0 < t < \pi/2$, let $\Sigma_t(\tau)$ be the image by the Gauss map of the cosmological level set $\tau^{-1}(t)$. According to Lemma 4.19, $\Sigma_t(\tau)$ is the space of realizing geodesics. In particular, it does not depend on t. We will denote by $\Sigma(\tau)$ this closed embedded submanifold and call it the *space of cosmological geodesics.*

4.6. Split AdS **spacetimes.** Let (p, q) be a pair of positive integers such that $p + q = n$. Let $\mathbb{R}^{2,n} = V \oplus W$ be a splitting so that

- V has dimension $p + 1$,
- W has dimension $q + 1$, and
- the restriction of $\mathrm{q}_{2,n}$ to V (respectively W) has signature $(1, p)$ (respectively $(1, q)$).

Let $(x_0, x_1, \dots, x_p, y_0, y_1, \dots, y_q)$ be a coordinate system for $\mathbb{R}^{2,n}$ such that V is the subspace $\{y_0 = y_1 = \cdots = y_q = 0\}$, W is the subspace $\{x_0 = x_1 = \cdots = x_q = 0\}$, and such that the quadratic form $\mathrm{q}_{2,n}$ is

$$-x_0^2 + x_1^2 + \cdots + x_p^2 - y_0^2 + y_1^2 + \cdots + y_q^2.$$

Let $G_{V,W} \approx \mathrm{SO}_0(1, p) \times \mathrm{SO}_0(1, q)$ be the subgroup of $\mathrm{SO}_0(2, n)$ preserving the splitting $\mathbb{R}^{2,n} = V \oplus W \approx \mathbb{R}^{1,p} \oplus \mathbb{R}^{1,q}$.

Let Λ_V (respectively Λ_W) be the subset $\mathbb{S}(\mathcal{C}_V)$ (respectively $\mathbb{S}(\mathcal{C}_V)$) of (the Klein model of) Ein_n where

$$\mathcal{C}_V := \{-x_0^2 + x_1^2 + \cdots + x_p^2 = 0, x_0 > 0, y_0 = y_1 = \cdots = y_q = 0\}$$

and

$$\mathcal{C}_W := \{-y_0^2 + y_1^2 + \cdots + y_q^2 = 0, y_0 > 0, x_0 = x_1 = \cdots = x_p = 0\}.$$

Observe that Λ_V, Λ_W are topological spheres of dimension, respectively, $p-1$, $q-1$. Moreover, for every pair of elements x, y in $\Lambda_V \cup \Lambda_W$ the scalar product $\langle \mathrm{x} \mid \mathrm{y} \rangle$ is non-positive. Hence, according to Corollary 2.11, $\Lambda_V \cup \Lambda_W$ is achronal. Moreover, every point in Λ_V is linked to every point in Λ_W by a unique lightlike geodesic segment contained in (the Klein model of) Ein_n.

Lemma 4.27. *The invisible domain $E(\Lambda_V \cup \Lambda_W)$ is the interior of the convex hull of $\Lambda_V \cup \Lambda_W$.*

Proof. Clearly,

$$\mathrm{Conv}(\mathcal{C}_V) = \{-x_0^2 + x_1^2 + \cdots + x_p^2 \leq 0, x_0 > 0, y_0 = y_1 = \cdots = y_q = 0\}.$$

Similarly,

$$\mathrm{Conv}(\mathcal{C}_W) = \{-y_0^2 + y_1^2 + \cdots + y_q^2 \leq 0, y_0 > 0, x_0 = x_1 = \cdots = x_p = 0\}.$$

Therefore, $\mathrm{Conv}(\Lambda_V \cup \Lambda_W)$ is the projection by $\mathbb{S}$ of the set of points $(x_0, x_1, \ldots, x_p, y_0, y_1, \ldots, y_q)$ satisfying the following inequalities:

$$\begin{array}{rcl} -x_0^2 + x_1^2 + \cdots + x_p^2 & \leq & 0, \\ -y_0^2 + y_1^2 + cldots + y_q^2 & \leq & 0, \\ x_0 & \geq & 0, \\ y_0 & \geq & 0. \end{array}$$

According to Remark 3.17, $\mathrm{Conv}(\Lambda_V \cup \Lambda_W)^\circ$ is contained in $E(\Lambda_V \cup \Lambda_W)$. Conversely, let $\mathsf{z} = (x_0, x_1, \ldots, x_p, y_0, y_1, \ldots, y_q)$ be an element of $\mathbb{R}^{2,n}$ representing an element of $E(\Lambda_V \cup \Lambda_W) \subset \mathbb{S}(\mathbb{R}^{2,n})$. Then, by definition of $E(\Lambda_V \cup \Lambda_W)$, the scalar product $\langle \mathsf{z} \mid \mathsf{x} \rangle$ is negative for every element x of $\mathcal{C}_V$. It follows that $(x_0, x_1, \ldots, x_p)$ must lie in the future cone of $V \approx \mathbb{R}^{1,p}$, *i.e.*,

$$\begin{array}{rcl} -x_0^2 + x_1^2 + \cdots + x_p^2 & < & 0, \\ x_0 & > & 0. \end{array}$$

Similarly, since $\langle \mathsf{z} \mid \mathsf{x} \rangle < 0$ for every element x of $\mathcal{C}_W$,

$$\begin{array}{rcl} -y_0^2 + y_1^2 + \cdots + y_q^2 & < & 0, \\ y_0 & > & 0. \end{array}$$

The lemma follows. q.e.d.

Let $\Lambda_{p,q}$ be the intersection in $\mathbb{S}(\mathbb{R}^{2,n})$ of $\mathrm{Conv}(\Lambda_V \cup \Lambda_W)$ and $\mathbb{S}(\mathcal{C}_n) \approx \mathrm{Ein}_n$. Let $(x_0, x_1, \ldots, x_p, y_0, y_1, \ldots, y_q)$ be an element of $\mathbb{R}^{2,n}$ representing an element of $\Lambda_{p,q}$. According to the proof of Lemma 4.27, we must have $-x_0^2 + x_1^2 + \cdots + x_p^2 \leq 0$ and $-y_0^2 + y_1^2 + \cdots + y_q^2 \leq 0$, and since $(x_0, x_1, \ldots, x_p, y_0, y_1, \ldots, y_q)$ lies in $\mathcal{C}_n$, these quantities must vanish. Hence the inequalities defining $\Lambda_{p,q}$ are

$$\begin{array}{rcl} -x_0^2 + x_1^2 + \cdots + x_p^2 & = & 0, \\ -y_0^2 + y_1^2 + \cdots + y_q^2 & = & 0, \\ x_0 & \geq & 0, \\ y_0 & \geq & 0. \end{array}$$

Therefore, $\Lambda_{p,q}$ is the union of Λ_V, Λ_W, and the lightlike segments joining a point of Λ_V to a point of Λ_W: it is achronal, but not acausal!

Topologically, $\Lambda_{p,q}$ is the join of two spheres; therefore, it is a sphere of dimension $1 + (p-1) + (q-1) = n-1$. It is not an easy task to figure out how it fits inside $\mathrm{Ein}_n = \partial\,\mathrm{AdS}_{n+1}$.

For that purpose, we consider the coordinates

$$(r, \theta, a_1, \dots, a_p, b_1, \dots, b_q)$$

on $\mathrm{AdS}_{n+1} = \{-x_0^2 + x_1^2 + ... + x_p^2 - y_0^2 + y_1^2 + \dots + y_q^2 = -1\} \subset \mathbb{R}^{2,n}$ such that $x_0 = r\cos\theta$, $y_0 = r\sin\theta$, $x_i = ra_i$, $y_i = rb_i$, and $r > 0$. Then we have

$$\begin{array}{rcl} V & = & \{\theta \equiv 0\ [\pi], b_1 = 0, \dots, b_q = 0\}, \\ W & = & \{\theta \equiv \pi/2\ [\pi], a_1 = 0, \dots, a_p = 0\}. \end{array}$$

According to Proposition 2.4, the $(n+1)$-tuple $(a_1, \dots, a_p, b_1, \dots, b_q, 1/r)$ describes the upper hemisphere $\mathbb{D}^n = \{a_1^2 + \dots + a_p^2 + b_1^2 + \dots + b_q^2 + 1/r^2 = 1, r > 0\}$ in the Euclidean sphere of $\mathbb{R}^{n+1}$ of radius 1. Furthermore AdS_{n+1} is conformally isometric to the product $\mathbb{S}^1 \times \mathbb{D}^n$ with the metric $-\,\mathrm{d}\theta^2 + \mathrm{ds}^2$, where ds^2 is the round metric on $\mathbb{D}^n$.

In these coordinates, the inequalities defining $E(\Lambda_V \cup \Lambda_W)$ established in the proof of Lemma 4.27 become

$$0 < \theta < \pi/2, \tag{2}$$

$$a_1^2 + \dots + a_p^2 < \cos^2\theta, \tag{3}$$

$$b_1^2 + \dots + b_q^2 < \sin^2\theta. \tag{4}$$

Let $\mathbb{D}_W$ be the subdisk of $\mathbb{D}^n$ defined by $a_1 = \dots = a_p = 0$, and let $\mathbb{D}_V$ be the subdisk defined by $b_1 = \dots = b_q = 0$. For every $\bar{x}$ in $\mathbb{D}^n$, let $d_W(\bar{x})$ be the distance in $\mathbb{D}^n$ of $\bar{x}$ to $\mathbb{D}_W$, and define similarly the "distance to $\mathbb{D}_V$" function $d_V : \mathbb{D}^n \to [0, +\infty[$. Observe that since $\mathbb{D}_V$ and $\mathbb{D}_W$ both contain the North pole $(0, \dots, 0, 1)$ of $\mathbb{D}^n$, and since every point in $\mathbb{D}^n$ is at distance at most $\pi/2$ of the North pole, d_W and d_V take value in $[0, \pi/2[$. Now, observe that the following identities hold

$$a_1^2 + \dots + a_p^2 = \sin^2 d_W(a_1, \dots, a_p, b_1, \dots, b_q, 1/r), \tag{5}$$

$$b_1^2 + \dots + b_q^2 = \sin^2 d_V(a_1, \dots, a_p, b_1, \dots, b_q, 1/r). \tag{6}$$

It follows that $E(\Lambda_V \cup \Lambda_W)$ is the domain in $\mathrm{AdS}_{n+1} \approx \mathbb{S}^1 \times \mathbb{D}^n$ consisting of the points $(\theta, \bar{x})$ such that

$$d_V(\bar{x}) < \theta < \pi/2 - d_W(\bar{x}).$$

In the terminology of Remark 3.3, it means that the lifting $\widetilde{E}(\widetilde{\Lambda_V \cup \Lambda_W})$ is defined by the functions $f^- = d_V$ and $f^+ = \pi/2 - d_W$. These functions extend uniquely as 1-Lipschitz maps $f^\pm : \overline{\mathbb{D}}^n \to [0, \pi/2]$.

The boundary $\partial\mathbb{D}^n = \mathbb{S}^{n-1}$ is totally geodesic in $\overline{\mathbb{D}}^n$, and $\partial\mathbb{D}_W$, $\partial\mathbb{D}_V$ are totally geodesic spheres of dimensions $p-1$, $q-1$, respectively. Let $\delta_W : \partial\mathbb{D}^n \to [0, \pi/2]$ (respectively $\delta_V : \partial\mathbb{D}^n \to [0, \pi/2]$) be the function "distance (in $\partial\mathbb{D}^n$, and also in $\mathbb{D}^n$) to $\partial\mathbb{D}_W$" (respectively "distance to

$\partial\mathbb{D}_V$"). It follows from equations (5) and (6) , which naturally extend to the boundary $\partial\mathbb{D}^n$, that every point of $\partial\mathbb{D}_W$ is at distance $\pi/2$ of $\partial\mathbb{D}_V$. Hence, on $\partial\mathbb{D}_V$,

$$\delta_W + \delta_V = \pi/2.$$

In other words, the restrictions of f^- and f^+ to $\partial\mathbb{D}^n$ coincide and are equal to $\delta_V = \pi/2 - \delta_W$. The restriction of $f^- = f^+$ to $\partial\mathbb{D}_V$ vanishes, and the graph of this restriction is Λ_V. The restriction of $f^- = f^+$ to $\partial\mathbb{D}_W$ is the constant map of value $\pi/2$, and the graph is Λ_W. The graph of $f^\pm : \partial\mathbb{D}^n \to \mathbb{S}^1$ is $\Lambda_{p,q}$, which is therefore an achronal sphere in Ein_n.

Clearly, $\Lambda_{p,q}$ is preserved by $G_{V,W}$. Let Γ be a cocompact lattice of $G_{V,W} \approx \mathrm{SO}_0(1,p) \times \mathrm{SO}_0(1,q)$. The inclusion $\Gamma \subset G_{V,W} \subset \mathrm{SO}_0(2,n)$ is a GH-regular representation, but not strictly since the invariant achronal limit set $\Lambda_{p,q}$ is not acausal. According to Proposition 4.13, the quotient space $M_{p,q}(\Gamma) := \Gamma\backslash E(\Lambda_{p,q})$ is a GH spacetime. Actually, the Cauchy surfaces of $M_{p,q}(\Gamma)$ are quotients by Γ of the graph of a 1-Lipschitz map $f : \mathbb{D}^n \to \mathbb{S}^1$, and hence they are $K(\Gamma,1)$ (since $\mathbb{D}^n$ is contractible). On the other hand, the quotient of $\mathbb{H}^p \times \mathbb{H}^q$ is a $K(\Gamma,1)$ too. Since Γ is a cocompact lattice, it follows that every n-dimensional manifold that is a $K(\Gamma,1)$—in particular, the Cauchy hypersurfaces in $M_{p,q}(\Gamma)$—is compact. The inclusion $\Gamma \subset \mathrm{SO}_0(2,n)$ is therefore GHC-regular.

Definition 4.28. The quotient space $M_{p,q}(\Gamma)$ is a *split* AdS *spacetime.* The representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ is a *split GHC-regular representation of type* (p,q).

Remark 4.29. The split AdS spacetimes of dimension $2+1$ are precisely the *Torus universes* studied in [**18**]. Observe indeed that the lattice in $\mathrm{SO}_0(1,1)\times\mathrm{SO}_0(1,1) \approx \mathbb{R}^2$ is isomorphic to $\mathbb{Z}^2$, and the Cauchy surfaces are indeed tori.

4.7. Crowns. A particular case of split AdS spacetime is the case $p = q = 1$ (and, therefore, $n = 2$). Then the topological spheres Λ_V and Λ_W have dimension 0, *i.e.,* are pairs of points $\Lambda_V = \{\mathrm{x}^-, \mathrm{y}^-\}$ and $\Lambda_W = \{\mathrm{x}^+, \mathrm{y}^+\}$. The topological circle $\Lambda_{p,q}$ is then piecewise linear; more precisely, it is the union of the four lightlike segments $[\mathrm{x}^-, \mathrm{x}^+]$, $[\mathrm{x}^+, \mathrm{y}^-]$, $[\mathrm{y}^-, \mathrm{y}^+]$, $[\mathrm{y}^+, \mathrm{x}^-]$. The invisible domain $E(\Lambda_{p,q})$ is then an ideal tetrahedron, interior of the convex hull of the four ideal points $\{\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+\}$. This tetrahedron has six edges; four of them are the lightlike segments forming $\Lambda_{p,q}$, and the other two are the spacelike geodesics $]\,\mathrm{x}^-, \mathrm{y}^-[$ and $]\,\mathrm{x}^+, \mathrm{y}^+[$ of AdS_{n+1} (see Figure 2). Observe that $[\mathrm{x}^-, \mathrm{x}^+]$ and $[\mathrm{y}^-, \mathrm{y}^+]$ are future oriented, whereas $[\mathrm{x}^+, \mathrm{y}^-]$ and $[\mathrm{y}^+, \mathrm{x}^-]$ are past oriented.

More generally:

Definition 4.30. For every integer $n \geq 2$, a *crown* of Ein_n is quadruple $\mathfrak{C} = (\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+)$ in Ein_n such that

- $\langle \mathrm{x}^- \mid \mathrm{x}^+ \rangle = \langle \mathrm{x}^- \mid \mathrm{y}^+ \rangle = 0,$

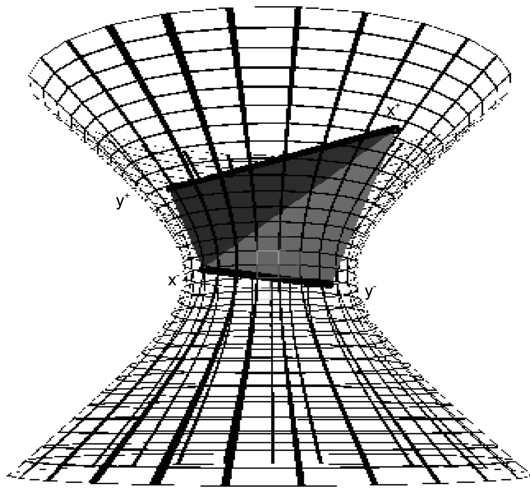

Figure 2. Picture of the realm of a crown in AdS_3. The hyperboloid represents the boundary of an affine domain of AdS_3 containing the realm of the crown. There are two triangular faces, one visible on the picture, which are in the past horizon. There are two triangular faces, one visible on the picture, forming the future horizon.

- $\langle \mathrm{y}^- \mid \mathrm{x}^+ \rangle = \langle \mathrm{y}^- \mid \mathrm{y}^+ \rangle = 0$,
- $\langle \mathrm{x}^- \mid \mathrm{y}^- \rangle < 0$,
- $\langle \mathrm{x}^+ \mid \mathrm{y}^+ \rangle < 0$, and
- the lightlike segment $[\mathrm{x}^-, \mathrm{x}^+]$ is future oriented.

The subset $\{\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+\}$ is then an achronal subset of Ein_n. The invisible domain $E(\{\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+\})$ is called the *realm of the crown* and denoted by $E(\mathfrak{C})$. The convex hull of $\{\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+\}$ is denoted by $\mathrm{Conv}(\mathfrak{C})$.

Observe that, for $n > 2$, the convex hull $\mathrm{Conv}(\mathfrak{C})$ and the realm $E(\mathfrak{C})$ do not coincide (see Remark 4.33).

Remark 4.31. Let $\mathfrak{C} = (\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+)$ be a crown in Ein_n, and let $\mathsf{x}^-, \mathsf{y}^-, \mathsf{x}^+, \mathsf{y}^+$ be elements of $\mathbb{R}^{2,n}$ representing the vertices of the crown. Let $V(\mathfrak{C})$ be the linear space spanned by $\mathsf{x}^+, \mathsf{x}^-, \mathsf{y}^-, \mathsf{y}^+$. The restriction of $\mathrm{q}_{2,n}$ to $V(\mathfrak{C})$ has signature $(2,2)$, and $\mathbb{S}(V(\mathfrak{C}))$ is the unique totally geodesic copy of Ein_2 in Ein_n containing $\mathfrak{C}$.

Remark 4.32. Let Z be the stabilizer in $\mathrm{SO}_0(2,n)$ of a crown. It preserves the orthogonal sum $V(\mathfrak{C}) \oplus V(\mathfrak{C})^\perp$. It is isomorphic to the product $A \times \mathrm{SO}(n-2)$, where A is a maximal $\mathbb{R}$-split semisimple abelian subgroup of $\mathrm{SO}_0(2,2)$, hence of $\mathrm{SO}_0(2,n)$. Therefore, Z is the centralizer in $\mathrm{SO}_0(2,n)$ of A, and it has finite index in the normalizer N of A. It follows that the space of crowns is naturally a finite covering over the space G/N

of maximal flats in the symmetric space $\mathcal{T}_{2n} = \mathrm{SO}_0(2,n)/SO(2)\times SO(n)$ of timelike geodesics.

Remark 4.33. In this remark, we go back to the coordinate system used in Section 2.1. Up to an isometry, one can assume that the crown $\mathfrak{C} = (\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+)$ is represented by

$$\begin{aligned} \mathsf{x}^+ &= (1,0,1,0,0,\ldots,0), \\ \mathsf{y}^+ &= (1,0,-1,0,0,\ldots,0), \\ \mathsf{x}^- &= (0,1,0,1,0,\ldots,0), \\ \mathsf{y}^- &= (0,1,0,-1,0,\ldots,0). \end{aligned}$$

According to Proposition 3.9, the realm $E(\mathfrak{C})$ is defined by the inequalities

$$\begin{gathered} x_1 - u < 0, \\ -x_1 - u < 0, \\ x_2 - v < 0, \\ -x_2 - v < 0, \\ -u^2 - v^2 + x_1^2 + \ldots + x_n^2 < 0, \end{gathered}$$

and hence by

$$|x_1| < u, \quad |x_2| < v, \quad -u^2 - v^2 + x_1^2 + \ldots + x_n^2 < 0.$$

Observe that the last inequality is implied by the two previous ones when $n = 2$. If $n = 2$, the realm of a crown $\mathfrak{C}$ coincides with the interior of $\mathrm{Conv}(\mathfrak{C})$ (Lemma 4.27), but this is obviously not true for $n > 2$, since $\mathrm{Conv}(\mathfrak{C})$ is always 3-dimensional.

5. Acausality of limit sets of Gromov hyperbolic groups

Throughout this section, Γ is a torsion-free Gromov hyperbolic group, and $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ a GHC-regular representation, with limit set $\Lambda \approx \mathbb{S}^n$. By hypothesis, $E(\Lambda)$ is not empty, and therefore Λ is not purely lightlike.

5.1. Non-existence of crowns.

Proposition 5.1. *The limit set Λ contains no crown.*

Proof. Recall that $\mathcal{T}_{2n}$ denotes the space of timelike geodesics (cf. Sect. 4.5). Let $\mathfrak{C} = (\mathrm{x}^-, \mathrm{y}^-, \mathrm{x}^+, \mathrm{y}^+)$ be a crown contained in Λ. Let $F(\mathfrak{C})$ be the subset of $\mathcal{T}_{2n}$ consisting of timelike geodesics containing a segment $[\mathrm{p}^-, \mathrm{p}^+]$ with $\mathrm{p}^\pm \in]\,\mathrm{x}^\pm, \mathrm{y}^\pm[$. Let A be the maximal $\mathbb{R}$-split abelian subgroup stabilizing $\mathfrak{C}$, *i.e.,* the subgroup of the stabilizer Z of $\mathfrak{C}$ acting trivially on $V(\mathfrak{C})^\perp$ (cf. Remark 4.32). Then, $F(\mathfrak{C})$ is an orbit of the action of A in $\mathcal{T}_{2n}$. Therefore, $F(\mathfrak{C})$ is a flat in the symmetric space $\mathcal{T}_{2n}$.

Let $\Sigma(\tau)$ be the space of cosmological geodesics in $E_0^-(\Lambda)$ (cf. Remark 4.26).

Claim: $\Sigma(\tau)$ *contains* $F(\mathfrak{C})$.

Let p^+, p^- be elements of the spacelike lines $]\,\mathrm{x}^+, \mathrm{y}^+[$, $]\,\mathrm{x}^-, \mathrm{y}^-[$. The closure of $E(\Lambda)$ contains $\mathrm{Conv}(\Lambda)$ (Remark 3.17); in particular, it contains $\mathrm{p}^\pm$. On the other hand, $\langle \mathrm{x}^+ \mid \mathrm{p}^- \rangle = 0$; hence, by Proposition 3.9, the point p^+ does not lie in $E(\Lambda)$. Therefore, p^- is an element of the past horizon $\mathcal{H}^-(\Lambda)$ (recall Definition 3.5).

Observe that $\langle \mathrm{p}^- \mid \mathrm{p}^+ \rangle = 0$. Hence p^- lies in the hyperplane $H^-(\mathrm{p}^+)$ past-dual to p^+. Now, since p^+ lies in $\mathrm{Conv}(\Lambda)$, we have $\langle \mathrm{p}^+ \mid \mathrm{y} \rangle \leq 0$ for every y in $E(\Lambda)$. Therefore, $H^-(\mathrm{p}^+)$ is a support hyperplane of $\mathcal{H}^-(\Lambda)$ at p^-, orthogonal to the timelike geodesic $[\mathrm{p}^-, \mathrm{p}^+]$. According to Proposition 4.17, $(\mathrm{p}^-, \mathrm{p}^+)$ is a realizing geodesic, hence an element of $\Sigma(\tau)$. The claim follows.

Consider now the Gauss metric on $\Sigma(\tau)$ (cf. Definition 4.25). According to the claim, $\Sigma(\tau)$ contains the Euclidean plane $F(\mathfrak{C})$. Since $F(\mathfrak{C})$ is totally geodesic in $\mathcal{T}_{2n}$, it is also totally geodesic in $\Sigma(\tau)$.

On the other hand, the group Γ acts on $\Sigma(\tau)$, and the quotient of this action is compact, since this quotient is the image by the Gauss map of a compact surface in $M_\rho(\Lambda)$. Hence $\Sigma(\tau)$ is quasi-isometric to Γ, and therefore, Gromov hyperbolic. This is a contradiction since a Gromov hyperbolic metric space cannot contain a 2-dimensional flat. q.e.d.

5.2. Compactness of the convex core. In Section 3.3, we have seen that, up to a lifting in $\widetilde{\mathrm{AdS}}_{n+1}$, the convex core $\mathrm{Conv}(\Lambda)$ (respectively the invisible domain $E(\Lambda)$) can be defined as the region between the graphs of functions $F^\pm : \mathbb{D}^n \to \mathbb{R}$ (respectively $f^\pm : \mathbb{D}^n \to \mathbb{R}$) such that (cf. Proposition 4.3)

$$f^- \leq F^- \leq F^+ \leq f^+, \tag{7}$$

where the inequality $F^- \leq F^+$ is strict as soon as $\rho : \Gamma \to \mathrm{SO}_0(2, n)$ is not Fuchsian.

Proposition 5.2. *The left and right inequalities in* (7) *are strict; i.e., for every* x *in* $\mathbb{D}^n$, *we have*

$$f^-(\bar{x}) < F^-(\bar{x}) \leq F^+(\bar{x}) < f^+(\bar{x}).$$

Proof. Assume not. Reverting the time orientation if necessary, it means that $f^-(\bar{x}) = F^-(\bar{x})$ for some $\bar{x}$ in $\mathbb{D}^n$. In other words, $(F^-(\bar{x}), \bar{x})$ represents an element $\mathrm{x} = \mathbb{S}(\mathsf{x})$ of $S^-(\Lambda)$ is on the boundary of $E(\Lambda) \subset \mathbb{ADS}_{n+1}$—more precisely, in the past horizon $\mathcal{H}^-(\Lambda)$). The representant x in $\mathbb{R}^{2,n}$ is a linear combination $\mathsf{x} = t_1\mathsf{x}_1 + \ldots + t_k\mathsf{x}_k$, where $k \geq 2$, t_i are positive real numbers and x_i are elements of $\mathcal{C}_n \subset \mathbb{R}^{2,n}$ such that the projections $\mathbb{S}(\mathsf{x}_i)$ belong to Λ. Moreover, since x lies in $\mathbb{ADS}_{n+1}$, we have $\langle \mathsf{x}_a \mid \mathsf{x}_b \rangle < 0$ for some integers a, b. Since x lies in the boundary of $E(\Lambda)$, there is an element x_0 in $\mathcal{C}_n$ representing an element of Λ such that

$$\begin{aligned} 0 &= \langle \mathsf{x}_0 \mid \mathsf{x} \rangle \\ &= t_1 \langle \mathsf{x}_0 \mid \mathsf{x}_1 \rangle + \ldots + t_k \langle \mathsf{x}_0 \mid \mathsf{x}_k \rangle. \end{aligned}$$

Since Λ is achronal, each $\langle x_0 \mid x_i \rangle$ is non-positive, and therefore vanishes. In particular,

- $\langle \mathsf{x}_0 \mid \mathsf{x}_a \rangle = \langle \mathsf{x}_0 \mid \mathsf{x}_b \rangle = 0$,
- $\langle \mathsf{x}_a \mid \mathsf{x}_b \rangle < 0$.

We can assume without loss of generality that x is actually equal to $\mathsf{x}_a + \mathsf{x}_b$, after rescaling if necessary x_a, x_b so that $\mathsf{x}_a + \mathsf{x}_b$ has norm -1, *i.e.,* lies in $\mathbb{ADS}_{n+1}$.

Consider now any element $\mathrm{y}_0 = \mathbb{S}(\mathsf{y}_0)$ of $E_0^-(\Lambda)$ in the future of x, *i.e.,* such that $]\,\mathrm{x}, \mathrm{y}_0[$ is a future oriented timelike segment. More precisely, we can select y_0 such that the timelike segment $[\mathsf{x}, \mathsf{y}_0]$ is orthogonal to the segment $[\mathsf{x}_a, \mathsf{x}_b]$. Let t_0 be the cosmological time at y_0, let S_0 be the cosmological level set $\tau^{-1}(t_0)$, and let d_0 be the induced metric on S_0: this metric is complete since S_0 admits a compact quotient.

Let P be the 3-subspace of $\mathbb{R}^{2,n}$ spanned by y_0, x and x_0: by construction, P is orthogonal to $\mathsf{x}_a - \mathsf{x}_b$. Then, $\mathbb{A} := \mathbb{S}(P) \cap \mathbb{ADS}_{n+1}$ is a totally geodesic copy of $\mathbb{ADS}_2$. The restriction of τ to $\mathbb{A} \cap E_0^-(\Lambda)$ is still a Cauchy time function, and $S_0 \cap \mathbb{A}$ is a spacelike path which contains y_0. Moreover, there is a sequence $\mathrm{y}_n := \mathbb{S}(\mathsf{y}_n)$ in $S_0 \cap \mathbb{A}$ converging to $\mathrm{x}_0 := \mathbb{S}(\mathsf{x}_0)$.

Let $K_0 \subset S_0$ be a compact fundamental domain for the action of $\rho(\Gamma)$ on S_0. There is a sequence $g_n = \rho(\gamma_n)$ in $\rho(\Gamma)$ such that $\mathrm{z}_n := g_n \, \mathrm{y}_n$ converge to $\bar{\mathrm{z}}$ in K_0. We define

$$\begin{aligned} \mathsf{a}_n &:= g_n \mathsf{x}_a, \\ \mathsf{b}_n &:= g_n \mathsf{x}_b, \\ \mathsf{q}_n &:= g_n \mathsf{x}_0, \\ \mathsf{x}_n &:= g_n \mathsf{x} = \mathsf{a}_n + \mathsf{b}_n. \end{aligned}$$

Up to a subsequence, we can assume that $\mathrm{a}_n := \mathbb{S}(\mathsf{a}_n)$, $\mathrm{b}_n := \mathbb{S}(\mathsf{b}_n)$, $\mathrm{q}_n := \mathbb{S}(\mathsf{q}_n)$ converge to elements $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$, $\bar{\mathrm{q}}$ of Λ, and that $\mathrm{x}_n := \mathbb{S}(\mathsf{x}_n)$ converges to an element $\bar{\mathrm{x}}$ of the segment $[\bar{\mathrm{a}}, \bar{\mathrm{b}}]$. At this level, it could happen that this segment is reduced to one point—*i.e.,* $\bar{\mathrm{a}} = \bar{\mathrm{b}}$—but we will prove that it is not the case.

Claim: $\bar{\mathrm{x}}$ *lies in* $\mathbb{ADS}_{n+1}$.

Indeed, since every x_n belongs to $\mathcal{H}^-(\Lambda)$, if the limit $\bar{\mathrm{x}}$ does not lie in $\mathbb{ADS}_{n+1}$, then, according to Proposition 4.1, it is an element of Λ. The segment $[\bar{\mathrm{x}}, \bar{\mathrm{z}}]$, limit of the timelike segments $[\mathrm{x}_n, \mathrm{z}_n]$, would be causal, and the element $\bar{\mathrm{z}}$ of $K_0 \subset E(\Lambda)$ would be causally related to the element $\bar{\mathrm{x}}$ of Λ: a contradiction.

Therefore, $\bar{\mathrm{x}}$ lies in $\mathcal{H}^-(\Lambda)$. It follows in particular that $\bar{\mathrm{a}} \neq \bar{\mathrm{b}}$. Consider now the iterates $\mathrm{p}_n := g_n \mathrm{y}_0$ of y_0. They belong to S_0. Up to a subsequence, we can assume that the sequence $(\mathrm{p}_n)_{n\in\mathbb{N}}$ admits a limit $\bar{\mathrm{p}}$. Since d_0 is complete and the y_n converge to a point in $\partial\mathbb{ADS}_{n+1}$, the distance $d_0(\mathrm{y}_n, \mathrm{y}_0)$ converge to $+\infty$. Therefore,

$$d_0(\mathrm{z}_n, \mathrm{p}_n) = d_0(g_n \mathrm{y}_n, g_n \mathrm{y}_0) = d_0(\mathrm{y}_n, \mathrm{y}_0)$$

is unbounded: the limit $\bar{\mathrm{p}}$ is at infinity, *i.e.,* an element of Λ.

The four points $\bar{\mathrm{q}}$, $\bar{\mathrm{a}}$, $\bar{\mathrm{b}}$, $\bar{\mathrm{p}}$ in Ein_n satisfy

- $\langle \bar{\mathrm{q}} \mid \bar{\mathrm{a}}\rangle = \langle \bar{\mathrm{q}} \mid \bar{\mathrm{b}}\rangle = 0$ (since $\langle \mathrm{x}_0 \mid \mathrm{x}_a\rangle = \langle \mathrm{x}_0 \mid \mathrm{x}_b\rangle = 0$),
- $\langle \bar{\mathrm{a}} \mid \bar{\mathrm{b}}\rangle < 0$ (since $]\bar{\mathrm{a}}, \bar{\mathrm{b}}[$ contains the element $\bar{\mathrm{x}}$ of $\mathbb{ADS}_{n+1}$), and
- $\langle \bar{\mathrm{p}} \mid \bar{\mathrm{a}}\rangle = \langle \bar{\mathrm{p}} \mid \bar{\mathrm{b}}\rangle = 0$ (since every p_n lies in $\mathsf{a}_n^\perp \cap \mathsf{b}_n^\perp$).

Now observe that in every iterate $\mathbb{A}_n = g_n\mathbb{A}$, the timelike geodesic Δ_n containing $[\mathrm{x}_n, \mathrm{z}_n]$ disconnects $\mathbb{A}_n$, and that the ideal points q_n, p_n lie on (the boundary of) different components of $\mathbb{A}_n \setminus \Delta_n$.

It follows that $\bar{\mathrm{p}} \neq \bar{\mathrm{q}}$. Observe that $\bar{\mathrm{q}}$, $\bar{\mathrm{p}}$ lie in the projection by $\mathbb{S}$ of the isotropic cone of $\bar{\mathrm{a}}^\perp \cap \bar{\mathrm{b}}^\perp$, which has signature $(1, n-1)$. Moreover, every p_n, q_n lies in the future of x_n: it follows that $\bar{\mathrm{q}}$, $\bar{\mathrm{p}}$ lies in the same connected component of the projection of the isotropic cone of $\bar{\mathrm{a}}^\perp \cap \bar{\mathrm{b}}^\perp$ (with the origin removed); therefore,

$$\langle \bar{\mathrm{p}} \mid \bar{\mathrm{q}}\rangle < 0.$$

It follows that $(\bar{\mathrm{a}}, \bar{\mathrm{b}}, \bar{\mathrm{p}}, \bar{\mathrm{q}})$ is a crown, contradicting Proposition 5.1. q.e.d.

5.3. Proof of Theorem 1.4. In this section, we prove the following:

Theorem 1.4. *Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a GHC-regular representation, where Γ is a Gromov hyperbolic group. Then the achronal limit set Λ is acausal,* i.e., *ρ is $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov.*

Proof. We equip the convex domain $E(\Lambda)$ with its *Hilbert metric:* for every element x, y in $E(\Lambda) \subset \mathbb{ADS}_{n+1}$, the Hilbert distance $d_h(\mathrm{x}, \mathrm{y})$ is defined to be $\frac{1}{2}[\mathrm{a}; \mathrm{x}; \mathrm{y}; \mathrm{b}]$, where the cross-ratio $[\mathrm{a}; \mathrm{x}; \mathrm{y}; \mathrm{b}]$ is defined so that $[0; 1; z; \infty] = z$ and where a, b are the intersections between $\partial E(\Lambda)$ and the projective line in $\mathbb{S}(\mathbb{R}^{2,n})$ containing x and y. The Hilbert metric is of course $\rho(\Gamma)$-invariant.

Assume by contradiction that Λ is *not* acausal. Then, since it is filled in the sense of Remark 3.19, it contains a lightlike segment $[\mathrm{x}, \mathrm{y}]$ with $\mathrm{x} \neq \mathrm{y}$. We can assume without loss of generality that this segment is maximal, *i.e.,* that $[\mathrm{x}, \mathrm{y}]$ is precisely the intersection between Λ and a projective line in $\mathrm{Ein}_n \subset \mathbb{S}(\mathbb{R}^{2,n})$. Let P be a projective 2-plane of $\mathbb{S}(\mathbb{R}^{2,n})$ containing $[\mathrm{x}, \mathrm{y}]$ and an element z of $\mathrm{Conv}(\Lambda)^\circ$. The intersection $P \cap \mathrm{Conv}(\Lambda)^\circ$ is a convex domain containing the ideal triangle x, y, z, with a side $[\mathrm{x}, \mathrm{y}]$ contained at infinity. Let u be an element in the segment $]\mathrm{x}, \mathrm{y}[$. For every $t > 0$, let x_t (respectively y_t) be the element of the segment $[\mathrm{z},\mathrm{x}[$ (respectively $[\mathrm{z},\mathrm{y}[$) such that $d_h(\mathrm{z},\mathrm{x}_t)=t$ (respectively $d_h(\mathrm{z},\mathrm{y}_t)=t$), and let u_t be the intersection $[\mathrm{z},\mathrm{u}]\cap[\mathrm{x}_t,\mathrm{y}_t]$. Observe that $[\mathrm{z},\mathrm{x}_t]\cup[\mathrm{x}_t,\mathrm{y}_t]\cup[\mathrm{y}_t,\mathrm{z}]$ is a geodesic triangle for d_h. Now, an elementary computation shows (see the proof of Proposition 2.5 in [**13**])

$$\lim_{t\to+\infty} d_h(\mathrm{u}_t,[\mathrm{z},\mathrm{x}_t]\cup[\mathrm{z},\mathrm{y}_t])=+\infty.$$

This implies that $\mathrm{Conv}(\Lambda)\setminus\Lambda$, equipped with the restriction of d_h, is not Gromov hyperbolic.

But, on the other hand, the quotient of $\mathrm{Conv}(\Lambda)\setminus\Lambda$ by $\rho(\Gamma)$ is compact. Indeed, according to Proposition 5.2, the future boundary $S^+(\Lambda)$ and the past boundary $S^-(\Lambda)$ of the convex core are contained in $E(\Lambda)$. Their projections in $M_\rho(\Lambda)$ are therefore compact achronal hypersurfaces, bounding a compact region C, which is precisely the quotient of $\mathrm{Conv}(\Lambda)\setminus\Lambda$.

Since Γ is Gromov hyperbolic, $(\mathrm{Conv}(\Lambda)\setminus\Lambda, d_h)$ is Gromov hyperbolic. Contradiction. q.e.d.

6. Limits of Anosov representations

This section is entirely devoted to the proof of the Theorem 1.2, wihich we restate here for the reader's convenience:

Theorem 1.2. *Let $n\geq 2$, and let Γ be a Gromov hyperbolic group of cohomological dimension $\geq n$. Then $\mathrm{Rep}_{an}(\Gamma,\mathrm{SO}_0(2,n))$ is open and closed in $\mathrm{Rep}(\Gamma,\mathrm{SO}_0(2,n))$.*

We recall that one important step of the proof will be be to show that under these hypotheses, Γ is the fundamental group of a closed manifold, and that its cohomological dimension is eventually n (cf. Remark 1.3).

Let Γ be as in the hypotheses of the Theorem a Gromov hyperbolic group of cohomological dimension $\geq n$. The fact that

$$\mathrm{Rep}_{an}(\Gamma,\mathrm{SO}_0(2,n))$$

is open in $\mathrm{Rep}(\Gamma,\mathrm{SO}_0(2,n))$ is well known (cf. Theorem 1.2 in [**28**], or [**29**]); hence our task is to prove that it is a closed subset.

Let $\rho_k:\Gamma\to\mathrm{SO}_0(2,n)$ be a sequence of $(\mathrm{SO}_0(2,n),\mathrm{Ein}_n)$-Anosov representations converging to a representation $\rho_\infty:\Gamma\to\mathrm{SO}_0(2,n)$.

Proposition 6.1. *The limit representation $\rho_\infty:\Gamma\to\mathrm{SO}_0(2,n)$ is discrete and faithful.*

Proof. Since Γ is Gromov hyperbolic and non-elementary, it contains no nilpotent normal subgroup (see [**22**]). Hence, by a classical argument, the limit $\rho_\infty:\Gamma\to\mathrm{SO}_0(2,n)$ is discrete and faithful (cf. Lemma 1.1 in [**24**]).

Actually, we give a sketch of the argument, since we will later need, in the proof of Lemma 6.2, a slightly more elaborate version of it. The key point is that $\mathrm{SO}_0(2,n)$, as any Lie group, contains a *Zassenhaus neighborhood*, *i.e.,* a neighborhood W_0 of the identity such that every discrete subgroup generated by elements in W_0 is contained in a nilpotent Lie subgroup of $\mathrm{SO}_0(2,n)$ (for a proof, see [**32**, Theorem 8.16] where this result is attributed to Zassenhaus and Kazhdan-Margulis). In particular, such a discrete subgroup is nilpotent, and there is a uniform bound N for the nilpotence class (*i.e.,* the length of the lower central series) of these nilpotent groups.

Assume that $\mathrm{Ker}(\rho_\infty) \subset \Gamma$ is non-trivial. Then it is a normal subgroup. For any finite subset F of $\mathrm{Ker}(\rho_\infty)$, there is an integer k_0 such that $k \geq k_0$ implies that $\rho_k(F)$ is contained in W_0, hence that the subgroup generated by $\rho_k(F)$ is nilpotent of nilpotence class $\leq N$. It follows that $\mathrm{Ker}(\rho_\infty)$ is nilpotent, therefore trivial: the representation ρ_∞ is faithful.

Let $\bar{G}_\infty$ be the closure of $\rho_\infty(\Gamma)$ in $\mathrm{SO}_0(2,n)$, and let $\bar{G}^0_\infty$ be the identity component of $\bar{G}_\infty$: it is a normal subgroup of $\bar{G}_\infty$, and it is generated by any neighborhood of the identity in $\bar{G}_\infty$. Therefore, $\rho_\infty(\Gamma)\cap W_0$ generates a dense subgroup of $\bar{G}^0_\infty$. On the other hand, any expression of the form

$$[\rho_\infty(\gamma_1), [\rho_\infty(\gamma_2), [\cdots[\rho_\infty(\gamma_N), \rho_\infty(\gamma_{N+1})]\cdots]]] \tag{8}$$

with every $\rho_k(\gamma_i) \in W_0$ is the limit for $k \to +\infty$ of

$$[\rho_k(\gamma_1), [\rho_k(\gamma_2), [\cdots[\rho_k(\gamma_N), \rho_k(\gamma_{N+1})]\cdots]]]. \tag{9}$$

For k sufficiently big, every $\rho_k(\gamma_i)$ belongs to W_0 and $\rho_k(\Gamma)$ is discrete; hence (9) is trivial. The limit (8) is trivial too. It follows that $\bar{G}^0_\infty$ is nilpotent. Then $\rho_\infty^{-1}(\rho_\infty(\Gamma)\cap\bar{G}^0_\infty)$ is a nilpotent normal subgroup of Γ, and hence trivial. It follows that $\bar{G}^0_\infty$ is trivial, *i.e.,* $\rho_\infty(\Gamma)$ is discrete.

q.e.d.

An immediate consequence of the representations ρ_k being Anosov is the existence of a $\rho_k(\Gamma)$-equivariant map $\xi : \partial_\infty\Gamma \to \mathrm{Ein}_n$ whose image is a closed $\rho_k(\Gamma)$-invariant acausal subset Λ_k (cf. [**10**, **28**]). According to the Remark 3.4, for every integer k there is a $\rho_k(\Gamma)$-invariant achronal topological $(n-1)$-sphere Λ^+_k, which is not purely lightlike since it contains the acausal subset Λ_k. Therefore, every ρ_k is a GH-regular representation. The Cauchy hypersurfaces of the associated GH spacetimes are contractible (since the universal coverings are topological disks embedded in regular domains of AdS_{n+1}) and have fundamental groups isomorphic to Γ. Since Γ has cohomological dimension $\geq n$, these Cauchy hypersurfaces are compact: the ρ_k are GHC-regular representations.

The $\rho_k(\Gamma)$-invariant spheres $\hat{\Lambda}_k$ are graphs of locally 1-Lipschitz maps $f_k : \mathbb{S}^{n-1} \to \mathbb{S}^1$. It follows easily by the Ascoli–Arzela Theorem that, up to a subsequence, $\rho_\infty(\Gamma)$ preserves the graph a of locally 1-Lipschitz map $f_\infty : \mathbb{S}^{n-1} \to \mathbb{S}^1$, *i.e.,* an achronal sphere Λ_∞.

Lemma 6.2. Λ_∞ *is not purely lightlike.*

Proof. Assume otherwise. Then Λ_∞ is the union of lightlike geodesics joining two antipodal points x_0 and $-\mathrm{x}_0$ in Ein_n (see Remark 3.19). Let G_0 be the stabilizer in $\mathrm{SO}_0(2,n)$ of $\pm\mathrm{x}_0$: the image $\rho_\infty(\Gamma)$ is a discrete subgroup of G_0.

According to Remark 2.13, the group G_0 is isomorphic to the group of conformal affine transformations of the Minkowski space $\mathrm{Mink}(\mathrm{x}_0) \approx \mathbb{R}^{1,n-1}$. There is an exact sequence

$$1 \to \mathbb{R}^{1,n-1} \to G_0 \to \mathbb{R} \times \mathrm{SO}_0(1, n-1) \to 1,$$

where the left term is the subgroup of translations of $\mathbb{R}^{1,n-1}$ and the right term the group of conformal linear transformations of $\mathbb{R}^{1,n-1}$. Let $L : G_0 \to \mathbb{R} \times \mathrm{SO}_0(1, n-1)$ be the projection morphism. Let $\bar{L}$ be the closure in $\mathbb{R} \times \mathrm{SO}_0(1, n-1)$ of $L(\rho_\infty(\Gamma))$, and let $\bar{L}_0$ be the identity component of $\bar{L}$. Considering as in the proof of Proposition 6.1 a Zassenhaus neighborhood V_0 of the identity in G_0, and using as a trick the fact that conjugacies in G_0 by homotheties in $\mathbb{R}^{1,n-1}$ can reduce at an arbitrary small-scale translations in $\mathbb{R}^{1,n-1}$, one proves that $\rho_\infty(\Gamma) \cap L^{-1}(L(\rho_\infty(\Gamma)) \cap \bar{L}_0)$ is a normal nilpotent subgroup of $\rho_\infty(\Gamma) \approx \Gamma$ (cf. [**19**, Theorem 1.2.1]). Therefore, it is trivial: $L(\rho_\infty(\Gamma))$ is a discrete subgroup of $\mathbb{R} \times \mathrm{SO}_0(1, n-1)$.

Now we consider $\mathbb{R} \times \mathrm{SO}_0(1, n-1)$ as the group of isometries of the Riemannian product $\mathbb{R} \times \mathbb{H}^{n-1}$. By what we have just proved, the action of $\rho_\infty(\Gamma)$ on $\mathbb{R} \times \mathbb{H}^{n-1}$ is properly discontinuous. On the other hand, Γ acts properly and cocompactly on a topological disk of dimension n (a Cauchy hypersurface in $E(\Lambda_k)$ for any k); hence its action on $\mathbb{R} \times \mathbb{H}^{n-1}$ is cocompact. This is a contradiction since $\mathbb{R} \times \mathbb{H}^{n-1}$ is not Gromov hyperbolic (it contains flats of dimension 2). q.e.d.

Proof of Theorem 1.2. According to Lemma 3.8, Proposition 6.1, and Lemma 6.2, $\rho_\infty : \Gamma \to \mathrm{SO}_0(2,n)$ is a GH-regular representation. It is actually a GHC-regular representation since Cauchy surfaces in $\rho_\infty(\Gamma)\backslash E(\Lambda_\infty)$ are $K(\Gamma,1)$ and thus compact since Cauchy surfaces in every $\rho_k(\Gamma)\backslash E(\Lambda_k)$ are compact. According to Theorem 1.4, the representation $\rho_\infty : \Gamma \to \mathrm{SO}_0(2,n)$ is $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov.

7. Bounded cohomology

This section is devoted to the proof of Theorem 1.5.

Theorem 1.5 *Let* $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ *be a faithful and discrete representation, where* Γ *is the fundamental group of a negatively curved closed manifold* M *of dimension* n*. The following assertions are equivalent:*

1) *ρ is $(\mathrm{SO}_0(2,n), \mathrm{Ein}_n)$-Anosov.*
2) *ρ is GHC-regular.*
3) *The bounded Euler class $\mathrm{eu}_b(\rho)$ vanishes.*

For a friendly introduction to bounded cohomology, close to our present concern, see [**23**, Section 6].

7.1. The bounded Euler class. We have the central exact sequence

$$1 \to \mathbb{Z} \to \widetilde{\mathrm{SO}}_0(2,n) \to \mathrm{SO}_0(2,n) \to 1, \tag{10}$$

where $\mathbb{Z}$ is the group of deck transformations of the covering $\hat{p} : \widetilde{\mathrm{Ein}}_n \to \mathrm{Ein}_n$, generated by the transformation δ (cf. Section 2.3). Observe that $\mathbb{Z}$ is not always the whole center of $\widetilde{\mathrm{SO}}_0(2,n)$, since $-\mathrm{Id}$ is an element of $\mathrm{SO}_0(2,n)$ when n is even. Fix an element $x_0 = (0, \bar{x}_0)$ in $\widetilde{\mathrm{Ein}}_n \approx \mathbb{R} \times \mathbb{S}^{n-1}$. In these coordinates, δ is the transformation $(\theta, \bar{x}) \mapsto (\theta + 2\pi, \bar{x})$. Hence we can define a section $\sigma : \mathrm{SO}_0(2,n) \to \widetilde{\mathrm{SO}}_0(2,n)$, called the *canonical section,* which maps every element g of $\mathrm{SO}_0(2,n)$ to the unique element $\sigma(g)$ of $\widetilde{\mathrm{SO}}_0(2,n)$ above g and such that $\sigma(g)(x_0)$ lies in the domain

$$\mathcal{W}_0 := \{(\theta, \bar{x}) \in \mathbb{R} \times \mathbb{S}^{n-1} \ / \ -\pi \leq \theta < \pi\}.$$

Observe that $\mathcal{W}_0$ is a fundamental domain for the action of $\langle \delta \rangle = \mathbb{Z}$ on $\widetilde{\mathrm{Ein}}_n$.

For any pair (g_1, g_2) of elements of $\mathrm{SO}_0(2,n)$, we define $c(g_1, g_2)$ as the unique integer k such that $\sigma(g_1 g_2) = \delta^k \sigma(g_1)\sigma(g_2)$.

Lemma 7.1 (Compare with Lemma 6.3 in [**23**])**.** *The 2-cocycle c takes only the values -1, 0, or 1.*

Proof. Let $x_1 = (\theta_1, \bar{x}_1)$ and $x_2 = (\theta_2, \bar{x}_2)$ be the images of x_0 by $\sigma(g_1)$, $\sigma(g_2)$, respectively. Let $x_3 = (\theta_3, \bar{x}_3)$ be the image of x_2 by $\sigma(g_1)$.

1) *If $|\theta_2| \leq d(\bar{x}_2, \bar{x}_0)$.* It means that x_2 is not in $I^{\pm}(x_0)$. Then $x_3 = \sigma(g_1)(x_2)$ is not in $I^{\pm}(x_1)$. Therefore
$$\mid \theta_3 - \theta_1 \mid \leq d(\bar{x}_3, \bar{x}_1) \leq \pi,$$
implying $\mid \theta_3 \mid \leq 2\pi$. It follows that if $x_3 = \sigma(g_1)\sigma(g_2)(x_0)$ is not already in $\mathcal{W}_0$, $\delta^{\epsilon}(x_0)$ for $\epsilon = \pm 1$ is. Hence $c(g_1, g_2) = \epsilon$ is 0, -1 or 1, as required.
2) *If $\theta_2 > d(\bar{x}_2, \bar{x}_0)$.* Then, $0 < \pi - \theta_2 < \pi - d(\bar{x}_2, \bar{x}_0) = d(\bar{x}_2, -\bar{x}_0)$ where $-\bar{x}_0$ is the antipodal point in $\mathbb{S}^{n-1}$ at distance π from $\bar{x}_0$. The point x_2 is not in $J^{\pm}((\pi, -\bar{x_0}))$, and hence its image x_3 by $\sigma(g_1)$ is not in $J^{\pm}((\pi + \theta_1, -\bar{x}_1)$. It follows that
$$\mid \theta_3 - (\pi + \theta_1) \mid < d(\bar{x}_3, -\bar{x}_1) \leq \pi.$$
Therefore,
$$\mid \theta_3 \mid < 3\pi.$$
Hence, for some $\epsilon = 0$ or ± 1, we have that $\delta^{\epsilon}(x_3)$ lies in $\mathcal{W}_0$, and $c(g_1, g_2) = \epsilon$ is 0, -1 or 1.

3) *If* $-\pi \leq \theta_2 < -d(\bar{x}_2, \bar{x}_0)$. We apply the same argument as in case (2), by observing that $\bar{x}_2$ is then not causally related to $(-\pi, -\bar{x}_0)$. Details are left to the reader.

q.e.d.

Definition 7.2. c is a bounded 2-cocycle. It represents an element of the bounded cohomology space $\mathrm{H}^2_b(\mathrm{SO}_0(2,n), \mathbb{Z})$ called the *bounded Euler class.*

For any representation $\rho : \Gamma \to \mathrm{SO}_0(2,n)$, the pull-back $\rho^*([c])$ is an element of $\mathrm{H}^2_b(\Gamma, \mathbb{Z})$, denoted by $\mathrm{eu}_b(\rho)$.

Of course, c also represents an element of the "classical" cohomology space $\mathrm{H}^2(\mathrm{SO}_0(2,n), \mathbb{Z})$. The associated 2-cocycle $\mathrm{eu}(\rho)$ represents the obstruction to lifting ρ to a representation $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,n)$. Indeed, $\mathrm{eu}(\rho) = 0$ means that there is a 1-cochain $a : \Gamma \to \mathbb{Z}$ such that for every γ_1, γ_2 in Γ we have

$$c(\rho(\gamma_1), \rho(\gamma_2)) = a(\gamma_1\gamma_2) - a(\gamma_1) - a(\gamma_2).$$

Then the map $\gamma \to \delta^{a(\gamma)}\sigma(\rho(\gamma))$ is a morphism, *i.e.,* a representation $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,n)$ that is a lift of ρ.

Now $\mathrm{eu}_b(\rho) = 0$ means precisely that $\mathrm{eu}(\rho) = 0$, but also that one can select the 1-cochain a so that it is *bounded.* The following proposition is a natural generalization of the fact that a group of orientation-preserving homeomorphisms of the circle has a vanishing bounded Euler class if and only if it has a global fixed point (see the end of Sect. 6.3 in [**23**]):

Proposition 7.3. *The bounded Euler class* $\mathrm{eu}_b(\rho)$ *vanishes if and only if* ρ *lifts to a representation* $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,n)$ *such that* $\tilde{\rho}(\Gamma)$ *preserves a closed* $(n-1)$*-dimensional achronal topological sphere in* $\widetilde{\mathrm{Ein}}_n$*.*

Proof. Recall that $\mathcal{W}_0$ is the domain $\{(\theta, \bar{x}) \in \mathbb{R} \times \mathbb{S}^{n-1} \ / \ -\pi \leq \theta < \pi\}$.

Invariant achronal sphere $\Rightarrow \mathrm{eu}_b(\rho) = 0$

Assume that ρ lifts to a representation $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,n)$ (*i.e.,* that $\mathrm{eu}(\rho) = 0$) and that $\tilde{\rho}(\Gamma)$ preserves a closed $(n-1)$-dimensional achronal topological sphere Λ in $\widetilde{\mathrm{Ein}}_n$, *i.e.,* the graph of a 1-Lipschitz map $f : \mathbb{S}^{n-1} \to \mathbb{R}$. Let $a : \Gamma \to \mathbb{Z}$ be the map associating to γ the unique integer k such that

$$\tilde{\rho}(\gamma) = \delta^k\sigma(\rho(\gamma)).$$

Then a is the 1-cochain whose coboundary represents the Euler class of ρ. The point is to prove that a is bounded.

The invariant achronal sphere Λ is contained in the closure of an affine domain of $\widetilde{\mathrm{Ein}}_n$ (cf. Lemma 2.5), *i.e.,* in a domain of the form $\{\theta_0 - \pi \leq \theta \leq \theta_0 + \pi\}$. More precisely, either it is contained in a domain

$\delta^q\mathcal{W}_0$ for some integer q, or it contains a point $(q\pi, \bar{x})$, in which case Λ is contained in the domain $\{(q-1)\pi \leq \theta < (q+1)\pi\}$. In both cases, there is an integer q such that Λ is contained in the union $\mathcal{Z}_q := \delta^{q-1}\mathcal{W}_0 \cup \delta^q\mathcal{W}_0$.

For every γ in Γ, the image of $x_0 = (0, \bar{x}_0)$ by $\sigma(\rho(\gamma))$ is a point $(\theta, \bar{y}_0)$ with $|\theta| \leq \pi$, and hence the intersection between $\mathcal{W}_0$ and $\sigma(\rho(\gamma))(\mathcal{W}_0)$ is non-trivial. Since δ commutes with $\sigma(\rho(\gamma))$, the intersection $\mathcal{W}_q \cap \sigma(\rho(\gamma))(\mathcal{W}_q)$ is non-empty. *A fortiori,* the same is true for the intersection $\mathcal{Z}_q \cap \sigma(\rho(\gamma))(\mathcal{Z}_q)$. However, since δ acts by adding 2π to the coordinate θ, the intersection $\mathcal{Z}_q \cap \delta^r\sigma(\rho(\gamma))(\mathcal{Z}_q)$ is empty as soon as r is an integer of absolute value > 2.

On the other hand, we know that $\mathcal{Z}_q \cap \tilde{\rho}(\gamma)\mathcal{Z}_q$ is non-empty since $\mathcal{Z}_q$ contains the invariant sphere Λ. It follows that the integer $a(\gamma)$ has absolute value at most 2.

$\mathrm{eu}_b(\rho) = 0 \Rightarrow$ *Invariant achronal sphere*

Assume now that $eu_b(\rho)$ vanishes, *i.e.,* that there is a bounded map $a : \Gamma \to \mathbb{Z}$ such that $\gamma \to \delta^{a(\gamma)}\sigma(\rho(\gamma))$ is a representation $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,n)$. Let α be an upper bound for $|a(\gamma)|$ $(\gamma \in \Gamma)$. Let $f_{id} : \mathbb{S}^n \to \mathbb{R}$ be the null map, and for every element γ of Γ, let $f_\gamma : \mathbb{S}^n \to \mathbb{R}$ be the 1-Lipschitz map whose graph is the image by $\tilde{\rho}(\gamma)$ of the graph of f_{id}. The graph of f_γ contains $\delta^{a(\gamma)}\sigma(\rho(\gamma))(0, \bar{x}_0)$, hence a point of θ-coordinate of absolute value bounded from above by $|a(\Gamma)| + \pi$. Since every f_γ is 1-Lipschitz and since the sphere has diameter π, there is a uniform upper bound for all the f_γ. For every $\bar{x}$ in $\mathbb{S}^n$ define

$$f_\infty(\bar{x}) := \mathrm{Sup}_{\gamma \in \Gamma} f_\gamma(\bar{x}).$$

Then f_∞ is a 1-Lipschitz map, whose graph is clearly $\rho(\Gamma)$-invariant.

q.e.d.

7.2. Proof of Theorem 1.5. Let $\rho : \Gamma \to \mathrm{SO}_0(2,n)$ be a faithful and discrete representation, where Γ is the fundamental group of a negatively curved closed manifold M. As we have already noticed, the equivalence between (1) and (2) follows from Theorem 1.4 and [**10**]. According to Proposition 7.3, the bounded Euler class $\mathrm{eu}_b(\rho)$ vanishes if and only if ρ lifts to a representation $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,n)$ such that $\tilde{\rho}(\Gamma)$ preserves a closed $(n-1)$-dimensional achronal topological sphere in $\widetilde{\mathrm{Ein}}_n$. Since by hypothesis the representation is assumed to be faithful and discrete, it means that item (3) is equivalent to the fact that the representation is GH-regular. Now since Γ is assumed to be the fundamental group of a negatively curved closed manifold, GH-regular representations of Γ are automatically GHC-regular. The equivalence between (2) and (3) follows. The theorem is proved.

7.3. The case $n = 2$. In this last section, we explain in which way one can deduce from Proposition 7.3 the following classical result:

Proposition 7.4. *Let $\rho_1, \rho_2 : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$ be two representations such that* $\mathrm{eu}_b(\rho_1) = \mathrm{eu}_b(\rho_2)$*. Then ρ_1 and ρ_2 are semi-conjugate, i.e., there is a monotone map $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ such that*

$$\forall \gamma \in \Gamma, \rho_1(\gamma) \circ f = f \circ \rho_2(\gamma).$$

Let us first recall the definition of the bounded Euler class for a representation $\rho : \Gamma \to \mathrm{PSL}(2,\mathbb{R})$: it is completely similar to the definition we have presented above.

Let $\mathrm{p} : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ be the universal covering. It acts naturally on the universal covering $\widetilde{\mathbb{RP}}^1$ of the projective line $\mathbb{RP}^1$, so that the kernel of p is the center of $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ and also the Galois group of $\widetilde{\mathbb{RP}}^1$. We fix a total order $<$ on $\widetilde{\mathbb{RP}}^1 \approx \mathbb{R}$ and a generator τ of $\ker \mathrm{p}$ so that $\tau(x) > x$ for every x in $\widetilde{\mathbb{RP}}^1$. Given an element x_0 of $\widetilde{\mathbb{RP}}^1$, there is still a canonical section $\sigma : \mathrm{PSL}(2,\mathbb{R}) \to \widetilde{\mathrm{SL}}(2,\mathbb{R})$, which is not a homomorphism, that associates to any element g of $\mathbb{RP}^1$ the unique element $\tilde{g}$ such that

$$x_0 \leq \tilde{g}x_0 < \tau(x_0).$$

Then the Euler class of the representation ρ is the bounded cohomology class represented by the cocycle c defined by

$$\sigma(\rho(\gamma_1\gamma_2)) = \tau^{c(\gamma_1,\gamma_2)}\sigma(\rho(\gamma_1))\sigma(\rho(\gamma_2)).$$

Proof of Proposition 7.4. Let ρ_1, ρ_2 be two representations of Γ into $\mathrm{PSL}(2,\mathbb{R})$ satisfying the hypothesis of Proposition 7.4: they have the same bounded cohomology class, meaning that, if c_1, c_2 are the two cocycles defined as above representing the bounded Euler classes of ρ_1, ρ_2, we have

$$c_2(\gamma_1,\gamma_2) = c_1(\gamma_1,\gamma_2) + a(\gamma_1\gamma_2) - a(\gamma_1) - a(\gamma_2), \tag{11}$$

where $a : \Gamma \to \mathbb{Z}$ is some bounded map.

It has the following consequence: Consider the map $\Gamma \times \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to \widetilde{\mathrm{SL}}(2,\mathbb{R})$ that associates to $(\gamma, \tilde{g})$ the element

$$\gamma * \tilde{g} := \tau^{-a(\gamma)}\sigma(\rho_2(\gamma))\tilde{g}\sigma(\rho_1(\gamma))^{-1}.$$

Then

$$\begin{aligned}
(\gamma_1\gamma_2) * \tilde{g} &= \tau^{-a(\gamma_1\gamma_2)}\sigma(\rho_2(\gamma_1\gamma_2))\tilde{g}[\sigma(\rho_1(\gamma_1\gamma_2))]^{-1} \\
&= \tau^{-a(\gamma_1\gamma_2)+c_2(\gamma_1,\gamma_2)}\rho_2(\gamma_1)\rho_2(\gamma_2)\tilde{g}[\tau^{c_1(\gamma_1,\gamma_2)} \\
&\quad \sigma(\rho_1(\gamma_1))\sigma(\rho_1(\gamma_2))]^{-1} \\
&= \tau^{-a(\gamma_1)-a(\gamma_2)}\tilde{g}[\sigma(\rho_1(\gamma_1))\sigma(\rho_1(\gamma_2))]^{-1} \text{(see (11))} \\
&= \gamma_1 * (\gamma_2 * \tilde{g}).
\end{aligned}$$

Now the key point is that $\widetilde{\mathrm{SL}}(2,\mathbb{R})$ is a model for the universal anti-de Sitter space $\widetilde{\mathrm{AdS}}_3$. Indeed, $-\det$ defines on the space $\mathrm{Mat}(2,\mathbb{R})$ of

2-by-2 matrices a quadratic form of signature $(2,2)$, which is preserved by the following action of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$:

$$\forall g_1, g_2 \in \mathrm{SL}(2,\mathbb{R}), \forall A \in \mathrm{Mat}(2,\mathbb{R}), (g_1,g_2).A := g_1 A g_2^{-1}.$$

The kernel of this action is the group I of order 2 generated by $(-\mathrm{Id}, -\mathrm{Id})$, where Id denotes the identity matrix. Hence there is a natural isomorphism between $\mathrm{SO}_0(2,2)$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/I$.

Therefore, the action $*$ we have defined is an isometric action of Γ on $\widetilde{\mathrm{AdS}}_3$, and hence induces a representation $\tilde{\rho} : \Gamma \to \widetilde{\mathrm{SO}}_0(2,2)$. Furthermore, the fact that the map a involved in the coboundary is bounded implies that this representation $\tilde{\rho}$ is the lifting of a representation into $\mathrm{SO}_0(2,2)$ whose bounded Euler class vanishes, *i.e.,* that the group $\tilde{\rho}(\Gamma)$ preserves a closed achronal circle in $\widetilde{\mathrm{Ein}}_2$.

We claim that the existence of such an invariant achronal circle is equivalent to the existence of a semi-conjugacy between ρ_1 and ρ_2 as stated in the conclusion of Proposition 7.4.

For the proof of this claim, it is convenient to consider the *projectivized* anti-de Sitter and Einstein spaces, *i.e.,* the quotients of AdS_3 and Ein_2 by $-\mathrm{Id}$. The projectivized anti-de Sitter space is then naturally identified with $\mathrm{PSL}(2,\mathbb{R})$. According to the identification between $(\mathrm{Mat}(2,\mathbb{R}), -\det)$ and $(\mathbb{R}^{2,2}, \mathrm{q}_{2,2})$, we obtain an identification between the projectivized Klein model $\overline{\mathrm{Ein}}_2$ and the space of non-zero non-invertible 2-by-2 matrices up to a non-zero factor. Such a class is characterized by the image and the kernel of its elements, *i.e.,* two lines in $\mathbb{R}^2$. In other words, $\overline{\mathrm{Ein}}_2$ is naturally isomorphic to the product $\mathbb{RP}^1 \times \mathbb{RP}^1$. The conformal action of $\mathrm{PO}(2,2) \approx \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R})$ on $\mathbb{RP}^1 \times \mathbb{RP}^1$ is the obvious one,

$$(g_1,g_2).(x,y) = (g_1 x, g_2 y)$$

since the image of $g_1 A g_2^{-1}$ is the image by g_1 of the image of A and its kernel is the image under g_2 of the kernel of A. The isotropic circles in $\overline{\mathrm{Ein}}_2 \approx \mathbb{RP}^1 \times \mathbb{RP}^1$ are the circles $\{*\} \times \mathbb{RP}^1$ and $\mathbb{RP}^1 \times \{*\}$. It follows quite easily that **acausal** circles in $\overline{\mathrm{Ein}}_2$ are graphs in $\mathbb{RP}^1 \times \mathbb{RP}^1$ of homeomorphisms from $\mathbb{RP}^1 \to \mathbb{RP}^1$. **Achronal** circles are allowed to follow during some time one segment in $\{*\} \times \mathbb{RP}^1$ or $\mathbb{RP}^1 \times \{*\}$. It follows that they are fillings (cf. Remark 3.19) of graphs of maps $f : \mathbb{RP}^1 \to \mathbb{RP}^1$ that are *monotone, i.e.,* of degree 1, preserving the cyclic order on $\mathbb{RP}^1$, but which can be constant on some intervals and which can be non-continuous at certain points. In other words, f lifts to a non-decreasing map $\tilde{f} : \widetilde{\mathbb{RP}}^1 \to \widetilde{\mathbb{RP}}^1$. For more details on this well-known geometric feature, we refer to [**30**] or [**8**].

In summary, we have proved that the representation $(\rho_1,\rho_2) : \Gamma \to \mathrm{PSL}(2,\mathbb{R}) \times \mathrm{PSL}(2,\mathbb{R}) \approx \mathrm{PO}(2,2)$ preserves a closed achronal circle Λ in $\overline{\mathrm{Ein}}_2 \approx \mathbb{RP}^1 \times \mathbb{RP}^1$, which is the filling of the graph of a monotone map $f : \mathbb{RP}^1 \to \mathbb{RP}^1$. The invariance of Λ means precisely that f is Γ-equivariant: Proposition 7.4 is proved. $\square$

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